ON THE MORSE INDEX WITH CONSTRAINTS FOR CAPILLARY SURFACES

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Abstract. In this paper we apply the abstract Morse index formulation developed in [27] to study several optimization set-ups with constraints. In each case, we classify how the general index is related to the index with a constraint. In addition, for capillary surfaces in a Euclidean ball, we obtain an index estimate which recovers stability results of G. Wang-C. Xia [29]and J. Gou- C. Xia [14] as special cases. By considering a family of examples, we show that the inequality is sharp. Furthermore, we precisely determine indices with constraints for important examples such as the critical catenoid, round cylinders in a ball, and CMC hypersurfaces with constant scalar curvature in a sphere.

1. Introduction

Let (Ω^{n+1}, g) be a complete orientable Riemannian manifold possibly with boundaries and $X : \Sigma^n \mapsto \Omega^{n+1}$, a smooth immersion of an n-dimensional compact manifold Σ. We frequently identify Σ with $X(\Sigma)$. In the presence of boundaries, we assume the immersion is proper, that is, $\partial\Sigma = \Sigma \cap \partial\Omega$. Then $\Sigma \subset \Omega$ is called a partitioning of Ω. One considers the set $\mathcal{I}(\Sigma, \Omega)$ of all such immersions and a functional, such as the area (n-volume) of Σ with the induced metric $i^*(g)$ for $i \in \mathcal{I}(\Sigma, \Omega)$.

In the calculus of variations, setting the first derivative of the functional to zero yields critical points and one then computes the second variation. At a two-sided critical point, it has the following structural formula, for smooth functions u and v,

$$Q(u, v) = \int_\Sigma \Big(\langle \nabla u, \nabla v \rangle - puv \Big) d\mu - \int_{\partial\Sigma} quvds. \tag{1.1}$$

Here, functions p and q are determined by the geometry of Σ and presumably depend on the particular variational problem we consider.

Let $\mathrm{MI}(Q)$ denote the Morse index of the bi-linear form Q on the vector space of smooth functions, $C^\infty(\Sigma)$, which is the maximal dimension of a subspace on which $Q(\cdot, \cdot)$ is negative definite. In the presence of a constraint, we only admit deformations preserving it. Consequently, the index with a constraint is the index of $Q(\cdot, \cdot)$ restricted to a smaller function space. The relation between these notions, with and without a constraint, has only been studied for special cases [2, 17, 15, 28, 24].

In this paper, we will use the abstract formulation developed in [27] to determine these relations for several variational problems. Due to the flexibility of our setup, the results would be applicable for many types of domains, .e.g., a wedge, a slab, a cylinder, a cone, or

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a half space. Also, all critical points considered belong to the family of capillary surfaces, defined as having constant mean curvature (CMC) and fixed intersecting angle. We refer to an article of Finn-McCuan-Wente [9], for the history of such surfaces, and Finn's book [8], for a survey of the mathematical theory. When the angle is a right angle and the mean curvature vanishes, they are called free boundary minimal surfaces (FBMS), a subject of great interest which has produced beautiful results; see, for example, A. Fraser-M.Li[10], A.Fraser-R. Schoen[11] and [12], M. Li and X. Zhou [16], and D. Maximo, I. Nunes, and G. Smith [18]. Next, we will describe our contributions in details.

1.1. **Index Relation.** We consider the functional E which is a linear combination of the area of the hypersurface and the wetted area. The latter is the the area of a portion of $\partial\Omega$ bounded by $\partial\Sigma$.

Definition 1.1. *A Type I constraint is to preserve the volume bounded by the partitioning hypersurface Σ and parts of $\partial\Omega$.*

Our definition follows from [19, 29] and is a slight generalization of [4]. Consequently, it can be shown that the corresponding Type I Morse index (Definition 2.10) is the index of $Q(\cdot,\cdot)$ in the space of smooth functions with zero average[1]. The relation of indices with and without the constraint is given by the following.

Theorem 1.2. *Let $\Sigma \subset \Omega$ be a critical point. Then the Type I Morse index is equal to $MI(Q)-1$ if and only if there is a smooth function u such that*

$$\begin{cases} (\Delta+p)u & = -1 \ \ on\ \Sigma, \\ \nabla_\eta u & = qu \ \ on\ \partial\Sigma, \\ \int_\Sigma u & \leq 0. \end{cases}$$

Otherwise, it is equal to $MI(Q)$.

Remark 1.3. *We note that our analysis is applicable (and easier) for the cases of closed hypersurfaces and the fixed boundary problem. See Section 3.2 for more details. In particular, we obtain a generalization of results from* [15] *and* [24].

Remark 1.4. *Either case might happen. See Subsection 4.1 and Proposition 4.7.*

Next, we consider the Type II constraint.

Definition 1.5. *A Type II constraint is to preserve the wetted area.*

Consequently, the type-II Morse index, Definition 2.10, is the index of $Q(\cdot,\cdot)$ in the space of smooth functions with zero boundary average. We have the following.

Theorem 1.6. *Let $\Sigma \subset \Omega$ be a critical point. Then its Type-II Morse index is equal to $MI(Q)-1$ if and only if there is a smooth function u such that*

$$\begin{cases} (\Delta+p)u & = 0 \ \ on\ \Sigma, \\ \nabla_\eta u - qu & = 1 \ \ on\ \partial\Sigma, \\ \int_{\partial\Sigma} u & \leq 0. \end{cases}$$

Otherwise, it is equal to $MI(Q)$.

[1]It is also called weak Morse index in literature.

1.2. **An Index Estimate.** When considering the partitioning of the Euclidean ball, $\Omega = \mathbb{B}^{n+1}$, we obtain more precise results. In particular, we introduce a generalization of Type I and Type II constraints, called Type I+II. To our knowledge, this is the first time such condition is considered in the literature of geometric variational problems. It corresponds to the partitioning of a convex body when preserving both the wetted area and enclosed volume. An intermediate consequence is that we streamline the Type I and Type II stability results of [29] and [14] as special cases.

The type-I+II Morse index, Definition 2.10, is the index of $Q(\cdot,\cdot)$ in the space of functions with zero boundary *and* interior average. We characterize the stability case (index zero) and give a lower bound. For $X : \Sigma^n \subset \mathbb{B}^{n+1}$, let $H, |\mathring{A}|, \theta, \nu$ be respectively the mean curvature, the norm of the traceless second fundamental form, the fixed contact angle, and a choice of an unit normal vector of Σ (for precise convention, see Section 2). For any coordinate system $\{e_1, \cdots, e_{n+1}\}$ of $\mathbb{R}^{n+1}$, we construct a matrix $\Upsilon = (\Upsilon_{ij})_{(n+1)\times(n+1)}$ with

$$\Upsilon_{ij} = \int_\Sigma n|\mathring{A}|^2[\langle (n - H\langle X, \nu\rangle)X + (n\cos\theta + \frac{H}{2}(|X|^2+1))\nu, e_i\rangle\langle X, e_j\rangle].$$

Definition 1.7. *A capillary hypersurface is called $|\mathring{A}|^2$-scale equivalent to a hyper-planar domain if $|\mathring{A}|^2 X$ is on a hyperplane.*

Remark 1.8. *A capillary hypersurface is $|\mathring{A}|^2$-scale equivalent to a hyper-planar then it is on a half-ball and the level sets of $|\boldsymbol{A}|^2$ are hyper-planar.*

Theorem 1.9. *Assume $X : \Sigma \to \mathbb{R}^{n+1}$ is an immersed capillary hypersurface in the Euclidean unit ball $\mathbb{B}^{n+1}$. Let ℓ be the number of nonnegative eigenvalues of the matrix Υ. Then*

(1) *If X has zero Type-I+II Morse index then it is totally umbilical.*
(2) *If it is $|\mathring{A}|^2$-scale equivalent to a hyper-planar domain then its Type-I+II Morse index is greater than or equal to $\ell - 1$.*
(3) *Otherwise, the Type-I+II Morse index is greater than or equal to ℓ.*

Remark 1.10. *The first part recovers results of Ros-Vergasta* [20]*, Wang-Xia* [29, Theorem 1.1]*, and* [14] *as special cases.*

Remark 1.11. *For a symmetrical surface, it is easy to compute ℓ and we'll show that the estimate is sharp for round cylinders of suitable radius (See Section 4).*

Remark 1.12. *ℓ only depends on the geometry of Σ. Comparatively, in the current literature, most lower bounds of an index come from some topology consideration; see Ambrozio-Carlotto-Sharp* [1] *and references therein.*

We also apply our technique to study the indices of FBMS. An FBMS is a critical point for either type I, type II, or type I+II partitioning problem. So the following result might be of independent interest.

Theorem 1.13. *Let $X : \Sigma \to \mathbb{B}^{n+1}$ be an immersed free boundary minimal hypersurface. If it has Type I+II Morse index less than $n+1$, then it must be totally geodesic.*

An immediate consequence is the following.

Corollary 1.14. *Let $X : \Sigma \to \mathbb{B}^{n+1}$ be an immersed free boundary minimal hypersurface. If it has Type I or Type II Morse index less than $n+1$, then it must be totally geodesic.*

Remark 1.15. *The Type I part of Cor. 1.14 can be deduced directly from the index estimate of* [12, 25]. *There is a general version for capillary minimal hypersurfaces (Cor. 5.1).*

1.3. **Precise Index Computations.** It is an interesting problem to determine precise Morse indices for important examples. The case of the critical catenoid as a free boundary minimal surface, without a constraint consideration, has received plenty of attention; see [23, 22, 6, 25, 26]. Combining the aforementioned index relation with a scheme to determine $\mathrm{MI}(Q)$ allows us to precisely determine Morse indices with constraints for several examples.

Corollary 1.16. *Let $\Sigma \subset \mathbb{S}^{n+1}$ be a closed CMC hypersurface of constant scalar curvature. Then, its weak Morse index is equal to*

$$MI(Q) - 1.$$

Remark 1.17. *When $n = 2$, the scalar curvature is a multiple of the intrinsic Gauss curvature. Thus, the reader can consult* [21] *for a precise computation when the Gaussian curvature is vanishing.*

For capillary cases, the simplest nontrivial examples are round cylinders and catenoids. It is observed in [22] that, for critical catenoids, their indices become surprisingly high when the dimensions are increasing. For the capillary round cylinders, even for a fixed dimension, the Morse indices can be arbitrarily large when the radii are close to 1 or 0. More precisely, we have the following statement.

