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RESEARCH ARTICLE

The fundamental inequality for cocompact Fuchsian groups

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Abstract
We prove that the hitting measure is singular with respect to the Lebesgue measure for random walks driven by finitely supported measures on cocompact, hyperelliptic Fuchsian groups. Moreover, the Hausdorff dimension of the hitting measure is strictly less than one. Equivalently, the inequality between entropy and drift is strict. A similar statement is proven for Coxeter groups.

1. Introduction

Let $G < SL_2(\mathbb{R})$ be a countable group and μ be a finitely supported, generating probability measure on G. We consider the random walk

$$w_n := g_1 g_2 \dots g_n,$$

where each (g_i) is independent and identically distributed with distribution μ. Let us fix a base point $o \in \mathbb{H}^2$. Then the *hitting measure* ν of the random walk on $S^1 = \partial\mathbb{D}$ is

$$\nu(A) := \mathbb{P}\Big(\lim_{n\to\infty} w_n o \in A\Big)$$

for any Borel set $A \subseteq \partial\mathbb{D}$. The hitting measure is also the unique μ-*harmonic*, or μ-*stationary*, measure, as it satisfies the convolution equation $\nu = \mu \star \nu$. On the other hand, the boundary circle $\partial\mathbb{D} = S^1$ also carries the *Lebesgue* measure, which is the unique rotationally invariant measure on S^1.

In the 1970s, Furstenberg [19] proved that for any discrete subgroup of $SL_2(\mathbb{R})$, there exists a measure μ such that the hitting measure of the corresponding random walk is absolutely continuous with respect to the Lebesgue measure. This was the first step to produce boundary maps, eventually leading to rigidity results. However, such measures μ are inherently infinitely supported, as they arise from discretisation of Brownian motion (see also [38]). Another construction of absolutely continuous hitting measures, still infinitely supported, on general hyperbolic groups is given by [12].

For finitely supported measures, though, the situation is quite different. For any finitely supported measure μ on $SL_2(\mathbb{Z})$, it is known since Guivarc'h-LeJan [27] that the hitting measure is singular. Kaimanovich-LePrince [31] produced on any countable Zariski dense subgroup of $SL_d(\mathbb{R})$ examples of finitely supported measures with singular hitting measure.

They also formulated the following *singularity conjecture*.

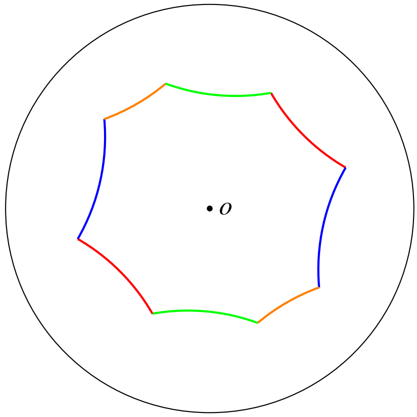

Figure 1. *A symmetric hyperbolic octagon. Sides of the same colour are identified by the Fuchsian group.*

Conjecture 1.1 ([31], page 259). *For any finitely supported measure μ on $SL_d(\mathbb{R})$ whose support generates a discrete subgroup, the hitting measure for the random walk driven by μ is singular with respect to the Lebesgue measure.*

This conjecture has been mentioned several times; see also [25, Remark 1.1], [30, page 817] and [5, Question (vi)]. In this paper, we focus on the case $d = 2$. Let $G < SL_2(\mathbb{R})$ be the subgroup generated by the support of μ. Recall that a discrete subgroup of $SL_2(\mathbb{R})$ is called a *Fuchsian group* and is *cocompact* if the quotient $\Sigma = \mathbb{D}/G$ is compact.

If G is discrete but not cocompact (which includes the case $G = SL_2(\mathbb{Z})$), the conjecture is known; in fact, there are many approaches to this result and several generalisations in many contexts with different proofs ([27], [7], [16], [31], [21], [22], [17], [42]), all of which exploit in various ways the fact that the cusp subgroup is highly distorted in G.

Note that if one drops the hypothesis that G be discrete, then Conjecture 1.1 no longer holds: there exist finitely supported measures on $SL_2(\mathbb{R})$ for which the hitting measure is absolutely continuous ([8], [4]), but the group generated by their support is not discrete (see also [31, Footnote 1]).

Thus the only case still open is when G is a cocompact Fuchsian group. In this case, the hyperbolic metric and the word metric on G are quasi-isometric to each other, and hence distortion arguments do not work. So far, the only known examples are the recent ones from [32] and [11], where the singularity of hitting measure is proven for cocompact Fuchsian groups whose fundamental domain is a *regular* polygon (except for a finite number of cases with few sides). These examples form a countable family.

In this paper, we prove Conjecture 1.1 for any hyperelliptic, cocompact Fuchsian group for measures supported on the canonical generating set.

Recall that a *hyperelliptic surface* is a Riemann surface Σ with a holomorphic involution $j : \Sigma \to \Sigma$. Any hyperelliptic surface can be uniformised as the quotient $\Sigma = \mathbb{D}/G$, where G is a Fuchsian group with fundamental domain a centrally symmetric hyperbolic polygon P, and generators of G are given by hyperbolic translations joining opposite sides of P (see Figure 1 and e.g., [23], [13]). We call such G a *hyperelliptic Fuchsian group* and such a generating set the *canonical generating set* of G. In order for G to be discrete, P needs to satisfy the *cycle condition* from Poincaré's theorem (see Definition 4.1). The space of hyperelliptic Fuchsian groups of genus g is a complex variety of dimension $2g - 1$. Our main result is the following.

Theorem 1.2. *Let P be a centrally symmetric hyperbolic polygon in the Poincaré disk $\mathbb{D}$, with $2m$ sides, satisfying the cycle condition, and let $S := \{t_1, t_2, \dots, t_{2m}\}$ be the hyperbolic translations that identify opposite sides of P. Then for any measure μ supported on the set S, the hitting measure ν on $S^1 = \partial\mathbb{D}$ is singular with respect to Lebesgue measure. Moreover, the Hausdorff dimension of ν is strictly less than* 1.

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If m is even, the above construction yields the standard presentation of a hyperelliptic Fuchsian group of genus $g = \frac{m}{2}$; if m is odd, we also obtain a discrete cocompact group of genus $g = \frac{m-1}{2}$.

To compare with [11] and [32], the authors of [11] use percolation to obtain a formula for the drift of the random walk, and then they obtain an asymptotic lower bound for the drift as the number of sides tends to ∞. Kosenko [32] obtains, in the regular case, explicit lower bounds for the translation lengths using hyperbolic geometry without resorting to approximation by percolation. When the fundamental polygon is not regular, an explicit bound on all translation lengths is not possible, as some translation lengths may be short, decreasing the drift. In particular, there is the risk that assigning a large probability to an element with a short translation length may result in the dimension of the measure going to 1. In this paper, we show that this phenomenon cannot happen, as the discreteness of the group forces at least some generators to have a large translation length. This subtle geometric balance is given by the inequality from Theorem 1.5.

Finally, if one replaces the random walk with a Brownian motion, then absolute continuity of harmonic measure only holds if the underlying manifold is highly homogeneous: to be precise, on a negatively curved surface, the hitting measure is absolutely continuous if and only if the curvature is constant ([36], [37]).

The fundamental inequality

This problem is closely related to the following 'numerical characteristics' of random walks. Recall that the *entropy* [2] of μ is defined as

$$h := \lim_{n\to\infty} \frac{-\sum_{g\in G} \mu^n(g) \log \mu^n(g)}{n}$$

and the *drift*, or *rate of escape*, is

$$\ell := \lim_{n\to\infty} \frac{d_{\mathbb{H}}(o, w_n o)}{n},$$

where $d_{\mathbb{H}}$ denotes the hyperbolic metric and the limit exists almost surely. The drift also equals the classical Lyapunov exponent for random matrix products [20]. Finally, the *volume growth* of G is

$$v := \limsup_{n\to\infty} \frac{1}{n} \log \#\{g \in G \ : \ d_{\mathbb{H}}(o, go) \le n\}.$$

The inequality

$$h \le \ell v \tag{1.1}$$

has been established by Guivarc'h [26] and is called the *fundamental inequality* by Vershik [44]. Several authors (e.g., [44, Question A]) have asked:

Question 1.3. Under which conditions is inequality (1.1) an equality?

