ON THE MORSE INDEX WITH CONSTRAINTS: AN ABSTRACT FORMULATION

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Abstract. The Morse index is an essential quantity in understanding the second variation of a geometric functional. In this paper, we study an abstract formulation of that concept in the context of a variational problem with constraints. Particularly, we examine the index and nullity of a symmetric bounded bilinear form in a Hilbert space and determine quantitatively how they change when restricting to a subspace of a finite codimension.

Keywords: Morse index, constraint, Hilbert space, second variation, stability.
Mathematics Subject Classification: 46C05, 53C21, 58E05, 49K40.

1. Introduction

Variational problems in geometry consider functionals, such as area or volume, on geometrical objects, such as a hypersurface or a manifold. From the viewpoint of Morse theory, it is essential to study the second variation which normally involves a symmetric bilinear form in a function space. One is interested in determining quantitatively how negative the form is, leading to the notion of the Morse index. Precisely, the index is defined to be the maximum dimension of a subspace on which the form is negative definite. Intuitively, it gives the number of distinct deformations which decrease the functional to the second order. The pioneered classical investigation has been done by H. Edwards [8], S. Smale [20], J. Simons [19], K. Uhlenbeck [24] and others. The theory has far-reaching applications; for example, it plays a crucial role in the recent resolution of the Willmore conjecture by F. Marques and A. Neves [17].

When there are constraints, one is led to examine reduced function spaces. For example, ancient mathematicians like Zenodorus and Princess Dido considered isoperimetric problems, finding the largest possible shape with a given perimeter. Thus, one considers only variations fixing the perimeter. Another example is the partitioning problem of a convex body by least-area hypersurfaces under a type I (prescribed volume) or type II (prescribed wetting area) constraint using the terminology from [6]. Consequently, it is crucial to study the index restricted to a subspace.

For the volume-preserving constraint, it is the so-called weak Morse index and the stability case, when it is zero, has received plenty of interests; see [15, 26, 4] and references therein. The difference between the weak Morse index and the general one (without any constraint) is zero or one. When the latter is zero, obviously so is the former. There are some special cases when they are equal to each other (and non zero), see [5] (for a non-compact, infinite volume, CMC immersion into a hyperbolic space) and [16] (for catenoids

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and Enneper surfaces with index one). However, one suspects that they are different in general [13], [25], [21].

In this paper, we will identify criteria determining the relation between these indices with and without constraints. Our approach is to interpret constraints as linear functionals on a vector space and the subspace as the intersection of their kernels. Consequently, to study the relation, one aims to relate these linear functionals to elements of the vector space. The Hilbert space formulation exactly provides that connection. Our work may be seen as a local version of the theory in [12]. In the consequent paper [23], we'll apply our abstract formulation to study critical points of several variational problems in geometry.

Let H be a separable Hilbert space with an inner product $(\cdot,\cdot)$. $S(\cdot,\cdot)$ is a symmetric continuous bilinear form. The inner product on H induces a linear map $\mathcal{S}$ from H to its continuous dual H^* such that, for all $v \in H$,

$$S(u,v) = (\mathcal{S}u)(v).$$

Via the Riesz representation theorem, H^* can be equipped with an inner product so that it is isometric to H.

Definition 1.1. *Let $ran(\mathcal{S})$ be the range of $\mathcal{S}$, $\overline{ran(\mathcal{S})}$ its closure by the induced norm. $\overline{ran(\mathcal{S})} - ran(\mathcal{S})$ is called the set of pure limit points.*

Remark 1.2. *In many PDE and geometric settings, one considers a self-adjoint operator with a finite dimensional kernel. It follows immediately that the operator is Fredholm with a closed range (*[3, Section 4.4]*) and the set of pure limit points is empty.*

Next, let $\phi \in H^*$ be a non-trivial continuous linear functional on H and let $\mathrm{MI}(S)$ and $\mathrm{MI}^\phi(S)$ denote the index of $S(\cdot,\cdot)$ with respect to H and $\mathrm{Ker}(\phi)$ respectively. We are now ready to state the main theorem.

Theorem 1.3. *Let H be a separable Hilbert space, $S(\cdot,\cdot)$ a continuous symmetric bilinear form, and ϕ a non-trivial continuous linear functional. If ϕ is not a pure limit point, then*

$$MI^\phi(S) = \begin{cases} MI(S) - 1 & \text{if there is } u \in H \text{ such that } \mathcal{S}u = \phi \text{ and } \phi(u) \leq 0, \\ MI(S) & \text{otherwise.} \end{cases}$$

Stability is normally associated with the case $\mathrm{MI}(S) = 0$. The following immediate consequence is an indicator of in-stability.

Corollary 1.4. *If there is a $u \in H$ such that $\mathcal{S}u = \phi \neq \vec{0}$ and $\phi(u) \leq 0$ then $MI(S) \geq 1$..*

It is also possible to generalize the result to several functionals. Let $\mathrm{MI}^{\phi_1,\ldots,\phi_n}(S)$ be the index of $S(\cdot,\cdot)$ in the intersection $\cap_{i=1}^n \mathrm{Ker}(\phi_i)$.

Theorem 1.5. *Let H be a separable Hilbert space, $S(\cdot,\cdot)$ a continuous symmetric bilinear form. Suppose that, for $i = 1, ...n$, $\mathcal{S}(u_i) = \phi_i$ and $\{\phi_i\}_{i=1}^n$ are linearly independent. Then,*

$$MI^{\phi_1,\ldots,\phi_n}(S) = MI(S) - c,$$

where c is the number of non-positive eigenvalues of the symmetric matrix $S(u_i,u_j)$. In particular,

$$MI(S) \geq c.$$

Similarly, the following statement addresses the nullity with a constraint (See Section 2 for a precise definition). They will not be used in the subsequent paper but are presented for independent interests.