Theorem 1.18. *For a round cylinder $Z := \{(x, z) \in \mathbb{R}^n \times \mathbb{R}, |x|^2 = r^2,\ r < 1\} \cap \mathbb{B}^{n+1}$,*

(1) *$MI(Q) \geq n + 2$,*
(2) *When $r \to 0$ or $r \to 1$, $MI(Q) \to \infty$;*
(3) *There is an interval $0 < a < r < b < 1$ such that $MI(Q) = n + 2$;*
(4) *The lower bound from Theorem 1.9 is $n + 1$. There are intervals on which the Type-I+II index is $n + 1$.*

Remark 1.19. *Thus, the index estimate is sharp.*

Finally, we compute the indices with different constraints for the critical catenoid.

Theorem 1.20. *Let $\Sigma \subset \mathbb{B}^3$ is the critical catenoid. Then its Type-I, Type-II, and Type I+II Morse indices are all equal to 3.*

In summary, the proof of index relation theorems follows from the interpretation of our abstract formulation [27] for concrete cases. The index estimate depends on a careful analysis making use of the domain's symmetry. The organization of the paper is as follows. The next section will collect some preliminaries and fix our notations. Then, we study the relation between a general index and one with a constraint proving Theorems 1.2 and 1.6 and Corollary 1.16 in Section 3. The index inequality in the context of an Euclidean domain will be investigated in Section 4, where there are proofs of Theorems 1.9 and 1.18. The last section is about free boundary minimal hypersurfaces and we obtain proofs of Theorem 1.13 and Theorem 1.20.

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2. Preliminaries

First, we record our notations, conventions, and collect useful results. We denote by $\bar{\nabla}$, $\bar{\Delta}$ and $\bar{\nabla}^2$ (correspondingly ∇, Δ and ∇^2) the gradient, the Laplacian and the Hessian on Ω (on Σ) respectively. One then considers a differentiable family of proper immersions $X(t,\cdot) : (-\epsilon, \epsilon) \times \Sigma \to \Omega$, that is,

$$X(t, \mathrm{int}\Sigma) \subset \mathrm{int}\Omega,$$
$$X(t, \partial\Sigma) \subset \partial\Omega.$$

For subsequent calculation, Σ is equipped with the pullback metric via $X(t,\cdot)$. We recall the area, volume, and wetted area functionals:

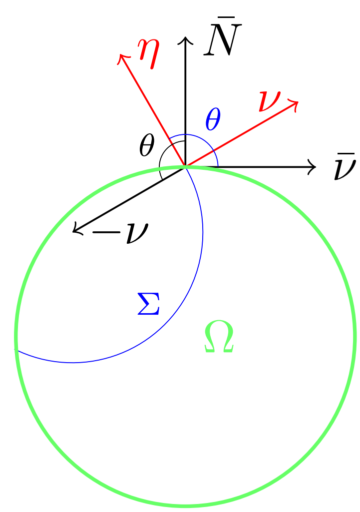
$$\mathbf{A}(t) = \int_\Sigma d\mu(t),$$
$$V(t) = \int_{[0,t]\times\Sigma} X^* d\mu_\Omega,$$
$$W(t) = \int_{[0,t]\times\partial\Sigma} X^* d\mu_{\partial\Omega}.$$

Also, it is relevant to consider the following energy functional, for a real number $\theta \in (0, \pi)$,

$$E(t) = A(t) - \cos(\theta) W(t).$$

This could be considered as a Lagrange multiplier setup where $A(t)$ is the functional to optimize and $W(t)$ is the functional representing a constraint.

Next, let $\Sigma = X(0, \Sigma)$ be a critical point of some functional and $d\mu = d\mu(0)$. For convenience, we will omit this term when writing integrals whenever the context removes all ambiguity. Let ν be a choice of a unit normal vector $\Sigma \subset \Omega$, η is the unit exterior conormal vector of $\partial\Sigma \subset \Sigma$. $\bar{N}$ is the outward pointing unit normal vector of $\partial\Omega \subset \Omega$. Let $\bar{\nu}$ be the unit normal to $\partial\Sigma$ in $\partial\Omega$ such that the bases $\{\nu, \eta\}$ and $\{\bar{\nu}, \bar{N}\}$ have the same orientation in the normal bundle of $\partial\Sigma \subset \Omega$. Thus, along $\partial\Sigma$, the pairs $(-\nu, \bar{N})$ and $(\eta, \bar{\nu})$ make the same angle.



The second fundamental form, for vector fields X, Y, is defined as, $\mathbf{A}(X,Y) = -\langle \bar{\nabla}_X Y, \nu \rangle$. Then, the mean curvature $H := \mathrm{Trace}(\mathbf{A})$. Let $\mathring{A}$ be the traceless second fundamental form and $|\mathring{A}|^2 = |\mathbf{A}|^2 - \frac{H^2}{n}$.

2.1. Variational Formulae and Constraints.

The first variations of the above functionals are well-known and collected below. For a velocity vector field

$Y = X'(0)$,

$$\begin{aligned} A'(0) &= \int_\Sigma H \langle \nu, Y \rangle \, d\mu + \int_{\partial\Sigma} \langle \eta, Y \rangle \, ds, \\ V'(0) &= \int_\Sigma \langle \nu, Y \rangle \, d\mu, \\ W'(0) &= \int_{\partial\Sigma} \langle \bar{\nu}, Y \rangle \, ds, \\ E'(0) &= \int_\Sigma H \langle \nu, Y \rangle \, d\mu + \int_{\partial\Sigma} \langle \eta - \cos(\theta)\bar{\nu}, Y \rangle \, ds. \end{aligned}$$

We consider functional E with either Type-I or Type-II constraint. The former is to preserve the prescribed volume V.

Definition 2.1. *A variation Y is volume-preserving and corresponds to a Type I partitioning if $\int_\Sigma \langle \nu, Y \rangle \, d\mu = 0$.*

It is immediate that Σ is a critical point of E with a Type I partitioning if and only if it has constant mean curvature H and the projection of η onto the tangent bundle of $\partial\Omega$ is exactly equal to $(\cos\theta)\bar{\nu}$. That is, H is constant and Σ intersects $\partial\Omega$ at fixed angle θ.

The Type II constraint, instead of preserving prescribed volume, preserves the wetted area. Following [14], we define:

Definition 2.2. *A variation Y is wetted-area-preserving and corresponds to a Type II partitioning if $\int_{\partial\Sigma} \langle \bar{\nu}, Y \rangle \, d\mu = 0$.*

It is clear that a Type II stationary surface is minimal and meets the boundary at a constant angle θ. On such a stationary surface, for $Y = Y_0 + u\nu$, with Y_0 tangential, it is observed that the wetted-area-preserving is equivalent to

$$\int_{\partial\Sigma} u ds = 0.$$

We also introduce the following extension of Type I and Type II constraints, called Type I+II. It essentially corresponds to the partitioning of a convex body while preserving both the wetted area and enclosed volume.

Definition 2.3. *A function u satisfies Type I+II constraint if simultaneously*

$$0 = \int_{\partial\Sigma} u ds = \int_\Sigma u d\mu.$$

Definition 2.4. *An orientable immersed smooth hypersurface $\Sigma^n \subset \Omega^{n+1}$ is called capillary if it has constant mean curvature and fixed intersecting angle.*

Furthermore, when Σ intersects $\partial\Omega$ at angle θ then

$$\begin{aligned} \eta &= \sin\theta \, \bar{N} + \cos\theta \, \bar{\nu}, \\ \nu &= -\cos\theta \, \bar{N} + \sin\theta \, \bar{\nu}. \end{aligned} \tag{2.1}$$

Next, we collect useful calculation when $\Omega = \mathbb{B}^{n+1}$, an Euclidean ball centered at the origin.

Lemma 2.5. *Let $X : \Sigma \to \mathbb{R}^{n+1}$ be an isometric immersion. The following identities hold:*

$$\begin{aligned}
&\Delta X = -H\nu,\\
&\Delta \frac{1}{2}|X|^2 = n - H\langle X, \nu\rangle,\\
&\Delta \nu = \nabla H - |\boldsymbol{A}|^2\nu,\\
&\Delta\langle X, \nu\rangle = \langle X, \nabla H\rangle + H - |\boldsymbol{A}|^2\langle X, \nu\rangle,\\
&\langle \bar\nabla X, \bar\nabla \nu\rangle = H.
\end{aligned}$$

Proof. We prove the last equation. For an orthonormal base $e_1, \cdots, e_{n+1}$ in $\mathbb{R}^{n+1}$, we have

$$X = (x_1, \cdots, x_{n+1}) = \sum_{i=1}^{n+1} x_i e_i \text{ and } \nu = (\nu_1, \cdots, \nu_{n+1}) = \sum_{i=1}^{n+1} \nu_i e_i.$$

Since $\langle \bar\nabla x_i, \bar\nabla \nu_j\rangle = \langle e_i, \bar\nabla \nu_j\rangle = e_i\langle e_j, \nu\rangle = \langle e_j, \bar\nabla_{e_i}\nu\rangle$, then

$$\langle \bar\nabla X, \nabla \nu\rangle = \sum_{i=1}^{n+1}\langle \bar\nabla x_i, \bar\nabla \nu_i\rangle = \sum_{i=1}^{n+1}\langle e_i, \bar\nabla_{e_i}\nu\rangle = H.$$

□

We denote the Jacobi operator $J := \Delta + |\mathbf{A}|^2$.

