Journal of Combinatorial Theory, Series B 157 (2022) 429–450

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

Chromatic index of dense quasirandom graphs

Songling Shan [1]

Illinois State University, Normal, IL 61790, United States of America

ARTICLE INFO

Article history:
Received 9 April 2021
Available online xxxx

Keywords:
Chromatic index
Overfull graph
Quasirandom graph

ABSTRACT

Let G be a simple graph with maximum degree $\Delta(G)$. A subgraph H of G is overfull if $|E(H)| > \Delta(G)\lfloor |V(H)|/2 \rfloor$. Chetwynd and Hilton in 1986 conjectured that a graph G on n vertices with $\Delta(G) > n/3$ has chromatic index $\Delta(G)$ if and only if G contains no overfull subgraph. Glock, Kühn and Osthus in 2016 showed that the conjecture is true for dense quasirandom graphs with even order, and they conjectured that the same should hold for such graphs with odd order. In this paper, we show that the conjecture of Glock, Kühn and Osthus is affirmative.

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1. Introduction

In this paper, a graph means a simple graph; and a multigraph may contain parallel edges but no loops. Let G be a multigraph. Denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. For $v \in V(G)$, $N_G(v)$ is the set of neighbors of v in G, and $d_G(v)$, the degree of v in G, is the number of edges of G that are incident with v. When G is simple, $d_G(v) = |N_G(v)|$. For $S \subseteq V(G)$, the subgraph of G induced on S is denoted by $G[S]$, and $G - S := G[V(G) \setminus S]$. For notational simplicity, we write $G - x$ for $G - \{x\}$. If $F \subseteq E(G)$, then $G - F$ is obtained from G by deleting all the edges of

E-mail address: sshan12@ilstu.edu.

[1] Partially supported by NSF grant DMS-2153938.

https://doi.org/10.1016/j.jctb.2022.08.001
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F. Let $V_1, V_2 \subseteq V(G)$ be two disjoint vertex sets. Then $E_G(V_1, V_2)$ is the set of edges in G with one end in V_1 and the other end in V_2, and $e_G(V_1, V_2) := |E_G(V_1, V_2)|$. We write $E_G(v, V_2)$ and $e_G(v, V_2)$ if $V_1 = \{v\}$ is a singleton. We also write $G[V_1, V_2]$ to denote the bipartite subgraph of G with vertex set $V_1 \cup V_2$ and edge set $E_G(V_1, V_2)$.

For two integers p, q, let $[p, q] = \{i \in \mathbb{Z} : p \leq i \leq q\}$. Let $k \geq 0$ be an integer. An *edge k-coloring* of G is a mapping φ from $E(G)$ to the set of integers $[1, k]$, called *colors*, such that no two adjacent edges receive the same color with respect to φ. The *chromatic index* of G, denoted $\chi'(G)$, is defined to be the smallest integer k so that G has an edge k-coloring. We denote by $\mathcal{C}^k(G)$ the set of all edge k-colorings of G. A graph G is *Δ-critical* if $\chi'(G) = \Delta(G) + 1 = \Delta + 1$ and $\chi'(H) < \Delta + 1$ for every proper subgraph H of G. In 1960's, Vizing [22] and, independently, Gupta [8] proved that for all simple graphs G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. This leads to a natural classification of graphs. Following Fiorini and Wilson [5], we say a graph G is of *class 1* if $\chi'(G) = \Delta(G)$ and of *class 2* if $\chi'(G) = \Delta(G) + 1$. Holyer [10] showed that it is NP-complete to determine whether an arbitrary graph is of class 1. Nevertheless, if a graph G has too many edges, i.e., $|E(G)| > \Delta(G)\lfloor |V(G)|/2 \rfloor$, then we have to color $E(G)$ using exactly $(\Delta(G) + 1)$ colors. Such graphs are *overfull*. An overfull subgraph H of G with $\Delta(H) = \Delta(G)$ is called a *$\Delta(G)$-overfull subgraph* of G.

Applying Edmonds' matching polytope theorem, Seymour [16] showed that whether a graph G contains an overfull subgraph of maximum degree $\Delta(G)$ can be determined in polynomial time. A number of long-standing conjectures listed in *Twenty Pretty Edge Coloring Conjectures* in [18] lie in deciding when a Δ-critical graph is overfull. Chetwynd and Hilton [2,3], in 1986, proposed the following conjecture.

Conjecture 1.1 *(Overfull conjecture). Let G be a simple graph with $\Delta(G) > \frac{1}{3}|V(G)|$. Then $\chi'(G) = \Delta(G)$ if and only if G contains no $\Delta(G)$-overfull subgraph.*

The degree condition $\Delta(G) > \frac{1}{3}|V(G)|$ in the conjecture above is best possible, as seen by the graph P^*, which is obtained from the Petersen graph by deleting one vertex. If the overfull conjecture is true, then the NP-complete problem of determining the chromatic index becomes polynomial-time solvable for graphs G with $\Delta(G) > \frac{|V(G)|}{3}$. Despite its importance, very little is known about its truth. It was confirmed only for graphs with $\Delta(G) \geq |V(G)| - 3$ by Chetwynd and Hilton [3] in 1989. By restricting the minimum degree, Plantholt [15] in 2004 showed that the overfull conjecture is affirmative for graphs G with even order n and minimum degree $\delta \geq \sqrt{7}n/3 \approx 0.8819n$. The 1-factorization conjecture is a special case of the overfull conjecture, which in 2016 was confirmed for large graphs by Csaba, Kühn, Lo, Osthus and Treglown [4]. The overfull conjecture is still wide open in general, and it seems extremely difficult even for graphs G with $\Delta(G) \geq |V(G)| - 4$.

Recently in 2016, Glock, Kühn and Osthus [6] showed that the overfull conjecture is true for dense quasirandom graphs of even order. Following their definition, for the notion of quasirandomness, the following one-sided version of ε-regularity will be considered.

Let $0 < \varepsilon, p < 1$. A graph G on n vertices is called *lower-(p,ε)-regular* if we have $e_G(S,T) \geq (p-\varepsilon)|S||T|$ for all disjoint $S, T \subseteq V(G)$ with $|S|, |T| \geq \varepsilon n$. In particular, the following result was proved in [6, Theorem 1.6].

Theorem 1.2. *For all $0 < p < 1$ there exist $\varepsilon, \eta > 0$ such that for sufficiently large n, the following holds: Suppose G is a lower-(p,ε)-regular graph on n vertices and n is even. Moreover, assume that $\Delta(G) - \delta(G) \leq \eta n$. Then $\chi'(G) = \Delta(G)$ if and only if G contains no $\Delta(G)$-overfull subgraph. Further, there is a polynomial-time algorithm which finds an optimal coloring.*

Glock, Kühn and Osthus [6] conjectured that the same result as in Theorem 1.2 should hold for such graphs G with odd order. We here confirm the conjecture.

Theorem 1.3. *For all $0 < p < 1$ there exist $\varepsilon, \eta > 0$ such that for sufficiently large n, the following holds: Suppose G is a lower-(p,ε)-regular graph on n vertices and n is odd. Moreover, assume that $\Delta(G) - \delta(G) \leq \eta n$. Then $\chi'(G) = \Delta(G)$ if and only if G is not overfull. Further, there is a polynomial-time algorithm which finds an optimal coloring.*

For a lower-(p,ε)-regular graph with odd order n, it is easy to see that for any subset $X \subseteq V(G)$ with $|X|$ odd and $3 \leq |X| \leq n-2$, we have $e_G(X, V(G) \setminus X) \geq \Delta(G)$. Thus $G[X]$ is not $\Delta(G)$-overfull. Therefore, the only possible $\Delta(G)$-overfull subgraph in G is G itself.

The remainder of this paper is organized as follows. In the next section, we list some preliminary results on quasirandom graphs and edge colorings. In Section 3, we study the chromatic index of a regular lower-(p,ε)-regular star-multigraph, which is obtained from a lower-(p,ε)-regular graph by adding a new vertex and some edges between the graph and the new vertex. In the last section, we prove Theorem 1.3.

2. Preliminaries

We will use the following notation: $0 < a \ll b \leq 1$. Precisely, if we say a claim is true provided that $0 < a \ll b \leq 1$, then this means that there exists a non-decreasing function $f : (0,1] \to (0,1]$ such that the statement holds for all $0 < a, b \leq 1$ satisfying $a \leq f(b)$.

2.1. Properties of lower-(p,ε)-regular graphs

A lower-(p,ε)-regular graph can be slightly modified so it is still lower-(p,ε')-regular for some $\varepsilon' \leq \varepsilon$, as listed in the following proposition.

Proposition 2.1 *([6], Proposition 3.1). Let $0 < 1/n_0 \ll \varepsilon, p < 1$, and let G be a lower-(p,ε)-regular graph on $n \geq n_0$ vertices. Then the following hold:*

(1) If G' is obtained from G by adding a new vertex w and arbitrary edges at w, then G' is lower-$(p, 2\varepsilon)$-regular.
(2) Let H be a graph on $V(G)$ such that $\Delta(H) \leq \eta n$ for some $0 < \eta < 1$, and let $\varepsilon' = \max\{2\varepsilon, 2\sqrt{\eta}\}$. Then $G - E(H)$ is lower-(p, ε')-regular.
(3) If $U \subseteq V(G)$ has size at least βn for some $0 < \beta < 1$, then $G[U]$ is lower-$(p, \varepsilon/\beta)$-regular.

A multigraph G is a *star-multigraph* if G has a vertex x that is incident with all multiple edges of G. In other words, $G - x$ is a simple graph. The vertex x is called the *multi-center* of G. For $0 < \varepsilon, p < 1$, a multigraph G is a lower-(p, ε)-regular *star-multigraph* if it is a star-multigraph such that its underlying simple graph is lower-(p, ε)-regular. Since we will deal with a lower-(p, ε)-regular graph of odd order, for convenient analyses, we will add a new vertex and some edges between the new vertex and the graph to form a star-multigraph of even order.

A path P connecting two vertices u and v is called a (u, v)-*path*, and we write uPv or vPu to specify the two endvertices of P. Let uPv and xQy be two disjoint paths. If vx is an edge, we write $uPvxQy$ as the concatenation of P and Q through the edge vx.

The following result was proved in [6] for lower-(p, ε)-regular graphs, and we here modify it for lower-(p, ε)-regular star-multigraphs.

