Received: 16 October 2021 | Revised: 17 June 2022 | Accepted: 23 August 2022

DOI: 10.1002/jgt.22889

ARTICLE

WILEY

Edge coloring graphs with large minimum degree

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Funding information
National Science Foundation, Grant/Award Number: DMS-2153938

Abstract

Let G be a simple graph with maximum degree $\Delta(G)$. A subgraph H of G is overfull if $|E(H)| > \Delta(G)\lfloor |V(H)|/2 \rfloor$. Chetwynd and Hilton in 1986 conjectured that a graph G with $\Delta(G) > |V(G)|/3$ has chromatic index $\Delta(G)$ if and only if G contains no overfull subgraph. The best previous results supporting this conjecture have been obtained for regular graphs. For example, Perković and Reed verified the conjecture for large regular graphs G with degree arbitrarily close to $|V(G)|/2$. We provide a similar result for general graphs asymptotically, showing that for any given $0 < \epsilon < 1$, there exists a positive integer n_0 such that the following statement holds: if G is a graph on $2n \geq n_0$ vertices with minimum degree at least $(1+\epsilon)n$, then G has chromatic index $\Delta(G)$ if and only if G contains no overfull subgraph.

KEYWORDS

1-factorization, chromatic index, overfull conjecture, overfull graph

1 | INTRODUCTION

In this paper, the terminology "graph" is used to mean a simple graph and a "multigraph" may contain parallel edges but no loops. Let G be a multigraph. Denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively, and by $e(G)$ the cardinality of $E(G)$. For $v \in V(G)$, $N_G(v)$ is the set of neighbors of v in G, and $d_G(v)$, the degree of v in G, is the number of edges of G that are incident with v. When G is simple, $d_G(v) = |N_G(v)|$. For $S \subseteq V(G)$, $N_G(S) = \bigcup_{v\in S} N_G(v)$, the subgraph of G induced on S is denoted by $G[S]$, and $G - S := G[V(G) \setminus S]$. If $F \subseteq E(G)$, then $G - F$ is obtained from G by deleting all the edges of F. Let $V_1, V_2 \subseteq V(G)$ be two disjoint vertex sets. Then $E_G(V_1, V_2)$ is the set of edges in G with one end in V_1 and the other end in V_2, and

$e_G(V_1, V_2) := |E_G(V_1, V_2)|$. We write $E_G(v, V_2)$ and $e_G(v, V_2)$ if $V_1 = \{v\}$ is a singleton. Define $\mu(G) = \max\{e_G(u, v) : u, v \in V(G)\}$ to be the multiplicity of G. We also write $G[V_1, V_2]$ to denote the bipartite subgraph of G with vertex set $V_1 \cup V_2$ and edge set $E_G(V_1, V_2)$.

For two integers p, q, let $[p, q] = \{i \in \mathbb{Z} : p \le i \le q\}$. Let $k \ge 0$ be an integer. An edge *k-coloring* of a multigraph G is a mapping φ from $E(G)$ to the set of integers $[1, k]$, called *colors*, such that no two adjacent edges receive the same color with respect to φ. The *chromatic index* of G, denoted $\chi'(G)$, is defined to be the smallest integer k so that G has an edge k-coloring. We denote by $\mathcal{C}^k(G)$ the set of all edge k-colorings of G.

In the 1960s, Gupta [11] and, independently, Vizing [26] proved that for all graphs G, $\Delta(G) \le \chi'(G) \le \Delta(G) + 1$. This leads to a natural classification of graphs. Following Fiorini and Wilson [8], we say a graph G is of *class 1* if $\chi'(G) = \Delta(G)$ and of *class 2* if $\chi'(G) = \Delta(G) + 1$. Holyer [13] showed that it is NP-complete to determine whether an arbitrary graph is of class 1. Nevertheless, if $|E(G)| > \Delta(G)\lfloor |V(G)|/2 \rfloor$, then we must use $(\Delta(G) + 1)$ colors to edge color G. Such graphs are *overfull*. An overfull subgraph H of G with $\Delta(H) = \Delta(G)$ is called a *$\Delta(G)$-overfull subgraph* of G. A number of long-standing conjectures listed in *Twenty Pretty Edge Coloring Conjectures* in [24] lie in deciding when a graph is overfull. Chetwynd and Hilton [3,4], in 1986, proposed the following conjecture.

Conjecture 1.1 (Overfull Conjecture). *Let G be a simple graph with $\Delta(G) > \frac{1}{3}|V(G)|$. Then $\chi'(G) = \Delta(G)$ if and only if G contains no $\Delta(G)$-overfull subgraph.*

The 3-critical graph P^*, obtained from the Petersen graph by deleting one vertex, has $\chi'(P^*) = 4$, satisfies $\Delta(P^*) = \frac{1}{3}|V(P^*)|$ but contains no 3-overfull subgraph. Thus the degree condition $\Delta(G) > \frac{1}{3}|V(G)|$ in the conjecture above is best possible. Applying Edmonds' matching polytope theorem, Seymour [22] showed that whether a graph G contains an overfull subgraph of maximum degree $\Delta(G)$ can be determined in polynomial time. Thus if the Overfull Conjecture is true, then the NP-complete problem of determining the chromatic index becomes polynomial-time solvable for graphs G with $\Delta(G) > \frac{1}{3}|V(G)|$. There have been some fairly strong results supporting the Overfull Conjecture in the case when G is regular. It is easy to verify that when G is regular with even order, G has no $\Delta(G)$-overfull subgraphs if its vertex degrees are at least $|V(G)|/2$. Thus the well-known 1-Factorization Conjecture stated below is a special case of the Overfull Conjecture.

Conjecture 1.2 (1-Factorization Conjecture). *Let G be a regular graph of order $2n$ with degree at least n if n is odd, or at least $n - 1$ if n is even. Then G is 1-factorable; equivalently, $\chi'(G) = \Delta(G)$.*

Hilton and Chetwynd [2] verified the 1-Factorization Conjecture if the vertex degree is at least $0.823|V(G)|$. Perković and Reed [19] showed in 1997 that the 1-Factorization Conjecture is true for large regular graphs with vertex degree at least $|V(G)|/(2 - \epsilon)$. In 2016, Csaba et al. [6] verified the conjecture for sufficiently large $|V(G)|$. Much less is known about the truth of the Overfull Conjecture if we no longer require that G is regular. It was confirmed for graphs with $\Delta(G) \ge |V(G)| - 3$ by Chetwynd and Hilton in 1989 [4]. Plantholt [20] in 2004 verified the conjecture for graphs of even order and minimum degree at least $0.8819|V(G)|$. More recently, Plantholt [21] showed the conjecture is true for sufficiently large even order graphs with minimum degree at least $2|V(G)|/3$. We extend these results and give an asymptotic result for general graphs that is similar to the Perković-Reed result for regular graphs, by obtaining the result below.

Theorem 1.3. *For all $0 < \varepsilon < 1$, there exists n_0 such that the following statement holds: if G is a graph on $2n \geq n_0$ vertices with $\delta(G) \geq (1 + \varepsilon)n$, then $\chi'(G) = \Delta(G)$ if and only if G contains no $\Delta(G)$-overfull subgraph. Furthermore, there is a polynomial time algorithm that finds an optimal coloring.*

Define $V_i(G) = \{v \in V(G) : d_G(v) = i\}$, and we write V_i for $V_i(G)$ if G is clear. Furthermore, $V_{\delta(G)}$ and $V_{\Delta(G)}$ are simply written as V_δ and V_Δ, respectively. The proof of Theorem 1.3 is based on the following result.

Theorem 1.4. *For all $0 < \varepsilon < 1$, there exists n_0 such that the following statement holds. If G is a graph on $2n \geq n_0$ vertices satisfying one of the following three conditions:*

(a) *G is regular with $\delta(G) \geq (1 + 4\varepsilon/5)n$,*
(b) *G has two distinct vertices x, y such that $d(x) = d(y) \geq (1/2 + 3\varepsilon/2)n$, for all $z \in V(G) \setminus \{x, y\}$, $d(z) = \Delta(G) \geq (1 + \varepsilon)n$, and $\Delta(G) - \delta(G) \leq (1/2 - \varepsilon/2)n$,*
(c) *$\Delta(G) - \delta(G) \geq n^{6/7}$, $|V_\delta| \geq n^{6/7}$ and $|V_\Delta| \geq n + 1$, and $\delta(G) \geq (1 + \varepsilon)n$,*

then $\chi'(G) = \Delta(G)$. Furthermore, there is a polynomial time algorithm that finds an optimal coloring.

The proof of Theorem 1.4 develops an approach to edge coloring even order large graphs G that have minimum degree arbitrarily close to $|V(G)|/2$ but are not regular. The approach is based on the proof scheme of Lemma 14 from [25] by Vaughan but the scheme there is only for regular graphs. The new technique is essentially different from the main ideas used in [23] by the second author and in [21] by the first author, where the graphs can be reduced into a regular graph still with good properties by taking off edge-disjoint linear forests or matchings.

The remainder of this paper is organized as follows. In the next section, we introduce some notation and preliminary results. In Section 3, we prove Theorem 1.3 by applying Theorem 1.4. Theorem 1.4 is then proved in the last section.

2 | NOTATION AND PRELIMINARIES

Let G be a multigraph and $\varphi \in \mathcal{C}^k(G)$ for some integer $k \geq 0$. For any $v \in V(G)$, the set of colors *present* at v is $\varphi(v) = \{\varphi(e) : e \in E(G)$ is incident with $v\}$, and the set of colors *missing* at v is $\overline{\varphi}(v) = [1, k] \setminus \varphi(v)$. For a subset X of $V(G)$ and a color $i \in [1, k]$, define $\overline{\varphi}_X^{-1}(i) = \{v \in X : i \in \overline{\varphi}(v)\}$, and we write $\overline{\varphi}^{-1}(i)$ for $\overline{\varphi}_{V(G)}^{-1}(i)$. An edge k-coloring of a multigraph G is said to be *equalized* if each color class contains either $\lfloor |E(G)|/k \rfloor$ or $\lceil |E(G)|/k \rceil$ edges.