For discrete, cocompact actions, Question 1.3 is equivalent to Conjecture 1.1: indeed, by [7] (see also [25] and Theorem 2.2), inequality (1.1) is strict if and only if the hitting measure is singular with respect to the Lebesgue measure.

If one replaces the hyperbolic metric $d_{\mathbb{H}}$ with a *word metric* d_w on G, then [25] prove that the inequality is strict unless the group G is virtually free. Observe that cocompact Fuchsian groups are not virtually free; however, the drift for $d_{\mathbb{H}}$ and the drift for d_w are not the same (in fact, one has $\ell_{d_{\mathbb{H}}} < \ell_{d_w}$), and hence the result from [25] does not settle Question 1.3 or Conjecture 1.1. Note that for a cocompact Fuchsian group, it is well-known that $v = 1$ (see, e.g., [41]).

Our result also has consequences on the Hausdorff dimension of the hitting measure. Recall that the Hausdorff dimension of a measure ν on a metric space is the infimum of the Hausdorff dimensions of

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subsets of full measure. Moreover, by [35], [43], [30], for cocompact Fuchsian groups, the Hausdorff dimension $\dim_H(\nu)$ of the hitting measure satisfies, for almost every $x \in S^1$,

$$\dim_H(\nu) = \lim_{r\to 0^+} \frac{\log \nu(B(x,r))}{\log r} = \frac{h}{\ell},$$

where $B(x,r)$ is a ball of centre x and radius r. Thus Theorem 1.2 implies:

Corollary 1.4. *Under the hypotheses of Theorem 1.2, the inequality $h < \ell$ is strict. Hence, the hitting measure ν has a Hausdorff dimension strictly less than one.*

A geometric inequality

The approach of this paper is based on the fact that cocompactness forces at least some of the generators to have long enough translation lengths (this is related to the *collar lemma*: two intersecting closed geodesics cannot be both short at the same time; also, the quotient Riemann surface has a definite positive area). Indeed, in Theorem 3.1, we prove a criterion for singularity in terms of the translation lengths of the generators, and then we show the following purely geometric inequality.

Theorem 1.5. *Let P be a centrally symmetric polygon with $2m$ sides, satisfying the cycle condition, and let $S := \{g_1, \dots, g_{2m}\}$ be the set of hyperbolic translations identifying opposite sides of P. Then we have*

$$\sum_{g\in S} \frac{1}{1+e^{\ell(g)}} < 1, \tag{1.2}$$

where $\ell(g)$ denotes the translation length of g in the hyperbolic metric.

Interestingly, our geometric inequality has exactly the same form as the main inequalities of [14], [1] for free Kleinian groups. However, it is not a consequence of theirs; see Section 4.

Coxeter groups

We also prove the following version of Theorem 1.2 for reflection groups.

Theorem 1.6. *Let P be a centrally symmetric, hyperbolic polygon with $2m$ sides and interior angles $\frac{\pi}{k_i}$, with $k_i \in \mathbb{N}^+$ for $1 \le i \le 2m$. Let μ be a probability measure supported on the set $R := \{r_1, \dots, r_{2m}\}$ of hyperbolic reflections on the sides of P, with $\mu(r_i) = \mu(r_{i+m})$ for all $1 \le i \le m$. Then the hitting measure for the random walk driven by μ is singular with respect to the Lebesgue measure. Moreover, the inequality $h < \ell$ is strict, and the hitting measure ν has a Hausdorff dimension strictly less than one.*

2. Preliminary results

Let μ be a probability measure on a countable group G. We assume that μ is *generating*: that is, the semigroup generated by the support of μ equals G. We define the *step space* as $(G^{\mathbb{N}}, \mu^{\mathbb{N}})$ and the map $\pi : G^{\mathbb{N}} \to G^{\mathbb{N}}$ as $\pi((g_n)_{n\in\mathbb{N}}) := (w_n)_{n\in\mathbb{N}}$, with for any n

$$w_n := g_1 g_2 \dots g_n.$$

The target space of π is denoted by Ω and called the *path space*; as a set, it equals $G^{\mathbb{N}}$ and is equipped with the measure $\mathbb{P}_\mu := \pi_\star(\mu^{\mathbb{N}})$.

Then we define the *first-passage function* $F_\mu(x, y)$ as

$$F_\mu(x,y) := \mathbb{P}_\mu(\exists n \,:\, w_n x = y)$$

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for any $x, y \in G$, and the *Green metric* d_μ on G, introduced in [6], as

$$d_\mu(x, y) := -\log F_\mu(x, y).$$

The following fact is well-known.

Lemma 2.1. *Let $p : G \to H$ be a group homomorphism, let μ be a probability measure on G, and let $\overline{\mu} := p_\star\mu$. Then for any $x, y \in G$,*

$$d_{\overline{\mu}}(p(x), p(y)) \leq d_\mu(x, y).$$

Proof. Since p induces a map from paths in G to paths in H, we have $\overline{\mu}^n(p(g)) \geq \mu^n(g)$ for any $g \in G$, any $n \geq 0$. Hence

$$\mathbb{P}_{\overline{\mu}}(p(x), p(y)) \geq \mathbb{P}_\mu(x, y)$$

for any $x, y \in G$, from which the claim follows. □

We shall use the following criterion, which relates the absolute continuity of the hitting measure to the fundamental inequality. Recall that a group action is *geometric* if it is isometric, properly discontinuous and cocompact.

Theorem 2.2 ([7, Corollary 1.4, Theorem 1.5], [43], [24]). *Let Γ be a non-elementary hyperbolic group acting geometrically on $\mathbb{H}^2$, endowed with the geometric distance $d = d_{\mathbb{H}}$ induced from the action. Consider a generating probability measure μ on Γ with finite support. Then the following conditions are equivalent:*

1. *The equality $h = \ell v$ holds.*
2. *The Hausdorff dimension of the hitting measure ν on S^1 is equal to* 1.
3. *The measure ν is equivalent to the Lebesgue measure on S^1.*
4. *For any $o \in \mathbb{H}^2$, there exists a constant $C > 0$ such that for any $g \in \Gamma$, we have*

$$|d_\mu(1, g) - d_{\mathbb{H}}(o, go)| \leq C.$$

For each $g \in G$, let $\ell(g)$ denote its translation length, namely

$$\ell(g) := \lim_{n\to\infty} \frac{d_{\mathbb{H}}(o, g^n o)}{n}.$$

Equivalently, $\ell(g)$ is the length of the corresponding closed geodesic on the quotient surface. The mechanism to utilise Theorem 2.2 is through the following lemma, similar to the one from [32].

Lemma 2.3. *Suppose that the hitting measure is absolutely continuous. Then for any $g \in G$, we have*

$$\ell(g) \leq d_\mu(1, g).$$

Proof. If not, then $\ell(g) > d_\mu(1, g) \geq 0$, and hence g is loxodromic. Let us pick some $o \in \mathbb{H}^2$, which lies on the axis of g, so that $d_{\mathbb{H}}(o, g^k o) = \ell(g^k) = k\ell(g)$ for any k. Moreover, by the triangle inequality for the Green metric, one has $d_\mu(1, g^k) \leq k d_\mu(1, g)$, and hence

$$d_{\mathbb{H}}(o, g^k o) - d_\mu(1, g^k) \geq k\ell(g) - k d_\mu(1, g) = k(\ell(g) - d_\mu(1, g));$$

thus, since $\ell(g) - d_\mu(1, g) > 0$,

$$\sup_{k\in\mathbb{N}} \left| d_{\mathbb{H}}(o, g^k o) - d_\mu(1, g^k) \right| = +\infty,$$

which contradicts Theorem 2.2. □

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Let F be a free group, freely generated by a finite set S. Recall that the (hyperbolic) *boundary* ∂F of F is the set of infinite, reduced words in the alphabet $S \cup S^{-1}$. Given a finite, reduced word g, we denote as $C(g) \subseteq \partial F$ the *cylinder* determined by g, namely the set of infinite, reduced words that start with g.