Lemma 2.2 *([6], Lemma 7.2). Let $0 < 1/n_0 \ll \varepsilon \ll \alpha, p < 1$, and G be a lower-(p, ε)-regular graph on $n \geq n_0$ vertices such that $\delta(G) \geq \alpha n$. Moreover, let $M = \{a_1b_1, \ldots, a_tb_t\}$ be a matching in the complete graph on $V(G)$ of size at most $\alpha n/5$. Then there exist vertex-disjoint path $P_1, \ldots, P_t$ in G such that $\bigcup V(P_i) = V(G)$ and P_i joins a_i to b_i, and these paths can be found in polynomial time.*

Lemma 2.3. *Let $0 < 1/n_0 \ll \varepsilon \leq \eta \ll \alpha, p < 1$, and G be a lower-(p, ε)-regular star-multigraph on $n \geq n_0$ vertices such that $\delta(G) \geq \alpha n$ and $e_G(x, v) \leq \eta n$ for any $v \in V(G)$, where x is the multi-center of G. Moreover, let $M = \{a_1b_1, \ldots, a_tb_t\}$ be a matching in the complete graph on $V(G)$ of size at most $\alpha n/6$. If $|N_G(x) \setminus \{a_1, b_1, \ldots, a_t, b_t\}| \geq 2$, then there exist vertex-disjoint paths $P_1, \ldots, P_t$ in G such that $\bigcup V(P_i) = V(G)$ and P_i joins a_i to b_i, and these paths can be found in polynomial time.*

Proof. By relabeling the matching edges if necessary, assume that if $x \in \{a_1, b_1, \ldots, a_t, b_t\}$, then $x = a_t$.

If $x \in \{a_1, b_1, \ldots, a_t, b_t\}$ and so $x = a_t$, then let $a'_t \in N_G(x) \setminus \{a_1, b_1, \ldots, a_t, b_t\}$, and $M' = (M \setminus \{a_tb_t\}) \cup \{a'_tb_t\}$. If $x \notin \{a_1, b_1, \ldots, a_t, b_t\}$, then let $x_1, x_2 \in N_G(x) \setminus \{a_1, b_1, \ldots, a_t, b_t\}$ be distinct, and let $M' = (M \setminus \{a_tb_t\}) \cup \{a_tx_1, x_2b_t\}$. Note that $\delta(G - x) \geq (\alpha - \eta)n \geq (\frac{5}{6}\alpha + \frac{5}{n})n$ and $|M'| \leq \frac{1}{6}\alpha n + 1 = (\frac{1}{6}\alpha + \frac{1}{n})n$, and $G - x$ is lower-$(p, 2\varepsilon)$-regular by Proposition 2.1(3). Applying Lemma 2.2 to $G - x$ with matching M', we find vertex-disjoint paths $P'_1, \ldots, P'_t$ in $G - x$ such that $\bigcup V(P'_i) = V(G - x)$. Furthermore, if $x = a_t$, P'_i joins a_i to b_i for $i \in [1, t-1]$, and P'_t joins a'_t to b_t; if $x \notin \{a_1, b_1, \ldots, a_t, b_t\}$,

P_i' joins a_i to b_i for $i \in [1, t-1]$, P_t' joins a_t to x_1, and P_{t+1}' joins x_2 to b_t. Letting $P_i = P_i'$ for $i \in [1, t-1]$, and $P_t = a_t a_t' P_t' b_t$ if $x = a_t$ and $P_t = a_t P_t' x_1 x x_2 P_{t+1}' b_t$ if $x \notin \{a_1, b_1, \ldots, a_t, b_t\}$ gives the desired paths for G and M. By Lemma 2.2 and the simple adjustment on M above, it is clear that these paths can be found also in polynomial time. □

Lemma 2.4. *Let $0 < 1/n_0 \ll \varepsilon, \gamma \ll \alpha, p < 1$, and G be a lower-(p, ε)-regular star-multigraph on $n \geq n_0$ vertices with multi-center x such that $\delta(G) \geq \alpha n$ and $e_G(x, v) \leq \gamma n$ for any $v \in V(G)$. Let $X, Y \subseteq V(G)$ be disjoint and $|X| = |Y|$, and H be a graph with $V(H) = X \cup Y$ and $E(H)$ obtained from $E_G(X, Y)$ by deleting some edges. If $\delta(H) > \frac{1}{4}\alpha n$ and each vertex $v \in V(H) \setminus \{x\}$ is incident with at most γn edges from $E_G(X, Y) \setminus E(H)$, then H has a perfect matching. Furthermore, a perfect matching of H can be found in polynomial time.*

Proof. If the multi-center of G is contained in $X \cup Y$, we may assume by symmetry that the multi-center is contained in X. To have a unified proof, if the multi-center of G is contained in X, we let x be the multi-center; otherwise we let x be an arbitrary vertex from X.

Since $\delta(H) > 0$, x has in H a neighbor $y \in Y$. Let $X_1 = X \setminus \{x\}$ and $Y_1 = Y \setminus \{y\}$. It suffices to show that $H_1 := H[X_1, Y_1]$ satisfies Hall's condition. For otherwise, there exists $A \subseteq X_1$ such that $B := N_{H_1}(A)$ satisfying $|B| < |A|$. Since $\delta(H_1) \geq \delta(H) - \max\{e_G(x, v) : v \in V(H)\} \geq \frac{1}{4}\alpha n - \gamma n$ and $|B| < |A|$, it follows that $|A| > \frac{1}{4}\alpha n - \gamma n$. On the other hand, since $E_{H_1}(A, Y_1 \setminus B) = \emptyset$, $Y_1 \setminus B \neq \emptyset$, and $\delta(H_1) \geq \frac{1}{4}\alpha n - \gamma n$, it follows that $|X_1 \setminus A| \geq \frac{1}{4}\alpha n - \gamma n$. Let $B_1 = Y_1 \setminus B$. Thus $|A| \leq |X_1| - (\frac{1}{4}\alpha n - \gamma n)$ and so $|B_1| = |Y_1| - |B| > |Y_1| - |A| \geq \frac{1}{4}\alpha n - \gamma n$. Now $|A| > \frac{1}{4}\alpha n - \gamma n > \varepsilon n$ and $|B_1| > \frac{1}{4}\alpha n - \gamma n > \varepsilon n$. By the lower-$(p, \varepsilon)$-regularity of G, we have $e_G(A, B_1) \geq (p - \varepsilon)|A||B_1|$. Since each vertex from B_1 is incident in G with at most γn edges from $E_G(X, Y) \setminus E(H)$ and at most γn edges with the other endvertex as x, we have $e_{G-E(H)}(A, B_1) \leq (\gamma n + \gamma n)|B_1|$. Therefore $e_{H_1}(A, B_1) \geq e_G(A, B_1) - e_{G-E(H)}(A, B_1) \geq (p - \varepsilon)|A||B_1| - 2\gamma n|B_1|$. Since

$$(p - \varepsilon)|A| > (p - \varepsilon)(\frac{1}{4}\alpha n - \gamma n) > 2\gamma n,$$

we get $e_{H_1}(A, B_1) > 0$, showing a contradiction.

There are polynomial-time algorithms such as the Hopcroft-Karp algorithm [11] in finding a maximum matching in any bipartite graph, thus a perfect matching of H can be found in polynomial time. □

2.2. Results on edges colorings

Let G be a multigraph and $\varphi \in \mathcal{C}^k(G)$ for some integer $k \geq 0$. For any $v \in V(G)$, the set of colors *present* at v is $\varphi(v) = \{\varphi(f) : f \in E(G) \text{ is incident to } v\}$, and the set of colors *missing* at v is $\overline{\varphi}(v) = [1, k] \setminus \varphi(v)$. For a vertex set $X \subseteq V(G)$, define

$\overline{\varphi}(X) = \bigcup_{v\in X}\overline{\varphi}(v)$. The set X is called *elementary* with respect to φ or simply φ-*elementary* if $\overline{\varphi}(u)\cap\overline{\varphi}(v)=\emptyset$ for every two distinct vertices $u,v\in X$. For two distinct colors $\alpha,\beta\in[1,\Delta]$, the components of the subgraph induced by edges with colors α or β are called (α,β)-chains. Clearly, each (α,β)-chain is either a path or an even cycle. If we interchange the colors α and β on an (α,β)-chain C of G, we get a new edge k-coloring of G, which is denoted by φ/C. This operation is called a *Kempe change*. For an (α,β)-chain P, if it is a path with an endvertex x, we also denote it by $P_x(\alpha,\beta,\varphi)$ to stress the endvertex x. An (α,β)-*swap* at x is just the Kempe change performed on $P_x(\alpha,\beta,\varphi)$.

Let $x,y\in V(G)$. If x and y are contained in the same (α,β)-chain of G with respect to φ, we say x and y are (α,β)-*linked* with respect to φ. Otherwise, x and y are (α,β)-*unlinked* with respect to φ.

The fan argument was introduced by Vizing [20,21] in his classic results on the upper bounds of chromatic indices for simple graphs and multigraphs. Multifans are generalized version of Vizing fans given by Stiebitz et al. [18].

Let G be a multigraph with maximum degree Δ. For an edge $e=rs_1\in E(G)$ and a coloring $\varphi\in\mathcal{C}^{\Delta}(G-e)$, a *multifan* centered at r w.r.t. e and φ is a sequence $F=(r,rs_1,s_1,rs_2,s_2,\ldots,rs_p,s_p)$ with $p\geq 1$ consisting of distinct vertices $r,s_1,s_2,\ldots,s_p$ and edges $rs_1,rs_2,\ldots,rs_p$ satisfying the following condition:

(F1) For every edge rs_i with $i\in[2,p]$, there exists $j\in[1,i-1]$ such that $\varphi(rs_i)\in\overline{\varphi}(s_j)$.

The set of vertices $r,s_1,\ldots,s_p$ contained in F is denoted by $V(F)$. The following result regarding a multifan can be found in [18, Theorem 2.1], where an edge e of G is critical if $\chi'(G-e)<\chi'(G)$.