Theorem 1.6. *Let H be a separable Hilbert space, $S(\cdot,\cdot)$ a continuous symmetric bilinear form, and ϕ a non-trivial continuous linear functional. We have the followings:*

(1) *If there is $u \in H$ such that $\mathcal{S}u = \phi$, $\phi(u) = 0$, then $n^{\phi}(S) = n(S) + 1$;*
(2) *If $\phi \notin \overline{ran(\mathcal{S})}$, $n^{\phi}(S) = n(S) - 1$;*
(3) *Otherwise, $n^{\phi}(S) = n(S)$.*

Theorem 1.7. *Let H be a separable Hilbert space and $S(\cdot,\cdot)$ is a continuous symmetric bilinear form. Suppose that, for $i = 1, ...n$,*

$$\mathcal{S}(u_i) = \phi_i,$$

and $\{\phi_i\}_{i=1}^n$ are linearly independent. Then,

$$n^{\phi_1,\ldots,\phi_n}(S) = n(S) + c,$$

where c is the dimension of the null space of the symmetric matrix $S(u_i, u_j)$.

The proof of these results is functional analysis in nature. We envision that our abstract theory will be applicable for several variational problems in geometry calculating and estimating Morse indices. The outline of the paper is as follows. The next Section will collect preliminaries results of the Hilbert space theory. Theorems 1.3 and 1.5 (1.6 and 1.7) will be presented in Section 3 (and Section 4, respectively). In Section 5, we establish an immediate consequence for a general index decomposition for a setup with boundaries.

1.1. **Acknowledgments.** H. T was partially supported by a Simons Foundation Grant [709791] and National Science Foundation Grant [DMS-2104988]. He also would like to thank Richard Schoen for valuable discussion on the topic and the Vietnam Institute for Advanced Study in Mathematics for its support and hospitality. D. Z. was partially supported by Faperj and CNPq of Brazil.

2. Preliminaries

We first recall the Morse index and nullity of a bilinear form.

Definition 2.1. *The Morse index of a bilinear form $S(\cdot,\cdot)$ in a vector space V, $MI(S)$, is the maximal dimension of a subspace of V on which $S(\cdot,\cdot)$ is negative definite. The nullity, $n(S)$, is the dimension of the radical of the bilinear form; that is, the set of all $u \in V$ such that $S(u, v) = 0$ for all $v \in V$.*

Let $S(\cdot,\cdot)$ be a symmetric continuous bilinear form on a separable Hilbert space H with an inner product $(\cdot,\cdot)$. Equivalently, $S(\cdot,\cdot)$ is bounded. That is, for any $u, v \in H$ there is a universal constant c such that

$$S(v, u) = S(u, v) \leq c||u||||v||.$$

Due to the Riesz representation theorem, we can identify H with its continuous dual of linear continuous functionals via the isomorphism

$$\pi : u \mapsto u^* : u^*(v) = (u, v).$$

The inner product on H^* is given by

$$(u^*, v^*) = (u, v).$$

Obviously, $||u^*||_{H^*} = ||u||_H$.

Also, it induces a linear map $\mathcal{S} : H \mapsto H^*$ and a self-adjoint operator $\mathbf{S} : H \mapsto H$ such that, for all $v \in H$,

$$S(u, v) = \mathcal{S}u(v) = (\mathbf{S}u, v).$$

It is immediate that $\text{Ker}(\mathbf{S}) = \text{Ker}(\mathcal{S})$ and $\mathbf{S}$ is a bounded self-adjoint operator. Consequently, the closure of its range is the S-orthogonal complement of its kernel:

$$\overline{\text{ran}(\mathbf{S})} = (\text{Ker}(\mathbf{S}))^{\perp},$$

$$H = \overline{\text{ran}(\mathbf{S})} \oplus \text{Ker}(\mathbf{S}). \tag{2.1}$$

The direct sum here is with respect to either the inner product or the bilinear form $S(\cdot, \cdot)$. Correspondingly, in the context of the isomorphism $\pi : H \mapsto H^*$

$$H^* = \pi(\text{Ker}(\mathbf{S})) \oplus \overline{\text{ran}(\mathcal{S})}.$$

Definition 2.2. *$\overline{ran(\mathcal{S})} - ran(\mathcal{S})$ is called the set of pure limit points.*

Remark 2.3. *For a normal operator with* 0 *in its spectrum, its range is closed if and only if* 0 *is not a limit point of the spectrum* [7, Proposition XI.4.5]*. For a general bounded operator, a generalized version is given by* [14] *considering the spectrum of the product of the operator and its adjoint.*

The following well-known result gives a decomposition of H with respect to $S(\cdot, \cdot)$.