Lemma 2.4. *Consider a random walk on the free group*

$$F_m = \langle s_1^{\pm 1}, \dots, s_m^{\pm 1} \rangle,$$

defined by a probability measure μ on the generators. If we denote $x_i := F_\mu(1, s_i)$, $\check{x}_i := F_\mu(1, s_i^{-1})$ and the hitting measure on the boundary of F_m by ν, then

$$\nu(C(s_i)) = \frac{x_i(1-\check{x}_i)}{1 - x_i \check{x}_i}.$$

A similar lemma is stated in [34, Exercise 5.14].

Proof. For any infinite word $w = s_{j_1} s_{j_2} s_{j_3} \dots$, there exist two possibilities:

1. There exists a subword $s_{j_1} \dots s_{j_k}$ such that it equals s_i in F_m.
2. No subword $s_{j_1} \dots s_{j_k}$ equals s_i, so it belongs to the set of paths that never hit s_i.

In the first case, we denote this subword by w_1, and we consider $w_1^{-1} w$; we apply the same procedure, but replacing s_i with s_i^{-1} at each subsequent step. This procedure yields the equality

$$\begin{aligned} \nu(C(s_i)) &= \mathbb{P}(1 \to s_i \nrightarrow 1) + \mathbb{P}(1 \to s_i \to 1 \to s_i \nrightarrow 1) + \dots = \\ &= \sum_{n=0}^{\infty} F_\mu(1, s_i)^{n+1} F_\mu(1, s_i^{-1})^n (1 - F_\mu(1, s_i^{-1})) \\ &= F_\mu(1, s_i)(1 - F_\mu(1, s_i^{-1})) \sum_{n=0}^{\infty} \Big(F_\mu(1, s_i) F_\mu(1, s_i^{-1}) \Big)^n = \frac{x_i(1-\check{x}_i)}{1 - x_i \check{x}_i}. \end{aligned}$$

□

3. A criterion for singularity

Theorem 3.1. *Let μ be a finitely supported measure on a cocompact Fuchsian group, and let S be the support of μ. Suppose that*

$$\sum_{g \in S \cup S^{-1}} \frac{1}{1 + e^{\ell(g)}} < 1. \tag{3.1}$$

Then the hitting measure ν on $\partial \mathbb{D}$ is singular with respect to the Lebesgue measure.

Proof. Denote as $(g_i^{\pm})_{i=1}^m$ the elements of $S \cup S^{-1}$, let F be a free group of rank m with generators $(h_i)_{i=1}^m$, and let $\widetilde{\mu}$ be a measure on F with $\widetilde{\mu}(h_i^{\pm}) = \mu(g_i^{\pm})$. Moreover, let us denote

$$\begin{aligned} x_i &:= F_{\widetilde{\mu}}(1, h_i) = \mathbb{P}_{\widetilde{\mu}}(\exists n \ : \ w_n = h_i) \\ \check{x}_i &:= F_{\widetilde{\mu}}(1, h_i^{-1}). \end{aligned}$$

Then we have

$$\sum_{i=1}^m \left(\frac{x_i(1-\check{x}_i)}{1 - x_i \check{x}_i} + \frac{\check{x}_i(1 - x_i)}{1 - x_i \check{x}_i} \right) = 1. \tag{3.2}$$

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Indeed, if $\widetilde{\nu}$ is the hitting measure on ∂F, by Lemma 2.4, the measure of the cylinder $C(h_i)$ starting with h_i is

$$\widetilde{\nu}(C(h_i)) = \frac{x_i(1-\check{x}_i)}{1-x_i\check{x}_i}, \qquad \widetilde{\nu}(C(h_i^{-1})) = \frac{\check{x}_i(1-x_i)}{1-x_i\check{x}_i},$$

from which, since the cylinders are disjoint and cover the boundary, equation (3.2) follows.

Then by equation (3.1), there exists an index i such that

$$\frac{2}{1+e^{\ell(g_i)}} < \frac{x_i(1-\check{x}_i)}{1-x_i\check{x}_i} + \frac{\check{x}_i(1-x_i)}{1-x_i\check{x}_i},$$

which is equivalent to

$$e^{\ell(g_i)} > \frac{2-x_i-\check{x}_i}{x_i+\check{x}_i-2x_i\check{x}_i}.$$

Finally, an algebraic computation yields

$$\frac{2-x_i-\check{x}_i}{x_i+\check{x}_i-2x_i\check{x}_i} \geq \min\left\{\frac{1}{x_i}, \frac{1}{\check{x}_i}\right\},$$

and thus we obtain

$$\ell(g_i) > \inf\{-\log x_i, -\log \check{x}_i\}. \tag{3.3}$$

If the hitting measure ν on $S^1 = \partial\mathbb{D}$ is absolutely continuous, then by Lemma 2.3 and Lemma 2.1, we get

$$\ell(g_i) \leq d_\mu(1, g_i) \leq d_{\widetilde{\mu}}(1, h_i) = -\log x_i$$

for any i. If we apply the same inequality to g_i^{-1}, we also have

$$\ell(g_i) = \ell(g_i^{-1}) \leq d_\mu(1, g_i^{-1}) \leq d_{\widetilde{\mu}}(1, h_i^{-1}) = -\log \check{x}_i,$$

and hence

$$\ell(g_i) \leq \inf\{-\log x_i, -\log \check{x}_i\},$$

which contradicts equation (3.3), showing that ν is singular with respect to Lebesgue measure. □

4. Parametrisation of the space of polygons

Let P be a convex, compact polygon in the hyperbolic disk $\mathbb{D}$, with $2m$ sides and interior angles $\{\gamma_1, \ldots, \gamma_{2m}\}$.

We say that P is *centrally symmetric* if there exists a point $o \in \mathbb{D}$ so that P is invariant under symmetry with respect to the point O. This clearly implies that opposite sides have equal length and opposite angles are equal.

Poincaré's theorem provides conditions to ensure that the group generated by side pairings is discrete (see [39]). In particular, one needs a condition on the angles, which in our setting can be formulated as follows.

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Definition 4.1. A centrally symmetric polygon P satisfies the *cycle condition* if there exists an integer $k \geq 1$ such that

$$\sum_{i=1}^{m} \gamma_{2i} = \sum_{i=1}^{m} \gamma_{2i-1} = \frac{2\pi}{k}.$$

Let $S := \{g_1, \dots, g_{2m}\}$ be the set of hyperbolic translations identifying opposite sides of P, with $g_{i+m} = g_i^{-1}$. By Poincaré's theorem [39], if the polygon P satisfies the cycle condition, then the group G generated by S is discrete.[1]

More precisely, denote as g_i the hyperbolic translation mapping the ith side of P to its $(i+m)$th side. If m is even, there is only one equivalence class of vertices, and it is fixed by the transformation

$$b := g_m g_{m-1}^{-1} \dots g_2 g_1^{-1} g_m^{-1} g_{m-1} \dots g_2^{-1} g_1,$$

which is called the *cycle transformation* in the language of [39]. If m is odd, then there are two equivalence classes of vertices, fixed, respectively, by the transformations

$$b_1 := g_m g_{m-1}^{-1} \dots g_2^{-1} g_1 \qquad \text{and} \qquad b_2 := g_m^{-1} g_{m-1} \dots g_2 g_1^{-1}.$$

Thus the presentations defining the group G are

$$\begin{array}{ll} \langle g_1, \dots, g_m \ : \ b^k = 1 \rangle & \text{if } m \text{ is even} \\ \langle g_1, \dots, g_m \ : \ (b_1)^k = (b_2)^k = 1 \rangle & \text{if } m \text{ is odd.} \end{array}$$

The following is our main geometric inequality.