Lemma 2.5. *Let G be a multigraph with $\chi'(G)=k\geq\Delta(G)+1$, $e=rs_1$ be a critical edge and $\varphi\in\mathcal{C}^k(G-e)$. If F is a multifan w.r.t. e and φ, then $V(F)$ is φ-elementary.*

In 1960's, Vizing [22] and, independently, Gupta [8] proved the following result, which can be proved by using the multifan arguments, where the multiplicity $\mu(G)$ of a multigraph G is $\max\{e_G(u,v) : u,v\in V(G)\}$.

Theorem 2.6. *For every multigraph G with multiplicity μ, $\chi'(G)\leq\Delta(G)+\mu$.*

Misra and Gries [14] described a polynomial-time algorithm for coloring the edges of any simple graph G with at most $\Delta(G)+1$ colors.

Proved by König [12] that every bipartite multigraph has chromatic index as its maximum degree.

Theorem 2.7. *Every bipartite multigraph G satisfies $\chi'(G)=\Delta(G)$.*

Lemma 2.8. *If G is a star-multigraph, then $\chi'(G) \leq \Delta(G)+1$. Furthermore, an edge coloring of G using at most $\Delta(G)+1$ colors can be found in polynomial time.*

Proof. Assume to the contrary that $\chi'(G) \geq \Delta(G)+2$. By deleting edges and vertices from G if necessary, we may assume that every edge e of G is critical, i.e., $\chi'(G-e) < \chi'(G)$. Assume first that there exists $u \in V(G)$ such that u is not incident with any multiple edges of G. Let $v \in N_G(u)$. Since uv is critical, $\chi'(G-uv) \leq \chi'(G)-1$. Let $\varphi \in \mathcal{C}^k(G-uv)$ where $k = \chi'(G)-1 \geq \Delta(G)+1$, and let F be a maximum multifan with respect to uv and φ and centered at u. Since there are $k \geq \Delta(G)+1$ colors, for every $w \in V(G)$, $\overline{\varphi}(w) \neq \emptyset$. By Lemma 2.5, if $w \in V(F) \setminus \{u\}$, then every color from $\overline{\varphi}(w)$ presents at u. Since $v \in V(F)$ and F is maximum, we conclude that $N_G(u) \subseteq V(F)$. Thus, as φ is an edge k-coloring of $G-uv$, and $V(F)$ is φ-elementary by Lemma 2.5, we have

$$\begin{aligned}
|\overline{\varphi}(V(F))| &= |\overline{\varphi}(u)| + |\overline{\varphi}(v)| + \sum_{w \in V(F)\setminus\{u,v\}} |\overline{\varphi}(w)| \\
&\geq (k-(d_G(u)-1)) + (k-(d_G(v)-1)) + (d_G(u)-1)(k-\Delta(G)) \\
&\geq 2 + (k-d_G(u)) + d_G(u)(k-\Delta(G)) \\
&\geq k+2,
\end{aligned}$$

where we used that facts that $|V(F) \setminus \{u,v\}| = d_G(u)-1$ and $k-\Delta(G) \geq 1$. The inequality gives a contradiction as $|\overline{\varphi}(V(F))| \leq k$ by $V(F)$ being φ-elementary.

Thus we assume that every vertex of G is incident with some multiple edges. Since G is a star-multigraph, it has a vertex x such that $V(G) = N_G(x) \cup \{x\}$. Let $N_G(x) = \{y_1, \ldots, y_t\}$, and let $d_i = e_G(x, y_i)$ for each $i \in [1,t]$. Since G is a star-multigraph, for each y_i, x is the only vertex such that there are possibly multiple edges between y_i and x. Thus $d_G(x) = \sum_{i=1}^{t} d_i = \Delta(G)$ and $d(y_i) \leq d_i + t - 1$. Let H be the underlying simple graph of G. It is readily check that $\Delta(H) \leq t$. By Theorem 2.6, $\chi'(H) \leq t+1$. By assigning a different color to each of the edges in $E(G) \setminus E(H)$, we see that $\chi'(G) \leq \chi'(H) + \sum_{i=1}^{t}(d_i - 1) \leq \Delta(G)+1$. This gives a contradiction to the assumption that $\chi'(G) \geq \Delta(G)+2$.

For the complexity of edge coloring G using at most $\Delta(G)+1$ colors, we analyze it below. The colors available will be $[1, \Delta(G)+1]$. Misra and Gries [14] described a polynomial-time algorithm for coloring the edges of any simple graph H with at most $(\Delta(H)+1)$ colors. Thus we first edge color the underlying simple graph of $G[N_G(x) \cup \{x\}]$ using at most $|N_G(x)|+1$ colors. This can be done in time of a polynomial in $|N_G(x)|$. Then we color the other edges of $G[N_G(x) \cup \{x\}]$ greedily, which results in a coloring using at most $\Delta(G)+1$ colors by the same argument as in the previous paragraph. It takes $O(|N_G(x)| + \Delta(G))$ steps to edge color $G[N_G(x) \cup \{x\}]$ using at most $\Delta(G)+1$ colors. If all edges of G are already colored, then we are done. Otherwise, we greedily

color the other edges of G until we encounter an edge uv_1 such that u and v_1 do not have a common missing color with respect to the current edge coloring, say φ. We may also assume that for any color $\alpha \in \overline{\varphi}(u)$ and any color $\beta \in \overline{\varphi}(v_1)$, u and v_1 are (α, β)-linked. By symmetry, assume $u \notin N_G(x)$. We construct a multifan centered at u with respect to uv_1 and φ. Assume $F = (u, uv_1, v_1, uv_2, v_2, \ldots, uv_s, v_s)$ is a maximum multifan centered at u. As shown in the first paragraph of this proof, the vertex set of the multifan is not φ-elementary.

Assume first that there exist v_i with $i \in [2, s]$ and a color γ such that $\gamma \in \overline{\varphi}(u) \cap \overline{\varphi}(v_i)$. By the construction of F, we may assume $v_1, \ldots, v_i$ is the sequence such that $\varphi(uv_j) \in \overline{\varphi}(v_{j-1})$ for $j \in [2, i]$. We color the edge uv_1 using the color on uv_2, recolor uv_j using the color on uv_{j+1} for each $j \in [2, i-1]$, and recolor uv_i by the color γ. Thus we assume that $\overline{\varphi}(u) \cap \overline{\varphi}(v_i) = \emptyset$ for every $i \in [1, s]$ and that there exist distinct $i, j \in [1, s]$ and a color γ such that $\gamma \in \overline{\varphi}(v_i) \cap \overline{\varphi}(v_j)$. Let $\alpha \in \overline{\varphi}(u)$. As at least one of v_i and v_j is (α, γ)-unlinked with u, say v_j, we do an (α, γ)-swap at v_j. (When both of v_i and v_j are (α, γ)-unlinked with u, we assume $j < i$.) Then $F^* = (u, uv_1, v_1, uv_2, v_2, \ldots, uv_j, v_j)$ is still a multifan with respect to uv_1 and the current edge coloring, but u and v_j have a common missing color. We can then again color uv_1 as in the first case. Thus we can color all the remaining edges of G in this way using at most $\Delta(G) + 1$ colors. For each uncolored edge such as uv_1, it takes $O(|V(G)|)$ steps to have it colored. As it takes $O(|N_G(x)| + \Delta(G))$ steps to edge color $G[N_G(x) \cup \{x\}]$, it then takes $O(|E(G)||V(G)|)$ steps to edge color G using at most $\Delta(G) + 1$ colors. □

Given an edge coloring of G, since all vertices not missing a given color α are saturated by the matching that consists of all edges colored by α in G, we have the Parity Lemma below, which has appeared in many papers, for example, see [7, Lemma 2.1].

Lemma 2.9 *(Parity Lemma). Let G be an n-vertex multigraph and $\varphi \in \mathcal{C}^k(G)$ for some integer $k \geq \Delta(G)$. Then for any color $\alpha \in [1, k]$, $|\{v \in V(G) : \alpha \in \overline{\varphi}(v)\}| \equiv n \pmod 2$.*

Let G be a multigraph, $k \geq 0$ be an integer and $\varphi \in \mathcal{C}^k(G)$. For a subset X of $V(G)$ and a color $i \in [1, k]$, define $\overline{\varphi}_X^{-1}(i) = \{v \in X : i \in \overline{\varphi}(v)\}$, and $e(X) = |E(G[X])|$. An edge k-coloring of a multigraph G is said to be *equalized* if each color class contains either $\lfloor |E(G)|/k \rfloor$ or $\lceil |E(G)|/k \rceil$ edges. McDiarmid [13] observed the following result.

Theorem 2.10. *Let G be a graph with chromatic index $\chi'(G)$. Then for all $k \geq \chi'(G)$, there is an equalized edge-coloring of G with k colors.*

We will need the following weaker version of "equalized" edge k-coloring.

Lemma 2.11. *Let G be a star-multigraph on $2n$ vertices with multi-center x, and let A and B be a partition of $V(G)$ with $|A| = |B|$, where we assume $x \in A$. If $e(A) = e(B)$, $E_G(A, B) = E_G(x, B)$, and G has an edge coloring using k colors, then there exists an*

edge coloring φ using k colors such that for each $i, j \in [1,k]$, $|\overline{\varphi}_A^{-1}(i)| = |\overline{\varphi}_B^{-1}(i)|$ and $\big||\overline{\varphi}_A^{-1}(i)| - |\overline{\varphi}_A^{-1}(j)|\big| \leq 2$. Furthermore, such a coloring φ can be found in $O(k^2n^2)$-time.