Theorem 2.4. [12, Theorem 7.1] *Given a symmetric bilinear form $S(\cdot, \cdot)$, H can be decomposed uniquely as the direct sum*

$$H = H_- \oplus H_0 \oplus H_+$$

satisfying the following properties

- *H_-, H_0, and H_+ are mutually perpendicular and S-perpendicular.*
- *$S(\cdot, \cdot)$ is negative definite on H_-, zero on H_0, and positive definite on H_+.*

That is,

$$H_0 = \text{Ker}(\mathbf{S}),$$
$$\text{MI}(S) = \dim(H_-)$$
$$n(S) = \dim(H_0).$$

Then, for the index analysis we'll assume that $\dim(H_-) < \infty$ because if it is infinity our result can be interpreted as vacuously true (similarly, we'll assume $\dim(H_0) < \infty$ when working with the nullity). Consequently, for any maximal space W on which $S(\cdot, \cdot)$ is negative definite, the S-projection from W to H_- is an isomorphism. Thus, $\dim(W) = \dim(H_-)$ and the following is immediate.

Lemma 2.5. *For a Hilbert space H, MI(S) is the dimension of a any maximal subspace on which $S(\cdot, \cdot)$ is negative definite.*

Remark 2.6. *It was brought to our attention that, in case the Morse index is infinite, there are certain approaches to generalize the Morse theory with important applications to the study of Hamiltonian systems, the wave equation, elliptic systems, and geodesics on Semi-Riemannian manifolds; see* [2, 1] *for examples. We plan to adapt our machinery to study such strongly indefinite bilinear forms somewhere else.*

In applications, the original function space might not be Hilbert and the following is useful and well-known. A proof is provided for completeness.

Lemma 2.7. *Let $V \subset H$ is a dense vector space inside a Hilbert space. For any continuous symmetric bilinear form $S(\cdot,\cdot)$ on H, its index on H is equal to that on V.*

Proof. Let m_H, m_V denotes the indices of $S(\cdot,\cdot)$ on H and V, respectively. By the definition,

$$m_H \geq m_V.$$

To prove the reverse inequality, we proceed by contradiction. Suppose that $m_H > m_V$. Let $W \subset V$ be a maximal space of m_V. Since, $m_H > m_V$, W is not a maximal space of m_H. That is, there is $u \in H$ such that u is perpendicular to W and $S(u,u) < 0$.

Since V is dense in H, there is a sequence $u_n \in V$ such that $||u_n - u||_H \to 0$. Furthermore, let v_n be the projection of u_n on W then

$$||u_n - u||^2 = ||v_n||^2 + ||(u_n - v_n) - u||^2.$$

Thus, for $u_n' = u_n - v_n \in V$, u_n' is perpendicular to W, $||u_n' - u||_H \to 0$. We have

$$S(u_n', u_n') = S(u,u) + S(u_n' - u, u + u_n').$$

Since $S(\cdot,\cdot)$ is continuous and $||u_n' - u||_H \to 0$, $S(u_n' - u, u + u_n') \to 0$. Hence, for sufficiently large n,

$$S(u_n', u_n') < 0,$$

which is a contradiction to the maximality of $W \subset V$. The proof is finished. □

Motivated by variational problems with constraints, we give the following definition. Let ϕ_i be a linear functional with kernel $\mathrm{Ker}(\phi_i)$.

Definition 2.8. *The Morse index of the bilinear form $S(\cdot,\cdot)$ with respect to $\phi_1, ...\phi_n$, $MI^{\phi_1,...\phi_n}(S)$, is the index of $S(\cdot,\cdot)$ in $\cap_{i=1}^n Ker(\phi_i)$. The nullity of the bilinear form $S(\cdot,\cdot)$ with respect to $\phi_1, ...\phi_n$, $n^{\phi_1,...\phi_n}(S)$, is the nullity of $S(\cdot,\cdot)$ in $\cap_{i=1}^n Ker(\phi_i)$.*

The following is immediate.

Lemma 2.9. *$MI^{\phi}(S)$ is either $MI(S)$ or $MI(S) - 1$.*

Proof. Obviously, by definition,

$$\mathrm{MI}^{\phi}(S) \leq \mathrm{MI}(S).$$

On the other hand, take W be a maximal subspace on which $S(\cdot,\cdot)$ is negative definite. Then $\mathrm{Ker}(\phi\mid_W)$ has co-dimension at most one. Obviously, $S(\cdot,\cdot)$ is positive definite on $\mathrm{Ker}(\phi\mid_W)$ which is certainly a subspace of $\mathrm{Ker}(\phi)$. Therefore

$$\mathrm{MI}^{\phi}(S) \geq \mathrm{MI}(S) - 1.$$

The result then follows. □

To determine the relation between $\mathrm{MI}^{\phi}(S)$ with $\mathrm{MI}(S)$, it is essential to consider the effect of ϕ on maximal subspaces. It leads to the following.

Definition 2.10. *A continuous linear function ϕ is called S-critical if for any maximal subspace W on which $S(\cdot,\cdot)$ is negative definite then $\phi(W)=\mathbb{R}$.*

3. Index of a Bilinear Form in a Hilbert Space

In this section, we prove an abstract theorem for the Morse index on a Hilbert subspace.

Theorem 3.1. *Let H be a separable Hilbert space and $S(\cdot,\cdot)$ be a continuous symmetric bilinear form. Assume that ϕ is a nonzero continuous linear functional which is not a pure limit point. Then the following assertions are equivalent:*

(1) $MI^{\phi}(S) = MI(S) - 1$;
(2) *There exists a $u \in H$ such that $\mathcal{S}u = \phi$ and $\phi(u) \leq 0$.*
(3) *ϕ is S-critical.*

Remark 3.2. *The assumption on the non-triviality of ϕ is necessary. If $\phi = \vec{0}$ then, obviously, $Ker(\phi) = H$ and $MI^{\phi}(S) = MI(S)$. However, for $u \in Ker(\mathcal{S})$, we have $\mathcal{S}(u) = \phi$ and $\phi(u) = 0$.*

Proof. First we prove that (1) is equivalent to (3). Suppose that ϕ is S-critical. If $\mathrm{MI}^{\phi}(S) = \mathrm{MI}(S)$ then there is a maximal subspace W of that dimension in $\mathrm{Ker}\phi$ on which $S(\cdot,\cdot)$ is negative definite. That is, $\phi(W) = 0$, a contradiction with the definition of S-criticality. Therefore, by Lemma 2.9, $\mathrm{MI}^{\phi}(S) = \mathrm{MI}(S) - 1$.

