Journal of Optimization Theory and Applications (2022) 194:636–642
https://doi.org/10.1007/s10957-022-02043-w

Cyclic Projections in Hadamard Spaces

Alexander Lytchak[1] **· Anton Petrunin**[2]

Received: 23 December 2021 / Accepted: 16 April 2022 / Published online: 16 June 2022

Abstract
We show that cyclic products of projections onto convex subsets of Hadamard spaces can behave in a more complicated way than in Hilbert spaces, resolving a problem formulated by Miroslav Bačák. Namely, we construct an example of convex subsets in a Hadamard space such that the corresponding cyclic product of projections is not asymptotically regular.

Keywords Hadamard spaces · Cyclic projections · Asymptotic regularity

1 Introduction

The method of cyclic projections is a classical algorithm seeking an intersection point of a finite family $C_1, \dots, C_k$ of closed convex subsets in a Hilbert space X. Denote by P_i the closest point projection $X \to C_i$; it sends a point $x \in X$ to the (necessarily unique) point $P_i(x)$ in C_i that minimizes the distance to x. Given a point $x \in X$, consider the sequence $x_n = P^n(x)$, where P is the cyclic composition of projections $P = P_1 \circ \dots \circ P_k$. The method of cyclic projections analyzes the sequence (x_n) and tries to find a limit point x_∞, to show $x_\infty \in C_1 \cap \dots \cap C_k$ and to understand the rate of convergence.

Let us list some results in the area. *If the intersection $C_1 \cap \dots \cap C_k$ is non-empty, then (x_n) always converges weakly to some point in $C_1 \cap \dots \cap C_k$* [14]. However, this convergence does not need to be strong [19]. *If, in addition, C_i are linear subspaces, then the convergence is strong* [18, 23]. If the intersection $C_1 \cap \dots \cap C_k$ is not assumed

Communicated by Aviv Gibali.

✉ Anton Petrunin
petrunin@math.psu.edu

Alexander Lytchak
alexander.lytchak@kit.edu

1 Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany

2 Math. Dept., PSU, University Park, State College, PA 16802, USA

to be non-empty, the analysis of the sequence (x_n) is more complicated. However, in [11] it has been established that the cyclic product $P = P_1 \circ \cdots \circ P_k$ is *asymptotically regular*; by definition, this means that for any starting point $x \in X$, we have $|x_n - x_{n+1}| \to 0$ as $n \to \infty$. The rates of convergence, respectively, and the rates of asymptotic regularity have been investigated in several works, see, for instance, [12, 20]. For further reference, see [4, 7, 8, 12, 17].

More recently, the method of cyclic projections has been investigated beyond the setting of Hilbert spaces in so-called Hadamard spaces (also known as CAT(0) spaces, or globally non-positively curved spaces in the sense of Alexandrov). This class of metric spaces includes hyperbolic spaces, metric trees, as well as complete simply connected Riemannian manifolds of non-positive curvature; it has played an important role in many areas of mathematics in the last decades. We assume some familiarity with Hadamard spaces, refer the reader to [2, 3, 9, 10, 15, 16] as general references on this subject. For the introduction and applications of the method of cyclic projections in Hadamard spaces, see [6], [7,Section 6.8], and the references therein.

Hadamard spaces are defined (loosely speaking) by the property that their distance function is at least as convex as the distance function on a Hilbert space. In particular, Hadamard spaces contain a huge variety of convex subsets; closest point projections to closed convex subsets are well defined and 1-Lipschitz, and the questions discussed above about cyclic projections are absolutely meaningful in a Hadamard space X.

Many results discussed above have been transferred from the linear setting of Hilbert spaces to general Hadamard spaces. For instance, *if the subsets C_i have a non-empty intersection, then the cyclic product of projections P is asymptotically regular, and for any initial point $x \in X$, the sequence $x_n = P^n(x)$ converges weakly to a point $x_\infty \in C_1 \cap \cdots \cap C_k$* [5, 8]. (The weak topology on Hadamard spaces is discussed in [6, 7, 22].) The rate of convergence in this setting has been studied in [21].