Proof. We first show that there exists an edge k-coloring φ of G such that $|\overline{\varphi}_A^{-1}(i)| = |\overline{\varphi}_B^{-1}(i)|$ for each $i \in [1,k]$. Among all edge k-colorings of G, we choose any φ such that

$$d_\varphi := \sum_{i=1}^{k} \big||\overline{\varphi}_A^{-1}(i)| - |\overline{\varphi}_B^{-1}(i)|\big|$$

is minimum. If $d_\varphi = 0$, then we are done. Thus $d_\varphi \geq 1$. By the Parity Lemma, for each i, $|\overline{\varphi}_A^{-1}(i)| + |\overline{\varphi}_B^{-1}(i)| \equiv 2n \pmod 2$. Thus $|\overline{\varphi}_A^{-1}(i)| - |\overline{\varphi}_B^{-1}(i)| \equiv 2n \pmod 2$. Since $d_\varphi > 0$, we assume, by symmetry, that there exists $i \in [1,k]$ such that $|\overline{\varphi}_A^{-1}(i)| \equiv |\overline{\varphi}_B^{-1}(i)| \pmod 2$ and $|\overline{\varphi}_A^{-1}(i)| - |\overline{\varphi}_B^{-1}(i)| \geq 2$. Since $e(A) = e(B)$, it follows that $\sum_{v\in A} d_G(v) = \sum_{v\in B} d_G(v)$. This together with the fact that $|A| = |B|$, implies

$$\mathrm{def}(A) := \sum_{v\in A}(k - d_G(v)) = \sum_{v\in B}(k - d_G(v)) =: \mathrm{def}(B).$$

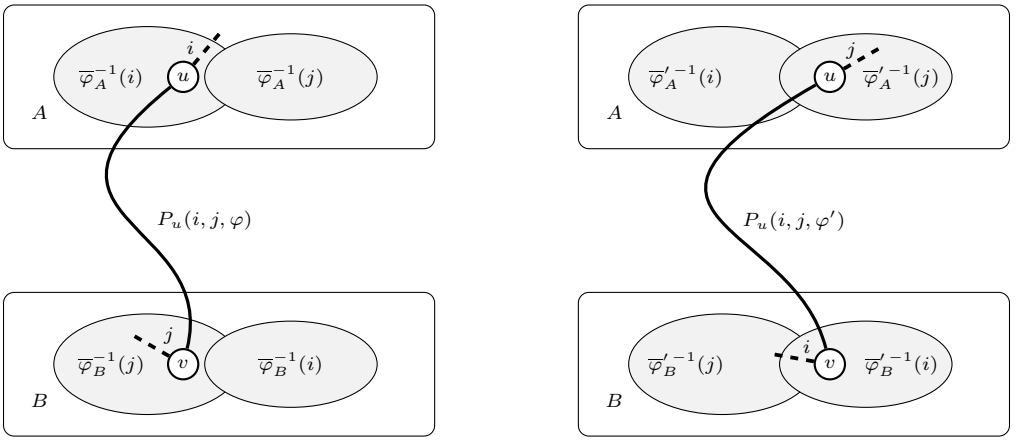
On the other hand,

$$\mathrm{def}(A) = \sum_{i=1}^{k} |\overline{\varphi}_A^{-1}(i)| \quad \text{and} \quad \mathrm{def}(B) = \sum_{i=1}^{k} |\overline{\varphi}_B^{-1}(i)|.$$

Therefore $\sum\limits_{i=1}^{k} |\overline{\varphi}_A^{-1}(i)| = \sum\limits_{i=1}^{k} |\overline{\varphi}_B^{-1}(i)|$. Consequently, by the existence of $i \in [1,k]$ such that $|\overline{\varphi}_A^{-1}(i)| - |\overline{\varphi}_B^{-1}(i)| \geq 2$, there exists $j \in [1,k]$ with $j \neq i$ such that $|\overline{\varphi}_B^{-1}(j)| - |\overline{\varphi}_A^{-1}(j)| \geq 2$. Next, we will do some Kempe changes on some (i,j)-chains to obtain another edge k-coloring φ' such that $d_{\varphi'} < d_\varphi$, which will lead a contradiction to the choice of φ.

If there exist distinct $u, v \in \overline{\varphi}_A^{-1}(i)$ such that u and v are (i,j)-linked with respect to φ, then we let $\varphi' = \varphi/P_u(i,j,\varphi)$. Clearly, $d_{\varphi'} = d_\varphi - 4$, showing a contradiction to the choice of φ. Thus for any two distinct $u, v \in \overline{\varphi}_A^{-1}(i)$, u and v are (i,j)-unlinked with respect to φ. Similarly, for any two distinct $u, v \in \overline{\varphi}_B^{-1}(j)$, u and v are (i,j)-unlinked with respect to φ. Thus, for every $u \in \overline{\varphi}_A^{-1}(i)$, the other end of $P_u(i,j,\varphi)$ is in $\overline{\varphi}_A^{-1}(j) \cup \overline{\varphi}_B^{-1}(i) \cup \overline{\varphi}_B^{-1}(j)$. Similarly, for every $u \in \overline{\varphi}_B^{-1}(j)$, the other end of $P_u(i,j,\varphi)$ is in $\overline{\varphi}_A^{-1}(i) \cup \overline{\varphi}_A^{-1}(j) \cup \overline{\varphi}_B^{-1}(i)$. Since $|\overline{\varphi}_A^{-1}(j)| \leq |\overline{\varphi}_B^{-1}(j)| - 2$ and $|\overline{\varphi}_B^{-1}(i)| \leq |\overline{\varphi}_A^{-1}(i)| - 2$ by the choices of the colors i and j, we have $|\overline{\varphi}_A^{-1}(j)| + |\overline{\varphi}_B^{-1}(i)| \leq |\overline{\varphi}_B^{-1}(j)| - 2 + |\overline{\varphi}_A^{-1}(i)| - 2$. Thus there exist $u \in \overline{\varphi}_A^{-1}(i)$ and $v \in \overline{\varphi}_B^{-1}(j)$ such that u and v are (i,j)-linked with respect to φ, see Fig. 1(a). We let $\varphi' = \varphi/P_u(i,j,\varphi)$, see Fig. 1(b). Again $d_{\varphi'} = d_\varphi - 4$, showing a contradiction to the choice of φ.

Thus, we assume that G has an edge k-coloring φ such that $|\overline{\varphi}_A^{-1}(i)| = |\overline{\varphi}_B^{-1}(i)|$ for every $i \in [1,k]$, which is called a *valid* coloring. We choose a valid edge k-coloring



(a): the existence of u and v (b): $\varphi' = \varphi/P_u(i,j,\varphi)$

Fig. 1. A Kempe change on an (i,j)-chain, where a dashed line indicates a missing color.

φ such that $g_\varphi := \max_{i,j} \left||\overline{\varphi}_A^{-1}(i)| - |\overline{\varphi}_A^{-1}(j)|\right|$ is the smallest and subject to this, the number h_φ of color pairs (i,j) such that $\left||\overline{\varphi}_A^{-1}(i)| - |\overline{\varphi}_A^{-1}(j)|\right| = g_\varphi$ is the smallest. If $g_\varphi \le 2$, then we are done. Thus $g_\varphi \ge 3$. We assume, without loss of generality, that there exist $i,j \in [1,k]$ such that $|\overline{\varphi}_A^{-1}(i)| - |\overline{\varphi}_A^{-1}(j)| = g_\varphi \ge 3$. As $|\overline{\varphi}_A^{-1}(i)| = |\overline{\varphi}_B^{-1}(i)|$ and $|\overline{\varphi}_A^{-1}(j)| = |\overline{\varphi}_B^{-1}(j)|$, we know $|\overline{\varphi}_B^{-1}(i)| - |\overline{\varphi}_B^{-1}(j)| = g_\varphi \ge 3$. If there exist $u \in \overline{\varphi}_A^{-1}(i)$ and $v \in \overline{\varphi}_B^{-1}(i)$ such that $P_u(i,j,\varphi) = P_v(i,j,\varphi)$, we let $\psi = \varphi/P_u(i,j,\varphi)$. Clearly ψ is still a valid coloring with $g_\psi \le g_\varphi$ and $h_\psi < h_\varphi$, showing a contradiction to the choice of φ.

Thus for any $u \in \overline{\varphi}_A^{-1}(i)$ and any $v \in \overline{\varphi}_B^{-1}(i)$, $P_u(i,j,\varphi) \ne P_v(i,j,\varphi)$. As $E_G(A,B) = E_G(x,B)$ and different (i,j)-chains are disjoint, there is at most one $u \in \overline{\varphi}_A^{-1}(i)$ and $v \in \overline{\varphi}_B^{-1}(j)$ such that u and v are (i,j)-linked; similarly, there is at most one $u \in \overline{\varphi}_B^{-1}(i)$ and $v \in \overline{\varphi}_A^{-1}(j)$ such that u and v are (i,j)-linked. As $|\overline{\varphi}_A^{-1}(i)| - |\overline{\varphi}_A^{-1}(j)| \ge 3$ and $|\overline{\varphi}_B^{-1}(i)| - |\overline{\varphi}_B^{-1}(j)| \ge 3$, there exist distinct $u_1,u_2 \in \overline{\varphi}_A^{-1}(i)$ and distinct $v_1,v_2 \in \overline{\varphi}_B^{-1}(i)$ such that $P_{u_1}(i,j,\varphi) = P_{u_2}(i,j,\varphi)$ and $P_{v_1}(i,j,\varphi) = P_{v_2}(i,j,\varphi)$. We now let $\varphi' = \varphi/P_{u_1}(i,j,\varphi)$. Note that $P_{v_1}(i,j,\varphi') = P_{v_1}(i,j,\varphi)$. We then let $\psi = \varphi'/P_{v_1}(i,j,\varphi')$. Again ψ is valid with $g_\psi \le g_\varphi$ and $h_\psi < h_\varphi$, showing a contradiction to the choice of φ. Thus we have a valid coloring φ such that $\left||\overline{\varphi}_A^{-1}(i)| - |\overline{\varphi}_A^{-1}(j)|\right| \le 2$ for all $i,j \in [1,k]$.

In the first step, it takes $O(kn^2)$ steps to find a valid edge coloring φ, as for each $i \in [1,k]$, it takes $O(n)$-time to decrease $|\overline{\varphi}_A^{-1}(i)| - |\overline{\varphi}_B^{-1}(i)|$ by 4; and it takes at most $n/2$ steps to eventually have $|\overline{\varphi}_A^{-1}(i)| = |\overline{\varphi}_B^{-1}(i)|$. In the second step, there are at most $\binom{k}{2}$ color pairs, each color pair takes $O(n)$-time to reduce the difference of the two corresponding color classes by 2, and $O(n)$ steps to make the two corresponding color classes close in size. Thus a desired edge coloring φ can be found in $O(k^2n^2)$-time. □

3. Edge coloring regular lower-(p, ε)-regular star-multigraphs

For a multigraph G, $S \subseteq V(G)$, and $v \in V(G)$, we let $N_S(v) = N_G(v) \cap S$, and let $d_S(v) = e_G(v, S \setminus \{v\})$. The proofs in this section follow and extend ideas of Vaughan from [19]. We will need the following version of Chernoff bound. (See e.g. [1, Theorem A.1.16].)