Conversely, suppose that $\mathrm{MI}^{\phi}(S) = \mathrm{MI}(S) - 1$ and ϕ is not S-critical. Then, there is a maximal subspace W on which $S(\cdot,\cdot)$ is negative definite and $\phi(W) = 0$. But that means $W \subset \mathrm{Ker}(\phi)$ and $\mathrm{MI}^{\phi}(S) = \mathrm{MI}(S)$, again a contradiction.

Now we prove (1) is equivalent to (2). Via the Riesz representation theorem, each ϕ uniquely corresponds to $\bar{\phi} \in H$ such that, for all $v \in H$,

$$\phi(v) = (\bar{\phi}, v).$$

There are two cases: (i) $\bar{\phi} \notin \mathrm{ran}(\mathbf{S})$ and (ii) $\bar{\phi} \in \mathrm{ran}(\mathbf{S})$.

Case (i). $\bar{\phi} \notin \mathrm{ran}(\mathbf{S})$. We will show that $\mathrm{MI}^{\phi}(S) = \mathrm{MI}(S)$.

By (2.1),

$$\bar{\phi} = u + s$$

with $u \in \mathrm{Ker}(\mathbf{S})$ and $s \in \overline{\mathrm{ran}(\mathbf{S})}$. Since ϕ is not a pure limit point, neither is $\bar{\phi}$ and $\vec{0} \neq u$. Then we have

$$\phi(u) = (\bar{\phi}, u) = (s+u, u) = (u,u) > 0.$$

If $\mathrm{MI}(S) = 0$, the result follows vacuously. Otherwise, let W be a maximal space on which $S(\cdot,\cdot)$ is negative definite. That is,

$$\dim(W) = \mathrm{MI}(S) > 0.$$

Since $u \neq \vec{0}$ and

$$S(u,v) = (\mathcal{S}u)v = \vec{0}v = 0,$$

we have $u \notin W$. Then let $W_1 = \mathrm{span}(W,u)$ and $W_2 = \mathrm{Ker}(\phi\mid_{W_1})$. It is clear that

$$\dim(W_1) = \mathrm{MI}(S) + 1.$$

Since $\phi(u) \neq 0$, the map $\phi : W_1 \mapsto \mathbb{R}$ is onto, then

$$\dim(W_2) = \dim(W_1) - 1 = \mathrm{MI}(S).$$

Let $\vec{0} \neq v \in W_2$. Then

$$v = w + cu,$$

for $\vec{0} \neq w \in W$ and some constant c. Thus,

$$S(v, v) = S(w, w) + 2cS(u, w) + c^2 S(u, u).$$

Since $u \in \mathrm{Ker}(\mathbf{S})$, $S(u, w) = S(u, u) = 0$. Thus, $S(v, v) = S(w, w) < 0$ and $S(\cdot, \cdot)$ is negative definite on W_2. Consequently, $\mathrm{MI}^{\phi}(S) = \mathrm{MI}(S)$.

Case (ii). $\phi \in \mathrm{ran}(\mathcal{S})$. Then there is $u \in H$ such that

$$\begin{aligned} \mathcal{S}u &= \phi, \\ \mathbf{S}u &= \bar{\phi}, \\ S(u, u) = (\mathbf{S}u, u) &= (\bar{\phi}, u) = \phi(u). \end{aligned}$$

In Propositions 3.3 we show that if $\phi(u) > 0$ then $\mathrm{MI}^{\phi}(S) = \mathrm{MI}(S)$ and in Propositions 3.6 and 3.7 we show that if $\phi(u) < 0$ or $\phi(u) = 0$ then $\mathrm{MI}^{\phi}(S) = \mathrm{MI}(S) - 1$. Therefore the equivalence between (1) and (2) follows from Propositions 3.3, 3.6, and 3.7 below. □

Proposition 3.3. *If $\phi(u) > 0$, then $MI^{\phi}(S) = MI(S)$.*

Proof. Let W be a maximal space on which $S(\cdot, \cdot)$ is negative definite. That is,

$$\dim(W) = \mathrm{MI}(S).$$

We have

$$S(u, u) = (\mathcal{S}u)u = \phi(u) > 0.$$

Therefore $u \notin W$. Let $W_1 = \mathrm{span}(W, u)$ and $W_2 = \mathrm{Ker}(\phi\mid_{W_1})$. It is readily checked that

$$\dim(W_2) = \dim(W_1) - 1 = \mathrm{MI}(S).$$

Furthermore, $v \in W_2$ if and only if $v = ku + w$ for $w \in W$ and $\phi(ku + w) = 0$. We calculate

$$\begin{aligned} S(v, v) &= k^2 S(u, u) + S(w, w) + 2kS(u, w) \\ &= k^2 \phi(u) + S(w, w) + 2k\phi(w) \\ &= -k^2 \phi(u) + S(w, w) < 0. \end{aligned}$$

The result then follows. □

So it remains to prove Propositions 3.6 and 3.7. For preparation, we observe several simple results. It is noted that u is generally not unique as one can replace it by $u + v$ for any $v \in \mathrm{Ker}(\mathcal{S})$. However, $\phi(u)$ is unique as

$$\phi(u + v) = S(u, u + v) = S(u, u).$$

Also, we have, for $v \in \mathrm{Ker}(\phi)$

$$S(u, v) = (\mathcal{S}u)(v) = \phi(v) = 0.$$

Hence u is S-perpendicular to $\mathrm{Ker}(\phi)$.