Corollary 2.6. *Let $X : \Sigma \to \mathbb{R}^{n+1}$ be an isometric immersion with constant mean curvature. Then the following identities hold:*

$$\begin{aligned}
&J\nu = 0,\\
&J\langle X, \nu\rangle = H.
\end{aligned}$$

Proposition 2.7. *Let $X : \Sigma \to \mathbb{B}^{n+1}$ be a capillary hypersurface with intersecting angle $\theta \in (0, \pi)$. Then along $\partial\Sigma$,*

$$\begin{aligned}
\bar\nabla_\eta(X + \cos\theta\,\nu) =& q(X + \cos\theta\,\nu)\\
\bar\nabla_\eta Y =& qY.
\end{aligned}$$

Here, Y and q are defined as

$$\begin{aligned}
Y &:= \langle X, \nu\rangle X - \frac{1}{2}(|X|^2 + 1)\nu,\\
q &:= \frac{1}{\sin\theta} + \cot(\theta)\boldsymbol{A}(\eta, \eta).
\end{aligned}$$

Furthermore,

$$\int_{\partial\Sigma} n\eta + H\bar\nu = 0.$$

Proof. We first prove the point-wise identities. On the boundary, $X = \bar N$ and $X + \cos\theta\,\nu = \sin\theta\,\eta$ by (2.1). Since $\partial\mathbb{B}^{n+1}$ is umbilical in $\mathbb{B}^{n+1}$, η is a principal direction of $\partial\Sigma$ in Σ, and

$$\bar\nabla_\eta \nu = \mathbf{A}(\eta, \eta)\eta.$$

Thus,

$$\bar\nabla_\eta(X + \cos\theta\,\nu) = \eta + \cos\theta\,\mathbf{A}(\eta, \eta)\eta = q\sin\theta\,\eta.$$

For the second identity, we observe, on the boundary, $\bar{\nabla}_\eta X = \eta$, $|X| = 1$ and

$$\bar{\nabla}_\eta \nu = \mathbf{A}(\eta,\eta)\eta,$$

$$\langle X, \nu\rangle X - \frac{1}{2}(|X|^2+1)\nu = -\cos\theta\, X - \nu = -(\cos\theta\, \sin\theta\,\eta + \sin^2\theta\,\nu).$$

Then,

$$\begin{aligned}
&\bar{\nabla}_\eta[\langle X,\nu\rangle X - \frac{1}{2}(|X|^2+1)\nu] \\
&= (\langle \bar{\nabla}_\eta X, \nu\rangle + \langle \bar{\nabla}_\eta \nu, X\rangle)X + \langle X,\nu\rangle \bar{\nabla}_\eta X - \langle \bar{\nabla}_\eta X, X\rangle \nu - \frac{1}{2}(|X|^2+1)\bar{\nabla}_\eta \nu \\
&= \langle \bar{\nabla}_\eta \nu, X\rangle X + \langle X,\nu\rangle \eta - \langle \eta, X\rangle \nu - \bar{\nabla}_\eta \nu \\
&= \mathbf{A}(\eta,\eta)(\sin\theta\, X - \eta) - \cos\theta\,\eta - \sin\theta\,\nu \\
&= -\mathbf{A}(\eta,\eta)\cot\theta(\cos\theta\,\sin\theta\,\eta + \sin^2\theta\,\nu) - \frac{1}{\sin\theta}(\cos\theta\,\sin\theta\,\eta + \sin^2\theta\,\nu) \\
&= q(\cos\theta\,\sin\theta\,\eta + \sin^2\theta\,\nu) \\
&= q(\nu + \cos\theta\ X).
\end{aligned}$$

The result follows. For the integral identity, let a be a constant vector field. By the divergence theorem

$$\int_\Sigma -H\langle \nu, a\rangle d\mu = \int_\Sigma \Delta\langle X, a\rangle = \int_{\partial\Sigma} \nabla_\eta \langle X, a\rangle = \int_{\partial\Sigma} \langle \eta, a\rangle.$$

On the other hand, one considers the tangential vector field $Y_a = \langle X,\nu\rangle a - \langle \nu, a\rangle X$. For a local normal coordinate system $\{E_i\}_{i=1}^n$ in Σ, we have

$$\begin{aligned}
\mathrm{div} Y_a &= \sum_i \nabla_{E_i}\langle Y_a, E_i\rangle \\
&= \sum_i \nabla_{E_i}\big(\langle X,\nu\rangle\langle a, E_i\rangle - \langle \nu, a\rangle\langle X, E_i\rangle\big) \\
&= \sum_i \big(\langle E_i,\nu\rangle\langle a, E_i\rangle + \langle X, \sum_j \mathbf{A}_{ij}E_j\rangle\langle a, E_i\rangle - \langle \sum_j \mathbf{A}_{ij}E_j, a\rangle\langle X, E_i\rangle - \langle \nu, a\rangle\langle E_i, E_i\rangle\big) \\
&= -n\langle \nu, a\rangle
\end{aligned}$$

Therefore, by the divergence theorem and (2.1),

$$\begin{aligned}
\int_\Sigma -n\langle \nu, a\rangle d\mu = \int_\Sigma \mathrm{div}Y_a = \int_{\partial\Sigma}\langle \eta, Y_a\rangle &= \int_{\partial\Sigma} -\cos\theta\langle \eta, a\rangle - \sin\theta\langle \nu, a\rangle \\
&= \int_{\partial\Sigma} -\langle \cos\theta\eta + \sin\theta\nu, a\rangle = \int_{\partial\Sigma} -\langle \bar{\nu}, a\rangle.
\end{aligned}$$

Combining the above identities yield the desired result. □

2.2. **The Index Form and An Abstract Formulation.** For $u = \langle Y, \nu\rangle$, [19, 3, 14] showed that, for appropriate variations satisfying Type I or Type II constraint, the second variation for functional E on a capillary surface Σ is given by,

$$E''(0) = \int_\Sigma |\nabla u|^2 - pu^2 d\mu - \int_{\partial\Sigma} qu^2 ds,$$

for

$$p := \mathrm{Rc}^{\Omega}(\nu,\nu) + |\mathbf{A}|^2,$$
$$q := \frac{1}{\sin\theta}\mathbf{A}^{\partial\Omega}(\bar{\nu},\bar{\nu}) + \cot\theta\mathbf{A}^{\Sigma}(\eta,\eta).$$

Here, $\mathbf{A}^{\partial\Omega}$ and $\mathbf{A}^{\Sigma}$ are the second fundamental forms with respect to $\bar{N}$ and ν respectively. Rc^{Ω} denotes the Ricci curuvature of Ω and our convention is such that the Ricci curvature of the round sphere $\mathbb{S}^n$ is n times the identity. For Jacobi operator $J = \Delta + p$ (for a Euclidean domain, $p = |\mathbf{A}|^2$ as the Ricci curvature vanishes), the index form is defined as

$$(2.2) \qquad Q(u,v) = \int_{\Sigma} \langle\nabla u, \nabla v\rangle - puv - \int_{\partial\Sigma} quv$$
$$= \int_{\Sigma} \langle -Ju, v\rangle + \int_{\partial\Sigma} (\nabla_{\eta}u - qu)v.$$

Definition 2.8. *The index of a symmetric bi-linear form on a vector space is the maximal dimension of a subspace on which the form is negative definite.*

Definition 2.9. *MI(Q) is the index of $Q(\cdot,\cdot)$ in the space of smooth functions on Σ.*

As we see in [27], it is possible to replace the space of smooth functions by $H^1(\Sigma) = W^{1,2}(\Sigma)$, the Sobolev space with one derivative and L^2-norm.

Definition 2.10. *The Type-I, Type-II, and Type-I+II Morse indices are with respect to the following subspaces, respectively,*

$$\mathfrak{F} := \left\{u \in H^1(\Sigma); \int_{\Sigma} u d\mu = 0\right\},$$
$$\mathfrak{G} := \{u \in H^1(\Sigma) : \int_{\partial\Sigma} u d\mu = 0\},$$
$$\mathfrak{L} := \mathfrak{F} \cap \mathfrak{G}.$$

Σ is called ($\cdot$)-stable if its corresponding index is zero.

Remark 2.11. *Obviously, for a given capillary hypersurface, the Type I and Type II Morse indices are greater than or equal to the Type I+II index.*

In [27], we developed an abstract formulation to relate indices on a Hilbert space and a finite-co-dimensional subspace. With appropriate interpretations, this theory will allow us to relate MI(Q) and an index with a constraint. We briefly recall the required results here. Let H be a separable Hilbert space. Let $S(\cdot,\cdot)$ be a symmetric continuous bi-linear form and associated with a closed self-adjoint operator with a finite index.

Definition 2.12. *For continuous linear forms $\{\phi_i\}_{i=1}^k$ on H, let $MI^{\phi_1,\ldots,\phi_k}(S)$ denote the index of S in the subspace $\cap_{i=1}^k Kernel(\phi_i)$.*

Theorem 2.13. *Let ϕ be a non-trivial continuous form. Then, $MI^{\phi}(S) = MI(S) - 1$ if and only if $\phi(\cdot) = S(u,\cdot)$ for some u satisfying $\phi(u) = S(u,u) \le 0$. Otherwise, $MI^{\phi}(S) = MI(S)$.*

Proof. It follows from [27, Theorem 1.1]. We'll sketch the key steps here for completeness.

By classical theory, $H = H_- \oplus H_0 \oplus H_+$ where $S(\cdot,\cdot)$ is negative on H_-, zero on H_0 and positive on H_+. Next, by the Riesz representation theorem, there is $\bar{\phi} \in H$ such that $(\bar{\phi}, v) = \phi(v) \forall v \in H$. As $S(\cdot,\cdot)$ is associated with a closed operator, consequently, $\bar{\phi} = \phi_- + \phi_0 + \phi_+$ for $\phi_- \in H_-$, $\phi_0 \in H_0$, $\phi_+ \in H_+$, and $(\phi_- + \phi_+, \cdot) = S(u,\cdot)$ for $u \in H$.

If $x=\phi_0+\phi_+\neq 0$ then we construct $W=\text{span}(H_-,x)$ and $S(\cdot,\cdot)$ is negative on the kernel of ϕ when restricted to W. Thus, $\text{MI}^{\phi}(S)=\text{MI}(S)$.

If $x=\phi_0+\phi_+=0$ then $\text{MI}^{\phi}(S)=\text{MI}(S)-1$ by direct calculation.

□

Also, there is a statement for several functionals and the following version for the bilinear form $Q(\cdot,\cdot)$ will be used to estimate the Morse indices of capillary hypersurfaces.