Lemma 3.1. *Let $X_1, \ldots, X_n$ be mutually independent random variables that satisfy $E(X_i) = 0$ and $|X_i| \leq 1$ for each $i \in [1, n]$. Set $S = X_1 + \ldots + X_n$. Then for any $a > 0$,*

$$\Pr(|S| > a) < 2e^{-a^2/2n}.$$

Lemma 3.2. *There exists a positive integer n_0 such that for all $n \geq n_0$ the following holds. Let G be a graph on $2n$ vertices, and $N = \{x_1, y_1, \ldots, x_t, y_t\} \subseteq V(G)$, where $1 \leq t \leq n$ is an integer. Then $V(G)$ can be partitioned into two parts A and B satisfying the properties below:*

(i) $|A| = |B|$;
(ii) $|A \cap \{x_i, y_i\}| = 1$ *for each* $i \in [1, t]$;
(iii) $|d_A(v) - d_B(v)| \leq n^{2/3} - 1$ *for each* $v \in V(G)$.

Furthermore, one such partition can be constructed in $O(2n^3 \log_2(2n^3))$-time.

Proof. Set A and B be two emptysets or "containers" for now. We first partition $V(G)$ into n pairs such that each pair (x_i, y_i) is partitioned into the same pair and the partition of $V(G) \setminus N$ is arbitrary. We then assign one vertex of each pair to A and the other to B uniformly at random. After the assignment, suppose the pairs are $(a_1, b_1), \ldots, (a_n, b_n)$ with $a_i \in A$ and $b_i \in B$. Fix a vertex v, and define the random variables $X_1, \ldots, X_n$ as below:

$$X_i = e_G(v, a_i) - e_G(v, b_i).$$

Clearly, $X_i \in \{-1, 0, 1\}$. So $|X_i| \leq 1$. If $a_i, b_i \in N(v)$ or $a_i, b_i \notin N(v)$, then $\Pr(X_i = 0) = 1$. If $|\{a_i, b_i\} \cap N(v)| = 1$, then $\Pr(X_i = 1) = \Pr(X_i = -1) = 1/2$. Thus $E(X_i) = 0$. Also it is easy to verify that for distinct $i, j \in [1, n]$, $\Pr(X_i = x | X_j = y) = \Pr(X_i = x)$ for all $x, y \in \{-1, 0, 1\}$. Thus $X_1, \ldots, X_n$ are mutually independent. Let $S = X_1 + \ldots + X_n$. Then $d_A(v) - d_B(v) = S$. By Lemma 3.1, for each $v \in V(G)$,

$$\begin{aligned}\Pr(|d_A(v) - d_B(v)| > n^{2/3} - 1) = \Pr(|S| > n^{2/3} - 1) &\leq \Pr(|S| > 0.9n^{2/3}) \\ &< 2e^{-0.4n^{1/3}},\end{aligned}$$

where for $n \geq 32$, we have $n^{2/3} - 1 > 0.9n^{2/3}$. There are $2n$ vertices, so the probability p that there is a vertex v for which the inequality in condition (iii) does not hold is less than

$$4ne^{-0.4n^{1/3}},$$

which is less than 0.481 when $n \geq 30000$. Thus for $n \geq n_0 := 30000$, there must be some partition of $V(G)$ into two equal parts A and B satisfying condition (ii) such that $|d_A(v) - d_B(v)| \leq n^{2/3} - 1$ for each $v \in V(G)$.

By a result of Srivastav and Stangier [17, Theorem 2.12], a partition that satisfies condition (iii) with probability ε for some $\varepsilon > 0$ can be constructed deterministically in $O(2n \times n^2 \log_2(\frac{2n \times n^2}{\varepsilon}))$-time, as desired. □

Theorem 3.3. *Let $0 < 1/n_0 \ll \varepsilon \leq \eta \ll \alpha \leq p < 1$, and let G be a regular lower-(p, ε)-regular star-multigraph on $2n \geq n_0$ vertices. Suppose x is the multi-center, $2 \leq |N_G(x)| \leq 2n - 2$, $e_G(x, v) \leq \eta n$ for every $v \in V(G)$, and $\delta(G) \geq 2\alpha n$. Then G is 1-factorizable or equivalently $\chi'(G) = \Delta(G)$. Furthermore, there is a polynomial-time algorithm that finds an optimal coloring.*

Proof. Let $y \in V(G) \setminus N_G(x)$. We take $2\lfloor \frac{|N_G(x)|}{2} \rfloor$ vertices from $N_G(x)$ and name them as $x_1, y_1, \ldots, x_t, y_t$, where $t := \lfloor \frac{|N_G(x)|}{2} \rfloor$. Applying Lemma 3.2 on the underlying simple graph of G and $N = \{x_1, y_1, \ldots, x_t, y_t, x_{t+1}, y_{t+1}\}$, where $x_{t+1} = x$ and $y_{t+1} = y$, we obtain a partition $\{A, B\}$ of $V(G)$ satisfying the following properties:

(i) $|A| = |B|$;
(ii) $|A \cap \{x_i, y_i\}| = 1$ for each $i \in [1, t]$;
(iii) $\big||N_A(v)| - |N_B(v)|\big| \leq n^{2/3} - 1$ for each $v \in V(G)$.

By switching x and y from their current partition if necessary, we assume that $d_B(x) \geq d_A(x)$. As G is a star-multigraph, we have

$$|d_A(v) - d_B(v)| \leq n^{2/3} + \eta n$$

for all $v \in V(G) \setminus \{x\}$.

Let

$$G_A = G[A], \quad G_B = G[B], \quad \text{and} \quad H = G[A, B].$$

Define $G_{A,B}$ to be the union of $G[A]$, $G[B]$ together with $(d_B(x) - d_A(x))/2$ edges incident with x from $E(H)$. Since G is regular and $|A| = |B|$, we have $\sum_{v \in A} d_G(v) = \sum_{v \in B} d_G(v) = \Delta(G)n$. Thus $2|E(G_A)| = 2|E(G_B)| = \Delta(G)n - e_G(A, B)$ and so $|E(G_A)| = |E(G_B)|$.

To prove the lemma, we will show that it is possible to find an edge coloring of G using $\Delta(G)$ colors, and we provide a procedure for constructing such an edge coloring. Below gives an overview of the steps. At the start of the process, $E(G)$ is assumed to be uncolored.

Step 1 Find a "near equalized" edge-coloring of $G_{A,B}$ using k colors, where $k = \Delta(G_{A,B}) + 1$ as guaranteed by Lemma 2.8 and Lemma 2.11. Call φ the current edge coloring of $G_{A,B}$. By Lemma 2.11, we can require $|\overline{\varphi}_A^{-1}(i)| = |\overline{\varphi}_B^{-1}(i)|$; furthermore, we require $|\overline{\varphi}_A^{-1}(i)|, |\overline{\varphi}_B^{-1}(i)| \leq \eta^{\frac{1}{2}} n$ for each color $i \in [1, k]$.

Step 2 Modify the partial edge-coloring of G obtained in Step 1 by exchanging alternating paths. When this step is completed, each of the k color classes will be a 1-factor of G. During the process of this step, a few edges of $H - E(G_{A,B})$ will be colored and a few edges of G_A and G_B will be uncolored.

Step 3 Let R_A and R_B be the subgraph of G_A and G_B that is induced by the uncolored edges, respectively. We can ensure both R_A and R_B to be simple, i.e., not contain the vertex x. We find equalized edge-colorings of R_A and R_B using exactly $\ell := \max\{\Delta(R_A), \Delta(R_B)\} + 1$ colors, which is possible by Theorem 2.6 and Theorem 2.10. At the end of Step 3, all the edges in G_A and G_B will be colored, and so will a few edges of $H - E(G_{A,B})$. The goal is to ensure that each of the $k + \ell$ color classes obtained so far is a 1-factor of G.

Step 4 At the start of Step 4, all of the uncolored edges of G belong to $H - E(G_{A,B})$. Also, each color class is a 1-factor, so the subgraph of G consisting of the uncolored edges is regular, of degree $\Delta(G) - k - \ell$. This subgraph is bipartite, so we can color its edges using $\Delta(G) - k - \ell$ colors.

At the conclusion of Step 4, we obtain an edge coloring of G using exactly $\Delta(G)$ colors. We now give the details of each step. Let $\mu(x) = \max\{e_G(x, v) : v \in V(G)\}$, which by the assumption is at most ηn.

Step 1

Let $k = \Delta(G_{A,B}) + 1$. Since G is regular, G_A and G_B have the same number of edges. Also by the construction of $G_{A,B}$, $E_{G(A,B)}(A, B) = E_{G_{A,B}}(x, B)$. By Lemma 2.8 and Lemma 2.11, $G_{A,B}$ has an edge k-coloring φ such that for each $i, j \in [1, k]$, $|\overline{\varphi}_A^{-1}(i)| = |\overline{\varphi}_B^{-1}(i)|$ and $\big||\overline{\varphi}_A^{-1}(i)| - |\overline{\varphi}_A^{-1}(j)|\big| \leq 2$. The coloring φ will be modified throughout the process and will still be named as φ. Note that $k > \delta(G_{A,B}) \geq \alpha n - n^{2/3} - \mu(x) \geq \alpha n - n^{2/3} - \eta n > \frac{2}{3}\alpha n$ and $d_{G_{A,B}}(x) = d_A(x) + (d_B(x) - d_A(x))/2 = (d_A(x) + d_B(x))/2 = \Delta(G)/2$. Since $\Delta(G_{A,B}) - \delta(G_{A,B}) \leq n^{2/3} + \mu(x) \leq n^{2/3} + \eta n$,

$$|\overline{\varphi}(v)| \leq \eta n + n^{2/3} + 1 < 2\eta n \quad \text{for each } v \in A \cup B. \tag{1}$$

So the average number of vertices in A that a color misses is less than

$$\frac{n(n^{2/3}+\eta n+1)}{k} \leq \frac{n(n^{2/3}+\eta n+1)}{\frac{2}{3}\alpha n} < \frac{2\eta n}{\frac{2}{3}\alpha} < \eta^{1/2}n - 2.$$

As any two color classes differ in size by at most two, in this partial edge-coloring of G, we have

$$|\overline{\varphi}_A^{-1}(i)| = |\overline{\varphi}_B^{-1}(i)| < \eta^{1/2}n \quad \text{for each } i \in [1,k]. \tag{2}$$

Step 2

By interchanging alternating paths, we will increase the size of the k color classes obtained in Step 1 until each color class is a 1-factor of G. During the procedure of Step 2, we will uncolor some of the edges of G_A and G_B, and will color some of the edges of $H - E(G_{A,B})$. Denote by R_A and R_B the subgraphs of G_A and G_B consisting of the uncolored edges, which will initially be empty, but one or two edges will be added to each of R_A and R_B each time we exchange an alternating path. We ensure that the following conditions are satisfied after the completion of Step 2:

(i) G_A and G_B have the same number of uncolored edges, which is less than $2\eta n^2+2\eta n$.
(ii) $\Delta(R_A)$ and $\Delta(R_B)$ are less than $2\eta^{\frac{1}{2}}n$.
(iii) Each vertex of $G-x$ is incident with fewer than $3\eta^{\frac{1}{2}}n$ colored edges of H.