For $x \in V(G)$, the *deficiency* of x in G is $\text{def}_G(x) := \Delta(G) - d_G(x)$. For $X \subseteq V(G)$, $\text{def}_G(X) = \sum_{x \in X} \text{def}_G(x)$. We simply write $\text{def}_G(V(G))$ as $\text{def}(G)$. A subgraph H of G with an odd order is *$\Delta(G)$-full* if $|E(H)| = \Delta(G)\lfloor |V(H)|/2 \rfloor$.

We will use the following notation: $0 < a \ll b \leq 1$. Precisely, if we say a claim is true provided that $0 < a \ll b \leq 1$, then this means that there exists a nondecreasing function $f: (0, 1] \to (0, 1]$ such that the statement holds for all $0 < a, b \leq 1$ satisfying $a \leq f(b)$.

In the 1960s, Gupta [11] and, independently, Vizing [26] provided an upper bound on the chromatic index of multigraphs, and König [15] gave an exact value of the chromatic index for bipartite multigraphs.

Theorem 2.1 (Gupta [11] and Vizing [26]). *Every multigraph G satisfies $\chi'(G) \leq \Delta(G) + \mu(G)$.*

Theorem 2.2 (König [15]). *Every bipartite multigraph G satisfies $\chi'(G) = \Delta(G)$.*

McDiarmid [16] observed the following result.

Theorem 2.3. *Let G be a multigraph with chromatic index $\chi'(G)$. Then for all $k \geq \chi'(G)$, there is an equalized edge-coloring of G with k colors.*

Let G be a multigraph, $k \geq 0$ be an integer and $\varphi \in \mathcal{C}^k(G)$. There is a polynomial time algorithm to modify φ into an equalized edge-coloring of G with k colors. To see this, suppose that φ is not equalized and so we take two colors $i, j \in [1, k]$ such that $||\overline{\varphi}^{-1}(i)| - |\overline{\varphi}^{-1}(j)||$ is largest. Since φ is not equalized, $||\overline{\varphi}^{-1}(i)| - |\overline{\varphi}^{-1}(j)|| \geq 4$. Assume by symmetry that $|\overline{\varphi}^{-1}(i)| - |\overline{\varphi}^{-1}(j)| \geq 4$. Consider the submultigraph of G induced on the set of edges colored by i or j, then the submultigraph must have a component that is a path P starting at an edge colored by j and ending at an edge colored by j. By swapping the colors i and j along this path P, we decreased $|\overline{\varphi}^{-1}(i)| - |\overline{\varphi}^{-1}(j)|$ by 4. Repeating this process, we can obtain an equalized edge-coloring of G with k colors after at most $k^2|V(G)|$ rounds.

Given an edge coloring of G and a given color i, since vertices presenting i are saturated by the matching consisting of all edges colored by i, we have the Parity Lemma below. The result had appeared in many papers, for example, see [10, lemma 2.1].

Lemma 2.4 (Parity Lemma). *Let G be a multigraph and $\varphi \in \mathcal{C}^k(G)$ for some integer $k \geq \Delta(G)$. Then $|\overline{\varphi}^{-1}(i)| \equiv |V(G)| \pmod 2$ for every color $i \in [1, k]$.*

We need the following classic result of Hakimi [12] on multigraphic degree sequence.

Theorem 2.5. *Let $0 \leq d_n \leq \cdots \leq d_1$ be integers. Then there exists a multigraph G on vertices $x_1, ..., x_n$ such that $d_G(x_i) = d_i$ for all i if and only if $\sum_{i=1}^{n} d_i$ is even and $\sum_{i>1} d_i \geq d_1$.*

Though it is not explicitly stated in [12], the inductive proof yields a polynomial time algorithm which finds an appropriate multigraph if it exists.

Theorem 2.6 (Dirac [7]). *Let G be a graph on $n \geq 3$ vertices. If $\delta(G) \geq \frac{n}{2}$, then G is hamiltonian; and if $\delta(G) \geq \frac{n+1}{2}$, then G is hamiltonian-connected.*

Following the proof of Dirac [7], a hamiltonian cycle can be constructed in polynomial time in n if $\delta(G) \geq \frac{n}{2}$. In fact, there is a polynomial time algorithm that constructs the closure of a graph G and finds a hamiltonian cycle of G if its closure is a complete graph (see [1, exercise 4.2.15, p. 62]).

Lemma 2.7. *Let G be an n-vertex graph such that all vertices of degree less than $\Delta(G)$ are mutually adjacent in G. Then $|V_\Delta| > \frac{n}{2}$.*

Proof. Suppose the set A of maximum degree vertices has cardinality k, and the number of vertices of degree less than maximum degree is $k + r$ with $r \geq 0$. Deleting r vertices not in A, we get a new graph H with $2k$ vertices, k of them forming A, and the remaining k forming a set of vertices B such that each vertex in B has degree less than each vertex of A in H. But B induces a complete graph in H so in H the sum of the vertex degrees in A is less than or equal to the degree sum of the vertices in B. Since every vertex of $V(G)\backslash V(H)$ is adjacent in G to every vertex of B, it follows that in G the sum of the vertex degrees in A is less than or equal to the degree sum of the vertices in B. This gives a contradiction. □

The two lemmas below concern existences of overfull subgraphs in graphs.

Lemma 2.8 (Plantholt [20])**.** *Let G be a graph of even order n with $\delta(G) > \frac{n}{2}$. If H is an induced proper subgraph of G such that H is either $\Delta(G)$-overfull or $\Delta(G)$-full, then $H = G - v$ for some vertex $v \in V_\delta$.*

Lemma 2.9. *Let G be a graph of even order n with $\delta(G) > \frac{n}{2}$. Then G contains no $\Delta(G)$-overfull subgraph if $|V_\delta| \geq 2$.*

Proof. Let $x, y \in V_\delta$ be distinct. Then $\sum_{v \in V(G-x)}(\Delta(G) - d_{G-x}(v)) = d_G(x) + (\Delta(G) - d_G(y)) + \text{def}_G(V(G)\backslash\{x, y\}) \geq \Delta(G)$. Thus $G - x$ is not $\Delta(G)$-overfull. By Lemma 2.8, G contains no $\Delta(G)$-overfull subgraph. □

Lemma 2.10. *Let $0 < \varepsilon < 1$, n_0 be a positive integer, and G be a graph on $2n \geq n_0$ vertices with $\delta(G) \geq (1 + \varepsilon)n$. If G contains a $\Delta(G)$-full subgraph, then G contains a spanning $\delta(G)$-regular subgraph obtained from G by deleting $\Delta(G) - \delta(G)$ matchings iteratively. As a consequence, $\chi'(G) = \Delta(G)$. Furthermore, there is a polynomial time algorithm that finds an optimal coloring.*

Proof. Define $g = \Delta(G) - \delta(G)$. If G is regular, then we are done by Theorem 1.4. Thus G is not regular and so $g \geq 1$. The graph G contains a $\Delta(G)$-full subgraph, which by Lemma 2.8 must be $G - x$ for some vertex $x \in V_\delta$. Also, if G contains a $\Delta(G)$-overfull subgraph, then $G - x$ must be $\Delta(G)$-overfull also by Lemma 2.8. Since $G - x$ is $\Delta(G)$-full, we conclude that G contains no $\Delta(G)$-overfull subgraph and so has another vertex of degree less than $\Delta(G)$. We let $y \in V(G)\backslash\{x\}$ such that $d_G(y)$ is smallest among all vertices in $V(G)\backslash\{x\}$. Since $G - x$ is $\Delta(G)$-full, we have $\Delta(G) = \text{def}(G - x) = d_G(x) + (\Delta(G) - d_G(y)) + \text{def}_G(V(G)\backslash\{x, y\})$. As $d_G(x) = \delta(G)$, if $d_G(y) = \delta(G)$, then $\text{def}_G(V(G)\backslash\{x, y\}) = 0$.

This implies that if $d_G(y) = \delta(G)$, then every vertex from $V(G)\setminus\{x, y\}$ has degree $\Delta(G)$ in G; and if $d_G(y) > \delta(G)$, then as y is chosen to have smallest degree in G among vertices from $V(G)\setminus\{x\}$, $V(G)\setminus\{x, y\}$ contains no vertex of degree $\delta(G)$ in G. Since $\delta(G - x - y) \geq n - 1$, $G - x - y$ has a hamiltonian cycle by Theorem 2.6. As $n - 2$ is even, we know that $G - x - y$ has a perfect matching M_1. Now we have $\delta(G - M_1) = \delta(G)$ and $\Delta(G - M_1) - \delta(G - M_1) = g - 1 < g$. Let $G_1 = G - M_1$. Since

$$\begin{aligned}\text{def}(G_1 - x) &= d_G(x) + (\Delta(G_1) - d_G(y)) + \text{def}_{G_1}(V(G)\setminus\{x, y\})\\ &= d_G(x) + \text{def}_G(V(G)\setminus\{x\}) - 1\\ &= \text{def}(G - x) - 1 = \Delta(G) - 1 = \Delta(G_1),\end{aligned}$$

we see that $G_1 - x$ is $\Delta(G_1)$-full. Thus we may repeat the procedure, and reach a $\delta(G)$-regular graph G^* after taking g matchings $M_1, \ldots, M_g$.

Now by Theorem 1.4, $\chi'(G^*) = \Delta(G^*) = \delta(G)$. Coloring each of the g matchings $M_1, \ldots, M_g$ using a different color together with an edge $\delta(G)$-coloring of G^* gives an edge $\Delta(G)$-coloring of G. Thus $\chi'(G) = \Delta(G)$.

