Graphs and Combinatorics (2022) 38:168
https://doi.org/10.1007/s00373-022-02578-9

ORIGINAL PAPER

Erdős–Gyárfás conjecture for P_8-free graphs

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Received: 26 May 2022 / Revised: 17 September 2022 / Accepted: 5 October 2022

Abstract
A graph is P_8-free if it contains no induced subgraph isomorphic to the path P_8 on eight vertices. In 1995, Erdős and Gyárfás conjectured that every graph of minimum degree at least three contains a cycle whose length is a power of two. In this paper, we confirm the conjecture for P_8-free graphs by showing that there exists a cycle of length four or eight in every P_8-free graph with minimum degree at least three.

Keywords Erdős–Gyárfás conjecture · P_8-free graph · Cycle

1 Introduction

All graphs considered in this paper are undirected and simple. Let G be a graph. The vertex set, the edge set, the maximum degree and the minimum degree of G are denoted by $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$, respectively. For a vertex $v \in V(G)$, the set of neighbors of v in G is denoted by $N_G(v)$ or $N(v)$ if G is understood. Let $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of G induced by S and $G - S$ to denote the subgraph $G[V(G) \setminus S]$. For any two vertices $u, v \in V(G)$, we write $u \sim v$ if $uv \in E(G)$ and $u \nsim v$ otherwise. A *uv-path* is a path having endvertices as u and v.

A path on k vertices is denoted by P_k. A cycle on k vertices is denoted by C_k and is called a *k-cycle*. A 3-cycle is also called a *triangle*. The *length* of a path or cycle is the number of edges it contains. The *girth* of a graph G, denoted by $g(G)$, is the length of a shortest cycle in G. The well-known Erdős–Gyárfás Conjecture [3] states that every graph of minimum degree at least three contains a 2^m-cycle for some integer $m \geq 2$. The conjecture is confirmed for some graph classes including $K_{1,m}$-free graphs of minimum degree at least $m+1$ or maximum degree at least $2m-1$ [8],

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Published online: 10 October 2022

3-connected cubic planar graphs [6], planar claw-free graphs [2] and some Cayley graphs [4, 5]. In [7], it is proved that every cubic claw-free graph contains a cycle whose length is 2^k, or $3 \cdot 2^k$, for some positive integer k.

Given a graph H, a graph G is *H-free* if G does not contain any induced subgraph isomorphic to H. In this paper, we confirm Erdős–Gyárfás Conjecture for P_8-free graphs by showing the following two theorems.

Theorem 1.1 *Every P_5-free graph with minimum degree at least three contains a C_4.*

Theorem 1.2 *Every P_8-free graph with minimum degree at least three contains a C_4 or C_8.*

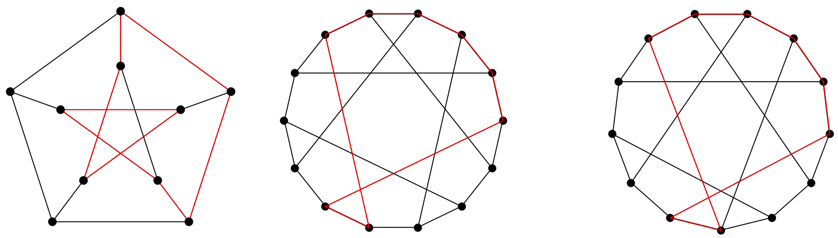
Theorem 1.1 is best possible in terms of the order of the forbidden path, as the Petersen graph (see Fig. 1a) is P_6-free and C_4-free. In confirming the Erdős–Gyárfás Conjecture for P_8-free graphs, Theorem 1.2 alone suffices. But we include Theorem 1.1 as it is stronger than the restriction of Theorem 1.2 on P_5-free graphs. The remainder of this paper is organized as follows. In Sect. 2, we prove Theorem 1.1. In Sect. 3, we prove Theorem 1.2.

2 Proof of Theorem 1.1

We will prove Theorem 1.1 by applying the following lemma which was shown in [7].