Theorem 4.2. *Let P be a centrally symmetric, hyperbolic polygon satisfying the cycle condition, with $2m$ sides, and let $S := \{g_1, \dots, g_{2m}\}$ be the set of hyperbolic translations identifying opposite sides of P. Then we have*

$$\sum_{g \in S} \frac{1}{1 + e^{\ell(g)}} < 1. \tag{4.1}$$

Remarks. The inequality (4.1) has the same form as the main inequality in [1] and [14] for free Kleinian groups; more recently, a stronger version for free Fuchsian groups has been obtained in [28], while generalisations in variable curvature (and any dimension) are due to [29], [3].

Equation (4.1) is also reminiscent of McShane's identity [40], where one obtains the equality by taking the infinite sum over all group elements of a punctured torus group. Our inequality, however, does not follow from any of them; in fact, it is in a way stronger than these, as a cocompact surface group can be deformed to a finite covolume group and then to a Schottky (hence free) group by increasing the translation lengths of the generators.

It is interesting to point out that the above inequalities have an interpretation in terms of hitting measures of stochastic processes (see, e.g., [33]). Here, we go along the opposite route: we prove the geometric inequality (4.1), and then we use it to conclude properties about the hitting measure.

Finally, there are generating sets of G for which equation (4.1) fails. Indeed, the mechanism behind the inequality is that since all curves corresponding to $(g_i)_{i=1}^m$ intersect each other, by the collar lemma, at most one of them can be short. In general, on a surface of genus g one can choose a configuration of $3g-3$ short curves and construct a Dirichlet domain for which the corresponding side pairing does not satisfy equation (4.1).

[1]Note that in the usual formulation of Poincaré's theorem, there are two cases: if m is even, all vertices of P are identified by G; if m is odd, there are two elliptic cycles corresponding to alternate vertices of P. If m is even and $k = 1$, the polygon P does not satisfy the classical version of Poincaré's theorem; but if P is symmetric, the group generated is still discrete, so all our arguments still apply.

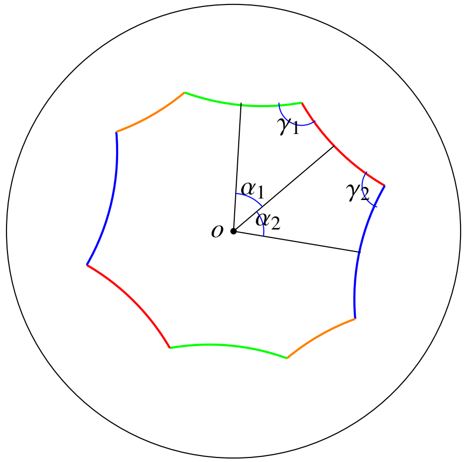

Figure 2. *Angles at the centre and vertices of a symmetric hyperbolic octagon.*

The proof of this inequality will take up most of the paper until Section 6. To begin with, let us note that a way to parametrise the space of all symmetric hyperbolic polygons is to write, by [10, Example 2.2.7],

$$\cos(\gamma_i) = -\cosh(a_i)\cosh(a_{i+1})\cos(\alpha_i) + \sinh(a_i)\sinh(a_{i+1}) \tag{4.2}$$

with $i = 1, \ldots, m$, where (a_i) are the distances between the base point and the ith side, (α_i) are the angles at the origin and (γ_i) are the angles at the vertices (see Figure 2). Since $\ell(g_i) \geq 2a_i$, it is enough to show

$$\sum_{i=1}^{m} \frac{1}{1+e^{2a_i}} < \frac{1}{2}$$

under the constraints $\sum_{i=1}^{m} \alpha_i = \pi$ and $\sum_{i=1}^{m} \gamma_i = \frac{2\pi}{k}$.

The fundamental geometric idea in our approach to Theorem 4.2 is that two intersecting curves cannot both be short, as a consequence of the *collar lemma* [9]. For instance, we get:

Lemma 4.3. *Suppose that there exists a_i such that* $\sinh(a_i) \leq \frac{2(m-1)}{m(m-2)}$. *Then the hitting measure is singular.*

Proof. From the collar lemma [9], we have

$$\sinh(a_i)\sinh(a_j) \geq 1$$

for all $i \neq j$. Recall that

$$\frac{2}{1+e^{2a}} = 1 - \tanh(a),$$

and hence, if we set $s := \sinh(a_1)$, we obtain for $i \neq 1$ that $\sinh(a_i) \geq \frac{1}{s}$; thus

$$\tanh(a_i) = \frac{\sinh(a_i)}{\sqrt{1+\sinh(a_i)^2}} = \frac{1}{\sqrt{1+\frac{1}{\sinh(a_i)^2}}} \geq \frac{1}{\sqrt{1+s^2}},$$

and hence

$$\sum_{i=1}^{m} \tanh(a_i) \geq \frac{s}{\sqrt{1+s^2}} + \frac{m-1}{\sqrt{1+s^2}} > m-1$$

if and only if $s < \frac{2(m-1)}{m(m-2)}$. □

To actually prove Theorem 4.2, however, we need an improvement on the previous estimate. Let us rewrite equation (4.2) above as

$$\cos(\alpha_i) = \tanh(a_i)\tanh(a_{i+1}) - \frac{\cos(\gamma_i)}{\cosh(a_i)\cosh(a_{i+1})};$$

and recalling that

$$\tanh^2(x) + \frac{1}{\cosh^2(x)} = 1,$$

we obtain, by setting $z_i = \tanh(a_i)$,

$$\cos(\alpha_i) = z_i z_{i+1} - \cos(\gamma_i)\sqrt{1-z_i^2}\sqrt{1-z_{i+1}^2} \tag{4.3}$$

with $0 \leq z_i \leq 1$. Finally, we want to show

$$\sum_{i=1}^{m} \frac{1}{1+e^{2a_i}} = \sum_{i=1}^{m} \frac{1-z_i}{2} \overset{?}{<} \frac{1}{2},$$

which is equivalent to

$$\sum_{i=1}^{m} z_i \overset{?}{>} m-1. \tag{4.4}$$

Now, let us first assume that $\gamma_i \leq \pi/2$ for all $1 \leq i \leq m$. Then equation (4.3) yields

$$\cos(\alpha_i) \leq z_i z_{i+1},$$

and hence the constraint becomes

$$\sum_{i=1}^{m} \arccos(z_i z_{i+1}) \leq \pi. \tag{4.5}$$

Note that $z_1 \to 0$ implies $\cos\alpha_1 \leq z_1 z_2 \to 0$ and thus $\alpha_1 \to \frac{\pi}{2}$, and $\cos\alpha_m \leq z_m z_1 \to 0$ and thus $\alpha_m \to \frac{\pi}{2}$, and hence also $\alpha_2, \alpha_3, \ldots, \alpha_{m-1} \to 0$, which implies $z_2, z_3, \ldots, z_m \to 1$.

5. An optimisation problem

By the above discussion, if we set $x_i = 1 - z_i$, we reduce the proof of Theorem 4.2 (at least in the case all angles of P are acute) to the following optimisation problem (see also Figure 3).

Theorem 5.1. *Let $m \geq 3$ and $0 \leq x_i \leq 1$ with $\sum_{i=1}^{m} x_i = 1$. Then*

$$\sum_{i=1}^{m} \arccos((1-x_i)(1-x_{i+1})) \geq \pi.$$

Moreover, equality holds if and only if there exists an index i such that $x_i = 1$ and $x_j = 0$ for all $j \neq i$.