It is polynomial-time to find a hamiltonian cycle in graphs H with $\delta(H) \geq \frac{1}{2}|V(H)|$ by the comments immediately after Theorem 2.6. Thus all the matchings $M_1, \ldots, M_g$ can be found in polynomial time. As an optimal edge coloring can be found in polynomial time for graphs satisfying the conditions in Theorem 1.4, we can find an edge $\Delta(G)$-coloring of G^* in polynomial time. Therefore, there is a polynomial time algorithm that finds an edge $\Delta(G)$-coloring for G. □

Lemma 2.11. *Let $G[X, Y]$ be bipartite graph with $|X| = |Y| = n$. Suppose $\delta(G) = t$ for some $t \in [1, n]$, and except at most t vertices all other vertices of G have degree at least $n/2$ in G. Then G has a perfect matching.*

Proof. We show that $G[X, Y]$ satisfies Hall's Condition. If not, we let $S \subseteq X$ with smallest cardinality such that $|S| > |N_G(S)|$. By this choice, $|S| = |N_G(S)| + 1$ and $|N_G(S)| < |Y|$. As $|S| > |N_G(S)|$, it follows that $|S| \geq \delta(G) + 1 \geq t + 1$. As G has at most t vertices of degree less than $n/2$, it then follows that $|S| > n/2$. Thus $|X\setminus S| < n/2$. Since $|N_G(S)| < |Y|$, there exists $y \in Y\setminus N_G(S)$ such that $N_G(y) \subseteq X\setminus S$. As $\delta(G) \geq t$, we have $|X\setminus S| \geq t$. As $|Y\setminus N_G(S)| = |Y| - |S| + 1 = |X| - |S| + 1 \geq t + 1$ and G has at most t vertices of degree less than $n/2$, $Y\setminus N_G(S)$ contains a vertex of degree at least $n/2$ in G. However $|X\setminus S| < n/2$, we obtain a contradiction. Hence G has a perfect matching. □

A path P connecting two vertices u and v is called a (u, v)-*path*, and we write uPv or vPu to specify the two endvertices of P. Let uPv and xQy be two disjoint paths. If vx is an edge, we write $uPvxQy$ as the concatenation of P and Q through the edge vx. If P is a path and $x, y \in V(P)$, then xPy is the subpath of P with endvertices x and y.

Lemma 2.12. *Let $0 < 1/n_0 \ll \varepsilon < 1$, and G be a graph on $n \geq n_0$ vertices such that $\delta(G) \geq (1 + \varepsilon)n/2$. Moreover, let $M = \{a_1b_1, \ldots, a_tb_t\}$ be a matching in the complete graph on $V(G)$ of size at most $\varepsilon n/8$. Then there exist vertex-disjoint path $P_1, \ldots, P_t$ in G such that $\bigcup V(P_i) = V(G)$ and P_i joins a_i to b_i, and these paths can be found in polynomial time.*

Proof. For $i \in [1, t-1]$, $|N_G(a_i) \cap N_G(b_i)| \geq \varepsilon n$, so we can greedily find vertices $c_i \in N_G(a_i) \cap N_G(b_i)$ such that $c_i \neq c_j$ for distinct $i, j \in [1, t-1]$. Thus we let $P_i = a_i c_i b_i$. Let $G^* = G - \bigcup_{i=1}^{t-1} V(P_i)$. Then $\delta(G^*) \geq (1+\varepsilon)n/2 - 3(t-1) \geq (1+\varepsilon/8)n/2$, and so G^* is hamiltonian-connected by Theorem 2.6. Thus we can find an (a_t, b_t)-hamiltonian path P_t in G^*.

It is clear that each of $P_1, \ldots, P_{t-1}$ can be found in polynomial time. For the path P_t, we construct it as follows. By the comments immediately after Theorem 2.6, we can find a hamiltonian cycle C of G^* in polynomial time. By taking a longer segment between a_t and b_t from C, we get in G^* an (a_t, b_t)-path Q_1 that contains at least $|V(G^*)|/2$ vertices. We will extend Q_1 into a hamiltonian (a_t, b_t)-path of G^*. Denote by Q_2 the remaining segment of C that is disjoint from Q_1 and let c and d be the endvertices of Q_2. Let $|V(Q_2)| = p$. Then as $\delta(G^*) \geq (1+\varepsilon/8)n/2$, each of c and d has on Q_1 at least $(1+\varepsilon/8)n/2 - (p-1) = (1+\varepsilon/8)n/2 - p + 1$ neighbors. Since $2((1+\varepsilon/8)n/2 - p) + p + |V(Q_2)| > |V(G^*)|$, it follows that one of the following two situations must happen: (a) there is a vertex $c_1 \in N_{G^*}(c) \cap V(Q_1)$ and a vertex $d_1 \in N_{G^*}(d) \cap V(Q_1)$ such that $c_1 Q_1 d_1$ contains less than $p+2$ vertices, and (b) c or d has on Q_1 two neighbors that are consecutive on Q_1. When (a) happens, assume by symmetry that c_1 is between a_t and d_1 on Q_1, then $Q_1^* = a_t Q_1 c_1 c Q_2 d d_1 Q_1 b_t$ is longer than Q_1 and the component of $G^* - V(Q_1^*)$ still contains a hamiltonian path. Similarly, we can extend Q_1 into a longer (a_t, b_t)-path such that the subgraph of G^* outside the path is hamiltonian if (b) happens. Repeating this procedure at most $n/2$ times, we obtain a hamiltonian (a_t, b_t)-path of G^*. Therefore, all the paths $P_1, \ldots, P_t$ can be found in polynomial time. □

3 | PROOF OF THEOREM 1.3

Theorem 1.3. *For all* $0 < \varepsilon < 1$, *there exists* n_0 *such that the following statement holds: if* G *is a graph on* $2n \geq n_0$ *vertices with* $\delta(G) \geq (1+\varepsilon)n$, *then* $\chi'(G) = \Delta(G)$ *if and only if* G *contains no* $\Delta(G)$*-overfull subgraph. Furthermore, there is a polynomial time algorithm that finds an optimal coloring.*

Proof. Choose positive integer n_0 such that $0 < 1/n_0 \ll \varepsilon$.

If G is regular, then we are done by Theorem 1.4. Thus we assume that G is not regular. If G contains a $\Delta(G)$-overfull subgraph, then $\chi'(G) = \Delta(G) + 1$. Thus we assume that G contains no $\Delta(G)$-overfull subgraph. As a consequence, $\text{def}(G) \geq \Delta(G)$. By Lemma 2.10, we may assume that G contains no $\Delta(G)$-full subgraph. Therefore, if two vertices with degree less than $\Delta(G)$ are not adjacent in G, we may add the edge between them without creating a $\Delta(G)$-overfull subgraph, or increasing $\Delta(G)$. We iterate this edge-addition procedure. If at some point we create a $\Delta(G)$-full subgraph, the result follows by Lemma 2.10. Otherwise, we reach a point where we may now assume that in G all vertices with degree less than $\Delta(G)$ are mutually adjacent, and so by Lemma 2.7, we have $|V_\Delta| \geq n + 1$.

Define $n_1 = |V_\delta|$. Note that $n_1 < n$. If $n_1 \geq n^{6/7}$ and $\Delta(G) - \delta(G) \geq n^{6/7}$, then we are done by Theorem 1.4. Thus we assume $n_1 < n^{6/7}$ or $\Delta(G) - \delta(G) < n^{6/7}$, and we consider the two cases below. We call a vertex of degree less than $\Delta(G)$ but greater than $\delta(G)$ a *middle degree* vertex.

Case 1. $n_1 < n^{6/7}$.

Note that for any $v \in V(G)\backslash V_\delta, \delta(G - v - V_\delta) \geq n$ and so both $G - V_\delta$ and $G - v - V_\delta$ are hamiltonian by Theorem 2.6. Thus if $G - V_\delta$ and $G - v - V_\delta$ have even order, then they each has a perfect matching. Hence if n_1 is even, we can decrease $\Delta(G) - \delta(G)$ but preserve $\delta(G)$ in deleting a perfect matching M of $G - V_\delta$. If n_1 is odd but G has a middle degree vertex v, we can decrease $\Delta(G) - \delta(G)$ but preserve $\delta(G)$ in deleting a perfect matching M of $G - v - V_\delta$. Denote by G_1 the reduced graph from G by deleting M in either of these two cases. If $|V_\delta| \geq 2$, then as $V_\delta \subseteq V_\delta(G_1)$, we know that G_1 still contains no $\Delta(G_1)$-overfull subgraph by Lemma 2.9. Thus $|V_\delta| = 1$. Let $V_\delta = \{u\}$. Note that $u \in V_\delta(G_1)$. Then $\text{def}(G_1 - u) = d_G(u) + (\Delta(G_1) - d_G(v)) + \text{def}_{G_1}(V(G_1)\backslash\{u, v\}) = d_G(u) + (\Delta(G) - 1 - d_G(v)) + \text{def}_G(V(G)\backslash\{u, v\}) = \sum_{w \in V(G-u)}(\Delta(G) - d_{G-u}(w)) - 1$. Since G contains no $\Delta(G)$-overfull subgraph, we have $\sum_{w \in V(G-u)}(\Delta(G) - d_{G-u}(w)) \geq \Delta(G)$. Thus $\text{def}(G_1 - u) \geq \Delta(G) - 1 = \Delta(G_1)$ and so G_1 contains no $\Delta(G_1)$-overfull subgraph by Lemma 2.8. Furthermore, $\chi'(G_1) = \Delta(G_1)$ implies that $\chi'(G) = \Delta(G)$. Thus in these two cases, we can consider G_1 in place of G and show that G_1 is a class 1 graph.