Lemma 2.1 [7] *Let G be a graph with $\delta(G) \geq 3$. If G does not contain C_4, then G has an induced cycle C_k for some $k \geq 5$.*

Proof of Theorem 1.1 Let G be a P_5-free graph with $\delta(G) \geq 3$. Suppose that G does not contain C_4. By Lemma 2.1, G contains an induced cycle $C = v_1 v_2 \dots v_k v_1$ for some $k \geq 5$. Since G is P_5-free, $k = 5$. As $\delta(G) \geq 3$ and G contains no C_4, v_1 has a neighbor $v_6 \notin \{v_1, v_2, \dots, v_5\}$ and $v_6 \nsim v_i$ for $i \in \{3, 4\}$. If $v_6 \nsim v_2$, then $v_6 v_1 v_2 v_3 v_4$ is



(a) Petersen graph (b) Heawood graph (c) Graph obtained by contracting one edge from the Heawood graph

Fig. 1 Illustration of Lemma 3.1

an induced P_5, a contradiction. Assume that $v_6 \sim v_2$, then $v_6 \nsim v_5$ otherwise there is a C_4. It follows that $v_6v_1v_5v_4v_3$ is an induced P_5, a contradiction. □

3 Proof of Theorem 1.2

We will need the lemma below in proving Theorem 1.2.

Lemma 3.1 [1] *Let G be a connected P_8-free graph with $g(G) \geq 5$ and $\delta(G) \geq 3$. Then one of the following holds*:

(i) *G is the Petersen graph (Fig. 1a)*;
(ii) *G is the Heawood graph (Fig. 1b)*;
(iii) *G is the graph obtained by contracting one edge in the Heawood graph (Fig. 1c)*.

Proof of Theorem 1.2 Let G be a P_8-free graph with $\delta(G) \geq 3$. We may assume that G is connected. Otherwise, we just consider a component of G. Suppose to the contrary that G contains neither C_4 nor C_8. Since each of the Petersen graph, the Heawood graph, and the graph obtained by contracting one edge in the Heawood graph contains a C_8 (see the red cycles in Fig. 1), it follows that $g(G) = 3$ by Lemma 3.1. Let $C = v_1v_2v_3v_1$ be a triangle in G. Then v_i has a neighbor v_{i+3} for $i \in \{1, 2, 3\}$ by $\delta(G) \geq 3$. Furthermore, v_4, v_5 and v_6 are all distinct and pairwise nonadjacent as G contains no C_4.

We say that two cycles are *adjacent* if they share a common edge. We claim that triangles in G are not adjacent to any 5-cycles or 6-cycles. □

Claim 3.2 *No C_3 and C_5 are adjacent.*

Proof We still use C to denote an arbitrary C_3 of G. It is sufficient to prove that v_4 and v_5 have no common neighbor. Suppose not, let $v_7 \notin \{v_1, v_2, \dots, v_6\}$ be a common neighbor of v_4 and v_5. As $\delta(G) \geq 3$ and G contains no C_4, v_i has a neighbor $v_{i+4} \notin \{v_1, \dots, v_7\}$ for $i \in \{4, 5\}$ and $v_8 \neq v_9$. If $v_8 \sim v_7$, $v_9 \sim v_7$, then $v_1v_3v_2v_5v_9v_7v_8v_4v_1$ is a C_8. So at least one of v_8 and v_9 is not adjacent to v_7. By symmetry, we assume that $v_8 \nsim v_7$. As G contains no C_4, $v_6 \nsim v_i$ for $i \in \{1, 2, 4, 5\}$. Furthermore, $v_6 \nsim v_8$, as otherwise $v_6v_3v_1v_2v_5v_7v_4v_8v_6$ is a C_8; and $v_6 \nsim v_9$, as otherwise $v_6v_3v_2v_1v_4v_7v_5v_9v_6$ is a C_8. We consider two cases regarding whether $v_6 \sim v_7$.

Case 1 $v_6 \nsim v_7$.