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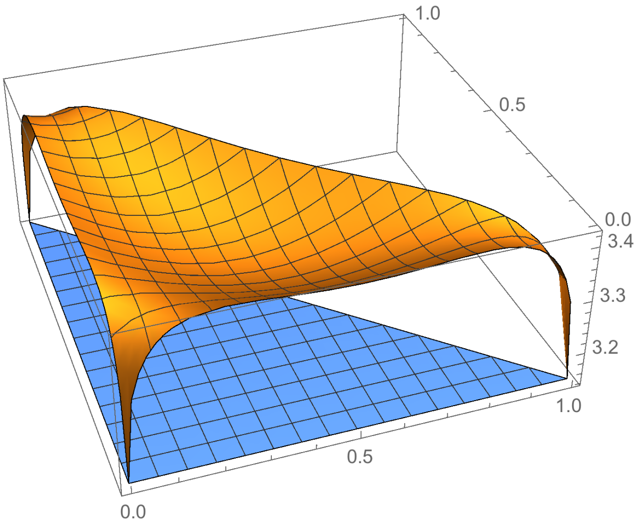

Figure 3. *The graph of* $f(x) := \sum_{i=1}^{3} \arccos((1-x_i)(1-x_{i+1}))$ *subject to the constraint* $\sum_{i=1}^{3} x_i = 1$*, compared with the constant function at height* π*. The lack of convexity (or concavity) of f makes the proof of Theorem 5.1 trickier.*

In the statement of Theorem 5.1 and elsewhere from now on, all indices i are meant modulo m. The next is the main technical lemma.

Lemma 5.2. *Let* $m \geq 3$ *and* $0 \leq x_i \leq 1$ *with* $\sum_{i=1}^{m} x_i = 1$. *Then*

$$\sum_{i=1}^{m} \sqrt{x_i + x_{i+1} - x_i x_{i+1}} \geq \sqrt{4 + 3\sum_{i=1}^{m} x_i x_{i+1}}.$$

Proof. Set $\Delta_i := x_i + x_{i+1} - x_i x_{i+1}$. Note that

$$\Delta_i \geq \max\{x_i, x_{i+1}\},$$

and hence

$$\sqrt{\Delta_i}\sqrt{\Delta_{i+1}} \geq x_{i+1}. \tag{5.1}$$

Moreover, since $m \geq 2$, we have $x_{i+1} + x_{i+2} \leq \sum_{i=1}^{m} x_i = 1$, and hence if we multiply by $(x_{i+1} + x_{i+2})$, we obtain

$$\begin{aligned} \Delta_i &= x_i + x_{i+1} - x_i x_{i+1} \\ &\geq (x_i + x_{i+1})(x_{i+1} + x_{i+2}) - x_i x_{i+1} \\ &\geq x_{i+1}^2 + x_{i+1} x_{i+2}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} \Delta_{i+2} &= x_{i+2} + x_{i+3} - x_{i+2} x_{i+3} \\ &\geq (x_{i+2} + x_{i+3})(x_{i+1} + x_{i+2}) - x_{i+2} x_{i+3} \\ &\geq x_{i+2}^2 + x_{i+1} x_{i+2}. \end{aligned}$$

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Thus Cauchy-Schwarz yields

$$\sqrt{\Delta_i}\sqrt{\Delta_{i+2}} \geq \sqrt{x_{i+1}^2 + x_{i+1}x_{i+2}}\sqrt{x_{i+2}^2 + x_{i+1}x_{i+2}} \geq 2x_{i+1}x_{i+2}. \tag{5.2}$$

By squaring both sides, our desired inequality is equivalent to

$$\sum_{i=1}^{m} \Delta_i + 2 \sum_{1 \leq i < j \leq m} \sqrt{\Delta_i}\sqrt{\Delta_j} \geq 4 + 3 \sum_{i=1}^{m} x_i x_{i+1};$$

thus, using $\sum_{i=1}^m \Delta_i = 2 - \sum_{i=1}^m x_i x_{i+1}$, it is enough to prove

$$\sum_{1 \leq i < j \leq m} \sqrt{\Delta_i}\sqrt{\Delta_j} \geq 1 + 2 \sum_{i=1}^{m} x_i x_{i+1}. \tag{5.3}$$

Now note that

$$\sum_{1 \leq i < j \leq m} \sqrt{\Delta_i}\sqrt{\Delta_j} = \sum_{i=1}^{m} \sqrt{\Delta_i}\sqrt{\Delta_{i+1}} + M$$

with

$$M = 0 \qquad \text{if } m = 3 \tag{5.4}$$

$$M = \sum_{i=1}^{2} \sqrt{\Delta_i}\sqrt{\Delta_{i+2}} \qquad \text{if } m = 4 \tag{5.5}$$

$$M \geq \sum_{i=1}^{m} \sqrt{\Delta_i}\sqrt{\Delta_{i+2}} \qquad \text{if } m \geq 5. \tag{5.6}$$

Thus for $m \geq 5$, we have, using equations (5.6), (5.1) and (5.2),

$$\begin{aligned} \sum_{1 \leq i < j \leq m} \sqrt{\Delta_i}\sqrt{\Delta_j} &\geq \sum_{i=1}^{m} \sqrt{\Delta_i}\sqrt{\Delta_{i+1}} + \sum_{i=1}^{m} \sqrt{\Delta_i}\sqrt{\Delta_{i+2}} \\ &\geq \sum_{i=1}^{m} x_{i+1} + 2\sum_{i=1}^{m} x_{i+1}x_{i+2} \\ &\geq 1 + 2\sum_{i=1}^{m} x_{i+1}x_{i+2}, \end{aligned}$$

which yields equation (5.3) and hence completes our proof. The cases $m = 3$ and $m = 4$ need to be dealt with separately. If $m = 3$, we obtain, by multiplying by $\sum_{i=1}^3 x_i = 1$,

$$\Delta_i = x_i^2 + x_{i+1}^2 + \sum_{i=1}^{3} x_i x_{i+1},$$

so by Cauchy-Schwarz, we get

$$\sqrt{\Delta_i}\sqrt{\Delta_{i+1}} \geq x_{i+1}^2 + x_i x_{i+2} + \sum_{i=1}^{3} x_i x_{i+1},$$

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and hence

$$\sum_{i=1}^{3}\sqrt{\Delta_i}\sqrt{\Delta_{i+1}} \ge \sum_{i=1}^{3}x_i^2 + 4\sum_{i=1}^{3}x_ix_{i+1}$$
$$= \left(\sum_{i=1}^{3}x_i\right)^2 + 2\sum_{i=1}^{3}x_ix_{i+1}$$
$$= 1 + 2\sum_{i=1}^{3}x_ix_{i+1},$$

which yields equation (5.3), as desired. Finally, if $m = 4$, then we note

$$\sum_{1\le i<j\le 4}\sqrt{\Delta_i}\sqrt{\Delta_j} = \sum_{i=1}^{4}\sqrt{\Delta_i}\sqrt{\Delta_{i+1}} + \sum_{i=1}^{2}\sqrt{\Delta_i}\sqrt{\Delta_{i+2}}$$

and, again by Cauchy-Schwarz,

$$\sqrt{\Delta_1}\sqrt{\Delta_3} \ge \sqrt{x_1^2 + x_2^2 + x_1x_4 + x_2x_3}\sqrt{x_3^2 + x_4^2 + x_1x_4 + x_2x_3} \ge 2x_1x_4 + 2x_2x_3$$

and similarly

$$\sqrt{\Delta_2}\sqrt{\Delta_4} \ge 2x_1x_2 + 2x_3x_4;$$

thus, using equation (5.1),

$$\sum_{1\le i<j\le 4}\sqrt{\Delta_i}\sqrt{\Delta_j} \ge \sum_{i=1}^{4}x_i + 2\sum_{i=1}^{4}x_ix_{i+1} = 1 + 2\sum_{i=1}^{4}x_ix_{i+1},$$

which is again equation (5.3). This completes the proof. □

Lemma 5.3. *For* $0 \le x \le 1$*, we have these inequalities:*

1.
$$\frac{2}{\pi}\arccos(1-x) \ge \frac{2}{3}\sqrt{x} + \frac{1}{3}x$$
with equality if and only if $x = 0$ *or* $x = 1$*;*

2.
$$\frac{2}{3}\sqrt{4+3x} + \frac{2-x}{3} \ge 2$$
with equality if and only if $x = 0$.

Proof. For the first inequality, let $f(x) := \frac{2}{\pi}\arccos(1-x^2) - \frac{2}{3}x - \frac{1}{3}x^2$. One checks that $f(0) = f(1) = 0$ and $f(\frac{1}{\sqrt{2}}) = \frac{1}{2} - \frac{\sqrt{2}}{3} > 0$; moreover, $f'(x)$ has a unique zero in $[0, 1]$. Hence, $f(x) \ge 0$ for all $0 \le x \le 1$, which implies (1).