Thus we assume that n_1 is odd and G has no middle degree vertex. This in particular, implies that $\delta(G)$ and $\Delta(G)$ have the same parity. As G has no $\Delta(G)$-overfull subgraph, $|V_\delta| \geq 3$. Let $x, y \in V_\delta$ be distinct. We find a perfect matching M_{11} in $G - (V_\delta\backslash\{x\})$ and a perfect matching M_{12} in $G - (V_\delta\backslash\{y\})$. The matchings exist by Theorem 2.6. Let $G_1 = G - M_{11} - M_{12}$. We repeat this same process and find a perfect matching M_{21} in $G_1 - (V_\delta\backslash\{x\})$ and a perfect matching M_{22} in $G_1 - (V_\delta\backslash\{y\})$. For $i \in [2, (\Delta(G) - \delta(G))/2]$, we let $G_i = G_{i-1} - M_{i1} - M_{i2}$. We have $d_{G_i}(x) = d_{G_i}(y) = \delta(G) - i$. As $\Delta(G) - \delta(G) \leq 2n - (1 + \varepsilon)n = (1 - \varepsilon)n$, we see that $d_{G_i}(x) = d_{G_i}(y) = \delta(G) - i \geq (1 + \varepsilon)n - \frac{1}{2}(1 - \varepsilon)n = (1/2 + 3\varepsilon/2)n$. For any vertex $z \in V(G_i)\backslash\{x, y\}, d_{G_i}(z) \geq \delta(G) \geq (1 + \varepsilon)n$. Let x^* be a neighbor of x in $G_i - (V_\delta\backslash\{x\})$ and y^* be a neighbor of y in $G_i - (V_\delta\backslash\{y\})$. Then $G_i - (V_\delta\backslash\{x\}) - \{x, x^*\}$ has a perfect matching $M^*_{(i+1)1}$, and $G_i - (V_\delta\backslash\{y\}) - \{y, y^*\}$ has a perfect matching $M^*_{(i+1)2}$. Let $M_{(i+1)1} = M^*_{(i+1)1} \cup \{xx^*\}$ and $M_{(i+1)2} = M^*_{(i+1)2} \cup \{yy^*\}$. Thus for each $i \in [2, (\Delta(G) - \delta(G))/2]$, we find matchings M_{i1} and M_{i2}, respectively from $G_{i-1} - (V_\delta\backslash\{x\})$ and $G_{i-1} - (V_\delta\backslash\{y\})$.

We claim $G^* := G_{(\Delta(G)-\delta(G))/2}$ satisfies Condition (b) of Theorem 1.4. By the analysis above, we have $d_{G^*}(x) = d_{G^*}(y) \geq (1/2 + 3\varepsilon/2)n, d_{G^*}(z) = \Delta(G^*) = \delta(G) \geq (1 + \varepsilon)n$ for all $z \in V(G^*)\backslash\{x, y\}$. Also $\Delta(G^*) - \delta(G^*) = \delta(G) - (\delta(G) - (\Delta(G) - \delta(G))/2) = \frac{1}{2}(\Delta(G) - \delta(G)) \leq \frac{1}{2}(2n - (1 + \varepsilon)n) = (1/2 - \varepsilon/2)n$. By Theorem 1.4, $\chi'(G^*) = \Delta(G^*) = \delta(G)$. Taking an edge $\delta(G)$-coloring of G^*, coloring edges in M_{i1} with color $\delta(G) + 2i - 1$ and coloring edges in M_{i2} with color $\delta(G) + 2i$ for each $i \in [1, (\Delta(G) - \delta(G))/2]$, we obtain an edge $\Delta(G)$-coloring of G.

Case 2. $\Delta(G) - \delta(G) < n^{6/7}$.

Let $V(G) = \{x_1, ..., x_{2n}\}$ and we assume $\mathrm{def}_G(x_1) \geq \cdots \geq \mathrm{def}_G(x_{2n}) = 0$. Since x_1 has the smallest degree in G and $G - x_1$ is not $\Delta(G)$-overfull by our assumption, $\sum_{i\geq 2}\mathrm{def}_G(x_i) \geq \mathrm{def}_G(x_1)$. Since $|V(G)| = 2n$ is even, $\sum_{i\geq 1}\mathrm{def}_G(x_i)$ is even. Then by Theorem 2.5, there exists a multigraph H on $V(G)$ such that $d_H(x) = \mathrm{def}_G(x_i)$ for each $i \in [1, 2n]$. This multigraph H will aid us to find a spanning regular subgraph of G.

Note that $\Delta(H) = \mathrm{def}_G(x_1) = \Delta(G) - \delta(G) < n^{6/7}$ and H contains isolated vertices. Thus $\chi'(H) \leq \Delta(H) + \mu(H) \leq 2\Delta(H) \leq 2n^{6/7}$. Hence we can greedily partition $E(H)$ into $k \leq 10n^{6/7}/\varepsilon$ matchings $M_1, ..., M_k$ each of size at most $\varepsilon n/5$. Now we take out linear forests from G by applying Lemma 2.12 with $M_1, ..., M_k$. More precisely, define spanning subgraphs $G_0, ..., G_k$ of G and edge-disjoint linear forests $F_1, ..., F_k$ such that

(1) $G_0 := G$ and $G_i = G_{i-1} - E(F_i)$ for $i \in [1, k]$,
(2) F_i is a spanning linear forest (each vertex of G_{i-1} has degree 1 or 2 in F_i) in G_{i-1} whose leaves are precisely the vertices in M_i.

Let $G_0 = G$ and suppose that for some $i \in [1, k]$, we already defined $G_0, ..., G_{i-1}$ and $F_1, ..., F_{i-1}$. As $\Delta(F_1 \cup \cdots \cup F_{i-1}) \leq 2(i-1) \leq 20n^{6/7}/\varepsilon$, it follows that $\delta(G_{i-1}) \geq (1+\varepsilon)n - 20n^{6/7}/\varepsilon \geq (1 + 4\varepsilon/5)n$. Since M_i has size at most $\varepsilon n/5$, we can apply Lemma 2.12 to G_{i-1} and M_i and obtain a spanning linear forest F_i in G_{i-1} whose leaves are precisely the vertices in M_i. Set $G_i := G_{i-1} - E(F_i)$.

We claim that G_k is regular. Consider any vertex $u \in V(G_k)$. For every $i \in [1, k], d_{F_i}(u) = 1$ if u is an endvertex of some edge of M_i and $d_{F_i}(u) = 2$ otherwise. Since $M_1, ..., M_k$ partition $E(H)$, we know that $\sum_{i=1}^{k} d_{F_i}(u) = 2k - d_H(u) = 2k - \mathrm{def}_G(u)$. Thus

$$d_{G_k}(u) = d_G(u) - \sum_{i=1}^{k} d_{F_i}(u) = d_G(u) - (2k - \mathrm{def}_G(u)) = \Delta(G) - 2k.$$

Note that $\Delta(G) \geq (1+\varepsilon)n - 20n^{6/7}/\varepsilon \geq (1 + 4\varepsilon/5)n$. Now $\chi'(G_k) = \Delta(G_k)$ by Theorem 1.4. We color the edges of F_i using two distinct colors from $[\Delta(G) - 2k + 1, \Delta(G)]$ for each $i \in [1, k]$. It is clear that any edge $\Delta(G_k)$-coloring of G_k together with this coloring of $\bigcup_{i=1}^{k} F_i$ gives an edge coloring of G using $\Delta(G_k) + 2k = \Delta(G)$ colors.

We lastly check that the procedure above yields a polynomial time algorithm. Given G, taking a vertex u of minimum degree in G, we first check if $G - u$ is $\Delta(G)$-overfull. If yes, then $\chi'(G) = \Delta(G) + 1$ and G can be edge colored using $\Delta(G) + 1$ colors in polynomial time [17]. Thus G contains no $\Delta(G)$-overfull subgraph. If G contains a $\Delta(G)$-full subgraph, then an edge $\Delta(G)$-coloring of G can be found in polynomial time by Lemma 2.10. Thus G contains no $\Delta(G)$-full subgraph. If there exist nonadjacent $u, v \in V(G)\setminus V_\Delta$, we add the edge uv in G. If we reach a point where the resulting graph contains a $\Delta(G)$-full subgraph, we then find an edge $\Delta(G)$-coloring of the graph in polynomial time by Lemma 2.10, which also gives an edge $\Delta(G)$-coloring of G. Thus we assume that every two vertices from $V(G)\setminus V_\Delta$ are adjacent in G. If G is in Condition (c) of Theorem 1.4, then we find an edge $\Delta(G)$-coloring of G in polynomial time by Theorem 1.4. Thus we have Case 1 or Case 2 as described in this proof. If G is in Case 1, it is polynomial time to find the desired matchings (basically find hamiltonian cycles of even length in graphs with large minimum degree by the comments immediately after

Theorem 2.6) to reduce G into a graph satisfying one of the conditions in Theorem 1.4. Then we find an edge $\Delta(G)$-coloring of G in polynomial time by Theorem 1.4. If G is in Case 2, then can construct an edge $\Delta(G)$-coloring of G through the process as described in Case 2. Since Theorem 2.5, Lemma 2.12 and Theorem 1.4 give appropriate running time statements, this can be achieved in time polynomial in n. □

4 | PROOF OF THEOREM 1.4

We will need the following result, which was proved using Chernoff bound.

Lemma 4.1 (Shan [23], lemma 3.2). *There exists a positive integer n_0 such that for all $n \geq n_0$ the following holds. Let G be a graph on $2n$ vertices, and $N = \{x_1, y_1, ..., x_t, y_t\} \subseteq V(G)$, where $t \in [1, n]$. Then $V(G)$ can be partitioned into two parts A and B satisfying the properties below*:

(i) $|A| = |B|$;
(ii) $|A \cap \{x_i, y_i\}| = 1$ *for each* $i \in [1, t]$;
(iii) $|d_A(v) - d_B(v)| \leq n^{2/3} - 1$ *for each* $v \in V(G)$, *where* $d_S(v) = |N_G(v) \cap S|$ *for any* $S \subseteq V(G)$.

Furthermore, one such partition can be constructed in $O(2n^3 \log_2(2n^3))$-time.

Theorem 1.4. *For all $0 < \varepsilon < 1$, there exists n_0 such that the following statement holds. If G is a graph on $2n \geq n_0$ vertices satisfying one of the following three conditions*:

(a) *G is regular with $\delta(G) \geq (1 + 4\varepsilon/5)n$,*
(b) *G has two distinct vertices x, y such that $d(x) = d(y) \geq (1/2 + 3\varepsilon/2)n$, for all $z \in V(G)\setminus\{x, y\}$, $d(z) = \Delta(G) \geq (1 + \varepsilon)n$, and $\Delta(G) - \delta(G) \leq (1/2 - \varepsilon/2)n$,*
(c) *$\Delta(G) - \delta(G) \geq n^{6/7}$, $|V_\delta| \geq n^{6/7}$ and $|V_\Delta| \geq n + 1$, and $\delta(G) \geq (1 + \varepsilon)n$,*

then $\chi'(G) = \Delta(G)$. Furthermore, there is a polynomial time algorithm that finds an optimal coloring.

Proof. Choose positive integer n_0 such that $0 < 1/n_0 \ll \varepsilon$.