Lemma 3.4. *For any $u \in H$ such that $S(u, u) < 0$, there is a maximal space W, $u \in W$, on which $S(\cdot, \cdot)$ is negative definite.*

Proof. It is possible to construct u_0, the S-projection of u on H_-. Since $S(\cdot,\cdot)$ is negative definite on H_-, we see that H_- decomposes into

$$H_- = \text{span}(u_0) \oplus W_1,$$

for u_0 S-perpendicular to W_1. As a consequence, u is S-perpendicular to W_1 and $W := \text{Span}(u, W_1)$ is a maximal space. □

Lemma 3.5. *For any nontrivial continuous linear functional ϕ, if MI$(S) > 0$ then there is a maximal space W on which $S(\cdot,\cdot)$ is negative definite and $W \not\subset$ Kerϕ.*

Proof. Due to Lemma 3.4, it suffices to find some $u \notin \text{Ker}\phi$ such that $S(u,u) < 0$. Recall, for all $v \in H$

$$\phi(v) = (v, \bar{\phi}).$$

Since ϕ is nontrivial, $\bar{\phi} \neq \vec{0}$. Next, for MI$(S) > 0$, there is x such that

$$S(x,x) < 0.$$

if $\phi(x) \neq 0$ then we are done. Otherwise, $\phi(x) = 0$, one considers the quadratic function

$$\begin{aligned} f(k) &= S(kx + \bar{\phi}, kx + \bar{\phi}) \\ &= k^2 S(x,x) + 2kS(x,\bar{\phi}) + S(\bar{\phi},\bar{\phi}) \to -\infty \text{ as } k \to \infty. \end{aligned}$$

Choose $u = kx + \bar{\phi}$, $\phi(u) = (\bar{\phi}, kx + \bar{\phi}) = (\bar{\phi},\bar{\phi}) > 0$, for sufficiently large k and the proof is finished. □

Proposition 3.6. *If $\phi(u) < 0$ then MI$^{\phi}(S) =$ MI$(S) - 1$.*

Proof. Since

$$S(u,u) = (\mathcal{S}u)u = \phi(u) < 0,$$

by Lemma 3.4, there is a maximal space W containing u such that $S(\cdot,\cdot)$ is negative definite on W. Let $W_1 = \text{Ker}(\phi\mid_W)$. By the discussion above u is S-perpendicular to W_1. Thus,

$$\begin{aligned} W &= \text{span}(u) \oplus W_1, \\ \dim(W_1) &= \dim(W) - 1 \\ &= \text{MI}(S) - 1. \end{aligned}$$

Consequently,

$$H = \text{span}(u) \oplus W_1 \oplus W^{\perp}.$$

The direct product is with respect to S-perpendicularity.

Claim: W_1 is a maximal subspace inside Ker(ϕ) such that $S(\cdot,\cdot)$ is negative definite on W_1.

Proof of the claim. The claim is proved by contradiction. Suppose that the claim is false. Then there exist an element $f \in \text{Ker}(\phi)$ which is S-perpendicular to W_1, and $S(f,f) < 0$. As f is S-perpendicular to W_1, by the decomposition above, we have

$$f = ku + f_1,$$

for $f_1 \in W^{\perp}$ and some constant k. Then we calculate

$$\begin{aligned} 0 &= S(u, f_1) \\ &= (\mathcal{S}u)(f_1) \\ &= \phi(f_1). \end{aligned}$$

Thus, $f_1 \in \mathrm{Ker}(\phi)$. Since $f = ku + f_1 \in \mathrm{Ker}(\phi)$, $ku \in \mathrm{Ker}(\phi)$. Recall that $\phi(u) < 0$, we conclude $k = 0$. Then, due to $f_1 \in W^\perp$,

$$S(f, f) = S(f_1, f_1) \geq 0.$$

That is a contradiction. So the proof is finished. □

Proposition 3.7. *If* $\vec{0} \neq \phi$ *and* $\phi(u) = 0$ *then* $MI^\phi(S) = MI(S) - 1$.

Proof. First, we observe that $\mathrm{MI}(S) \geq 1$. Suppose the claim is false. Then $H_- = \emptyset$ from Theorem 2.4. Then, $u \in H_0$ and

$$\phi = \mathcal{S}(u) = \vec{0},$$

which is contradiction. So the claim is true.

By Lemma 3.5, there is a maximal subspace W on which $S(\cdot, \cdot)$ is negative definite and $W \not\subset \mathrm{Ker}(\phi)$. Let $W_1 = \mathrm{Ker}(\phi \mid_W)$ then W_1 has codimension one. Thus, there is a nonzero vector $u_0 \in W$ such that it is S-perpendicular to W_1, $\phi(u_0) \neq 0$, and

$$\begin{aligned} W &= \mathrm{span}(u_0) \oplus W_1, \\ \dim(W_1) &= \dim(W) - 1 \\ &= \mathrm{MI}(S) - 1. \end{aligned}$$

Consequently,

$$H = \mathrm{span}(u_0) \oplus W_1 \oplus W^\perp.$$

Claim: W_1 is a maximal subspace inside $\mathrm{Ker}(\phi)$ such that $S(\cdot, \cdot)$ is negative definite on W_1.