To prove (2), let $g(x) := \frac{2}{3}\sqrt{4+3x} + \frac{2-x}{3}$. Then one checks $g(0) = 2$ and $g'(x) = \frac{1}{\sqrt{4+3x}} - \frac{1}{3} > 0$ for $0 \le x \le 1$, which implies $g(x) \ge 2$ for all $0 \le x \le 1$. □

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Proof of Theorem 5.1. By setting $f(x) := \frac{2}{\pi}\arccos(1-x)$, our claim is equivalent to

$$\sum_{i=1}^{m} f(x_i + x_{i+1} - x_i x_{i+1}) \geq 2$$

under the constraint $\sum_{i=1}^m x_i = 1$, with $m \geq 3$ and $0 \leq x_i \leq 1$.

Let us set $\Delta_i := x_i + x_{i+1} - x_i x_{i+1}$ and $\sigma := \sum_{i=1}^m x_i x_{i+1}$. Observe that $2\sigma \leq (\sum_{i=1}^m x_i)^2 = 1$. Then we have, by Lemma 5.3,

$$\sum_{i=1}^{m} f(\Delta_i) \geq \frac{2}{3}\sum_{i=1}^{m}\sqrt{\Delta_i} + \frac{1}{3}\sum_{i=1}^{m}\Delta_i$$

and using Lemma 5.2 and the fact $\sum_{i=1}^m \Delta_i = 2 - \sigma$, we obtain

$$\geq \frac{2}{3}\sqrt{4+3\sigma} + \frac{1}{3}(2-\sigma) \geq 2,$$

where in the last step, we apply Lemma 5.3 (2). This completes the proof of the inequality. By Lemma 5.3 (1), equality implies that $\Delta_i = 0, 1$ for every i, which in turn implies that $x_i = 0, 1$ for all i. Since $\sum_{i=1}^m x_i = 1$, this can only happen if $x_i = 1$ for exactly one index i. □

6. The obtuse angle case

The proof in the previous section works as long as all angles γ_i are less than or equal to $\pi/2$. If one of them is obtuse, we have a geometric argument to reduce ourselves to that case.

6.1. *Neutralising pairs*

We call a *neutralising pair* for P a pair $\{\gamma_i, \gamma_{i+1}\}$ of adjacent interior angles of P with $\gamma_i + \gamma_{i+1} \leq \pi$. Whenever we have a neutralising pair, we can apply the following lemma (see Figure 4).

Lemma 6.1. *Let $ABCDE$ be a hyperbolic pentagon with right angles $\widehat{B}$ and $\widehat{E}$, and suppose that $\widehat{C} < \pi/2$ and $\widehat{C} + \widehat{D} \leq \pi$. Then there exist points F on the line $\overline{BC}$ and G on the line $\overline{ED}$ such that the hyperbolic pentagon $ABFGE$ satisfies $\widehat{F} \leq \pi/2$ and $\widehat{G} \leq \pi/2$.*

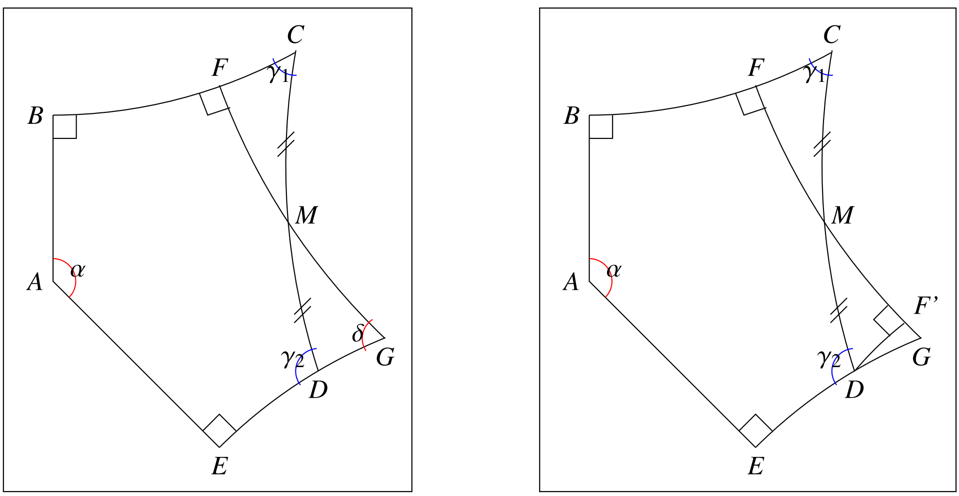

Figure 4. *The hyperbolic pentagon of Lemma 6.1.*

Proof. Let M be the midpoint of $\overline{CD}$, and let F be the foot of the orthogonal projection of M to $\overline{BC}$. If the intersection, which we denote as G, of the lines $\overline{FM}$ and $\overline{ED}$ exists, then we claim that the angle $\delta = D\widehat{G}F$ satisfies $\delta \le \pi/2$.

To see this, let F' be the symmetric point to F with respect to M. Then CFM and DMF' are equal triangles. Hence $E\widehat{D}F' = E\widehat{D}M + M\widehat{D}F' = E\widehat{D}C + B\widehat{C}D \le \pi$, and hence F' lies on the segment $\overline{MG}$. Moreover, $D\widehat{F'}M = C\widehat{F}M = \pi/2$, and hence $\delta = D\widehat{G}M \le \pi/2$.

If the lines $\overline{FM}$ and $\overline{ED}$ do not intersect, we take as G the foot of the perpendicular from M to the line $\overline{ED}$, and we define as F'' the intersection of $\overline{GM}$ with the line $\overline{BC}$. Then the hyperbolic pentagon $ABF'GE$ satisfies $\widehat{F''} < \pi/2$ and $\widehat{G} = \pi/2$, so the claim follows by replacing F by F''. □

We say that P has *disjoint neutralising pairs* if there is a set $\Pi := \{\{\gamma_{i_1}, \gamma_{i_1+1}\}, \dots, \{\gamma_{i_k}, \gamma_{i_k+1}\}\}$ of neutralising pairs such that each obtuse angle of P belongs to some pair in Π, and the pairs are disjoint: that is, $\{\gamma_{i_r}, \gamma_{i_r+1}\} \cap \{\gamma_{i_s}, \gamma_{i_s+1}\} = \emptyset$ for any $r \ne s$.

Let us use the notation

$$\varphi(x_1, x_2, \dots, x_m) := \sum_{i=1}^{m} \frac{1}{1 + e^{2x_i}}.$$

Proposition 6.2. *Let P be a centrally symmetric hyperbolic polygon with $2m$ sides and centre o, and let $\ell_1, \dots, \ell_m$ be the distances between o and the midpoints of the sides. If P has disjoint neutralising pairs, there exists a centrally symmetric hyperbolic $2m$-gon P' with no obtuse angles and such that*

$$\varphi(\ell_1, \ell_2, \dots, \ell_m) \le \varphi(d'_1, d'_2, \dots, d'_m),$$

where d'_i is the distance between o and the ith side of P'.

Proof. Let us denote as d_i the distance between o and the ith side of P. Note that by definition, $d_i \le \ell_i$ for all i.

If the polygon P only has acute angles, we take $P = P'$ and note that by definition, $d'_i = d_i \le \ell_i$, which yields the claim.

Suppose now that the hyperbolic polygon P has one obtuse angle, say γ_1, which belongs to a neutralising pair, and let ℓ_1 correspond to the side adjacent to the obtuse angle and the other angle, say γ_2, in the neutralising pair. Consistently with this choice, let us denote as $s_1, s_2, \dots, s_{2m}$ the sides of P.

Let us now consider the hyperbolic pentagon delimited by s_{2m}, s_1, s_2 and the orthogonal projections from o to s_2 and s_{2m}. Let us call this pentagon $ABCDE$, where $o = A$, the side s_1 is denoted $\overline{DC}$, the orthogonal projection from o to s_2 is B and the orthogonal projection from o to s_{2m} is E.