If G is in Condition (a), we let $N = \varnothing$. If G is in Condition (b), we let $N = \{x_1, y_1\}$, where $x_1 = x$ and $y_1 = y$. If G is in Condition (c), we take $2\lfloor(2n - |V_\Delta|)/2\rfloor$ vertices from $V(G)\setminus V_\Delta$ and name them as $x_1, y_1, ..., x_t, y_t$, where $t := \lfloor(2n - |V_\Delta|)/2\rfloor$ and we assume that the first $\lfloor|V_\delta|/2\rfloor$ pairs of vertices x_i, y_i are all from V_δ. Let $N = \{x_1, y_1, ..., x_t, y_t\}$. Applying Lemma 4.1 on G and N, we obtain a partition $\{A, B\}$ of $V(G)$ satisfying the following properties:

P.1 $|A| = |B|$;
P.2 $|A \cap \{x_i, y_i\}| = 1$ for each $i \in [1, t]$;
P.3 $|d_A(v) - d_B(v)| \leq n^{2/3} - 1$ for each $v \in V(G)$.

Thus when G is in Condition (b), we may assume $x \in A$ and $y \in B$. When G is in Condition (c), we know that $|A \cap V_\delta| \geq \frac{1}{2}(|V_\delta| - 1)$, $|B \cap V_\delta| \geq \frac{1}{2}(|V_\delta| - 1)$, $|A \cap V_\Delta| \geq n/2$ and $|B \cap V_\Delta| \geq n/2$. By P.3, for any $v \in V(G)$, we have

$$\frac{1}{2}(d_G(v) - n^{2/3}) \leq d_A(v), d_B(v) \leq \frac{1}{2}(d_G(v) + n^{2/3}).$$

Let

$$G_A = G[A], G_B = G[B], \text{ and } \quad H = G[A, B].$$

To prove the theorem, we will construct an edge coloring of G using $\Delta(G)$ colors. We provide below an overview of the steps. At the start of the process, $E(G)$ is assumed to be uncolored, and throughout the process, the partial edge coloring of G is always denoted by φ, which is updating step by step.

Step 1 Define $S = \{v \in V(G) \ : \ \Delta(G) - d_G(v) \geq 7n^{2/3}\}$. Let $k = \max\{\Delta(G_A), \Delta(G_B)\} + 1$. By Theorem 2.1, we find an edge k-coloring φ of $G_A \cup G_B$. If there exist distinct $u, v \in S \cap A$ or distinct $u, v \in S \cap B$ such that $\overline{\varphi}(u) \cap \overline{\varphi}(v) \neq \emptyset$, we add an edge joining u and v and color the new edge by a color in $\overline{\varphi}(u) \cap \overline{\varphi}(v)$. The edge coloring φ is updated and we still call it φ. We iterate this process of adding and coloring edges and call the multigraphs resulting from G_A and G_B, respectively, G_A^* and G_B^*, and call G^* the union of G_A^*, G_B^* and H. We will modify the current edge coloring, which is still named φ, such that the following properties are satisfied:

S1.1 When G is in Conditions (a) or (b),

$$\left|\overline{\varphi}_A^{-1}(i)\right| = \left|\overline{\varphi}_B^{-1}(i)\right| \quad \text{for every } i \in [1, k].$$

When G is in Condition (c), assume by symmetry that $e(G_A^*) \leq e(G_B^*)$, then

$$\left|\overline{\varphi}_A^{-1}(i)\right| \geq \left|\overline{\varphi}_B^{-1}(i)\right| \quad \text{for every } i \in [1, k].$$

S1.2

$$\begin{aligned} &\sum_{u \in A} |\overline{\varphi}(u)| \leq \frac{n}{2}(n^{2/3} + 1) + \frac{n}{2}(6n^{2/3}) + k \leq 4n^{5/3} - 2n, \\ &\sum_{u \in B} |\overline{\varphi}(u)| \leq \frac{n}{2}(n^{2/3} + 1) + \frac{n}{2}(6n^{2/3}) + k \leq 4n^{5/3} - 2n, \qquad \text{(S1.I)} \\ &\left|\overline{\varphi}_A^{-1}(i)\right| \leq 4n^{2/3} \quad \text{and} \quad \left|\overline{\varphi}_B^{-1}(i)\right| \leq 4n^{2/3} \quad \text{for every } i \in [1, k]. \end{aligned}$$

Step 2 Modify the partial edge-coloring of G^* obtained in Step 1 by exchanging alternating paths. When this step is finished, each of the k color class will be a 1-factor of G^*. During the process of this step, a few edges of H will be colored and a few edges of G_A^* and G_B^* will be uncolored. Denote by R_A and R_B, respectively, the submultigraphs of G_A^* and G_B^* consisting of the uncolored edges. The two multigraphs R_A and R_B will initially be empty, but one, two or three edges will be added to at least one of them when each time we exchange an alternating path. The conditions below will be satisfied at the completion of this step:

S2.1 The number of uncolored edges in each of G_A^* and G_B^* is less than $12n^{5/3}$. When G is in Conditions (a) or (b), G_A^* and G_B^* have the same number of uncolored edges; and when G is in Condition (c), the number of uncolored edges in G_B^* is greater than or equal to the number of uncolored edges in G_A^* (this follows from our assumption that $e(G_A^*) \leq e(G_B^*)$).

S2.2 $\Delta(R_A)$ and $\Delta(R_B)$ are less than $n^{5/6} + 1$.

S2.3 Define

$$S_A = \left\{u \in S \cap A \ : \ d_{G_A^*}(u) \leq k - 2n^{2/3}\right\},$$
$$S_B = \left\{u \in S \cap B \ : \ d_{G_B^*}(u) \leq k - 2n^{2/3}\right\}.$$

We require

S2.3.1 Every vertex in $V(G^*)\setminus(S_A \cup S_B)$ is incident in G^* with fewer than $2n^{5/6}$ colored edges of H.

S2.3.2 When G is in Condition (b), each of the vertex from $S_A \cup S_B$ is incident in G^* with fewer than $\left(\frac{1}{4} - \frac{1}{5}\varepsilon\right)n$ colored edges of H.

S2.3.3 When G is in Condition (c), each of the vertex from $S_A \cup S_B$ is incident in G^* with fewer than $\left(\frac{1}{2} - \frac{1}{3}\varepsilon\right)n$ colored edges of H.

Step 3 We will edge color R_A and R_B and a few uncolored edges of H using another ℓ colors, where $\ell = \lceil 2n^{5/6} \rceil$. The goal is to ensure that each of these ℓ new color classes obtained at the completion of Step 3 presents at all vertices from $V(G^*)\setminus V_\delta$ while preserving the k 1-factors already obtained through Steps 1 and 2.

Step 4 At the start of Step 4, all of the uncolored edges of G^* belong to H. Denote by R the subgraph of G^* consisting of the uncolored edges. It will be shown that $\Delta(R) = \Delta(G^*) - k - \ell$. This subgraph is bipartite, so we can color its edges using $\Delta(G^*) - k - \ell$ colors by Theorem 2.2.

When Step 4 is completed, we obtain an edge coloring of G^* using exactly $\Delta(G^*)$ colors. We now give the details of each step, and for concepts that were already defined in the outline above, we will use them directly.

Step 1: Coloring G_A and G_B

Recall $S = \{v \in V(G) \ : \ \Delta(G) - d_G(v) \geq 7n^{2/3}\}$. Note that when G is in Condition (a), $S = \varnothing$; when G is in Condition (b), then $S \subseteq \{x, y\}$; and when G is in Condition (c), then $V_\delta \subseteq S$. Following the operations described in the outline of Step 1, for the current edge coloring φ of $G_A^* \cup G_B^*$, the following statement holds: $\overline{\varphi}(u) \cap \overline{\varphi}(v) = \varnothing$ for any two distinct $u, v \in S \cap A$ or any two distinct $u, v \in S \cap B$. Therefore,

$$\sum_{u\in A\cap S}\left|\bar{\varphi}(u)\right| = \sum_{u\in A\cap S}\left(k - d_{G_A^*}(u)\right) \le k,\ \sum_{u\in B\cap S}\left|\bar{\varphi}(u)\right| = \sum_{u\in B\cap S}\left(k - d_{G_B^*}(u)\right) \le k. \quad \text{(S1.II)}$$

We will in the rest of the proof show that $\chi'(G^*) = \Delta(G^*)$, this is because G is a subgraph of G^* and $\Delta(G^*) = \Delta(G)$. The latter is seen as follows: for any $u \in S \cap A$, we have

$$\begin{aligned} d_{G^*}(u) &\le k + e_G(u, B) \le \frac{1}{2}(\Delta(G) + n^{2/3}) + 1 + \frac{1}{2}(\Delta(G) - 7n^{2/3} + n^{2/3}) \\ &\le \Delta(G). \end{aligned}$$

Similarly, we have $d_{G^*}(u) \le \Delta(G)$ for any $u \in S \cap B$. In particular, if $u \in V_\delta$, as $\Delta(G) - \delta(G) \ge n^{6/7}$, we have

$$d_{G^*}(u) \le \frac{1}{2}(\Delta(G) + n^{2/3}) + 1 + \frac{1}{2}(\Delta(G) - n^{6/7} + n^{2/3}) \le \Delta(G) - \frac{1}{3}n^{6/7}. \quad \text{(S1.III)}$$

Let φ_A and φ_B be the restrictions of φ on G_A^* and G_B^*, respectively. By Lemma 2.3 and the comments immediately below the lemma, we modify φ_A and φ_B into equitable edge k-colorings of G_A^* and G_B^*, respectively, and still call φ the edge k-coloring of $G_A^* \cup G_B^*$ consisting of the modifications of φ_A and φ_B. Note that under the new colorings, it is possible that $\overline{\varphi}(u) \cap \overline{\varphi}(v) \ne \emptyset$ for some distinct $u, v \in S \cap A$ or distinct $u, v \in S \cap B$. However the inequalities in (S1.II) still hold.