Therefore, it is somewhat surprising that the fundamental result of Heinz Bauschke [11] for (possibly) non-intersecting convex subsets C_i does not admit a generalization to the setting of Hadamard spaces. The following main result of this paper provides a negative answer to the question of Miroslav Bačák [7, Problem 6.13].

Theorem 1.1 *There exist a Hadamard space X and compact convex subsets $C_1, \dots, C_k$ in X such that the composition of the closest point projections $P = P_1 \circ \cdots \circ P_k$ is not asymptotically regular.*

We provide an explicit example with X being a product of two trees, proving the theorem for $k = 3$. Setting $C_3 = \cdots = C_k$ defines examples for any $k \geq 3$.

In this example, all subsets C_i are isometric to the unit interval, and the projections P_i map all of these segments isometrically onto C_i and the composition $P = P_1 \circ P_2 \circ P_3$ maps C_1 to itself isometrically but exchanges the endpoints of this interval. A stronger version of the theorem is proved in the appendix; it requires a somewhat deeper understanding of the geometry of Hadamard spaces. It seems possible, but would require some non-trivial technical work, to adapt the example from the appendix so that the Hadamard space becomes a smooth Hadamard manifold.

On the other hand, in the case $k = 2$, the result of Heinz Bauschke [11] does admit a generalization; in this case, the algorithm is sufficiently simple to be controlled explicitly, even providing an optimal rate of asymptotic regularity. As it was pointed out by an anonymous referee, the following statement follows from [5,Theorem 3.3], under the additional assumption of the existence of a fixed point of the composition P.

Proposition 1.2 *Let C_1, C_2 be two closed convex subsets of a Hadamard space X. Then the composition $P = P_1 \circ P_2$ is asymptotically regular.*

Moreover, $|x_n - x_{n+1}| = o\left(\frac{1}{\sqrt{n}}\right)$ for any $x \in X$ and $x_n = P^n(x)$.

Here and further, we denote by $|x - y|$ the distance between points x and y in any metric space, even without linear structure.

Examples given by the real axis $C_1 \subset \mathbb{R}^2$ and the set

$$C_2 = \{ (x, y) : x > 0, y \geq 1 + x^{-\epsilon} \}$$

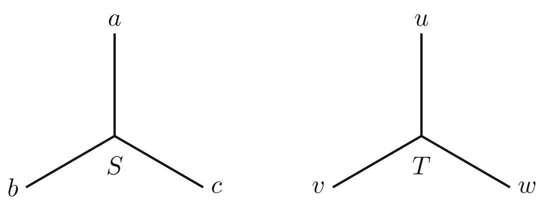
reveal that the convergence rate in Proposition 1.2 cannot be improved to $O(n^{-\frac{1}{2}-\epsilon})$ for any $\epsilon > 0$.

This also shows that the optimal rate of asymptotic regularity for cyclic product of projections on two convex subsets is the same for the Euclidean plane and general Hadamard spaces.

2 Three Segments in a Product of Two Tripods

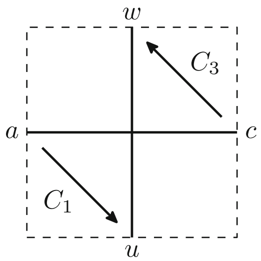
In this section, we prove Theorem 1.1.

Proof A union of three unit segments that share one endpoint with the induced length metric will be called a *tripod*. Consider two tripods S and T and the product space $X = S \times T$. Our space X is a product of two trees and thus of two Hadamard spaces. Hence, X is a Hadamard space.



Denote by a, b, c and u, v, w the sides of S and T, respectively.

The following diagram shows 3 isometric copies of $2{\times}2$-square in X; they are obtained as the products of two pairs of sides in S and T as labeled.