Theorem 2.14. *Assume $X:\Sigma\to\mathbb{R}^{n+1}$ is an immersed capillary hypersurface in the Euclidean unit ball $\mathbb{B}^{n+1}$. Let $\varphi_1,...,\varphi_m$ be independent continuous linear functionals over $H^1(\Sigma)$ and $G=\cap_{i=1}^m Ker(\varphi_i)$. Suppose that $\psi_1,\psi_2,\cdots,\psi_k$ are functions such that there exist C^2 functions $u_1,u_2,\cdots,u_k$ satisfy*

$$\begin{cases} u_i\in G; \\ Ju_i=\psi_i, \ \text{ on } \Sigma; \\ \nabla_\eta u_i = qu_i, \ \text{ on } \partial\Sigma. \end{cases}$$

Furthermore, assume that $\varphi_1,...,\varphi_m,\bar{\psi}_1,...,\bar{\psi}_k$ are linearly independent where $\bar{\psi}_i$ is the linear functional defined by $L^2(\Sigma)$-multiplication by ψ_i. Then

$$MI^{\{\varphi_1,...\varphi_m,\bar{\psi}_1,\cdots,\bar{\psi}_k\}}(Q)=MI^{\{\varphi_1,...\varphi_m\}}(Q)-i_k,$$

where i_k is the number of nonnegative eigenvalues of the matrix

$$\Upsilon:=\left(\int_\Sigma u_j\psi_i\right).$$

In particular, $MI^{\{\varphi_1,...\varphi_m\}}(Q)\geq i_k$.

Proof. Since G is a subspace of finite codimension, it is a Hilbert space. Because $\varphi_1,...,\varphi_m$, $\bar{\psi}_1,...,\bar{\psi}_k$ are linearly independent, $\bar{\psi}_1,....\bar{\psi}_k$ are linearly independent as linear functionals on G. Recall

$$Q(u,v):=\int_\Sigma \nabla u\nabla v-puv-\int_{\partial\Sigma}quv.$$

By integration by parts,

$$\begin{aligned} Q(u_i,u_j) &= \int_\Sigma -Ju_iu_j+\int_{\partial\Sigma}(\nabla_\eta u_i-qu_i)u_j \\ &= -\int_\Sigma u_j\psi_i. \end{aligned}$$

Applying [27][Theorem 1.3] for linear independent functionals $\bar{\psi}_1,...\bar{\psi}_k$ finishes the proof.

□

2.3. **Fredholm Alternative with Robin Boundary Condition.** In this subsection, we will derive a Fredholm alternative for an elliptic operator with a Robin boundary condition. The presentation here follows [7]. First, we recall the abstract Fredholm alternative.

Theorem 2.15 (Abstract Fredholm Alternative)**.** [7, Appendix D] *Let $K:H\mapsto H$ be a compact linear operator on a Hilbert space and K^* its adjoint. Then*

- *The null space of $Id-K$ is finite dimensional,*
- *The range of $Id-K$ is closed,*
- *The range of $Id-K$ is the orthogonal complement of the null space of $Id-K^*$,*
- *The null space of $Id-K$ is trivial if and only if $Id-K$ is onto,*

- *The dimension of the null space of $Id - K$ is equal to that of $Id - K^*$.*

Let Σ be a compact domain with a smooth boundary and outward conormal vector η. We define the trace operator $T : H^1(\Sigma) \mapsto L^2(\partial\Sigma)$ and the normal derivative $D_\eta : H^1(\Sigma) \mapsto L^2(\partial\Sigma)$ such that, for $u \in C^\infty(\Sigma)$,

$$Tu = u\mid_{\partial\Sigma},\ \ D_\eta u = \nabla_\eta u.$$

We consider a symmetric bilinear form, for $u, v \in H^1(\Sigma)$,

$$Q(u,v) := \int_\Sigma \nabla u \nabla v - puv - \int_{\partial\Sigma} qT(u)T(v).$$

This bilinear form induces an abstract linear map $\mathcal{Q} : H^1(\Sigma) \mapsto (H^1)^*(\Sigma)$ (continuous dual) such that $Q(u,v) := \mathcal{Q}u(v)$. A pair $(-f, g)$, for $f \in L^2(\Sigma)$ and $g \in L^2(\partial\Sigma)$, defines an element in $(H^1)^*(\Sigma)$ via $(-f,g)v = (-f,v)_{L^2(\Sigma)} + (g, Tv)_{L^2(\partial\Sigma)}$. $(\mathcal{Q}u) = (-f, g)$ if and only if there exists a weak solution to system

$$\begin{cases} Ju &= f \text{ in } \Sigma, \\ \nabla_\eta u - qu &= g \text{ on } \partial\Sigma. \end{cases} \tag{2.3}$$

Indeed, for $u, v \in C^\infty(\Sigma)$, by part integration, one observes

$$\begin{aligned} Q(u,v) &= -\int_\Sigma vJu + \int_{\partial\Sigma} (D_\eta(u) - qT(u))T(v), \\ &= (-Ju, v)_{L^2(\Sigma)} + (D_\eta u - qT(u), v)_{L^2(\partial\Sigma)}. \end{aligned}$$

Also, the associated homogeneous problem is given by

$$\begin{cases} Ju &= 0 \text{ on } \Sigma, \\ D_\eta u - qu &= 0 \text{ on } \partial\Sigma. \end{cases} \tag{2.4}$$

By Cauchy-Schwarz inequalities and the trace theorem $||Tu||_{L^2(\partial\Sigma)} \le c(\Sigma)||u||_{H^1(\Sigma)}$, $Q(\cdot,\cdot)$ is bounded. Since $|p|$ and q are bounded, following classical theory, one modifies, for some constant $\gamma > 0$

$$Q_\gamma(u,v) = Q(u,v) + \gamma\langle u, v\rangle_{L^2(\Sigma)}$$

such that Q_γ is coercive, for some constant $\beta > 0$,

$$Q_\gamma(u,u) \ge \beta||u||^2_{H^1(\Sigma)}.$$

Then Lax-Milgram theorem implies that the corresponding operator $\mathcal{Q}_\gamma$ is an isomorphism from $H^1(\Sigma)$ to $(H^1)^*(\Sigma)$. Thus, $K := \gamma\mathcal{Q}_\gamma^{-1}(\cdot, 0)$ is well-defined and by the natural identification $L^2(\Sigma) \subset (H^1)^*(\Sigma)$ and $H^1(\Sigma) \subset L^2(\Sigma)$,

$$K : L^2(\Sigma) \mapsto L^2(\Sigma).$$

It is immediate that for $Kf = v$, $\mathcal{Q}_\gamma(v) = \gamma(f, 0)$,

$$\begin{aligned} \beta||Kf||^2_{H^1(\Sigma)} = \beta||v||^2_{H^1(\Sigma)} &\le Q_\gamma(v,v) = (\mathcal{Q}_\gamma v)(v) = \gamma(f,0)(v) = \gamma(f,v)_{L^2(\Sigma)} \\ &\le \gamma||f||_{L^2(\Sigma)}||v||_{L^2(\Sigma)} \le \gamma||f||_{L^2(\Sigma)}||v||_{H^1(\Sigma)}. \end{aligned}$$

By Reillich-Kondrachov compactness, theorem, K is a compact operator. Furthermore, u is a weak solution of (2.3) if and only if

$$(\text{Id} - K)u = f_1 \text{ for } f_1 = \mathcal{Q}_\gamma^{-1}(-f, g). \tag{2.5}$$

So the abstract Fredholm alternative theorem is applicable and yields the following.

Theorem 2.16. *The operator $\mathcal{Q}$ has closed range. For any $f \in L^2(\Sigma)$ and $g \in L^2(\partial\Sigma)$, either (2.3) has a unique weak solution or the homogeneous problem (2.4) has a non-trivial space N of weak solutions. Furthermore, if the latter holds, then N has finite dimension and (2.3) has a weak solution if and only if $(f, v)_{L^2(\Sigma)} = (g, v)_{L^2(\partial\Sigma)}$ for all $v \in N$.*

Proof. From 2.5, we have

$$\mathcal{Q}_\gamma^{-1}\mathcal{Q}u = (Id - K)u.$$

As $\mathcal{Q}_\gamma$ is coercive, for $v = \mathcal{Q}_\gamma^{-1}w$,

$$\beta||\mathcal{Q}_\gamma^{-1}w||^2_{H^1(\Sigma)} = \beta||v||^2_{H^1(\Sigma)} \leq Q_\gamma(v, v) = (\mathcal{Q}_\gamma v)(v) = w(v) \leq ||w|_{(H^1)^*(\Sigma)}||v||_{H^1(\Sigma)}.$$

Thus, $||\mathcal{Q}_\gamma^{-1}w||_{H^1(\Sigma)} \leq \frac{1}{\beta}||w|_{(H^1)^*(\Sigma)}$ and $\mathcal{Q}_\gamma^{-1}$ is a bounded operator. Consequently, the closed range of $\mathcal{Q}$ comes from the close range of $\mathrm{Id} - K$.

For the Fredholm alternative, one first observes that $K = K^*$. With the discussion above, the only remaining nontrivial part is to interpret $f_1 = \mathcal{Q}_\gamma^{-1}(-f, g)$ in the orthogonal complement of the null space of $\mathrm{Id} - K^* = \mathrm{Id} - K$. For v in the null space, we have $(\mathrm{Id} - K)v = 0$. That is, $\mathcal{Q}_\gamma(v) = \gamma(v, 0)$. Thus, f_1 is in orthogonal complement if and only if

$$\begin{aligned} 0 &= (f_1, v)_{L^2(\Sigma)} = (v, 0)(f_1) = \mathcal{Q}_\gamma(v)(f_1) = \mathcal{Q}f_1(v) = (-f, g)v \\ &= -(f, v)_{L^2(\Sigma)} + (g, v)_{L^2(\partial\Sigma)}. \end{aligned}$$

The result then follows. □

Remark 2.17. *When there is no boundaries, Theorem 2.16 just recovers the Fredholm alternative of an elliptic operator on a compact closed manifold. When $q = 0$, it recovers the Fredholm alternative of an elliptic operator with a Neumann boundary condition.*

3. Index Relations

In this section, we characterize in index with either a Type I or Type II constraint.