Proof of the claim. The claim is proved by contradiction. If it were false then there would be $f \in \mathrm{Ker}(\phi)$, such that f is S-perpendicular to W_1, and $S(f, f) < 0$. Since f is S-perpendicular to W_1, by the decomposition above,

$$f = k_0 u_0 + f_1,$$

for some constant k_0 and $f_1 \in W^\perp$. We also observe that

$$S(u, u_0) = (\mathcal{S}u)u_0 = \phi(u_0) \neq 0.$$

Let $v = u - \frac{S(u,u_0)}{S(u_0,u_0)} u_0$ then it is readily checked that $S(v, u_0) = 0$ and v is S-perpendicular to W_1. Therefore, $v \in W^\perp$ and, consequently,

$$f = ku + f_2,$$

for some constant k and $f_2 \in W^\perp$. We calculate

$$\begin{aligned} S(f, f) &= S(ku + f_2, ku + f_2) \\ &= k^2 S(u, u) + 2kS(u, f_2) + S(f_2, f_2) \\ &= k^2 \phi(u) + 2k\phi(f_2) + S(f_2, f_2). \end{aligned}$$

Since we have $u, f \in \mathrm{Ker}(\phi)$ so $f_2 \in \mathrm{Ker}(\phi)$. Thus, for $f \in W^\perp$,

$$S(f, f) = S(f_2, f_2) \geq 0.$$

That is a contradiction and the claim is proved. The result then follows. □

Now the S-criticality is characterized by Theorem 3.1. Theorems 1.3 and 1.5 will follow immediately.

Proof of Theorem 1.3. It follows from Theorem 3.1 and Lemma 2.9. □

Towards Theorem 1.5, let $\phi_1, ...\phi_n$ be non-trivial continuous linear functionals. Via the Riesz representation theorem, each ϕ_i corresponds to $\bar{\phi}_i \in H$ such that, for every $u \in H$,

$$\phi_i(u) = (\bar{\phi}_i, u).$$

Lemma 3.8. *$\phi_1, ...\phi_n$ are linearly independent if and only if, for each i, ϕ_i is a non-trivial linear functional on $\cap_{j=1, j\neq i}^n Ker\phi_j$.*

Proof. In this proof, all perpendicularity is with respect to the Hilbert space inner product. By the closeness of a subspace of finite codimension in a Hilbert space,

$$\big(\cap_{j=1,j\neq i}^n \text{Ker}\phi_j\big)^{\perp} = \text{span}(\bar{\phi}_j, j \neq i).$$

Thus, ϕ_i is a trivial linear functional on $\cap_{j=1,j\neq i}^n \text{Ker}\phi_j$ if and only if $\bar{\phi}_i \in \text{span}(\bar{\phi}_j, j \neq i)$. The proof then follows. □

Proof of Theorem 1.5. Since $\phi_1, \phi_2, \cdots, \phi_n$ are linearly independent, each is non-trivial on H. Furthermore, $u_1, u_2, ...u_n$ are also linear independent. As the matrix $S(u_i, u_j)$ is symmetric, it can be diagonalized by a basis $u_1', ...u_n'$ so that for $i \neq j$

$$S(u_i', u_j') = 0.$$

Note that $u_1, u_2, ...u_n$ and $u_1', ...u_n'$ span the same subspace. By linear algebra, for $\mathcal{S}u_i' = \phi_i'$,

$$\cap_{i=1}^n \text{Ker}\phi_i = \cap_{i=1}^n \text{Ker}\phi_i'.$$

Then

$$\text{MI}^{\phi_1,...,\phi_n}(S) = \text{MI}^{\phi_1',...,\phi_n'}(S) = (...(\text{MI}^{\phi_1'})...)^{\phi_n'}(S)$$

Since u_i' and u_j' are S-perpendicular for $i \neq j$, $\phi_i'(u_j') = S(u_i', u_j') = 0$. Therefore,

$$u_j' \in \cap_{i=1,i\neq j}^n \text{Ker}\phi_i'.$$

Since $\phi_1, \phi_2, \cdots, \phi_n$ are linearly independent, so are $\phi_1', \phi_2', \cdots, \phi_n'$.

We next proceed by induction. The statement is true for $n = 1$ by Theorem 1.3. Suppose then it is true for $n = k$ and we consider the case $n = k + 1$. Let

$$H_k = \cap_{i=1}^k \text{Ker}\phi_i'$$

then H_k is subspace of finite co-dimension in a Hilbert space. Thus, H_k is a Hilbert space. By observations above, $u_{k+1}' \in H_k$ and $\mathcal{S}u_{k+1}' = \phi_{k+1}'$. Also, as $\phi_1', \phi_2', \cdots, \phi_n'$ are linearly independent, ϕ_{k+1}' is a non-trivial continuous linear functional on H_k. Applying Theorem 1.3 finishes the proof. □

4. Nullity

The purpose of this section is to prove Theorems 1.6 and 1.7. Before giving their proof, we will collect useful results. The nullity counting the dimension of the kernel of $\mathcal{S}$ is assumed to be finite throughout this section. Consequently, $\mathcal{S}$ is a Fredholm operator with a closed range [3]. Each element in the kernel is S-perpendicular to every element in the Hilbert space. So it makes sense to consider the perpendicularity, projection, and direct sum with respect to the inner product.