When G is in Conditions (a) or (b), we have $|S| \le 2$ and $e(G_A) = e(G_B)$ by the partition $\{A, B\}$ of $V(G)$. Since $|S \cap A| = |S \cap B| \le 1$, it follows that $G_A^* = G_A$ and $G_B^* = G_B$. Thus $e(G_A^*) = e(G_B^*)$. Since φ_A and φ_B are equitable edge k-colorings of G_A^* and G_B^*, by renaming some color names in G_A^* if necessary, we assume

$$\left|\overline{\varphi}_A^{-1}(i)\right| = \left|\overline{\varphi}_B^{-1}(i)\right| \quad \text{for every } i \in [1, k].$$

When G is in Condition (c), by symmetry, we assume $e(G_A^*) \le e(G_B^*)$. For the same reasoning as above, we assume

$$\left|\overline{\varphi}_A^{-1}(i)\right| \ge \left|\overline{\varphi}_B^{-1}(i)\right| \quad \text{for every } i \in [1, k].$$

By the Parity Lemma, $|\overline{\varphi}_A^{-1}(i)| - |\overline{\varphi}_B^{-1}(i)|$ is even for every $i \in [1, k]$. Therefore, we have the statement S1.1 as stated in the outline of Step 1.

Next, we verify that every color $i \in [1, k]$ is missing at a small number of vertices. Property P.2 of the partition $\{A, B\}$ implies $|A \cap V_\Delta| \ge n/2$ and $|B \cap V_\Delta| \ge n/2$, and each vertex $u \in V_\Delta$ satisfies $|\overline{\varphi}(u)| \le n^{2/3} + 1$, call this Fact 1. By the definition of S, for every $u \in V(G^*)\backslash S$, $d_{G^*}(u) = d_G(u) > \Delta(G^*) - 7n^{2/3}$, and Property P.3 of the partition $\{A, B\}$ implies $d_{G_A^*}(u) \ge \frac{1}{2}d_G(u) - n^{2/3}$ for every $u \in A\backslash S$ and $d_{G_B^*}(u) \ge \frac{1}{2}d_G(u) - n^{2/3}$ for every

$u \in B \backslash S$. Thus $|\overline{\varphi}(u)| \le k - (\frac{1}{2}d_G(u) - n^{2/3}) < 6n^{2/3}$ for every $u \in V(G^*)\backslash S$, call this Fact 2. These two facts together with the fact in (S1.II), give

$$\sum_{u \in A} |\overline{\varphi}(u)| \le \frac{n}{2}(n^{2/3}+1) + \frac{n}{2}(6n^{2/3}) + k \le 4n^{5/3} - 2n.$$

Similarly,

$$\sum_{u \in B} |\overline{\varphi}(u)| \le \frac{n}{2}(n^{2/3}+1) + \frac{n}{2}(6n^{2/3}) + k \le 4n^{5/3} - 2n.$$

Since φ_A and φ_B are equitable edge k-colorings of G_A^* and G_B^*, we get

$$\left|\overline{\varphi}_A^{-1}(i)\right| \le 4n^{2/3} \quad \text{and} \quad \left|\overline{\varphi}_B^{-1}(i)\right| \le 4n^{2/3}.$$

Therefore, we have the statement S1.2 as stated in the outline of Step 1.

Step 2: Extending existing color classes into 1-factors

Each of the k color classes obtained in Step 1 will be extended into k 1-factors of G^* through exchanging of alternating paths, which consist of colored edges and uncolored edges. The colored edges and uncolored edges of these alternating paths are from $G_A^* \cup G_B^*$ and H, respectively. Thus during the procedure of Step 2, we will uncolor some of the edges of G_A^* and G_B^*, and will color some of the edges of H. Recall that R_A and R_B are the submultigraphs of G_A^* and G_B^* consisting of the uncolored edges, which are empty initially.

To ensure Condition S2.2 is satisfied, we say that an edge $e = uv \in E(G_A^* \cup G_B^*)$ is *good* if $e \notin E(R_A \cup R_B)$ and the degree of u and v in both R_A and R_B is less than $n^{5/6}$ (actually, note that when $uv \in E(G_A^*)$, then the degree of u and v is zero in R_B and vice versa). Thus a good edge can be added to R_A or R_B without violating S2.2.

By S1.1, for each color $i \in [1, k]$, we pair up each vertex from $\overline{\varphi}_B^{-1}(i)$ with a vertex from $\overline{\varphi}_A^{-1}(i)$, and then pair up the remaining unpaired vertices from $\overline{\varphi}_A^{-1}(i)$ as $|\overline{\varphi}_A^{-1}(i)| - |\overline{\varphi}_B^{-1}(i)|$ is even and we assumed $|\overline{\varphi}_A^{-1}(i)| \ge |\overline{\varphi}_B^{-1}(i)|$. Each of those pairs is called a *missing-common-color pair* or *MCC-pair* in short with respect to the color i. In particular, when G is in Conditions (a) or (b), every vertex from $\overline{\varphi}_A^{-1}(i)$ is paired up with a vertex from $\overline{\varphi}_B^{-1}(i)$.

For every MCC-pair (a, b) with respect to some color $i \in [1, k]$, we will exchange an alternating path P from a to b with at most 11 edges, where, if exist, the first, third, fifth, seventh, ninth, and eleventh edges are uncolored and the second, fourth, sixth, eighth, and tenth edges are good edges colored by i. After P is exchanged, a and b will be incident with edges colored by i, and at most three good edges will be added to each of R_A and R_B. With this information at hand, before demonstrating the existence of such paths, we show that Conditions S2.1, S2.2, and S2.3 can be guaranteed at the end of Step 2. After the completion of Step 1, by (S1.I), the total number of missing colors from vertices in A or from vertices in B is at most $4n^{5/3} - 2n$. Thus there are at most $4n^{5/3} - 2n$ MCC-pairs. For each MCC-pair (a, b) with $a, b \in V(G^*)$, at most three edges will be added to each of

R_A and R_B when we exchange an alternating path from a to b. Thus there will always be fewer than

$$3(4n^{5/3} - 2n) < 12n^{5/3}$$

edges in each of R_A and R_B. Each of the k color classes is a 1-factor of G^* at the end of Step 2. Thus the number of colored edges in G_A^* is the same as that in G_B^*. Since $e(G_A^*) = e(G_B^*)$ when G is in Conditions (a) or (b), and $e(G_B^*) \geq e(G_A^*)$ when G is in Condition (c), we have $e(R_A) = e(R_B)$ when G is in Conditions (a) or (b), and $e(R_B) \geq e(R_A)$ when G is in Condition (c). Thus Condition S2.1 will be satisfied at the end of Step 2. And as we only ever add good edges to R_A and R_B, Condition S2.2 will hold automatically. We now show that Condition S2.3 will also be satisfied. Recall

$$S_A = \left\{u \in S \cap A \ : \ d_{G_A^*}(u) \leq k - 2n^{2/3}\right\} \quad \text{and} \quad S_B = \left\{u \in S \cap B \ : \ d_{G_B^*}(u) \leq k - 2n^{2/3}\right\}.$$

Since $\sum_{u \in A \cap S}(k - d_{G_A^*}(u)), \sum_{u \in B \cap S}(k - d_{G_B^*}(u)) \leq k \leq \frac{1}{2}(\Delta(G) + n^{2/3}) + 1 < 2n$, it follows that

$$|S_A| < n^{1/3} \quad \text{and} \quad |S_B| < n^{1/3}.$$

Thus for every vertex $u \in S \backslash (S_A \cup S_B)$, $|\overline{\varphi}(u)| < 2n^{2/3}$. For every vertex $u \in V(G^*) \backslash S$, as $d_G(u) = d_{G^*}(u) > \Delta(G) - 7n^{2/3}$, it follows that $|\overline{\varphi}(u)| < k - (\frac{1}{2}(\Delta(G) - 7n^{2/3}) - n^{2/3}) < 6n^{2/3}$. Thus for any $u \in V(G^*) \backslash (S_A \cup S_B)$, we have $|\overline{\varphi}(u)| < 6n^{2/3}$. In the process of Step 2, the number of newly colored edges of H that are incident with a vertex $u \in V(G^*) \backslash (S_A \cup S_B)$ will equal the number of alternating paths containing u that have been exchanged. The number of such alternating paths of which u is the first vertex will equal the number of colors that missed at u at the end of Step 1, which is less than $6n^{2/3}$. The number of alternating paths in which u is not the first vertex will equal the degree of u in $R_A \cup R_B$, and so will be less than $n^{5/6} + 1$. Hence the number of colored edges of H that are incident with u will be less than

$$6n^{2/3} + n^{5/6} + 1 < 2n^{5/6}.$$

This applies to all vertices in $V(G^*) \backslash (S_A \cup S_B)$, and so Condition S2.3.1 will be satisfied.

When G is in Condition (b), for any vertex $u \in S_A \cup S_B = \{x, y\}$, since $\Delta(G) - \delta(G) \leq \frac{1}{2}(1 - \varepsilon)n$, we have

$$\begin{aligned} |\overline{\varphi}(u)| &\leq k - \frac{1}{2}(d_G(u) - n^{2/3}) \leq k - \frac{1}{2}\left(\Delta(G) - \frac{1}{2}(1 - \varepsilon)n - n^{2/3}\right) \\ &\leq \frac{1}{4}(1 - \varepsilon)n + n^{2/3} + 1. \end{aligned}$$

Hence the number of colored edges of H that are incident with u will be less than

$$\frac{1}{4}(1-\varepsilon)n + n^{2/3} + 1 + n^{5/6} + 1 < \left(\frac{1}{4} - \frac{1}{5}\varepsilon\right)n.$$

Therefore, Condition S2.3.2 will be satisfied.

When G is in Condition (c), for any vertex $u \in S_A \cup S_B$, since $\Delta(G) - \delta(G) \leq (1-\varepsilon)n$, we have

$$|\overline{\varphi}(u)| \leq k - \frac{1}{2}(d_G(u) - n^{2/3}) \leq k - \frac{1}{2}(\Delta(G) - (1-\varepsilon)n - n^{2/3})$$
$$\leq \frac{1}{2}(1-\varepsilon)n + n^{2/3} + 1.$$

Hence the number of colored edges of H that are incident with u will be less than

$$\frac{1}{2}(1-\varepsilon)n + n^{2/3} + 1 + n^{5/6} + 1 < \left(\frac{1}{2} - \frac{1}{3}\varepsilon\right)n.$$

Therefore, Condition S2.3.3 will be satisfied.