3.1. **Type-I Partitioning.**

The Type I Morse index is at most equal to $\mathrm{MI}(Q)$ and is finite. We are ready to exactly determine their difference.

Proof of Theorem 1.2. The Type I Morse index is equal to $\mathrm{MI}^\phi(Q)$ where ϕ is the functional

$$\phi(u) = (1, u)_{L^2(\Sigma)}.$$

As $\mathcal{Q}$ has closed range, Theorem 2.13 implies that $\mathrm{MI}^\phi(Q) = \mathrm{MI}(Q) - 1$ if and only if there exists a weak solution

$$\begin{cases} -Ju &= 1 \text{ in } \Sigma, \\ Du &= qu \text{ on } \partial\Sigma, \\ (1, u)_{L^2(\Sigma)} &\leq 0. \end{cases}$$

The elliptic regularity theory (see, for example, [13, Chapter 8]) then asserts that the solution is smooth. □

3.2. **Weak Morse index.** Here we consider a special case of Type I partitioning when there is no boundary (the main examples are boundary-less hypersurfaces in a sphere). The second variation is simplified [3][Proposition 2.5]:

$$Q(u,v) = \int_\Sigma \Big(\langle \nabla u, \nabla v\rangle - puv\Big) d\mu,$$
$$p := \mathrm{Rc}^\Omega(\nu,\nu) + |\mathbf{A}^\Sigma|^2.$$

Definition 3.1. *The weak Morse index of the hypersurface $\Sigma^n \subset \Omega^{n+1}$ is the index of $Q(\cdot,\cdot)$ on $\mathfrak{F} = \{u \in H^1(\Sigma) : \int_\Sigma u d\mu = 0\}$.*

An immediate consequence of Theorem 1.2 is the following.

Theorem 3.2. *Let $\Sigma \subset \Omega$ be a closed, orientable, CMC hypersurface. Its weak Morse index is equal to MI$(Q) - 1$ if and only if there is a smooth function u such that*

$$\begin{cases} (\Delta + p)u & = -1 \text{ on } \Sigma, \\ \int_\Sigma u & \leq 0. \end{cases} \tag{3.1}$$

Otherwise, it is equal to MI(Q).

In some cases, (3.1) can be determined from the geometry of the surface. For instance, Corollary 1.16 says that the difference of Morse indices is one for CMC surfaces with constant scalar curvature in a unit sphere.

Proof of Corollary 1.16. For a CMC hypersurface in $\mathbb{S}^{n+1}$,

$$J = \Delta + n + |\mathbf{A}|^2.$$

Denoting by S the scalar curvature, by [5, Chapter 1], we recall the following Gauss equation

$$S_\Sigma = S_{\mathbb{S}^{n+1}} - 2\mathrm{Rc}_{\mathbb{S}^{n+1}}(\nu,\nu) + H^2 - |\mathbf{A}|^2.$$

Therefore

$$|\mathbf{A}|^2 = H^2 - S_\Sigma + n(n-1).$$

Since S_Σ is constant,

$$p = n + |\mathbf{A}|^2 = n^2 + H^2 - S = \text{constant}.$$

We have, $\mathrm{Ker}(\mathcal{Q})$ is the set of solutions

$$(\Delta + p)u = 0.$$

It is non-trivial if and only if $p > 0$ is an eigenvalue of Δ. In that case, since the constant function 1 is the first eigenfunction of Δ with eigenvalue zero, for any $v \in \mathrm{Ker}(\mathcal{Q})$

$$(1, v)_{L^2(\Sigma)} = 0.$$

By the Fredhom alternative, Theorem 2.16, there is always a solution of

$$Ju = -1.$$

For such u, we have,

$$\int_\Sigma pud\mu = p\int_\Sigma ud\mu = \int_\Sigma (-1 - \Delta u) d\mu = -\mathrm{Area}(\Sigma) < 0.$$

The assertion then follows from Theorem 3.2. $\square$

Remark 3.3. *For $\Sigma^2 \subset \mathbb{S}^3$, we can replace the scalar curvature by the intrinsic Gauss curvature K*

$$|\boldsymbol{A}|^2 = H^2 - 2(K-1) = 2 + H^2 - 2K.$$

An almost identical analysis replacing $H^1(\Sigma)$ by $H^1_0(\Sigma)$ is applicable for the fixed boundary problem. The statement goes as follows.

Theorem 3.4. *Let $\Sigma \subset \Omega$ be a CMC hypersurface with boundaries. Then its weak Morse index with respect to the fixed boundary problem is equal to MI$(Q) - 1$ if and only if there is a smooth function u such that*

$$\begin{cases} Ju & = -1 \text{ on } \Sigma, \\ u & = 0 \text{ on } \partial\Sigma, \\ \int_\Sigma u & \leq 0. \end{cases} \tag{3.2}$$

Otherwise, it is equal to MI(Q).

3.3. **Type-II Partitioning.** In this section, we investigate Type II Morse indices for stationary hypersurfaces. It is at most MI(Q) and finite. We are ready to determine it precisely.

Proof of Theorem 1.6. The Type II Morse index is equal to MI$^\varphi(Q)$ where φ is the functional

$$\varphi(u) = (1, u)_{L^2(\partial\Sigma)}.$$

As $\mathcal{Q}$ has closed range, Theorem 2.13 implies that MI$^\phi(Q) =$ MI$(Q) - 1$ if and only if there exists a weak solution

$$\begin{cases} -Ju & = 0 \text{ on } \Sigma, \\ Du - qu & = 1 \text{ on } \partial\Sigma, \\ (1, u)_{L^2(\partial\Sigma)} & \leq 0. \end{cases}$$

The elliptic regularity theory (see, for example, [13, Chapter 8]) then asserts that the solution is smooth. □

4. Capillary hypersurfaces in a Euclidean ball

In this section, we study capillary hypersurfaces in a Euclidean ball $\Omega^{n+1} = B^{n+1}$, center at the origin, with respect to Type-I+II constraint. An advantage is to streamline stability results of [29] and [14] as special cases. The key is to construct a family of functions satisfying Type I+II constraint. We would like to highlight that our computation is valid and simpler if the boundary $\partial\Sigma$ is empty. So our results can be applied to closed CMC hypersurfaces in Euclidean spaces. First, one recalls the Jacobi operator, $J = \Delta + p = \Delta + |\mathbf{A}|^2$, from Section 2.

Proposition 4.1. *Let $X : \Sigma \to \mathbb{B}^{n+1}$ be an isometric immersion with constant mean curvature H and fixed intersecting angle θ. We consider*

$$\begin{aligned} \phi : &= \frac{|X|^2}{2} + \frac{1}{2} \\ Z : &= (\Delta\phi)X - \phi\Delta X, \\ \Phi : &= H\phi - n\langle X, \nu\rangle - n\cos\theta - H. \end{aligned}$$

Then, for any constant c, the following identities hold:

$$J(Z + c\nu) = n|\mathring{A}|^2 X, \tag{4.1}$$

$$\Delta\Phi = n|\mathring{A}|^2 \langle X, \nu\rangle, \tag{4.2}$$

$$\langle (Z + n\cos\theta\nu), J(Z + n\cos\theta\nu)\rangle = n^2|\mathring{A}|^2|X^T|^2 - \Phi\Delta\Phi. \tag{4.3}$$

$$\bar{\nabla}_\eta (Z + n\cos\theta\nu)|_{\partial\Sigma} = q(Z + n\cos\theta\nu)|_{\partial\Sigma}, \tag{4.4}$$

Proof. We first prove (4.1). First, $Z = (\Delta\phi)X - \phi\Delta X = X\Delta\phi + \phi H\nu$. Since $\phi = \frac{|X|^2}{2} + \frac{1}{2}$, $\Delta\phi = n - H\langle X, \nu\rangle$. Therefore,

$$\begin{aligned}
\Delta Z &= X(-H\Delta\langle X, \nu\rangle) + H\phi\Delta\nu \\
&= -HX(H - |\mathbf{A}|^2\langle X, \nu\rangle) + H\phi(-|\mathbf{A}|^2\langle \nu, a\rangle) \\
&= -H^2 X + H|\mathbf{A}|^2\langle X, \nu\rangle X + \phi|\mathbf{A}|^2\Delta X \\
&= (n|\mathbf{A}|^2 - H^2)X - |\mathbf{A}|^2[(n - H\langle X, \nu\rangle)X - \phi\Delta X]
\end{aligned}$$

Then

$$\Delta Z + |\mathbf{A}|^2 Z = n|\mathring{A}|^2 X.$$

Next,

$$\Delta\Phi = H(n - H\langle X, \nu\rangle) + n(|\mathbf{A}|^2\langle X, \nu\rangle - H) = n|\mathring{A}|^2\langle X, \nu\rangle.$$

Then,

$$\begin{aligned}
\langle (Z + n\cos\theta\nu), X\rangle &= |X|^2\Delta\phi + \phi H\langle X, \nu\rangle + n\cos\theta\langle X, \nu\rangle \\
&= n|X|^2 - H|x|^2\langle X, \nu\rangle + \phi H\langle X, \nu\rangle + n\cos\theta\langle X, \nu\rangle \\
&= n|X^T|^2 + n\langle X, \nu\rangle^2 - H(2\phi - 1)\langle X, \nu\rangle + \phi H\langle X, \nu\rangle + n\cos\theta\langle X, \nu\rangle \\
&= n|X^T|^2 - \langle X, \nu\rangle\Phi.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle (Z + n\cos\theta\nu), J(Z + n\cos\theta\nu)\rangle &= n^2|\mathring{A}|^2|X^T|^2 - n\Phi|\mathring{A}|^2\langle X, \nu\rangle \\
&= n^2|\mathring{A}|^2|x^T|^2 - \Phi\Delta\Phi
\end{aligned}$$

The last identity follows from Proposition 2.7. □

We now fix a coordinate system of $\mathbb{R}^{n+1}$ and choose, for $i = 1, 2, \cdots, n+1$,

$$\begin{aligned}
u_i &= (Z + n\cos\theta\nu)_i, \\
Ju_i &= \psi_i = n|\mathring{A}|^2 X_i.
\end{aligned}$$