As $\delta(G) \geq 3$, v_6 has two neighbors $v_{10}, v_{11} \notin \{v_1, v_2, \dots, v_9\}$. For each $i \in \{10, 11\}$, we have the following nonadjacencies:

$v_i \nsim v_2$: otherwise $v_i v_6 v_3 v_2 v_i$ is a C_4;
$v_i \nsim v_4$: otherwise $v_i v_6 v_3 v_1 v_2 v_5 v_7 v_4 v_i$ is a C_8;
$v_i \nsim v_5$: otherwise $v_i v_6 v_3 v_2 v_1 v_4 v_7 v_5 v_i$ is a C_8;
$v_i \nsim v_8$: otherwise $v_i v_6 v_3 v_2 v_5 v_7 v_4 v_8 v_i$ is a C_8.

If one of v_{10} and v_{11}, say v_{11}, is adjacent to v_3, then $v_{10} \nsim v_3$, as otherwise $v_{10} v_6 v_{11} v_3 v_{10}$ is a C_4; and $v_{10} \nsim v_7$, as otherwise $v_{10} v_6 v_{11} v_3 v_2 v_1 v_4 v_7 v_{10}$ is a C_8. As G has no C_4, it then follows that $v_{10} v_6 v_3 v_2 v_5 v_7 v_4 v_8$ is an induced P_8, a contradiction. Now assume that $v_i \nsim v_3$ for each $i \in \{10, 11\}$. As G contains no C_4, at most one of v_{10} and v_{11} is adjacent to v_7. Assume by asymmetry that v_{10} is nonadjacent to v_7. Then $v_{10} v_6 v_3 v_2 v_5 v_7 v_4 v_8$ is an induced P_8, a contradiction. (See Fig. 2a).

Case 2 $v_6 \sim v_7$.

As $\delta(G) \geq 3$, v_6 has a neighbor $v_{10} \notin \{v_1, v_2, \ldots, v_9\}$. Note that $v_{10} \nsim v_i$ for $i \in \{1, 2, 4, 5\}$ as G contains no C_4. Furthermore, $v_{10} \nsim v_8$, as otherwise $v_{10} v_6 v_3 v_2 v_5 v_7 v_4 v_8 v_{10}$ is a C_8; and $v_{10} \nsim v_9$, as otherwise $v_{10} v_6 v_3 v_1 v_4 v_7 v_5 v_9 v_{10}$ is a C_8.

Subcase 2.1 $v_{10} \sim v_3$.

In this case, $v_{10} \nsim v_7$, as otherwise $v_{10} v_3 v_6 v_7 v_{10}$ is a C_4. So v_{10} has a neighbor $v_{11} \notin \{v_1, v_2, \ldots, v_{10}\}$. Then

$v_{11} \nsim v_3$: otherwise $v_{11} v_{10} v_6 v_3 v_{11}$ is a C_4;
$v_{11} \nsim v_2$: otherwise $v_{11} v_{10} v_3 v_2 v_{11}$ is a C_4;
$v_{11} \nsim v_5$: otherwise $v_{11} v_{10} v_3 v_2 v_1 v_4 v_7 v_5 v_{11}$ is a C_8;
$v_{11} \nsim v_7$: otherwise $v_{11} v_{10} v_6 v_7 v_{11}$ is a C_4;
$v_{11} \nsim v_4$: otherwise $v_{11} v_{10} v_6 v_3 v_2 v_5 v_7 v_4 v_{11}$ is a C_8;
$v_{11} \nsim v_8$: otherwise $v_{11} v_{10} v_3 v_2 v_5 v_7 v_4 v_8 v_{11}$ is a C_8.

However $v_{11} v_{10} v_3 v_2 v_5 v_7 v_4 v_8$ is an induced P_8, a contradiction. (See Fig. 2b).

Subcase 2.2 $v_{10} \nsim v_3$.