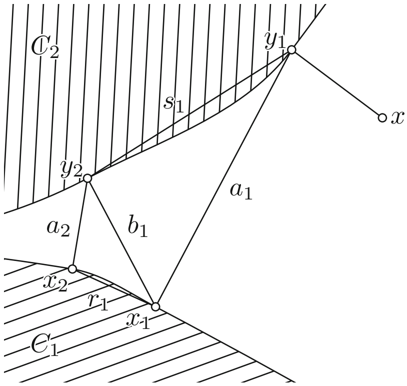


Consider the segments C_1, C_2, and C_3 shown on the diagram; they all have slope -1 and project to each other isometrically. Note that each projection P_i reverses the shown orientation. It follows that the composition $P = P_1 \circ P_2 \circ P_3$ sends the segment C_1 to itself isometrically and changes the orientation of the segment. In particular, P exchanges the ends of the segment; hence, P is not asymptotically regular. (In fact, for an end e of C_1, and any n, we have $|P^n(e) - P^{n+1}(e)| = 1$.)

Finally, setting $C_3 = \cdots = C_k$ defines examples for any $k \geq 3$. □

3 Two Sets

In this section, we prove Proposition 1.2.



Proof By definition, $x_n \in C_1$ for all n. Set $y_{n+1} = P_2 \circ P^n(x)$, so $y_1 = P_2(x)$, $x_1 = P_1(y_1)$, $y_2 = P_2(x_1)$, and so on. Further set

$$r_n := |x_n - x_{n+1}|,$$
$$s_n := |y_n - y_{n+1}|.$$

Since the closest point projection is non-expanding, we get

$$s_1 \geq r_1 \geq s_2 \geq r_2 \geq \cdots \tag{1}$$

Set

$$a_n := |x_n - y_n| = \mathrm{dist}_{C_1} y_n,$$
$$b_n := |y_{n+1} - x_n| = \mathrm{dist}_{C_2} x_n.$$

Note that

$$a_1 \geq b_1 \geq a_2 \geq b_2 \geq \cdots \tag{2}$$

Since C_1 is convex and $x_n \in C_1$ lies at the minimal distance from y_n, we have $\measuredangle[x_n\,{}^{x_{n-1}}_{y_n}] \geq \frac{\pi}{2}$. Since X is a Hadamard space,

$$r_n^2 \leq b_n^2 - a_{n+1}^2.$$

Therefore, (2) implies that

$$\sum_n r_n^2 \leq b_1^2.$$

By (1), r_n is non-increasing. Therefore, $r_n = o(\frac{1}{\sqrt{n}})$. □

4 Conclusions

We have shown that a cyclic product of $k \geq 3$ projections onto convex subsets of a Hadamard space does not need to be asymptotically regular, even if the convex subsets involved are compact. This should be seen in contrast to the asymptotic regularity of such maps in Hilbert spaces and to the fact that many other results about cyclic projections generalize easily from the linear setting to the setting of Hadamard spaces. On the other hand, we show that a cyclic product of two projections to convex subsets of Hadamard spaces must always be asymptotically regular.

Acknowledgements We thank Miroslav Bačák and Nina Lebedeva for helpful comments and conversations. Let us also thank the anonymous referees for careful reading and useful suggestions. Alexander Lytchak was partially supported by the DFG Grant, No. 281071066, TRR 191. Anton Petrunin was partially supported by the NSF Grant, DMS-2005279. Data sharing was not applicable to this article as no datasets were generated or analyzed during the current study.

Appendix: Three Disks

While the cyclic product of projections P constructed in Sect. 2 is not asymptotically regular, its square P^2 is the identity on C_1; in particular, P^2 *is* asymptotically regular. The construction in Sect. 2 produces a Möbius band B divided into three rectangles and a map from B to a Hadamard space that is distance-preserving on each rectangle.

In this appendix, we produce a Hadamard space that contains an embedding of a twisted solid torus with arbitrary twisting angle, such that the solid torus consists of 3 isometrically embedded flat cylinders. In this case, we obtain again 3 projections onto convex sets, each of them isometric to a Euclidean disk, the bases of the cylinders. Then the cyclic product of these projections is the rotation of a disk by the prescribed twisting angle α. In particular, if $\frac{\alpha}{\pi}$ is irrational, then any power of this cyclic product of projections may not be asymptotically regular.