Lemma 4.2. *Each u_i satisfies the Type I+II constraint.*

Proof. Let $w = \langle X, \nu\rangle$. We have

$$\begin{aligned}
Z_i &= (\Delta\phi)X_i - \phi\Delta X_i = \mathrm{div}(X_i\nabla\phi - \phi\nabla X_i), \\
n\nu_i &= \mathrm{div}(\nu_i X - w e_i).
\end{aligned}$$

Therefore, by the divergence theorem,

$$\begin{aligned}\int_\Sigma u_i &= \int_\Sigma \operatorname{div}(X_i\nabla\phi - \phi\nabla X_i + \cos\theta(\nu_i X - we_i))d\mu\\ &= \int_{\partial\Sigma}(X_i\langle X,\eta\rangle - \eta_i + \cos\theta(\nu_i\langle X,\eta\rangle - w\eta_i))ds\end{aligned}$$

Applying (2.1) yields

$$\int_\Sigma u_i = \int_{\partial\Sigma}\sin\theta(-\cos\theta\nu_i + \cos\theta\nu_i)ds = 0.$$

Next, we consider u_i on the boundary $\partial\Sigma$. Using the above calculation and (2.1) we have

$$\begin{aligned}u_i\mid_{\partial\Sigma} &= (n - H\langle X,\nu\rangle)X_i + \frac{1}{2}(X^2+1)H\nu_i + n\cos\theta\nu_i\\ &= (n + H\cos\theta)X_i + H\nu_i + n\cos\theta\nu_i\\ &= n\langle X + \cos\theta\nu, e_i\rangle + H\langle\cos\theta X + \nu, e_i\rangle\\ &= n\sin\theta\langle\eta, e_i\rangle + H\sin\theta\langle\bar{\nu}, e_i\rangle.\end{aligned}$$

Thus, by Proposition 2.7,

$$\int_{\partial\Sigma} u_i = 0.$$

□

We consider the following matrix

$$\Upsilon := \left(\int_\Sigma u_i\psi_j\right)_{(n+1)\times(n+1)}.$$

Alternatively, Υ can be seen as a bilinear form on $\mathbb{R}^{n+1}$. Namely, for $v_1, v_2 \in \mathbb{R}^{n+1}$,

$$\begin{aligned}\Upsilon(v_1,v_2) :&= \int_\Sigma (n|\mathbf{A}|^2 - H^2)\langle Z + n\cos\theta\nu, v_1\rangle\langle X, v_2\rangle\\ &= \int_\Sigma n|\mathring{A}|^2[\langle(n - H\langle X,\nu\rangle)X + (n\cos\theta + \frac{H}{2}(|X|^2+1))\nu, v_1\rangle\langle X, v_2\rangle]\end{aligned}$$

Then, we can choose an orthonormal basis to diagonalize Υ as $\operatorname{diag}(\lambda_1, ..., \lambda_{n+1})$ with

$$\lambda_i = \int_\Sigma n|\mathring{A}|^2(n - H\langle X,\nu\rangle)X_i^2 + \int_\Sigma n|\mathring{A}|^2(n\cos\theta + \phi H)X_i\nu_i. \tag{4.5}$$

Lemma 4.3. *The trace of Υ, $tr(\Upsilon)$, satisfies:*

$$tr(\Upsilon) = \int_\Sigma n^2\left(|\mathring{A}|^2|X^T|^2 + |\mathring{A}(X^T)|^2\right).$$

Proof. We use (4.3) in Proposition 4.1:

$$\sum_{i=1}^{n+1} u_i\psi_i = \langle(Z + c_1\nu), J(Z + c_1\nu)\rangle = n(n|\mathbf{A}|^2 - H^2)|X^T|^2 - \Phi\Delta\Phi.$$

Since $\phi(X) = \frac{1}{2}(|X|^2+1)$, on the boundary,

$$\begin{aligned}\Phi &= H\phi - nw - n\cos\theta - H\\ &= \frac{1}{2}H(|X|^2+1) - n\langle X,\nu\rangle - n\cos\theta - H\\ &= 0.\end{aligned}$$

Therefore,

$$\int_\Sigma \Phi\Delta\Phi = -\int_\Sigma |\nabla\Phi|^2.$$

Consequently,

$$\mathrm{tr}(\Upsilon) = \int_\Sigma \left(n(n|\mathbf{A}|^2 - H^2)|X^T|^2 + |\nabla\Phi|^2\right).$$

Finally,

$$\begin{aligned}\nabla\Phi &= H\nabla\frac{|X|^2}{2} - n\nabla\langle X,\nu\rangle\\ &= HX^T - n\mathbf{A}(X^T).\end{aligned}$$

The result then follows. □

We are ready to prove Theorem 1.9.

Proof of Theorem 1.9. By Lemma 4.2, $u_i \in \mathcal{L}$ (see Definition 2.10). Thus, if Σ is Type I+II stable then Υ is nonpositive definite, and $\mathrm{tr}(\Upsilon_1) \leq 0$. From Lemma 4.3

$$\int_\Sigma \left(|\mathring{A}|^2|X^T|^2 + |\mathring{A}(X^T)|^2\right) \leq 0. \tag{4.6}$$

Thus, $|X^T|^2|\mathring{A}|^2 \equiv 0 \equiv \mathring{A}(X^T)$ and, consequently, $\nabla\Phi \equiv 0$. Since $\Phi \equiv 0$ on $\partial\Sigma$, $\Phi \equiv 0$ everywhere. Therefore, $\langle X,\nu\rangle(n|\mathbf{A}|^2 - H^2) = \Delta\Phi \equiv 0$. If $\langle X,\nu\rangle$ vanishes at a point different from the origin, then, since $|X|^2 = \langle X,\nu\rangle^2 + |X^T|^2$, $|X|^T \neq 0$ and $|\mathring{A}|^2 = 0$. Consequently, $\mathring{A} \equiv 0$ over a dense subset and, by unique continuation, Σ is totally umbilical.

Second, if Σ is not umbilical then $(n|\mathbf{A}|^2 - H^2) \neq 0$ at some point, then there exists an open subset U on which $(n|\mathbf{A}|^2 - H^2) \neq 0$. Let ψ_i' be the linear functional defined by $L^2(\Sigma)$ multiplication of the function ψ_i.

Claim: $\psi_1, \cdots, \psi_{n+1}$ are linearly independent and, thus, the dimension of the space $\mathrm{span}\{\psi_1, \cdots, \psi_{n+1}\}$ is $n+1$.

Otherwise there exists $c_1, \cdots, c_{n+1} \in \mathbb{R}$ such that $c_1^2 + \cdots + c_{n+1}^2 \neq 0$ and, by Proposition 4.1,

$$c_1X_1 + \cdots c_{n+1}X_{n+1} = 0.$$

This implies that U is contained in a hyperplane and thus is totally geodesic which contradicts to $(n|\mathbf{A}|^2 - H^2) \neq 0$. Thus, the claim is true,

Claim: The space $\mathrm{span}(\bar{\psi}_1, \cdots, \bar{\psi}_{n+1})$, as linear functionals over $\mathcal{L}$ has dimension n if Σ is $|\mathring{A}|^2$-scale equivalent to a hyper-planar. Otherwise it has dimension $n+1$.

Let χ and Ψ be linear functionals on $H^1(\Sigma)$ defined respectively by $L^2(\Sigma)$ and $L^2(\partial\Sigma)$-multiplication by the constant function 1. It is immediate that

$$\{\bar{\psi}_1, \cdots, \bar{\psi}_{n+1}, \Psi\}$$

are linearly independent. Then, $\{\bar{\psi}_1, \cdots, \bar{\psi}_{n+1}, \chi\}$ are linearly dependent if and only if, by Proposition 4.1,

$$c_1|\mathring{A}|^2 X_1 + ...c_{n+1}|\mathring{A}|^2 X_{n+1} + 1 = 0.$$

That is, $|\mathring{A}|^2\langle X, a\rangle = 1$ for some constant vector a. In that case, $|\mathring{A}|^2 X$ is hyper-planar. Using the previous claim, the proof is finished.

Finally, applying Theorem 2.14 yields the desired results.

□

4.1. **Cylinder in a ball.** We consider a flat cylinder of radius $0 < r < 1$ inside $\Omega = \mathbb{B}^{n+1}$, $X : \Sigma = [-T, T] \times \mathbb{S}^{n-1} \mapsto \Omega = \mathbb{B}^{n+1}$

$$X(t, z) = (t, rz). \tag{4.7}$$

Here $T = \sqrt{1 - r^2}$. Let $\{w_i, i = 1, ...n - 1\}$ be a basis of tangent vectors on $\mathbb{S}^{n-1}$. It is straightforward to compute tangent vectors: $X_{,t} = (1, \vec{0}), X_{,i} = (0, rw_i)$. Consequently,

$$\nu = (0, z),$$
$$\eta = \frac{t}{|t|}\partial_t.$$

Thus, the boundary derivative becomes

$$\nabla_\eta = \partial_{\pm t}.$$

By our convention,

$$\sin\theta = \langle X.\eta\rangle = \sqrt{1 - r^2} = T,$$
$$\cos\theta = -\langle X, \nu\rangle = -r.$$

Thus,

$$q = \frac{1}{\sin\theta} + \cot\theta \mathbf{A}(\eta, \eta) = \frac{1}{\sqrt{1 - r^2}} = \frac{1}{T}.$$

The second fundamental form is

$$\mathbf{A}(X_t, X_t) = -\langle X_{tt}, \nu\rangle = 0,$$
$$\mathbf{A}(X_i, X_i) = -\langle X_{ii}, \nu\rangle = r.$$

Thus,

$$H = (n - 1)\frac{r}{|X_i|^2} = \frac{n - 1}{r},$$
$$|\mathbf{A}|^2 = \frac{n - 1}{r^2},$$
$$|\mathring{A}|^2 = |\mathbf{A}|^2 - \frac{H^2}{n} = \frac{n - 1}{nr^2}.$$

The Jacobi operator is

$$J = \Delta + |\mathbf{A}|^2 = \partial_t^2 + \frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}} + \frac{n - 1}{r^2}.$$

We will first consider the index without any constraint, MI(Q).