Using Lemma 6.1, let us replace P by a new polygon P' obtained by substituting the pentagon $ABCDE$ by the pentagon $ABFGE$, which satisfies $\widehat{F} = \pi/2$ and $\widehat{G} \le \pi/2$. If we denote by d'_1 the distance between $o = A$ and $\overline{FG}$, then we have

$$d'_1 = d(A, \overline{FG}) \le \ell_1.$$

On the other hand, note that for $i = 2, \dots, m$, the distance between o and the ith side is the same for P and P'. That is, $d_i = d'_i$ for $i = 2, \dots, m$. Hence,

$$\varphi(\ell_1, \ell_2, \dots, \ell_m) \le \varphi(\ell_1, d_2, \dots, d_m) \le \varphi(d'_1, d_2, \dots, d_m) = \varphi(d'_1, d'_2, \dots, d'_m).$$

If there is more than one neutralising pair (by symmetry, the number of neutralising pairs is even), we can analogously replace each side adjacent to the pair by rotating it around its midpoint. This proves the claim. □

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6.2. The general case

Let $(p_i)_{i=1}^{2m}$ denote the vertices of P and $(q_i)_{i=1}^{2m}$ denote the midpoints of the sides, indexed so that q_i lies between p_{i-1} and p_i. Let o denote the centre of symmetry of P. Let $\alpha_i = q_i \widehat{o} q_{i+1}$ be the angles at the origin and $\gamma_i = q_i \widehat{p}_i q_{i+1}$ the angles at the vertices of P. By the cycle condition and symmetry, we have

$$\sum_{i=1}^{m} \alpha_i = \pi, \qquad \sum_{i=1}^{m} \gamma_i = \frac{2\pi}{k},$$

where $k \geq 1$ is an integer. Note that if $k \geq 2$, at most one of the γ_i is obtuse, and hence P has disjoint neutralising pairs. However, if $k = 1$, P need not have disjoint neutralising pairs; in particular, it may have three consecutive obtuse angles. To deal with this case, we need the notion of a *dual polygon*.

6.3. Dual polygons

Given a centrally symmetric polygon P with centre o and sum of angles 4π, we construct its *dual polygon* $\widehat{P}$ as follows.

Let Q_i be the quadrilateral delimited by o, q_i, p_i, q_{i+1}. As in Figure 5, we can cut and rearrange the Q_is with $1 \leq i \leq m$ as follows. Glue all vertices $(p_i)_{i=1}^{m}$ to a single point, which we now denote as v, so that the Q_is with $1 \leq i \leq m$ lie in counterclockwise order around v. Since $\sum_{i=1}^{m} \gamma_i = 2\pi$, the copies of Q_i fit together, creating a new polygon $\widehat{P}$ with m sides. By construction, the sides of $\widehat{P}$ have lengths $2\ell_1, \ldots, 2\ell_m$, with $\ell_i = d(o, q_i)$. Also by construction, the angles of $\widehat{P}$ are $\alpha_1, \ldots, \alpha_m$, and hence their sum is $\sum_{i=1}^{m} \alpha_i = \pi$.

We define the pair $(\widehat{P}, v)$ to be the dual polygon to (P, o).

The duality relation

$$(P, o) \leftrightarrow (\widehat{P}, v)$$

defines a bijective correspondence between centrally symmetric $2m$-gons with sum of angles 4π and m-gons with the sum of angles π together with a choice of a point inside them.

To see that this is a bijection, let us construct the inverse map as follows: given an m-gon $\widehat{P}$ and a point v inside it, denote as $v_1, \ldots, v_m$ its vertices. Decompose $\widehat{P}$ as the union of m quadrilaterals $R_1, \ldots, R_m$ by drawing the segments joining v and the midpoints of the sides of $\widehat{P}$; then take copies of the polygons $R_1, \ldots, R_m, R_1, \ldots, R_m$ and glue them in this order by identifying all v_is to a point, which we call o. This will create a polygon P whose sum of angles is twice the sum of internal angles of $\widehat{P}$, and hence 4π. Moreover, by construction, this polygon is centrally symmetric about the point o.

Given a polygon P with $2m$ sides and a point o inside P, we define

$$\Sigma(P) := \sum_{i=1}^{m} \frac{1}{1 + e^{2\ell_i}},$$

where ℓ_i are the segments connecting o and the midpoint of the ith side. Let us also define

$$\widehat{\Sigma}(P) := \sum_{i=1}^{m} \frac{1}{1 + e^{s_i}},$$

where s_i are the lengths of the sides of P. Then note that we have

$$\Sigma(P) = \widehat{\Sigma}(\widehat{P}).$$

In particular, $\Sigma(P)$ does *not* depend on v but only on $\widehat{P}$.

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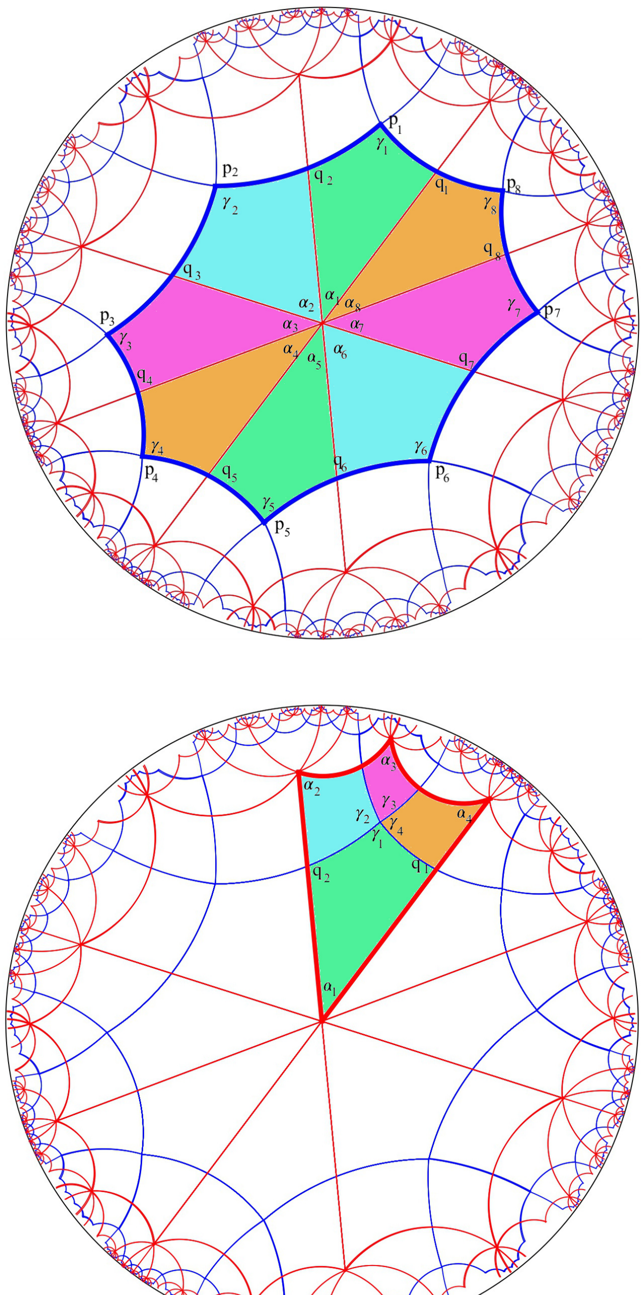

Figure 5. *The construction of the dual polygon. Top: the polygon P, in blue. Bottom: the dual polygon $\widehat{P}$, in red. The angles α_i at the origin in P become the angles at the vertices of $\widehat{P}$; on the other hand, the angles γ_i at the vertices of P become the angles at the point $\widehat{v}$ in the interior of $\widehat{P}$. Quadrilaterals of the same colour are congruent. The point $\widehat{v}$ is the common intersection of the four coloured regions in the bottom picture.*

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Lemma 6.3. *Let P be a centrally symmetric hyperbolic polygon with $2m$ sides and the total sum of its interior angles 4π. Then there exists a centrally symmetric hyperbolic polygon P' with the same number of sides so that $\Sigma(P) = \Sigma(P')$ so that P' has at most four obtuse angles, which belong to disjoint neutralising pairs.*

Proof. Let $\widehat{P}$ be the dual polygon to P, as defined above. We claim that we can pick another point v' inside $\widehat{P}$ so that at most two of the angles at v' are obtuse.