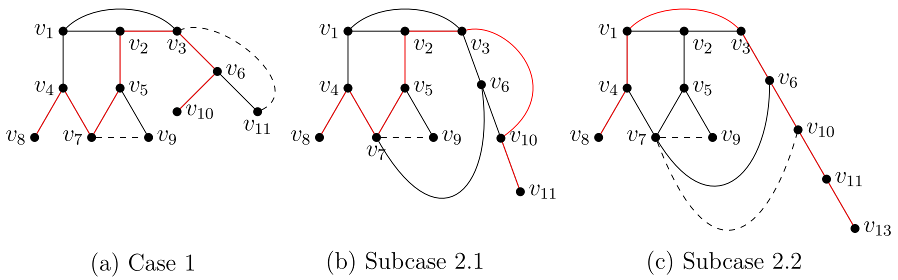


Fig. 2 Illustration of Claim 3.2

We claim that v_{10} has a neighbor v_{11} such that $v_{11} \nsim v_6$. If $v_{10} \sim v_7$, then $v_{11} \nsim v_6$, as otherwise $v_{11}v_{10}v_7v_6v_{11}$ is a C_4. If $v_{10} \nsim v_7$, then v_{10} has two neighbors $v_{11}, v_{12} \notin \{v_1, v_2, \ldots, v_{10}\}$. At least one of v_{11} and v_{12}, say v_{11}, is nonadjacent to v_6. Then

$v_{11} \nsim v_1$: otherwise $v_{11}v_{10}v_6v_7v_5v_2v_3v_1v_{11}$ is a C_8;
$v_{11} \nsim v_2$: otherwise $v_{11}v_{10}v_6v_7v_4v_1v_3v_2v_{11}$ is a C_8;
$v_{11} \nsim v_3$: otherwise $v_{11}v_{10}v_6v_3v_{11}$ is a C_4;
$v_{11} \nsim v_4$: otherwise $v_{11}v_{10}v_6v_3v_2v_5v_7v_4v_{11}$ is a C_8;
$v_{11} \nsim v_5$: otherwise $v_{11}v_{10}v_6v_3v_1v_4v_7v_5v_{11}$ is a C_8;
$v_{11} \nsim v_i$ for each $i \in \{7, 8\}$: otherwise $v_{11}v_{10}v_6v_3v_2v_1v_4v_iv_{11}$ is a C_8;
$v_{11} \nsim v_9$: otherwise $v_{11}v_{10}v_6v_3v_1v_2v_5v_9v_{11}$ is a C_8.

Then v_{11} has a neighbor v_{13} such that $v_{13} \nsim v_{10}$. Furthermore,

$v_{13} \nsim v_i$ for $i \in \{1, 3\}$: otherwise $v_{13}v_{11}v_{10}v_6v_7v_5v_2v_iv_{13}$ is a C_8;
$v_{13} \nsim v_2$: otherwise $v_{13}v_{11}v_{10}v_6v_7v_4v_1v_2v_{13}$ is a C_8;
$v_{13} \nsim v_4$: otherwise $v_{13}v_{11}v_{10}v_6v_3v_2v_1v_4v_{13}$ is a C_8;
$v_{13} \nsim v_5$: otherwise $v_{13}v_{11}v_{10}v_6v_3v_1v_2v_5v_{13}$ is a C_8;
$v_{13} \nsim v_6$: otherwise $v_{13}v_{11}v_{10}v_6v_{13}$ is a C_4;
$v_{13} \nsim v_i$ for $i \in \{7, 8\}$: otherwise $v_{13}v_{11}v_{10}v_6v_3v_1v_4v_iv_{13}$ is a C_8;
$v_{13} \nsim v_9$: otherwise $v_{13}v_{11}v_{10}v_6v_3v_2v_5v_9v_{13}$ is a C_8.

If $v_8 \nsim v_1$, then $v_{13}v_{11}v_{10}v_6v_3v_1v_4v_8$ is an induced P_8, a contradiction. (See Fig. 2c). Now assume that $v_8 \sim v_1$, then $v_9 \nsim v_2$ as otherwise $v_1v_3v_2v_9v_5v_7v_4v_8v_1$ is a C_8. Furthermore, $v_9 \nsim v_3$, otherwise $v_9v_5v_2v_3v_9$ is a C_4. It follows that $v_{13}v_{11}v_{10}v_6v_3v_2v_5v_9$ is an induced P_8, a contradiction. □