Lemma 4.1. *$n^{\phi}(S) \geq n(S) - 1$.*

Proof. It follows from an argument similar to the proof of Lemma 2.9. □

For a non-trivial continuous linear functional ϕ, let

$$\begin{aligned} H_1 &:= \text{Ker}\phi, \\ N &:= H_0 = \text{Ker}(\mathcal{S}), \\ N_1 &:= (H_1)_0 = \text{Ker}(\mathcal{S} \mid_{H_1}). \end{aligned}$$

Via the Riesz representation theorem, each ϕ uniquely corresponds to $\bar{\phi} \in H$ such that, for all $v \in H$,

$$\phi(v) = (\bar{\phi}, v).$$

Lemma 4.2. *$v \notin N$ and $v \in N_1$ if and only if $\phi(v) = 0$, $\mathbf{S}(v) = k\bar{\phi}$ for some non-zero constant k.*

Proof. If $\mathbf{S}(v) = k\bar{\phi}$, $0 \neq k$, then immediately $v \notin N$. Since $\phi(v) = 0$, $v \in H_1$. For every $u \in H_1$,

$$S(v, u) = \mathcal{S}v(u) = (\mathbf{S}v, u) = k(\bar{\phi}, u) = 0.$$

Thus, $v \in N_1$.

For the other direction, let $v \in N_1$, $v \notin N$. Thus, $v \in H_1$ and $\phi(v) = 0$. Consider, for every $u \in H_1$,

$$0 = S(v, u) = (\mathbf{S}v, u).$$

Therefore, the projection (by the inner product) of $\mathbf{S}v$ on H_1 is $\vec{0}$. Since $v \notin N$, $\mathbf{S}v \neq \vec{0}$. As H is a direct sum (by the inner product) of $H_1 \oplus \text{span}(\bar{\phi})$, the result follows. □

Proof of Theorem 1.6. There are two cases.

Case (i). If $\phi \in \text{ran}(\mathcal{S})$, then $\bar{\phi}$ is S-perpendicular to $\text{Ker}(\mathcal{S})$ and also H-inner product perpendicular to $\text{Ker}(\mathcal{S})$. Therefore $\text{Ker}(\mathcal{S}) \subset \text{Ker}(\phi)$, $N \subset N_1$, and $n^{\phi}(S) \geq n(S)$.

By Lemma 4.2, the inequality is proper if and only if there is $u \in H$ such that

$$\begin{aligned} \mathcal{S}u &= \phi, \\ \phi(u) &= 0. \end{aligned}$$

u is unique up to an addition of an element from N. As a consequence, in this case,

$$n^{\phi}(S) = n(S) + 1.$$

Otherwise,

$$n^{\phi}(S) = n(S).$$

Case (ii). If $\phi \notin \text{ran}(\mathcal{S})$. Then $\bar{\phi} = u + s$ for $\vec{0} \neq u \in N$ and $s \in \text{ran}(\mathcal{S})$. By Lemma 4.2, $N_1 \subset N$. Furthermore,

$$\phi(u) = (u, u + s) = (u, u) > 0.$$

Thus, $u \notin \text{Ker}\phi$ and, consequently, $u \notin N_1$. So the inclusion $N_1 \subset N$ is proper and by Lemma 4.1,

$$n^{\phi}(S) = n(S) - 1.$$

□

Proof of Theorem 1.7. We argue as in the proof of Theorem 1.5 using Theorem 1.6 as the base case. □

5. An Index Formula for Manifolds with Boundaries

In this section, using our abstract formulation, we prove a general index formula for manifolds with boundaries, generalizing a theorem from [22]. Let Σ be a smooth, orientable Riemannian manifold with boundaries. Let ∇, Δ be the covariant derivative and Laplace operator on Σ.

Generally, a second variation of some functional is associated with the following structurally general bilinear form, for smooth functions p, q determined by the geometry of Σ,

$$\begin{aligned} Q(u,v) &= \int_\Sigma \Big(\langle \nabla u, \nabla v\rangle - puv \Big) d\mu - \int_{\partial\Sigma} quvds \\ &= \int_\Sigma \Big(-(Ju)v \Big) d\mu + \int_{\partial\Sigma} (\nabla_\eta u - qu)vds. \end{aligned}$$

Here $J := \Delta + p$ is the so-called Jacobi operator and η is an out-ward conormal vector along $\partial\Sigma$. By classical PDE theory, Q is associated with a Fredholm operator on an appropriate space; see [23] for more discussion on this perspective. The index, $\mathrm{MI}(Q)$, is precisely the number of negative eigenvalues of a Robin boundary problem:

$$\begin{cases} Ju &= -\lambda u \text{ on } \Sigma, \\ \nabla_\eta u &= qu \text{ on } \partial\Sigma. \end{cases} \tag{5.1}$$

In [22], the first author shows that $\mathrm{MI}(Q)$ can be precisely determined by data of simpler problems. First, one consider only variations fixing the boundary corresponding to the following Dirichlet problem:

$$\begin{cases} Jv &= -\delta v \text{ on } \Sigma, \\ v &= 0 \text{ on } \partial\Sigma. \end{cases} \tag{5.2}$$

The Dirichlet eigenvalues can be characterized variationally. Let $H^1_0(\Sigma)$ be the Sobolev space with one derivative, L^2-norm, and zero trace (intuitively, zero on the boundary). Let $V_k \subset H^1_0(\Sigma)$ denote a k-dimensional subspace. Then

$$\delta_k = \min_{V_k} \max_{\vec{0} \neq v \in V_k} \frac{\int_\Sigma |\nabla v|^2 - pv^2}{\int_\Sigma v^2}, \tag{5.3}$$

The influence of the boundary is, then, captured by the Jacobi-Steklov problem. The Steklov eigenvalue problem, associated with the Laplace operator instead of J, in a geometric setup also received tremendous interests recently, for example, [10, 11, 18].