To do this, let $(r_i)_{i=1}^m$ be the midpoints of the sides of $\widehat{P}$; join r_1 and r_3 by a segment, and pick any point v' in the interior of that segment. Then consider the angles $\gamma_i' := r_i\widehat{v'}r_{i+1}$ with $1 \le i \le m$. Since $\sum_{i=1}^2 \gamma_i' = \pi$ and $\sum_{i=3}^m \gamma_i' = \pi$, at most two of the angles γ_i' with $1 \le i \le m$ can be obtuse.

Then we define P' to be the dual of $(\widehat{P}, v')$. Since P and P' have the same dual, we have $\Sigma(P) = \Sigma(P')$. By our previous choice of v', in P', there are at most 4 obtuse angles γ_i', and there are no three consecutive obtuse angles; hence, for all of them there exists another adjacent angle $\gamma_{i\pm1}'$ so that $\gamma_i' + \gamma_{i\pm1}' < \pi$. Hence, P' has neutralising pairs. □

By putting together these reductions, we can complete the proof of Theorem 4.2. Let us see the details.

Proof of Theorem 4.2. Let us first suppose that $\gamma_i \le \pi/2$ for all i. We know by equation (4.5) that $\sum_{i=1}^m \arccos(z_i z_{i+1}) \le \pi$ with $0 < z_i < 1$. Then we need to show that $\sum_{i=1}^m z_i > m-1$. Suppose not; then there exists z_i with $\sum_{i=1}^m z_i \le m-1$. Then there exists $(z_i')_{i=1}^m$ with $0 \le z_i \le z_i' \le 1$ for all i, so that $\sum z_i' = m-1$. Then we have, by Theorem 5.1, $\pi \le \sum_{i=1}^m \arccos(z_i' z_{i+1}') \le \sum_{i=1}^m \arccos(z_i z_{i+1}) \le \pi$, and hence $\sum_{i=1}^m \arccos(z_i' z_{i+1}') = \pi$, which by the second part of Theorem 5.1 implies $z_i' = 0$ for some i, and hence also $z_i = 0$, which is a contradiction.

In the general case, we first apply Lemma 6.3 to reduce to the case where P has disjoint neutralising pairs. Then by applying Proposition 6.2, we reduce to the case of P having no obtuse angles, which we can deal with as above. This completes the proof. □

Proof of Theorem 1.2. Theorem 4.2 shows that the criterion of Theorem 3.1 holds, proving the singularity of hitting measure. □

7. Coxeter groups

Let P be a centrally symmetric convex polygon with $2m$ sides in $\mathbb{H}^2$, with each angle γ_i at the vertices being equal to $\frac{\pi}{k_i}$ for some natural $k_i > 1$, for $1 \le i \le 2m$. Then due to [15, Theorem 6.4.3], the group of isometries generated by hyperbolic reflections $R := \{r_1, \dots, r_{2m}\}$ with respect to the sides of P acts geometrically on $\mathbb{H}^2$. Therefore, it is a hyperbolic group, so Theorem 2.2 can be applied to it. Such groups are referred to as *hyperbolic Coxeter groups*.

Below, we will show that Theorem 1.2 can be quickly generalised to hyperbolic Coxeter groups.

Lemma 7.1. *Let $m > 1$. Consider a random walk on the free product of $2m$ copies of $\mathbb{Z}/2\mathbb{Z}$*

$$F_{2m}' = \langle s_1, \dots, s_{2m} \mid s_i^2 = 1 \rangle,$$

defined by a probability measure μ on the generators. If we denote $x_i := F_\mu(1, s_i)$ for $1 \le i \le 2m$ and the hitting measure on the boundary of F_{2m}' by ν, then

$$\nu(C(s_i)) = \frac{x_i}{1 + x_i}.$$

Proof. The proof of this lemma can be obtained similarly to the proof of Lemma 2.4 for F_m because the Cayley graphs for F_m and F_{2m}' are isometric.

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More precisely, a sample path converges to the boundary of the cylinder $C(s_i)$ if and only if it crosses the edge s_i an odd number of times. This leads to the following computation:

$$\begin{aligned}\nu(C(s_i)) &= \mathbb{P}(1 \to s_i \nrightarrow 1) + \mathbb{P}(1 \to s_i \to 1 \to s_i \nrightarrow 1) + \cdots = \\ &= \sum_{n=0}^{\infty} F_\mu(1, s_i)^{2n+1}(1 - F_\mu(1, s_i)) \\ &= \sum_{k=1}^{\infty} (-1)^{k+1} x_i^k = \frac{x_i}{1 + x_i}. \end{aligned}$$ □

A measure μ on the set $R = \{r_1, \dots, r_{2m}\}$ of reflections through the sides of P is called *geometrically symmetric* if $\mu(r_i) = \mu(r_{i+m})$ for each $1 \le i \le m$.

Theorem 7.2. *Let μ denote a geometrically symmetric measure supported on the generators $R = \{r_1, \dots, r_{2m}\}$ of a hyperbolic Coxeter group. Suppose that*

$$\sum_{i=1}^{m} \frac{1}{1 + e^{\ell(r_i r_{i+m})/2}} < \frac{1}{2}. \tag{7.1}$$

Then the hitting measure ν in $\partial\mathbb{D}$ is singular with respect to the Lebesgue measure.

Proof. The proof of this theorem is quite similar to the proof of Theorem 3.1. We consider a measure $\tilde{\mu}$ on a free product $\langle h_1, \dots, h_{2m} \mid h_i^2 = 1\rangle$ of $2m$ copies of $\mathbb{Z}/2\mathbb{Z}$ uniquely defined by $\tilde{\mu}(h_i) = \mu(r_i)$.

If ν were to be absolutely continuous, then a similar argument would yield that

$$\begin{aligned}\ell(r_i r_{i+m}) &\le d_\mu(1, r_i r_{i+m}) \le d_\mu(1, r_i) + d_\mu(1, r_{i+m}) \\ &\le d_{\tilde{\mu}}(1, h_i) + d_{\tilde{\mu}}(1, h_{i+m}) = 2 d_{\tilde{\mu}}(1, h_i) = -2 \log x_i.\end{aligned}$$

Keep in mind that $d_{\tilde{\mu}}(1, h_i) = d_{\tilde{\mu}}(1, h_{i+m})$ due to $\tilde{\mu}$ being geometrically symmetric as well. Therefore,

$$\frac{x_i}{1 + x_i} \le \frac{1}{1 + e^{\ell(r_i r_{i+m})/2}},$$

and due to Lemma 7.1, we obtain

$$1 = \sum_{i=1}^{2m} \frac{x_i}{1 + x_i} \le 2 \sum_{i=1}^{m} \frac{1}{1 + e^{\ell(r_i r_{i+m})/2}} < 1,$$

which delivers a contradiction. □

Theorem 7.3. *The hitting measure of a nearest-neighbour random walk generated by a geometrically symmetric measure on a Coxeter group associated with a centrally symmetric polygon is singular with respect to Lebesgue measure on $\partial\mathbb{D}$.*

Proof. Let us recall that $(g_i)_{i=1}^m$ denotes the translations identifying the opposite sides of P. It is easily seen that $\ell(r_i r_{i+m}) = 2\ell(g_i) = 2\ell(g_{i+m})$ for every $1 \le i \le m$. However, we can apply Theorem 4.2, because there are no obtuse angles, to get

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$$\sum_{i=1}^{m} \frac{2}{1+e^{\ell(r_i r_{i+m})/2}} = \sum_{g\in S} \frac{1}{1+e^{\ell(g)}} < 1.$$

We conclude the proof by applying Theorem 7.2. □

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