We now show below the existence of alternating paths for MCC-pairs. For a given color $i \in [1, k]$, and vertices $a \in A$ and $b \in B$, let $N_B(a)$ be the set of vertices in B that are joined with a by an uncolored edge and are incident with a good edge colored i such that the good edge is not incident with any vertex of S_B, and let $N_A(b)$ be the set of vertices in A that are joined with b by an uncolored edge and are incident with a good edge colored i such that the good edge is not incident with any vertex of S_A. To estimate the sizes of $N_A(b)$ and $N_B(a)$, we show that A and B contain only a few vertices that either miss the color i or are incident with a non-good edge colored i. By S2.1, there are at most $12n^{5/3}$ edges in R_B, so there are fewer than $24n^{5/6}$ vertices of degree at least $n^{5/6}$ in R_B. Each non-good edge is incident with one or two vertices of R_B through the color i, so there are fewer than $48n^{5/6}$ vertices in B that are incident with a non-good edge colored i. Furthermore, there are at most $2|S_B| \leq 2n^{1/3}$ vertices in B that are either contained in S_B or adjacent to a vertex from S_B through an edge with color i. Finally, there are fewer than $4n^{2/3}$ vertices in B that are missed by the color i. So the number of vertices in B that are not incident with a good edge colored i such that the good edge is not incident with any vertex from S_B is less than

$$48n^{5/6} + 2n^{1/3} + 4n^{2/3} < 49n^{5/6}.$$

By symmetry, the number of vertices in A that are not incident with a good edge colored i such that the good edge is not incident with any vertex from S_A is less than $49n^{5/6}$. By S2.3.1, when $\{a, b\} \cap (S_A \cup S_B) = \varnothing$,

$$|N_A(b)|, |N_B(a)| \geq \frac{1}{2}((1 + 4\varepsilon/5)n - n^{2/3}) - 2n^{5/6} - 49n^{5/6} > \left(\frac{1}{2} + \frac{1}{3}\varepsilon\right)n. \quad \text{(S2.I)}$$

When G is in Condition (b) and $\{a, b\} \cap \{x, y\} \neq \varnothing$, by S2.3.2, we have

$$|N_A(b)|, |N_B(a)| \geq \frac{1}{2}((1/2 + 3\varepsilon/2)n - n^{2/3}) - \left(\frac{1}{4} - \frac{1}{5}\varepsilon\right)n - 49n^{5/6} > \frac{3}{4}\varepsilon n. \quad \text{(S2.II)}$$

When G is in Condition (c) and $\{a, b\} \cap (S_A \cup S_B) \neq \emptyset$, by S2.3.3, we have

$$|N_A(b)|, |N_B(a)| \geq \frac{1}{2}((1 + \varepsilon)n - n^{2/3}) - \left(\frac{1}{2} - \frac{1}{3}\varepsilon\right)n - 49n^{5/6} > \frac{1}{2}\varepsilon n. \quad \text{(S2.III)}$$

Let $M_B(a)$ be the set of vertices in B that are joined with a vertex in $N_B(a)$ by an edge of color i, and let $M_A(b)$ be the set of vertices in A that are joined with a vertex in $N_A(b)$ by an edge of color i. Note that $(S_A \cup S_B) \cap (M_A(b) \cup M_B(a)) = \emptyset$ by the choice of $N_A(b)$ and $N_B(a)$. Note also that $|M_B(a)| = |N_B(a)|$ but some vertices may be in both. Similarly $|M_A(b)| = |N_A(b)|$.

For a MCC-pair (a, b), to have a unified discussion as in the case that $\{a, b\} \cap (S_A \cup S_B) = \emptyset$, if necessary, by exchanging an alternating path of length 2 from a to another vertex a^*, and exchanging an alternating path from b to another vertex b^*, we will replace the pair (a, b) by (a^*, b^*) such that $\{a^*, b^*\} \cap (S_A \cup S_B) = \emptyset$. Precisely, we will implement the following operations to vertices in $S_A \cup S_B$. For any vertex $a \in S_A$, and for each color $i \in \overline{\varphi}(a)$, we take an edge b_1b_2 with $b_1 \in N_B(a)$ and $b_2 \in M_B(a)$ such that b_1b_2 is colored by i, where the edge b_1b_2 exists by (S2.II)–(S2.III) and the fact that $|M_B(a)| = |N_B(a)|$. Then we exchange the path ab_1b_2 by coloring ab_1 with i and uncoloring the edge b_1b_2 (See Figure 1A). After this, the edge ab_1 of H is now colored by i, and the uncolored edge b_1b_2 is added to R_B. We then update the original MCC-pair that contains a with respect to the color i by replacing the vertex a with b_2. We do this at the vertex a for every color $i \in \overline{\varphi}(a)$ and then repeat the same process for every vertex in S_A. Similarly, for any vertex $b \in S_B$, and for each color $i \in \overline{\varphi}(b)$, we take an edge a_1a_2 with $a_1 \in N_A(b)$ and $a_2 \in M_A(b)$ such that a_1a_2 is colored by i, where the edge a_1a_2 exists by (S2.II)–(S2.III) and the fact that $|M_B(b)| = |N_B(b)|$. Then we exchange the path ba_1a_2 by coloring ba_1 with i and uncoloring the edge a_1a_2. The same, we update the original MCC-pair that contains b with respect to the color i by replacing the vertex b with a_2.

After the procedure above, we have now three types MCC-pair (u, v): $u, v \in A$, $u, v \in B$, and A contains exactly one of u and v and B contains the other. However, in either case, $\{u, v\} \cap (S_A \cup S_B) = \emptyset$. We will exchange alternating path for each of such pairs.

We deal with each of the colors from $[1, k]$ in turn. Let $i \in [1, k]$ be a color. We consider first an MCC-pair (a, a^*) with respect to i such that $a, a^* \in A$. By (S2.I), we have $|M_B(a^*)| > \left(\frac{1}{2} + \frac{1}{3}\varepsilon\right)n$. We take an edge $b_1^*b_2^*$ colored by i with $b_1^* \in N_B(a^*)$ and $b_2^* \in M_B(a^*)$. Then again, by (S2.I), we have $|M_B(a)|, |M_A(b_2^*)| > \left(\frac{1}{2} + \frac{1}{3}\varepsilon\right)n$. Therefore, as each vertex $c \in M_A(b_2^*)$ satisfies $|N_B(c)| > \left(\frac{1}{2} + \frac{1}{3}\varepsilon\right)n$, we have $|N_B(c) \cap M_B(a)| \geq \frac{2}{3}\varepsilon n$. We take $a_2a_2^*$ colored by i with $a_2^* \in N_A(b_2^*)$ and $a_2 \in M_A(b_2^*)$. Then we let $b_2 \in N_B(a_2) \cap M_B(a)$, and let b_1 be the vertex in $N_B(a)$ such that b_1b_2 is colored by i. Now we get the alternating path $P = ab_1b_2a_2a_2^*b_2^*b_1^*a^*$ (See Figure 1C). We exchange P by coloring ab_1, b_2a_2, $a_2^*b_2^*$ and $b_1^*a^*$ with color i and uncoloring the edges b_1b_2, $b_1^*b_2^*$ and

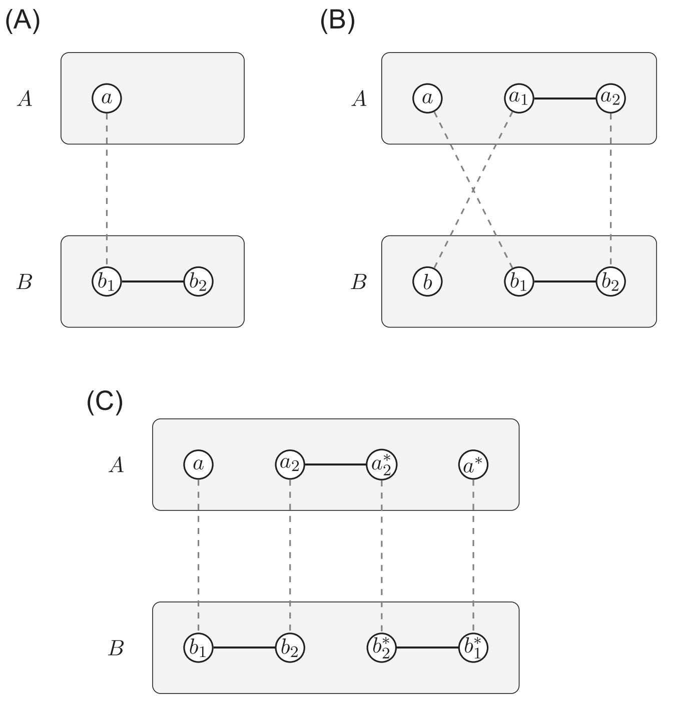

FIGURE 1 The alternating path P. Dashed lines indicate uncolored edges, and solid lines indicate edges with color i. (A) P with 3 edges (B) P with 5 edges (C) P with 7 edges

$a_2a_2^*$. After the exchange, the color i appears on edges incident with a and a^*, the edges b_1b_2 and $b_1^*b_2^*$ are added to R_B and the edge $a_2a_2^*$ is added to R_A. We added at most one edge to each of R_A and R_B when we updated the original MCC-pair corresponding to (a, a^*). Thus we added at most three edges to each of R_A and R_B when we modify φ to have the color i present at both of the vertices in the original MCC-pair corresponding to (a, a^*). By symmetry, we can deal with an MCC-pair (b, b^*) with respect to i such that $b, b^* \in B$ similarly as above.