Proposition 4.4. *For a round cylinder of radius r in the unit ball, we have*

(1) *MI(Q) $\geq n + 2$;*

(2) *When* $r \to 0$ *or* $r \to 1$ *MI*$(Q) \to \infty$*;*
(3) *There is an interval* $0 < a < r < b < 1$ *such that MI*$(Q) = n + 2$*.*

Proof. Recall that $\mathrm{MI}(Q)$ is the number of negative eigenvalues of the system:

$$Ju + \lambda u = 0 \text{ on } \Sigma,$$
$$\nabla_\eta u = qu \text{ on } \partial\Sigma.$$

By separation of variables, the eigenfunctions have the form $u = f(t)g(z)$, where g is a spherical Laplacian eigenfunction. That is,

$$\Delta_{\mathbb{S}^{n-1}} g + \alpha g = 0,$$

and f satisfies, for $\beta := \lambda + \frac{n-1}{r^2} - \frac{\alpha}{r^2}$,

$$\begin{cases} f'' + \beta f = 0, \\ \frac{f(T)}{f'(T)} = -\frac{f(-T)}{f'(-T)} = T. \end{cases}$$

Therefore, we exhaust all possible cases which may produce indices:

$$\begin{cases} \beta = 0, \\ \beta > 0, \quad \tan(\sqrt{\beta}T) = \sqrt{\beta}T \text{ or } \cot(\sqrt{\beta}T) = -\sqrt{\beta}T, \\ \beta < 0, \quad \coth(\sqrt{-\beta}T) = \sqrt{-\beta}T. \end{cases}$$

Here we exclude the case $\tanh(\sqrt{-\beta}T) = \sqrt{-\beta}T$ since $\tanh x > x$ for all $x \neq 0$. By spherical harmonic analysis, $\alpha = k(k+n-2)$ with multiplicity

$$m_0 = 1, \ m_1 = n,$$
$$m_k = \frac{(n+k-1)!}{(n-1)!k!} - \frac{(n+k-3)!}{(n-1)!(k-2)!}.$$

and corresponding eigenfunctions, $g_0 = 1, g_1, g_2, \cdots, g_n, \cdots, \cdots$. Recall that $\mathrm{MI}(Q)$ is the number of negative eigenvalues λ for

$$\lambda = \beta + \frac{1}{r^2}(k-1)(k+n-1),$$

for $k = 0, 1, 2....$ Next, we analyse all possible cases which may contribute to the indices.

Case 1. If $\beta = 0$ then $f(t) = t$. If $k \geq 1$ then $\lambda \geq 0$. Hence only choosing $k = 0$ yields $\lambda < 0$. We know that in this case $\beta = 0$ contributes 1 to the index whenever $r \in (0,1)$. Namely the eigenfunction is $u(t) = t$ and the corresponding eigenvalue is $\lambda = -\frac{n-1}{r}$.

Case 2. If $\beta < 0$, choosing $k = 0, 1$ yields $\lambda < 0$. Since $\coth(\sqrt{-\beta}T) = \sqrt{-\beta}T$, then $\sqrt{-\beta}T = T_0$, where T_0 be the unique positive number such that

$$\coth(T_0) = T_0.$$

Then, for $k \geq 2$,

$$\lambda = \frac{1}{r^2}(k-1)(k+n-1) - \frac{T_0^2}{T^2}.$$

In particular, $k = 2$,

$$\lambda = \frac{n+1}{r^2} - \frac{T_0^2}{1-r^2}.$$

Here we have used $T^2 = 1 - r^2$. So $\lambda < 0$ if and only if

$$r > r_0 := \sqrt{\frac{n+1}{T_0^2 + n + 1}}.$$

Therefore $\lambda < 0$ for all k such that

$$(k-1)(k+n-1) < \frac{r^2 T_0^2}{1-r^2}.$$

When $r \to 1$, the right hand side of above inequality goes to infinity and k can be sufficiently large when r is close to 1. This implies $\mathrm{MI}(Q) \to \infty$ as $r \to 1$. Also we know that the case $\beta < 0$ contributes exactly $n+1$ to the index when $r < r_0$. Namely the eigenfunctions are $u_0(t) = \cosh \frac{T_0 t}{T}$, and $u_1(t) = \cosh \frac{T_0 t}{T} g_1, \cdots, u_n(t) = \cosh \frac{T_0 t}{T} g_n$, corresponding eigenvalues are $\lambda = -\frac{n-1}{r} - \frac{T_0^2}{1-r^2}$ and $\lambda = -\frac{T_0^2}{1-r^2}$ with multiplicity n.

Case 3. When $\beta > 0$, one observes that, for $k \geq 1$, $\lambda > 0$. So we consider only $k = 0$. Note that here are infinitely many periodic positive values $T_1 < T_2 < \cdots \to +\infty$, satisfying the equation $\tan x = x$ or $\cot x = -x$. In this case we can choose β_j such that $\sqrt{\beta_j} T = T_j$. Since $k = 0$,

$$\lambda_j = \beta_j - \frac{n-1}{r^2} = \frac{T_1^2}{T^2} - \frac{n-1}{r^2}.$$

Therefore, for each β_j we can find a small r such that $\lambda < 0$. So $\mathrm{MI}(Q) \to \infty$ when $r \to 0$.

Moreover each β_j have multiplicity one, we have the largest r_1 such that $\mathrm{MI}(Q) = n+3$. Therefore the case $\beta > 0$ contributes at most 1 to the index when $r > r_1$, where r_1 is defined as

$$r_1^2 = \frac{n-1}{T_1^2 + n - 1}.$$

When $r > r_1$, a direct computation shows that the only eigenvalue is $\lambda = \frac{T_1^2}{1-r^2} - \frac{n-1}{r^2}$ which is positive and the corresponding eigenfunction is $u(t) = \sin \frac{T_1 t}{T}$.

Finally, combining with the case $\beta < 0$, $\mathrm{MI}(Q) = n+2$ if and only if

$$n - 1 < \frac{r^2 T_1^2}{1-r^2},$$
$$n + 1 > \frac{r^2 T_0^2}{1-r^2}.$$

□

Remark 4.5. *Numerically,* $T_0 \approx 1.19968$ *and* $T_1 \approx 2.79838$*.*

Next, we'll show that both cases of Theorem 1.2 might arise. As r varies from 0 to 1,

$$x = T\sqrt{n-1}/r = \sqrt{n-1}\frac{\sqrt{1-r^2}}{r}$$

goes from $+\infty$ to 0. Thus, $\cos(x) + x \sin x$ fluctuates and assumes all possible real values.

Lemma 4.6. *Let* $x = T\sqrt{n-1}/r$*. If* $\cos(x) + x\sin x = 0$ *then the system*

$$Ju = -1,$$
$$\nabla_\eta u = qu$$

has no solution.

Proof. By our analysis above, $\lambda = 0$ (when $\beta = 0$ and $k = 1$) is an eigenvalue of

$$Ju + \lambda u = 0,$$
$$\nabla_\eta u = qu.$$

Under the hypothesis, let $\sqrt{\beta} = \sqrt{n-1}/r$, then $\cot(\sqrt{\beta}T) = -\sqrt{\beta}T$ and $u = \cos(\sqrt{\beta}t)$ is an eigenfunction with eigenvalue 0 of the homogeneous system above. It is observed that $\int_\Sigma u d\mu \neq 0$. So, by the Fredholm alternative, Theorem 2.16, the result follows. □

Proposition 4.7. *Let $x = T\sqrt{n-1}/r$, the Type-I Morse index of a cylinder is*

$$\begin{cases} MI(Q) \text{ if either } \cos(x) + x\sin x = 0 \text{ or } x < \frac{\sin x}{x\sin x + \cos x}, \\ MI(Q) - 1 \text{ otherwise.} \end{cases}$$

Proof. If $\cos(x) + x\sin x = 0$ then the result follows from Lemma 4.6 and Theorem 1.2.

Otherwise, $\cos(x) + x\sin x \neq 0$ and let

$$u = c\cos(t\sqrt{n-1}/r) - \frac{r^2}{n-1}.$$

It is readily verified that u solves the system

$$Ju = -1,$$
$$\nabla_\eta u = qu$$

for $c(x\sin x + \cos x) = \frac{r^2}{n-1}$. As $u_{tt} + \frac{n-1}{r^2}u = -1$,

$$\frac{n-1}{r^2}\int_\Sigma u d\mu = -2u'(T) - 2T.$$

We have

$$\begin{aligned} \sqrt{n-1}/r(u'(T) + T) &= -c\frac{n-1}{r^2}\sin x + x \\ &= -\frac{\sin x}{x\sin x + \cos x} + x. \end{aligned}$$

Applying Theorem 1.2 again yields the conclusion. □

Next we check how Theorem 1.9 is applicable in this case. For these cylinders,

$$\langle X, \nu\rangle = r.$$

Thus components of the matrix from Theorem 1.9 become

$$\begin{aligned} \Upsilon_{ij} &= \int_\Sigma n\frac{n-1}{nr^2}\langle (n - \frac{n-1}{r}r)X + (n(-r) + \frac{n-1}{2r}(t^2 + r^2 + 1))\nu, e_i\rangle\langle X, e_j\rangle \\ &= \frac{n-1}{r^2}\int_\Sigma \langle X + \frac{(n-1)(t^2+1) - (n+1)r^2}{2r}\nu, e_i\rangle X_j \\ &= \frac{n-1}{r^2}\int_\Sigma X_iX_j + \frac{(n-1)(t^2+1) - (n+1)r^2}{2r}\nu_iX_j. \end{aligned}$$

For $X_1 = t, X_i = rz_{i-1}$, $\nu_1 = 0$, $\nu_i = z_{i-1}$, and $i \neq j$,

$$\Upsilon_{ij} = 0.$$

It remains to calculate the diagonal terms. First, we have

$$\Upsilon_{11} = \frac{n-1}{r^2}\int_\Sigma t^2 > 0.$$

Then, for $j = i+1 > 1$,

$$\begin{aligned}\Upsilon_{jj} &= \frac{n-1}{r^2}\int_\Sigma r^2 z_i^2 + \frac{(n-1)(t^2+1)-(n+1)r^2}{2r} r z_i^2 \\ &= \frac{(n-1)^2}{2r^2}\int_\Sigma (t^2+1-r^2) r^2 z_i^2 > 0.\end{aligned}$$

Thus, Theorem 1.9 implies the Type I+II index is at least $n+1$.