To ensure that Condition (ii) is satisfied, we say that an edge $e = uv$ is *good* if $e \notin E(R_A) \cup E(R_B)$ and the degree of each u and v in both R_A and R_B is less than $2\eta^{\frac{1}{2}}n$. Thus a good edge can be added to R_A or R_B without violating Condition (ii).

We will consider the k colors one by one. For each $i \in [1,k]$, since $|\overline{\varphi}_A^{-1}(i)| = |\overline{\varphi}_B^{-1}(i)|$, we can pair up vertices in $\overline{\varphi}_A^{-1}(i)$ with vertices in $\overline{\varphi}_B^{-1}(i)$, and will exchange exactly one alternating path for each such pair. Suppose (a,b) is one of the pairs, where $a \in A$, $b \in B$, and $i \in \overline{\varphi}(a) \cap \overline{\varphi}(b)$. If $x \notin \{a,b\}$, we will exchange an alternating path P from a to b, consisting of five edges with the first, third and fifth edges uncolored and with the second and fourth edges good and colored i. (See Fig. 2(a).) After P is exchanged, a and b will be incident with edges of color i, and one good edge will be added to each of R_A and R_B. If $x \in \{a,b\}$, we will exchange an alternating path P from a to b, consisting of nine edges, where the first, third, fifth, seventh, and ninth edges are uncolored and the second, fourth, sixth, and eighth edges are good edges colored by i. (See Fig. 2(b).) After P is exchanged, a and b will be incident with edges of color i, and two good edges will be added to each of R_A and R_B.

Before demonstrating how such paths can be found, we show that Conditions (i), (ii) and (iii) can be ensured at the end of Step 2. After the completion of Step 1, for each $v \in V(G)$, we have $|\overline{\varphi}(v)| < 2\eta n$ by (1). For each $i \in \overline{\varphi}(a)$ with $a \in A$ and $a \neq x$, exactly one edge will be added to each of R_A and R_B; when $x = a$, exactly two edges will be added to each of R_A and R_B. Thus there will always be fewer than

$$2\eta n^2 - 2\eta n + 4\eta n = 2\eta n^2 + 2\eta n$$

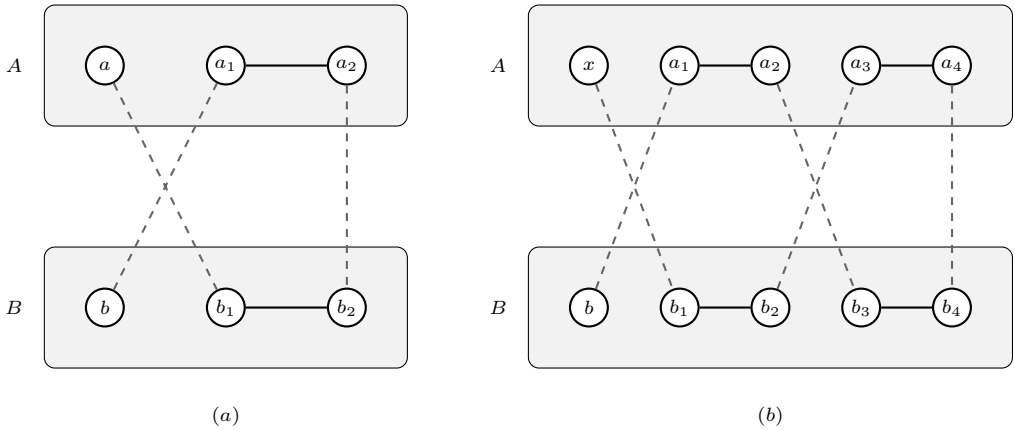


Fig. 2. The alternating path P. Dashed lines indicate uncolored edges, and solid lines indicate edges with color i.

edges in each of R_A and R_B. Therefore Condition (i) will hold at the end of Step 2. As we only ever add good edges to R_A and R_B, Condition (ii) will hold automatically. We now show that Condition (iii) will also be satisfied. Let $v \in V(G) \setminus \{x\}$ be any vertex. After Step 1, the only colored edges in H are those incident with x, and so v is incident with at most ηn colored edges at the beginning of Step 2. In the process of Step 2, the number of newly colored edges of H that are incident with v will be equal to the number of alternating paths of length 5 or 9 containing v that have been exchanged. The number of such alternating paths of which v is the first vertex will be equal to the number of colors that missed v at the end of Step 1, which is less than $2\eta n$. The number of alternating paths in which v is not the first vertex will be equal to the degree of v in R_A, and so will be less than $2\eta^{\frac{1}{2}} n$. Hence the number of colored edges of H that are incident with v will be less than

$$\eta n + 2\eta n + 2\eta^{\frac{1}{2}} n < 3\eta^{\frac{1}{2}} n.$$

This applies to all vertices in $G - x$, and so Condition (iii) will be satisfied.

We now show the existence of such alternating paths. For a pair (a, b) with $a \in A$ and $b \in B$ such that $i \in \overline{\varphi}(a) \cap \overline{\varphi}(b)$, in order to deal with the two cases regarding whether $x = a$ using a unified approach, we deal with colors from $\overline{\varphi}(x)$ first. This initial operation is different from what is outlined in the beginning of Step 2 but the outline applies after this initial step. Note that $|\overline{\varphi}(x)| < 2\eta n$ by (1) at the end of Step 1. For each $i \in \overline{\varphi}(x)$, we will add at most one edge to each of R_A and R_B to replace the pair (x, b) by another pair that both of its vertices miss the color i. After we deal with colors from $\overline{\varphi}(x)$, all the edges of R_A and R_B are still good edges.

So we assume $a = x$ and let $i \in \overline{\varphi}(x)$. If there is a vertex $y \in N_B(x)$ with xy uncolored and $i \in \overline{\varphi}(y)$, we may assume $y = b$ by repairing up vertices in $\overline{\varphi}_A^{-1}(i)$ with vertices in $\overline{\varphi}_B^{-1}(i)$ if necessary. In this case, we simply color xy by the color i. Thus we assume that

for every $y \in N_B(x)$ with xy uncolored, it holds that $i \notin \overline{\varphi}(y)$. Let N_B be the set of vertices in B that are joined with x by an uncolored edge and are incident with a good edge colored i. Since only $(d_B(x) - d_A(x))/2$ edges between x and B are assigned to $G_{A,B}$, x is incident to at least $d_B(x) - (d_B(x) - d_A(x))/2 = d_G(x)/2 \geq \alpha n$ uncolored edges with the other endvertex in B at the end of Step 1. Furthermore, as $|\overline{\varphi}(x)| < 2\eta n$, we know that in H, x is incident to at least $\alpha n - 2\eta n$ edges that are uncolored during this procedure of dealing with colors from $\overline{\varphi}(x)$. Since at this stage all the edges in G_B colored i are good edges, $N_B \neq \emptyset$. We choose $b_1 \in N_B$ and $b_2 \in B$ such that $\varphi(b_1b_2) = i$. Note that such vertex b_2 exists since no edges between A and B are colored by i: during this process, when a color is used on an edge between A and B, the color is already present at x. Likewise, let N_A be the set of vertices in A that are joined with b by an uncolored edge and are incident with a good edge colored i. Note that $x \notin N_A$ as the color i is missing at x. By the same reasoning as above, $N_A \neq \emptyset$. We choose $a_1 \in N_A$ and $a_2 \in A$ such that $\varphi(a_1a_2) = i$. Since the color i was missing at x, $a_2 \neq x$. We now color xb_1 by i and uncolor b_1b_2, color ba_1 by i and uncolor a_1a_2, and we pair up a_2 and b_2 as a pair that both of its vertices miss the color i to replace the original pair (x, b). We do this for every color from $\overline{\varphi}(x)$. After this step, all the k colors are present at x, and each of R_A and R_B contains at most $2\eta n$ edges, and at most $4\eta n$ edges of $H - E(G_{A,B})$ are colored.

Thus we assume (a, b) is a pair with $a \neq x$. The same as before, let N_B be the set of vertices in B that are joined with a by an uncolored edge and are incident with a good edge colored i, and let N_A be the set of vertices in $A \setminus \{x\}$ that are joined with b by an uncolored edge and are incident with a good edge colored i such that the edge is not incident with x. Note that we exclude x from N_A to make R_A simple. There are fewer than $2\eta n^2 + 2\eta n$ edges in R_B, so there are fewer than $2(\eta^{\frac{1}{2}}n + \eta^{\frac{1}{2}})$ vertices of degree at least $2\eta^{\frac{1}{2}}n$ in R_B. Each non-good edge with color i is incident with one or two vertices of R_B, so there are fewer than

$$4(\eta^{\frac{1}{2}}n + \eta^{\frac{1}{2}}) \tag{3}$$

vertices in B that are incident with a non-good edge colored i. In addition, there are fewer than $\eta^{1/2}n$ vertices in B that are missed by the color i by (2). So the number of vertices in B that are not incident with a good edge colored i is less than

$$4(\eta^{\frac{1}{2}}n + \eta^{\frac{1}{2}}) + \eta^{1/2}n < 6\eta^{\frac{1}{2}}n. \tag{4}$$

By symmetry, the number of vertices in $A \setminus \{x\}$ that are not incident with a good edge colored i or an edge from $E_G(x, A \setminus \{x\})$ colored i is less than $4(\eta^{\frac{1}{2}}n + \eta^{\frac{1}{2}}) + \eta^{1/2}n + 2 < 6\eta^{\frac{1}{2}}n$.