Theorem A.1 *There is a cyclic projection P as in Theorem 1.1 such that any of its powers P^m is not asymptotically regular.*

Proof of A.1 Fix an angle α and a small $\epsilon > 0$. Consider the closed ϵ-neighborhood A of a closed geodesic γ in the unit sphere $\mathbb{S}^3$. Note that the boundary of A is a saddle surface in $\mathbb{S}^3$; hence, it has curvature bounded from above by 1. Thus, A is a compact Riemannian manifold with boundary, such that the curvature of the interior and of the boundary is bounded from above by 1. Therefore, by the result of Stephanie Alexander, David Berg and Richard Bishop [1], *A equipped with the induced intrinsic metric is locally* CAT(1). The universal cover $\tilde{A}$ of A with its induced metric is locally CAT(1) as well. Since $\tilde{A}$ does not contain closed geodesics, it is CAT(1), by the generalized Hadamard–Cartan theorem [3,8.13.3], [10,6.8+6.9], [13].

Denote by E the inverse image of γ in $\tilde{A}$. The isometry group of $\tilde{A}$ contains the group of translations along E and the rotations that fix E. Let T be the composition of translation along E of length $2 \cdot \pi + 10 \cdot \epsilon$ and the rotation by angle α. The element T generates a discrete subgroup Γ in the group of isometries of $\tilde{A}$ that acts freely and discretely on $\tilde{A}$.

Set $Y = \tilde{A}/\Gamma$. Since ε is small, any non-trivial element of Γ moves every point of $\tilde{A}$ by more than $2 \cdot \pi$. Therefore, Y is a compact locally CAT(1) space that does not contain closed geodesics of length less than $2 \cdot \pi$. Hence, by the generalized Hadamard–Cartan theorem [3], Y is CAT(1). By construction, Y is locally isometric to $\mathbb{S}^3$ outside its boundary B. The projection of E to Y is a closed geodesic G of length $2 \cdot \pi + 10 \cdot \epsilon$.

Denote by X the Euclidean cone over Y; since Y is CAT(1), we get that X is CAT(0); see [3]. Moreover, X is locally Euclidean outside its *boundary* — the cone over B.

The cone Z over the closed geodesic G is the Euclidean cone over a circle of length $2 \cdot \pi + 10 \cdot \epsilon$. By construction, Z is a locally convex subset of X. Hence, Z is a convex subset of X [2,2.2.12]. Let us consider a geodesic triangle $[q_1 q_2 q_3]$ in Z that surrounds the origin o of the cone Z.

By construction, the sides of the triangle $[q_1 q_2 q_3]$ lie in the flat part of X. Thus, we can find a small $r > 0$ such that the $2 \cdot r$-neighborhood U_1 of the geodesic $[q_1 q_2]$ is isometric to a convex subset of the Euclidean space. We can assume that $2 \cdot r$-neighborhoods U_2 of $[q_2 q_3]$ and U_3 of $[q_3 q_1]$ have the same property.

Denote by C_i the disk of radius r centered at q_i and being orthogonal to Z. By construction, C_i and C_{i+1}, for $i = 1, 2, 3 \pmod 3$ are contained in U_i. Since Z is convex, C_i and C_{i+1} are *parallel* inside U_i, thus their convex hull Q_i is isometric to the cylinder $C_i \times [q_i, q_{i+1}]$ with bottom and top C_i and C_{i+1}. In particular, the projection P_i defines an isometry $C_{i+1} \to C_i$.

By construction, the composition $P = P_1 \circ P_2 \circ P_3 \colon C_1 \to C_1$ rotates C_1 by angle α. If $\frac{\alpha}{\pi}$ is irrational, then P, as well as all its powers, are *not* asymptotically regular.