Thus we consider an MCC-pair (a, b) with respect to i such that $a \in A$ and $b \in B$. By (S2.I), we have $|M_B(a)|, |M_A(b)| > \left(\frac{1}{2} + \frac{1}{3}\varepsilon\right)n$. We choose a_1a_2 with color i such that $a_1 \in N_A(b)$ and $a_2 \in M_A(b)$. Now as $|M_B(a)|, |N_B(a_2)| > \left(\frac{1}{2} + \frac{1}{3}\varepsilon\right)n$ by (S2.I), we know that $N_B(a_2) \cap M_B(a) \neq \emptyset$. We choose $b_2 \in N_B(a_2) \cap M_B(a)$ and let $b_1 \in N_B(a)$ such that b_1b_2 is colored by i. Then $P = ab_1b_2a_2a_1b$ is an alternating path from a to b (See Figure 1B). We exchange P by coloring ab_1, b_2a_2 and a_1b with color i and uncoloring the edges a_1a_2 and b_1b_2. After the exchange, the color i appears on edges incident with a and b, the edge a_1a_2 is added to R_A and the edge b_1b_2 is added to R_B. We added at most one edge to each of R_A and R_B when we updated the original MCC-pair corresponding to (a, b). Thus we added at most three edges to each of R_A and R_B when we modify φ to have the color i present at both of the vertices in the original MCC-pair corresponding to (a, b). By finding such paths for all MCC-pairs with respect to the color i, we can increase the number of edges colored i until the color class is a 1-factor of G^*. By doing this for all colors, we can make each of the k color classes a 1-factor of G^*.

Step 3: Coloring R_A and R_B and extending the new color classes

Each of the color classes for the colors from $[1, k]$ is now a 1-factor of G^*. We now consider the multigraphs R_A and R_B that consist of the uncolored edges of G_A^* and G_B^*. By Condition S2.1, R_A and R_B each has fewer than $12n^{5/3}$ edges, and $\Delta(R_A), \Delta(R_B) < n^{5/6} + 1$. Note that R_A and R_B might contain parallel edges with endvertices in S. By Theorem 2.1 and Theorem 2.3, R_A and R_B each have an equalized edge-coloring with exactly $\ell := \lceil 2n^{5/6} \rceil$ colors $k + 1, ..., k + \ell$.

If G is in Conditions (a) or (b), then we have $e(R_A) = e(R_B)$. Under these two conditions, by renaming some color classes of R_A if necessary, we can assume that in the edge colorings of R_A and R_B, each color appears on the same number of edges in R_A as it does in R_B. When G is in Condition (c), by our assumption that G_B^* has more edges than G_A^* does, we have $e(R_A) \le e(R_B)$. In this case, we can assume that in the edge colorings of R_A and R_B, the number of edges with a color $i \in [k + 1, k + \ell]$ in R_B is at least the number of edges with a color $i \in [k + 1, k + \ell]$ in R_A.

There are fewer than $12n^{5/3}$ edges in each of R_A and R_B, and $\ell > n^{5/6}$, so each of the color $i \in [k + 1, k + \ell]$ appears on fewer than $12n^{5/6} + 1$ edges in each of R_A and R_B. We will now color some of the edges of H with the ℓ colors from $[k + 1, k + \ell]$ so that each of these color classes present at vertices from $V(G^*)\setminus V_\delta$. We perform the following procedure for each of the ℓ colors in turn.

Given a color i with $i \in [k + 1, k + \ell]$, we let A_i and B_i be the sets of vertices in A and B respectively that are incident with edges colored i. Note that $|A_i| \le |B_i| < 2(12n^{5/6} + 1)$ as R_A and R_B each contains fewer than $12n^{5/6} + 1$ edges colored i. Note that if G is in Conditions (a) or (b), we have $|A_i| = |B_i|$; and we might have $|B_i| \ge |A_i|$ when G is in Condition (c). When G is in Condition (c) and $|B_i| > |A_i|$, we let

$$A_i^* \subseteq (V_\delta \cap A)\setminus A_i$$

such that $|A_i^*| + |A_i| = |B_i|$, and just let $A_i^* = \emptyset$ otherwise. Note that such A_i^* exists as $|V_\delta \cap A| \ge \frac{1}{2}n^{6/7} - 1$ and $|A_i|, |B_i| < 2(12n^{5/6} + 1)$. Let H_i be the subgraph of H obtained by deleting the vertex sets $A_i \cup A_i^*$ and B_i and removing all colored edges. We will show next that H_i has a perfect matching and we will color the edges in the matching by the color i.

Each vertex in $V(G^*)\setminus(S_A \cup S_B)$ is incident with fewer than $2n^{5/6} + \ell \le 5n^{5/6}$ edges of H that are colored, since fewer than $2n^{5/6}$ were colored in Step 2 by S2.3.1 and at most $2n^{5/6} + 2 < 3n^{5/6}$ have been colored in Step 3. Also each vertex in G^* has fewer than $2(12n^{5/6} + 1)$ edges that join it with a vertex in A_i or B_i. So each vertex from $V(H_i)\setminus(S_A \cup S_B)$ is adjacent in H_i to more than

$$\frac{1}{2}((1 + 4\varepsilon/5)n - n^{2/3}) - 5n^{5/6} - 2(12n^{5/6} + 1) > \frac{1}{2}(1 + \varepsilon/2)n$$

vertices.

When G is in condition (b), each vertex in $S_A \cup S_B$ is incident with fewer than $\left(\frac{1}{4} - \frac{1}{5}\varepsilon\right)n + 3n^{5/6}$ edges of H that are colored, since fewer than $\left(\frac{1}{4} - \frac{1}{5}\varepsilon\right)n$ were colored in Step 2 by S2.3.2 and at most $3n^{5/6}$ have been colored in Step 3. Also each vertex in G^*

has fewer than $2(12n^{5/6}+1)$ edges that join it with a vertex in A_i or B_i. So when G is in Condition (b), each vertex from $S_A \cup S_B$ is adjacent in H_i to more than

$$\frac{1}{2}((1/2+3\varepsilon/2)n-n^{2/3})-\left(\left(\frac{1}{4}-\frac{1}{5}\varepsilon\right)n+3n^{5/6}\right)-2(12n^{5/6}+1)>\frac{3}{4}\varepsilon n$$

vertices.

When G is in condition (c), each vertex in $S_A \cup S_B$ is incident with fewer than $(\frac{1}{2}-\frac{1}{3}\varepsilon)n+3n^{5/6}$ edges of H that are colored, since fewer than $(\frac{1}{2}-\frac{1}{3}\varepsilon)n$ were colored in Step 2 by S2.3.3 and at most $3n^{5/6}$ have been colored in Step 3. Also each vertex in G^* has fewer than $2(12n^{5/6}+1)$ edges that join it with a vertex in $A_i \cup A_i^*$ or B_i. So when G is in Condition (c), each vertex from $S_A \cup S_B$ is adjacent in H_i to more than

$$\frac{1}{2}((1+\varepsilon)n-n^{2/3})-\left(\left(\frac{1}{2}-\frac{1}{3}\varepsilon\right)n+3n^{5/6}\right)-2(12n^{5/6}+1)>\frac{1}{2}\varepsilon n$$

vertices.

Thus $\delta(H_i)\geq\frac{1}{2}\varepsilon n$ in either case and H_i has at most $|S_A \cup S_B|\leq 2n^{1/3}<\frac{1}{2}\varepsilon n$ vertices of degree less than $\frac{1}{2}n$. So H_i has a 1-factor F by Lemma 2.11. If we color the edges of F with the color i, then every vertex in $V(G^*)\backslash A_i^*$ is incident with an edge of color i. We repeat this procedure for each of the colors from $[k+1,k+\ell]$. After this has been done, each of these ℓ colors presents at all vertices from $V(G^*)\backslash V_\delta$. So at the conclusion of Step 3, all of the edges in G_A^* and G_B^* are colored, some of the edges of H are colored, each of the k color classes for colors from $[1,k]$ is a 1-factor of G^*, and each of the ℓ colors from $[k+1,k+\ell]$ presents at all vertices from $V(G^*)\backslash V_\delta$.

Step 4: Coloring the graph R

Let R be the subgraph of G^* consisting of the remaining uncolored edges. These edges all belong to H, so R is a subgraph of H and hence is bipartite. We claim that $\Delta(R)=\Delta(G^*)-k-\ell$. Note that every vertex from $V(G^*)\backslash V_\delta$ presents every color from $[1,k+\ell]$ and so those vertices have degree at most $\Delta(G^*)-k-\ell$ in R. For the vertices from V_δ, they present all the colors from $[1,k]$. Thus by (S1.III), those vertices have degree at most

$$\Delta(G^*)-\frac{1}{3}n^{6/7}-k<\Delta(G^*)-k-\ell$$

in R. By Theorem 2.2 we can color the edges of R with $\Delta(R)$ colors from $[k+\ell+1,\Delta(G^*)]$. Thus $\chi'(G^*)\leq k+\ell+(\Delta(G^*)-k-\ell)=\Delta(G^*)$ and so $\chi'(G^*)=\Delta(G^*)$, as desired.

Lastly, we check that there is a polynomial time algorithm to obtain an edge $\Delta(G)$-coloring of G. By Lemma 4.1, we can obtain a desired partition $\{A,B\}$ of $V(G)$ in polynomial time. Also, it is polynomial time to edge color G_A and G_B by an algorithm described in [17]. Modifying G_A and G_B into G_A^* and G_B^* and the corresponding edge colorings into equalized edge-colorings can be done in polynomial time too. In Step 2, the construction of the alternating paths and swaps of the colors on the paths can be done in $O(n^3)$-time, as the total number of colors missing at vertices is $O(n^2)$ and it takes $O(n)$-time to find an alternating path for a MCC-pair. In Step 3, there is a polynomial time algorithm (see e.g. [14]) to edge color R_A and R_B using at most ℓ colors. Then by doing Kempe changes as mentioned in the comments immediately after Theorem 2.3, these edge colorings can be modified into equalized edge-colorings in polynomial time. The last step is to edge color the bipartite graph R using $\Delta(R)$ colors, which can be done in polynomial-time in n, for example, using an algorithm from [5]. Thus, there is a polynomial time algorithm that gives an edge coloring of G using $\Delta(G)$ colors. □

ACKNOWLEDGMENTS

The second author wishes to thank Dr. Mark Ellingham for bringing up reference [1] to her attention. The authors are very grateful to the two anonymous referees for their careful reading and valuable comments.

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How to cite this article: M. J. Plantholt and S. Shan, *Edge coloring graphs with large minimum degree*, J. Graph Theory. (2022), 1–22. https://doi.org/10.1002/jgt.22889