So for any vertex $v \in V(G) \setminus \{x\}$, the number of uncolored edges that join v with a vertex w in the other part, where w is incident with a good edge colored i and $x \notin \{v, w\}$, is more than

$$d_H(v) - 3\eta^{\frac{1}{2}}n - 6\eta^{\frac{1}{2}}n \geq \alpha n - n^{2/3} - \eta n - 3\eta^{\frac{1}{2}}n - 6\eta^{\frac{1}{2}}n > \alpha n - 10\eta^{\frac{1}{2}}n > \frac{1}{2}\alpha n,$$

by Condition (iii) and (4). Since $G-x$ is simple and $x \notin \{a, b\}$, it follows that $|N_A| > \frac{1}{2}\alpha n$ and $|N_B| > \frac{1}{2}\alpha n$. Let M_B be the set of vertices in B that are joined with a vertex in N_B by an edge of color i, and let M_A be the set of vertices in A that are joined with a vertex in N_A by an edge of color i. Note that $x \notin M_A$ by the choice of N_A. Note also that $|M_B| = |N_B|$ as each vertex of N_B is incident with an edge colored i, even though some vertices may be in both. Similarly $|M_A| = |N_A|$.

Since $|M_A| \geq \frac{1}{2}\alpha n > 2\varepsilon n$ and $|M_B| \geq \frac{1}{2}\alpha n > 2\varepsilon n$ and G is lower (p, ε)-regular, it follows that $e_G(M_A, M_B) \geq (p-\varepsilon)|M_B||M_A|$. By Condition (iii) of this step, the number of colored edges in H that are incident with a vertex other than x is at most $3\eta^{\frac{1}{2}}n$, we know that the number of colored edges between M_A and M_B is at most $3\eta^{\frac{1}{2}}n|M_A| < (p-\varepsilon)|M_B||M_A|$. Therefore, there is an uncolored edge $a_2b_2 \in E_G(M_A, M_B)$ with $a_2 \in A$ and $b_2 \in B$. We now let $a_1 \in N_A$ and $b_1 \in N_B$ such that a_1a_2 and b_1b_2 are colored by i. Then $P = ab_1b_2a_2a_1b$ is a desired alternating path of five edges, where the first, third and fifth edges are uncolored and the second and fourth edges are good edges colored by i. We color ab_1, b_2a_2 and a_1b by the color i and uncolor b_1b_2 and a_1a_2. After the exchange, the color i appears on edges incident with a and b. By finding such paths for all pairs of vertices (a, b) that miss i, we can increase the number of edges of color i until the color class is a 1-factor of G. By doing this for all colors, we make each of the k color classes a 1-factor of G.

Step 3

Each of the color classes for the colors from $[1, k]$ is now a 1-factor of G. We now consider the graphs R_A and R_B that consist of the uncolored edges of G_A and G_B. Since the vertex x was excluded from N_A and M_A in Step 2, $x \notin V(R_A)$. Thus both R_A and R_B are simple graphs. By Conditions (i) and (ii) of Step 2, R_A and R_B each has fewer than $2\eta n^2 + 2\eta n$ edges, and $\Delta(R_A), \Delta(R_B) < 2\eta^{\frac{1}{2}}n$. By Theorem 2.6 and Theorem 2.10, R_A and R_B each has an equalized edge-coloring with exactly $\ell := \lceil 2\eta^{\frac{1}{2}}n \rceil$ colors $k+1, \ldots, k+\ell$. Since R_A and R_B have the same number of edges, by renaming some color classes of R_A if necessary, we can assume that in the edge colorings of R_A and R_B, each color appears on the same number of edges in R_A as it does in R_B. There are fewer than $2\eta n^2 + 2\eta n$ edges in each of R_A and R_B, and $\ell \geq 2\eta^{\frac{1}{2}}n$, so each of the color $i \in [k+1, k+\ell]$ appears on fewer than $\eta^{\frac{1}{2}}n + \eta^{\frac{1}{2}} + 1 < \eta^{\frac{1}{2}}n + 2$ edges in each of R_A and R_B. We will now color some of the edges of H with the ℓ colors $[k+1, k+\ell]$ so that each of these color classes becomes a 1-factor of G. We perform the following procedure for each of the ℓ colors in turn.

Given a color i with $i \in [k+1, k+\ell]$, we let A_i and B_i be the sets of vertices in A and B respectively that are incident with edges colored i. Note that

$$|A_i| = |B_i| < 2(\eta^{\frac{1}{2}}n + 2)$$

as R_A and R_B each contains fewer than $\eta^{\frac{1}{2}}n+2$ edges colored i. Also $x \notin A_i$ as the vertex x was excluded in R_A in Step 2. Let H_i be the subgraph of H obtained by deleting the vertex sets A_i and B_i and removing all colored edges. Each vertex in $G-x$ is incident with fewer than

$$3\eta^{\frac{1}{2}}n + 2\eta^{\frac{1}{2}}n + 1 < 6\eta^{\frac{1}{2}}n$$

edges of H that are colored, since fewer than $3\eta^{\frac{1}{2}}n$ were colored in Step 2 and at most $2\eta^{\frac{1}{2}}n+1$ have been colored already in Step 3. Each vertex in $G-x$ has fewer than $2(\eta^{\frac{1}{2}}n+2)$ edges that join it with a vertex in A_i or B_i. So for $v \in V(H_i) \setminus \{x\}$, $d_{H_i}(v)$ is more than

$$\alpha n - n^{2/3} - \eta n - 6\eta^{\frac{1}{2}}n - 2(\eta^{\frac{1}{2}}n+2) > \alpha n - 9\eta^{\frac{1}{2}}n > \frac{1}{2}\alpha n.$$

For the vertex x, recall that the missing colors at x after Step 1 were dealt with at the beginning of Step 2, and x was not involved in the rest of Step 2. Thus $d_{H_i}(x)$ is at least

$$\begin{aligned} &d_B(x) - (d_B(x) - d_A(x))/2 - 2\eta n - 2(\eta^{\frac{1}{2}}n+2) \\ \geq\ &(d_B(x) + d_A(x))/2 - 2\eta n - 2(\eta^{\frac{1}{2}}n+2) \\ >\ &\alpha n - 3\eta^{\frac{1}{2}}n > \frac{1}{2}\alpha n. \end{aligned}$$

Therefore $\delta(H_i) \geq \frac{1}{2}\alpha n$. Furthermore, by the analysis above, each vertex of $H_i - x$ is incident with at most $9\eta^{\frac{1}{2}}n$ edges from $E(H) \setminus E(H_i)$. So H_i has a 1-factor F by Lemma 2.4 (taking $\gamma = 9\eta^{\frac{1}{2}}$). If we color the edges of F with the color i, then every vertex in G is incident with an edge of color i, and so the color class is now a 1-factor of G. We repeat this procedure for each of the colors from $[k+1, k+\ell]$. After this has been done, each of these ℓ color classes is a 1-factor of G. So at the conclusion of Step 3, all of the edges in G_A and G_B are colored, some of the edges of H are colored, and each of the $k+\ell$ color classes is a 1-factor of G.

Step 4

Let R be the subgraph of G consisting of the remaining uncolored edges. These edges all belong to H, so R is a subgraph of H and hence is bipartite. As each of the $k+\ell$ color classes is a 1-factor of G, R is regular of degree $\Delta(R) = \Delta(G) - k - \ell$. Note that since $k \leq \alpha n + n^{2/3} + \eta n$ and $\ell \leq 2\eta^{\frac{1}{2}}n + 1$, $\Delta(R) \geq 2\alpha n - k - \ell > (\alpha - 3\eta^{\frac{1}{2}})n$. By Theorem 2.7 we can color the edges of R with $\Delta(R)$ colors from $[k+\ell+1, \Delta(G)]$. Clearly each of these color classes is a 1-factor of G. This completes our edge coloring of G with $\Delta(G)$ colors. Each of the color classes is a 1-factor, so G is 1-factorizable.

We check that there is a polynomial-time algorithm to obtain a 1-factorization of G. By Lemma 3.2, we can obtain a desired partition $\{A, B\}$ of $V(G)$ in polynomial time. Also, $G_{A,B}$ can be edge colored with the required properties in polynomial time

by Lemma 2.8. In Step 2, the construction of the alternating paths and swaps of the colors on the paths can be done in $O(n^3)$-time, as the total number of colors missing at vertices is $O(n^2)$ and it takes $O(n)$-time to find an alternating path for a pair of vertices (a, b) with a common missing color. In Step 3, there is polynomial-time algorithm (see e.g. [14]) to edge color R_A and R_B using at most ℓ colors; then by doing Kempe changes as in the proof of Lemma 2.8, one can obtain an equalized edge-coloring in polynomial time. The remaining procedures are only about finding perfect matchings in bipartite graphs, which can be done in polynomial time too such as applying the Hopcroft-Karp algorithm [11]. Thus, there is a polynomial-time algorithm that gives a 1-factorization of G. □

4. Proof of Theorem 1.3

We need the following classical result of Hakimi [9] on multigraphic degree sequence.

Theorem 4.1. *Let $0 \leq d_n \leq \ldots \leq d_1$ be integers. Then there exists a multigraph G on vertices $x_1, \ldots, x_n$ such that $d_G(x_i) = d_i$ for all i if and only if $\sum_{i=1}^{n} d_i$ is even and $\sum_{i>1} d_i \geq d_1$.*

Though it is not explicitly stated in [9], the inductive proof yields a polynomial-time algorithm which finds a desired multigraph if it exists.

Let G be a graph and $v \in V(G)$. Define $\mathrm{def}_G(v) = \Delta(G) - d_G(v)$ to be the *deficiency* of v in G. It is clear that G is overfull if and only if $|V(G)|$ is odd and $\sum_{v \in V(G)} \mathrm{def}_G(v) \leq \Delta(G) - 2$. The proof below follows the ideas of Glock, Kühn, and Osthus in the proof of Theorem 1.2 from [6].