As before, setting $C_3 = \cdots = C_k$ defines examples for any $k \geq 3$. ☐

References

1. Alexander, S., Berg, D., Bishop, R.: Geometric curvature bounds in Riemannian manifolds with boundary. Trans. Am. Math. Soc. **339**(2), 703–716 (1993)
2. Alexander, S., Kapovitch, V., Petrunin, A.: An Invitation to Alexandrov Geometry. SpringerBriefs in Mathematics. Springer, Cham (2019)
3. Alexander, S., Kapovitch, V., Petrunin, A.: Alexandrov geometry: foundations. arXiv:1903.08539 [math.DG] (2022)
4. Ariza-Ruiz, D., Fernáandez-Leóon, A., López-Acedo, G., Nicolae, A.: Chebyshev sets in geodesic spaces. J. Approx. Theory **207**, 265–282 (2016)
5. Ariza-Ruiz, D., López-Acedo, G., Nicolae, A.: The asymptotic behavior of the composition of firmly nonexpansive mappings. J. Optim. Theory Appl. **167**(2), 409–429 (2015)
6. Bačák, M.: Convex Analysis and Optimization in Hadamard Spaces. De Gruyter Series in Nonlinear Analysis and Applications, vol. 22. De Gruyter, Berlin (2014)
7. Bačák, M.: Old and new challenges in Hadamard spaces. arXiv:1807.01355 [math.FA] (2018)
8. Bačák, M., Searston, I., Sims, B.: Alternating projections in CAT(0) spaces. J. Math. Anal. Appl. **385**(2), 599–607 (2012)
9. Ballmann, W.: Lectures on Spaces of Nonpositive Curvature, Vol. 25. DMV Seminar, with an Appendix by Misha Brin (1995)
10. Ballmann, W.: On the geometry of metric spaces. MPIM. https://people.mpim-bonn.mpg.de/hwbllmnn/archiv/sin40827.pdf (2004)
11. Bauschke, H.: The composition of projections onto closed convex sets in Hilbert space is asymptotically regular. Proc. Am. Math. Soc. **131**(1), 141–146 (2003)
12. Bauschke, H., Borwein, J., Lewis, A.: The method of cyclic projections for closed convex sets in Hilbert space. In: Censor, Y., Reich, S. (eds.) Recent Developments in Optimization Theory and Non-linear Analysis (Jerusalem 1995). Vol. 204. Contemporary Mathematics. American Mathematical Society, Providence, RI, pp. 1–38 (1997)
13. Bowditch, B.H.: Notes on Locally CAT(1) Spaces. In: Charney, R., Davis, M., Shapiro, M. (eds.) Geometric Group Theory (Columbus, OH, 1992). Vol. 3. Ohio State University, Mathematics Research Institute, Publ. de Gruyter, Berlin, pp. 1–48 (1995)
14. Brègman, L.M.: Finding the common point of convex sets by the method of successive projection. Dokl. Akad. Nauk SSSR **162**, 487–490 (1965)
15. Bridson, M., Haeiger, A.: Metric spaces of non-positive curvature. Vol. 319. Grundlehren der Mathematischen Wissenschaften (1999)
16. Burago, D., Burago, Y., Ivanov, S.: A Course in Metric Geometry. Vol. 33. Graduate Studies in Mathematics (2001)
17. Deutsch, V.: The method of alternating orthogonal projections. In: Charron, R., Watson, B. (eds.) Approximation Theory, Spline Functions and Applications (Maratea, 1991). Vol. 356. NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci. Kluwer Acad. Publ., Dordrecht, pp. 105–121 (1992)
18. Halperin, I.: The product of projection operators. Acta Sci. Math. (Szeged) **23**, 96–99 (1962)
19. Hundal, H.S.: An alternating projection that does not converge in norm. Nonlinear Anal. **57**(1), 35–61 (2004)
20. Kohlenbach, U.: A polynomial rate of asymptotic regularity for compositions of projections in Hilbert space. Found. Comput. Math. **19**(1), 83–99 (2019)
21. Kohlenbach, U., López-Acedo, G., Nicolae, A.: Quantitative asymptotic regularity results for the composition of two mappings. Optimization **66**(8), 1291–1299 (2017)
22. Lytchak, A., Petrunin, A.: Weak topology on CAT(0) spaces. arXiv:2107.09295 [math.MG] (2021) **(To appear in Israel J. Math)**
23. von Neumann, J.: On rings of operators. Reduction theory. Ann. Math. (2) **50**, 401–485 (1949)