Claim 3.3 *No C_3 and C_6 are adjacent.*

Proof Suppose that $C' = u_1u_2u_3u_1$ and $C'' = u_1u_2u_4u_5u_6u_7u_1$ are two cycles sharing a common edge u_1u_2. As G contains no C_4, we have $u_1 \nsim u_4$ and $u_3 \nsim u_i$ for $i \in \{4, \ldots, 7\}$. Then by Claim 3.2 and the fact that $u_1 \nsim u_4$, we know that C'' is an induced cycle. Since $\delta(G) \geq 3$, u_3 has a neighbor $u_8 \notin \{u_1, \ldots, u_7\}$. It can be seen that $u_8 \nsim u_i$ for $i \in \{1, 2, 4, 7\}$ as G contains no C_4, and $u_8 \nsim u_i$ for $i \in \{5, 6\}$ by Claim 3.2. So u_8 has two neighbors $u_9, u_{10} \notin \{u_1, \ldots, u_8\}$. At least one of u_9 and u_{10}, say u_9, is nonadjacent to u_3, as otherwise $u_9u_8u_{10}u_3u_9$ is a C_4. Furthermore, $u_9 \nsim u_i$ for $i \in \{1, 2\}$ as G contains no C_4, $u_9 \nsim u_i$ for $i \in \{4, 7\}$ by Claim 3.2, $u_9 \nsim u_i$ for $i \in \{5, 6\}$ as G contains no C_8. It follows that $u_9u_8u_3u_1u_7u_6u_5u_4$ is an induced P_8, a contradiction. (See Fig. 3a). □

Since $\delta(G) \geq 3$, v_i has two neighbors v_{2i-1}, v_{2i} for $i \in \{4, 5\}$. By Claim 3.2, v_7, v_8, v_9, v_{10} are pairwise distinct. At least one of v_7 and v_8, say v_7, is nonadjacent to v_1. Furthermore, $v_7 \nsim v_i$ for $i \in \{2, 3\}$ since G contains no C_4, and $v_7 \nsim v_i$ for $i \in \{5, 6, 9, 10\}$ by Claim 3.2 and Claim 3.3. As $\delta(G) \geq 3$ and G has no C_4, v_7 has

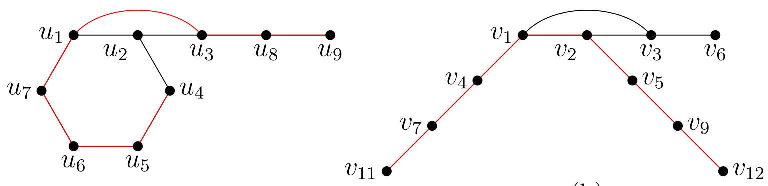


Fig. 3 Illustration of finding an induced P_8

a neighbor $v_{11} \notin \{v_1, v_2, \ldots, v_{10}\}$ for which $v_{11} \nsim v_4$. Furthermore, $v_{11} \nsim v_1$, as otherwise $v_{11}v_7v_4v_1v_{11}$ is a C_4; $v_{11} \nsim v_i$ for $i \in \{2, 3, 5, 6\}$ by Claim 3.2 and Claim 3.3; and $v_{11} \nsim v_9$, as otherwise $v_{11}v_7v_4v_1v_3v_2v_5v_9v_{11}$ is a C_8. Similarly, we can assume that $v_9 \nsim v_2$ and v_9 has a neighbor $v_{12} \notin \{v_1, v_2, \ldots, v_{11}\}$ such that $v_{12} \nsim v_i$ for $i \in \{1, 2, \ldots, 8\}$. It can be seen that $v_{11} \nsim v_{12}$, as otherwise $v_{11}v_7v_4v_1v_2v_5v_9v_{12}v_{11}$ is a C_8. It follows that $v_{11}v_7v_4v_1v_2v_5v_9v_{12}$ is an induced P_8, a contradiction. (See Fig. 3b).

Acknowledgements The authors are very grateful to the referees' careful reading and valuable comments.

Funding Yuping Gao is partially supported by NSFC (No. 11901263, 12071194, 12271228), NSFC of Gansu Province (No. 20JR5RA229, 21JR7RA511). Songling Shan is supported by NSF grant DMS-2153938.

Availability of Data and Material Not applicable.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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