Theorem 1.3. *For all $0 < p < 1$ there exist $\varepsilon, \eta > 0$ such that for sufficiently large n, the following holds: Suppose G is a lower-(p, ε)-regular graph on n vertices and n is odd. Moreover, assume that $\Delta(G) - \delta(G) \leq \eta n$. Then $\chi'(G) = \Delta(G)$ if and only if G is not overfull. Further, there is a polynomial-time algorithm which finds an optimal coloring.*

Proof. Choose constants ε, η and positive integer n_0 such that $0 < 1/n_0 \ll \varepsilon \leq \eta \ll p$. Let G be a lower (p, ε)-regular graph on $n \geq n_0$ vertices such that $\Delta(G) - \delta(G) \leq \eta n$, where n is odd. For any $X \subseteq V(G)$ with $|X| = \lceil \varepsilon n \rceil$, we have $E_G(X, V(G) \setminus X) \geq (p - \varepsilon)(n - \varepsilon n - 1)|X|$. Thus the average degree of a vertex from X is at least $(p-\varepsilon)(n-\varepsilon n-1)$, and therefore $\delta(G) \geq (p - \varepsilon)(n - \varepsilon n - 1) - \eta n > \frac{p}{2} n$. We let $\alpha = p/2$ and use αn as a lower bound on $\delta(G)$ in the following.

If G is overfull, then $\chi'(G) = \Delta(G) + 1$. Thus assume G is not overfull. We will now add a new vertex to G to form a multigraph with even order. Let x be a new vertex and let G' be obtained from G by adding x to G and adding some edges between x and vertices $y \in V(G)$ with $d_G(y) < \Delta(G)$ with the following constraints:

(1) $d_{G'}(x) = \delta(G') = \delta(G)$ and $\Delta(G') = \Delta(G)$;

(2) subject to (1), at most one neighbor of x in G' is not a maximum degree vertex of G'.

Since G is not overfull and n is odd, we have $2e(G) = \sum_{y\in V(G)} d_G(y) \le \Delta(G)(n-1)$. Consequently, $\sum_{y\in V(G)} \mathrm{def}_G(y) \ge \Delta(G)$. Let $V(G) = \{x_1, \dots, x_n\}$ and assume that $d_G(x_1) \le \dots \le d_G(x_n)$. Note that $d_G(x_n) = \Delta(G)$. Assume $s \in [1, n-1]$ is the smallest integer such that $\sum_{i=1}^{s} \mathrm{def}_G(x_i) \ge \delta(G)$. For each $i \in [1, s-1]$, we add exactly $\mathrm{def}_G(x_i)$ edges between x and x_i, and we add $\delta(G) - \sum_{i=1}^{s-1} \mathrm{def}_G(x_i)$ edges between x and x_s, and call the resulting multigraph G'. It is clear that G' satisfies both of the constraints (i) and (ii). Since $\Delta(G) - \delta(G) \le \eta n$, we know that

$$|N_{G'}(x)| \ge \frac{\alpha}{\eta} \ge \frac{1}{\alpha^3} \ge 3.$$

We claim that G' contains no $\Delta(G)$-overfull subgraph. (Note that G' maybe a multigraph with parallel edges but we still define a $\Delta(G)$-overfull subgraph H of G' the same way as for simple graphs.) Let $X \subseteq V(G)$ with $|X|$ odd. We show that $G[X]$ is not $\Delta(G)$-overfull. Assume first that $3 \le |X| \le n-2$. If $|X| < \varepsilon n + 1$ or $|X| > n - \varepsilon n - 1$, then $e_G(X, V(G)\setminus X) \ge 2(\delta(G) - \varepsilon n) > \Delta(G)$. If $1 + \varepsilon n \le |X| \le n - \varepsilon n - 1$, by the lower-$(p,\varepsilon)$-regularity of G, we have $e_G(X, V(G)\setminus X) \ge e_G(X\setminus\{x\}, V(G)\setminus(X\cup\{x\})) \ge (p-\varepsilon)(|X|-1)(n-|X|-1) > \Delta(G)$. Therefore $2e(X) < \Delta(G)|X| - \Delta(G)$ and so $G[X]$ is not $\Delta(G)$-overfull. Thus, the only possible $\Delta(G)$-overfull subgraph of G' is obtained from G' by deleting a single vertex $y \in V(G)$. Note that if $G' - y$ is $\Delta(G)$-overfull for some $y \in V(G')$, then so is $G' - x$ as x has minimum degree in G'. However, $G' - x = G$, which is not overfull by our assumption. Thus $G' - y$ is not $\Delta(G)$-overfull for any $y \in V(G')$.

Now $V(G') = \{x, x_1, \dots, x_n\}$. Since x has the smallest degree in G' and $G = G' - x$ is not overfull, $\sum_{i\ge 1} \mathrm{def}_{G'}(x_i) \ge \mathrm{def}_{G'}(x)$. Since $|V(G')| = n+1$ is even, $\mathrm{def}_{G'}(x) + \sum_{i\ge 1} \mathrm{def}_{G'}(x_i)$ is even. Then by Theorem 4.1, there exists a multigraph H on $V(G')$ such that $d_H(x) = \mathrm{def}_{G'}(x)$ and $d_H(x_i) = \mathrm{def}_{G'}(x_i)$ for each $i \in [1, n]$. This multigraph H will aid us to find a spanning regular subgraph of G'.

Note that $\Delta(H) = \mathrm{def}_{G'}(x) = \Delta(G) - \delta(G) \le \eta n$ and H contains isolated vertices. Thus $\chi'(H) \le \Delta(H) + \mu(H) \le 2\Delta(H) \le 2\eta n < 7\eta n/\alpha$. Hence we can greedily partition $E(H)$ into $k \le 7\eta n/\alpha$ matchings $M_1, \dots, M_k$ each of size at most $\alpha n/7$. Now we take out linear forests (forests with each component being a path) from G' by applying Lemma 2.3 with $M_1, \dots, M_k$. As at most one neighbor of x is not a maximum degree vertex of G', it follows that x is adjacent in G' to at most one vertex of those vertices from M_i for each $i \in [1, k]$. More precisely, define spanning subgraphs $G_0, \dots, G_k$ of G' and edge-disjoint linear forests $F_1, \dots, F_k$ such that

(1) $G_0 := G'$ and $G_i = G_{i-1} - E(F_i)$ for $i \in [1, k]$,

(2) F_i is a spanning linear forest (each vertex of G_{i-1} has degree 1 or 2 in F_i) in G_{i-1} whose leaves are precisely the vertices in M_i.

Let $G_0 = G'$ and suppose that for some $i \in [1,k]$, we already defined $G_0, \dots, G_{i-1}$ and $F_1, \dots, F_{i-1}$. As $\Delta(F_1 \cup \dots \cup F_{i-1}) \le 2(i-1) \le 14\eta n/\alpha$, it follows that $\delta(G_{i-1}) \ge (\alpha - 14\eta/\alpha)n \ge 6\alpha n/7$. Moreover, let $\varepsilon' = 2\sqrt{14\eta/\alpha}$. Since G' is lower-$(p, 2\varepsilon)$-regular by Proposition 2.1(1) and $\varepsilon' > 4\varepsilon$, G_{i-1} is lower-(p,ε')-regular by Proposition 2.1(2). Note that $e_{G_{i-1}}(x,v) \le e_{G'}(x,v) \le \eta n$ for every $v \in V(G')$ and $|N_{G_{i-1}}(x)| \ge \frac{6\alpha/7}{\eta} \gg 3$. Thus x has in G_{i-1} at least two neighbors outside the vertices of M_i. Hence, since M_i has size at most $\alpha n/7$, we can apply Lemma 2.3 to G_{i-1} and M_i and obtain a spanning linear forest F_i in G_{i-1} whose leaves are precisely the vertices in M_i. Set $G_i := G_{i-1} - E(F_i)$.

We claim that G_k is regular. Consider any vertex $u \in V(G_k)$. For every $i \in [1,k]$, $d_{F_i}(u) = 1$ if u is an endvertex of some edge of M_i and $d_{F_i}(u) = 2$ otherwise. Since $M_1, \dots, M_k$ partition $E(H)$, we know that $\sum_{i=1}^{k} d_{F_i}(u) = 2k - d_H(u) = 2k - \mathrm{def}_{G'}(u)$. Thus

$$d_{G_k}(u) = d_{G'}(u) - \sum_{i=1}^{k} d_{F_i}(u) = d_{G'}(u) - (2k - \mathrm{def}_{G'}(u)) = \Delta(G') - 2k.$$

Let $d = \Delta(G') - 2k$ and $\alpha' = 6\alpha/7$. We have shown that G_k is a d-regular, lower-(p,ε')-regular star-multigraph with $d \ge \alpha' n$. Furthermore, $e_{G_k}(x,v) \le \eta n$ for every $v \in V(G_k)$, $|N_{G_k}(x)| \ge \frac{6\alpha/7}{\eta} \gg 3$, and $|N_{G_k}(x)| \le n-1$ as $e_{G'}(x, x_n) = 0$. By Theorem 3.3, $\chi'(G_k) = d$. Now we color the edges of F_i using 2 distinct colors from $[d+1, d+2k]$ for each $i \in [1,k]$. It is clear that any edge d-coloring of G_k together with this coloring of $\bigcup_{i=1}^{k} F_i$ gives an edge coloring of G' using $d + 2k = \Delta(G')$ colors. As G is a proper subgraph of G' with $\Delta(G) = \Delta(G')$, it follows that $\chi'(G) = \Delta(G)$.

We lastly check that the procedure above yields a polynomial-time algorithm. Given G, we first check $\sum_{v \in V(G)} \mathrm{def}_G(v)$. If it is at most $\Delta(G) - 2$, then we conclude that $\chi'(G) = \Delta(G) + 1$, and G can be edge colored using $\Delta(G) + 1$ colors in polynomial time [14]. If $\sum_{v \in V(G)} \mathrm{def}_G(v) > \Delta(G) - 2$, then we add a new vertex x to G and form G' as described in the beginning of this proof. Then we can construct an edge $\Delta(G')$-coloring of G' through the process. Since Theorem 4.1, Lemma 2.3 and Theorem 3.3 give appropriate running time statements, this can be achieved in time polynomial in n. □

Acknowledgment

The author wishes to thank Drs. Daniela Kühn and Deryk Osthus for bringing up the concept of derandomizing Chernoff bound, which helps to obtain a polynomial-time algorithm that gives an optimal edge coloring of the dense quasirandom graphs of odd order. The author is very grateful to the two anonymous referees for their careful reading and valuable comments.

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