Proposition 4.8. *Let $a < r < b$ as in part (3) of Proposition 4.4. Let $x = T\sqrt{n-1}/r$ and assume that $\cos(x) + x\sin x \neq 0$ and $x > \frac{\sin x}{x\sin x+\cos x}$. Then, the Type I+II index is equal to the Type I index and equal to $n+1$.*

Proof. Let m_{I+II} and m_I be the indices with respect to constrains I+II and I respectively. First, it is clear that

$$m_{I+II} \leq m_I.$$

By the calculation above and Theorem 1.9, m_{I+II} is at least $n+1$. By Propositions 4.4 and 4.7, m_I is at most $n+1$. The result then follows. ☐

Remark 4.9. *It is clearly possible to do an analogous analysis between Type I+II and Type II indices. We leave it for the reader.*

Proof of Theorem 1.18. It is immediate from Propositions 4.4 and 4.8. ☐

5. Capillary Minimal Hypersurfaces

In this section, we study the case $H = 0$. First, we have the following.

Corollary 5.1. *Assume that $X : \Sigma \to \mathbb{R}^{n+1}$ is a properly immersed capillary minimal hypersurface in the Euclidean unit ball $\mathbb{B}^{n+1}$. If it is not $|\mathring{A}|^2$-scale to a hyper-planar domain and*

$$\int_\Sigma |\boldsymbol{A}|^2 \langle X, v\rangle^2 \geq \int_\Sigma |\boldsymbol{A}|^2 \cos^2\theta,$$

for any unit vector $v \in \mathbb{R}^{n+1}$, then either it is totally geodesic or it has Type-I+II Morse index bigger than $n+1$.

Proof. Assume that Σ is not totally geodesic. Then there exists a coordinate $X_1, X_2, \cdots, X_{n+1}$, such that $|\mathbf{A}|^2 X_1, |\mathbf{A}|^2 X_2, \cdots, |\mathbf{A}|^2 X_{n+1}$ are linearly independent. From (4.5) we know

$$\lambda_i = \int_\Sigma (n|\mathbf{A}|^2)(nX_i^2 + n\cos\theta X_i \nu_i).$$

By Cauchy inequality and $|\nu_i| \leq 1$, we have

$$|\int_\Sigma |\mathbf{A}|^2 \cos\theta X_i\nu_i| \leq \int_\Sigma |\mathbf{A}|^2 |\cos\theta||X_i| \leq \left(\int_\Sigma |\mathbf{A}|^2\cos^2\theta\right)^{\frac{1}{2}} \left(\int_\Sigma |\mathbf{A}|^2 X_i^2\right)^{\frac{1}{2}}.$$

Hence

$$\begin{aligned}\lambda_i &= n^2\left[\int_\Sigma |\mathbf{A}|^2 X_i^2 + \int_\Sigma |\mathbf{A}|^2 \cos\theta X_i \nu_i\right]\\ &\geq n^2\left[\int_\Sigma |\mathbf{A}|^2 X_i^2 - \left(\int_\Sigma |\mathbf{A}|^2\cos^2\theta\right)^{\frac{1}{2}}\left(\int_\Sigma |\mathbf{A}|^2 X_i^2\right)^{\frac{1}{2}}\right]\\ &= n^2\left(\int_\Sigma |\mathbf{A}|^2 X_i^2\right)^{\frac{1}{2}}\left[\left(\int_\Sigma |\mathbf{A}|^2\langle X, e_i\rangle^2\right)^{\frac{1}{2}} - \left(\int_\Sigma |\mathbf{A}|^2\cos^2\theta\right)^{\frac{1}{2}}\right]\\ &\geq 0.\end{aligned}$$

From Theorem 1.9, we know that the Type-I+II Morse index is at least $n+1$. The proof is complete. □

Corollary 5.2. *Assume that $X : \Sigma \to \mathbb{R}^{n+1}$ is a capillary minimal hypersurface in the Euclidean unit ball $\mathbb{B}^{n+1}$. If it is not $|\mathring{A}|^2$-scale to a hyper-planar domain and is a polar-graph then either it is totally geodesic or it has Type-I+II Morse index bigger than $n+1$.*

Proof. Since Σ is a polar graph, by choosing the unit normal conveniently we may assume the function $\langle X, \nu\rangle$ is non-negative. We may also assume it is positive somewhere because the embedding is not totally geodesic. Notice that by Corollary 2.5 we have

$$(\Delta + |\mathbf{A}|^2)\langle X, \nu\rangle = 0.$$

By the maximum (or rather minimum) principle, we conclude that the function $\langle X, \nu\rangle$ must attain its minimum at the boundary. Since at the boundary we have, by (2.1)

$$\langle X, \nu\rangle = |\cos\theta|$$

we conclude that

$$|\langle X, \nu\rangle| \geq |\cos\theta|.$$

everywhere on Σ, and the result now follows from Corollary 5.1. □

When $H = 0$ and $\theta = \pi/2$, Σ is called a FBMS. As $\cos\theta = 0$, the functional E no longer depends on the wetted area and the boundaries are so-called free.

Proof of Theorem 1.13. Assume for sake of contradiction that the immersion is not totally geodesic. First, we claim that Σ is not $|\mathring{A}|^2$-scale to a hyper-planar domain. If it is, then for some constant unit vector e_i and some fixed number $c > 0$

$$|\mathring{A}|^2\langle X, e_i\rangle = c.$$

That is X_i is positive. On the other hand, by Lemma 4.2,

$$\int_\Sigma X_i = 0,$$

which is a contradiction.

Then from the proof of Theorem 1.9, we know that $\phi_1, \cdots, \phi_{n+1}$ are linearly independent and the dimension of span$\{\phi_1, \cdots, \phi_{n+1}\}$ is $n+1$ as linear functionals acting on $H_1(\Sigma)$. Since Σ a FBMS, then $H = 0$, $\cos\theta = 0$. (4.5) implies

$$\lambda_i = \int_\Sigma n^2|\mathbf{A}|^2 X_i^2 \geq 0.$$

If the Type-I+II Morse index is less than $n+1$, there exists at least one $i \in \{1, 2, ..., n+1\}$ such that $X_i \equiv 0$ on some open subset U. Hence $|\mathbf{A}| \equiv 0$ on U. This is a contradiction to the unique continuation of minimal surface and assumption that Σ is not totally geodesic. The proof is complete. □

5.1. **Critical Catenoid in a ball.** In this section, we'll determine precisely the indices with different constraints for the embedded critical catenoid, the unique (up to isometry) rotationally symmetric FBMS in $\mathbb{B}^3$. Here $\theta = \frac{\pi}{2}$ and $\Omega = \mathbb{B}^3$. Then,

$$p = |\mathbf{A}|^2 \text{ and } q = 1.$$

The surface can be parametrized by a conformal harmonic map $X : \Sigma \mapsto \mathbb{B}^3$, for $t \in [-T, T]$ and $0 \leq \tau \leq 2\pi$,

$$X(t, \tau) = c(\cosh t \cos \tau, \cosh t \sin \tau, t). \tag{5.1}$$

T and c are determined by

$$\cosh T = T \sinh T \text{ and } c = \frac{1}{T \cosh T}.$$

Following a straightforward calculation, we have

$$|\mathbf{A}|^2 = \frac{2}{c^2 \cosh^4 t} \text{ and } \nabla_\eta = \frac{1}{c \cosh T} \partial_{\pm t} = T \partial_{\pm t}.$$

The Jacobi operator is given by

$$J = \Delta + p = \Delta + |\mathbf{A}|^2 = \frac{1}{c^2 \cosh^2(t)} (\partial_t^2 + \partial_\tau^2 + \frac{2}{\cosh^2(t)}).$$

Proposition 5.3. *The Type-I Morse index of the critical catenoid is 3.*

Proof. We consider

$$u = -a \cosh^2 t + b(1 - t \tanh(t)).$$

One then calculate

$$Ju = \frac{-4a}{c^2},$$

$$D_\eta u(T) = T(-a \sinh(2T) + b(-\tanh(T) - \frac{T}{\cosh^2(T)})).$$

Solving $D_\eta u(T) = u(T)$ gives

$$b = -a \cosh(T) \sinh(T).$$

Therefore, choosing $a = \frac{c^2}{4}$ and $b = -a \cosh(T) \sinh(T)$ yields

$$\begin{cases} Ju & = -1 \text{ on } \Sigma \\ \nabla_\eta u & = u \text{ on } \partial\Sigma, \\ \int_\Sigma u & < 0. \end{cases}$$

By Theorem 1.2, its Type-I Morse index is equal to $\text{MI}(Q) - 1$. By recent results of [25, 23, 6], $\text{MI}(Q) = 4$ and the proof is complete. □

Now, we complete the second part of Corollary 1.20.

Proposition 5.4. *The Type II Morse index of the critical catenoid is 3.*

Proof. We consider

$$u = a(1 - t\tanh(t)).$$

One then calculate

$$Ju = 0,$$

$$D_\eta u = -Ta(\tanh(T) + \frac{T}{\cosh^2(T)}).$$

Solving $D_\eta u - u = 1$ gives

$$a = -\coth^2(T).$$

For that choice of a

$$\begin{cases} Ju & = 0 \text{ on } \Sigma \\ D_\eta u - u & = 1 \text{ on } \partial\Sigma, \\ \int_\Sigma u & < 0. \end{cases}$$

By Theorem 1.6, its Type II Morse index is equal to $\mathrm{MI}(Q) - 1$. By recent results of [25, 23, 6], $\mathrm{MI}(Q) = 4$ and the proof is complete. □

Theorem 5.5. *Let $\Sigma \subset \mathbb{B}^3$ is the critical catenoid. Then its Type-I, Type-II, and Type I+II Morse indices are all equal to 3.*

Proof. By Corollary 1.13, the Type I+II index is at least 3. Since it is smaller than or equal to either the Type I or type II one, by Propositions 5.3 and 5.4, the result follows. □

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