Suppose that $q \in C^\infty(\partial\Sigma)$ be a non-zero non-negative function. We consider:

$$\begin{cases} Jh &= 0 \text{ on } \Sigma, \\ \nabla_\eta h &= \mu qh \text{ on } \partial\Sigma. \end{cases} \tag{5.4}$$

The J-Steklov eigenvalues can be characterized variationally. Let $V_k \subset H^1(\Sigma) \cap \mathrm{Ker}(J)$ denote a k-dimensional subspace, then

$$\mu_k = \min_{V_k} \max_{0 \neq h \in V_k} \frac{\int_\Sigma |\nabla h|^2 - ph^2}{\int_{\partial\Sigma} qh^2}, \tag{5.5}$$

Remark 5.1. *Ker*($\mathbf{Q}$) *is essentially the eigenspace of eigenvalue* 0 *of (5.1) and is also the eigenspace of eigenvalue* 1 *of (5.4).*

The following result, a slight generalization of [22][Theorem 3.3], basically relates MI(Q), the number of negative eigenvalues of (5.1) with spectral data of (5.2) and (5.4).

Theorem 5.2. *Let $(\Sigma, \partial\Sigma)$ be a smooth compact Riemannian manifold with boundary and $q \geq 0$, $q \not\equiv 0$. Then MI(Q) is equal to*

$$a + b.$$

Here a is the number of non-positive eigenvalues of (5.2) counting multiplicity; b is the number of eigenvalues smaller than 1 of (5.4) counting multiplicity.

We give a proof based on the theory we just developed.

Proof. By Lemma 2.7, it suffices to consider the index form $Q(\cdot,\cdot)$ on the Hilbert space $H^1(\Sigma)$. Let $\mathcal{Q}$ be the associated operator from that Hilbert space to its continuous dual.

Let $u_1, ...u_a$ be a maximal set of independent orthonormal eigenfunctions, $L^2(d\mu)$ mutually perpendicular, of (5.2) with non-positive eigenvalues. Let $h_1, ..., h_b$ be a maximal set of independent orthonormal eigenfunctions, $L^2(qds)$ mutually perpendicular, of (5.4) with eigenvalues smaller than 1. Let

$$\phi_i = \mathcal{Q}u_i,$$
$$\varphi_j = \mathcal{Q}h_j.$$

One readily checks that $\phi_1, ..., \phi_a$, $\varphi_1, ...\varphi_b$ are linearly independent and mutually Q-perpendicular to each other. Then, it follows from Theorem 1.5 that

$$\mathrm{MI}^{\phi_1,...,\phi_a,\varphi_1,...,\varphi_b}(Q) = \mathrm{MI}(Q) - a - b.$$

So the rest of the proof is to prove $\mathrm{MI}^{\phi_1,...,\phi_a,\varphi_1,...,\varphi_b}(Q) = 0$.

Indeed let $v \in \cap_{i=1}^{a}\mathrm{Ker}(\phi_i) \cap \cap_{j=1}^{b}\mathrm{Ker}(\varphi_j)$. We have

$$0 = \phi_i(v) = \mathcal{Q}(u_i)v = Q(u_i, v).$$

Claim: There is a function $h \in H^1(\Sigma)$ such that

$$Jh = 0 \text{ on } \Sigma,$$
$$h = v \text{ on } \partial\Sigma.$$

Proof of the claim. If 0 is not an eigenvalue of (5.2), then the associated homogeneous system has no solution. Then the claim follows from the Fredholm alternative [9].

If 0 is an eigenvalue of (5.2) with eigenfunction u_i, then by the calculation above,

$$\int_{\partial\Sigma}(\nabla_\eta u_i)v\,ds = 0.$$

By a variation of the Fredholm alternative [22, Lemma 2.5], the claim also follows.

Thus, the claim holds and let $u = v - h$. Then $u = 0$ on $\partial\Sigma$ and, without loss of generality, we can assume that u is $L^2(d\mu)$-perpendicular to $\mathrm{Ker}(\mathcal{Q})$.

$$\begin{aligned}\varphi_j(h) = Q(h_j, h) &= \int_{\partial\Sigma}(\nabla_\eta h_j - qh_j)h = (\mu_j - 1)\int_{\partial\Sigma} qh_jh \\ &= \varphi_j(v) - \varphi_i(u) = -\varphi_i(u) = -Q(u, h_j) \\ &= \int_\Sigma uJh_j\,d\mu - \int_{\partial\Sigma}(\nabla_\eta h_j - qh_j)u\,ds = 0.\end{aligned}$$

Thus, h is $L^2(qds)$ perpendicular to each h_j $j = 1, ...b$. Similarly, u is $L^2(d\mu)$ perpendicular to each u_i $i = 1, ...a$.

By the variational characterizations (5.3), (5.5), and $Q(u, h) = 0$,

$$Q(v, v) = Q(u + h, u + h) = Q(u, u) + Q(h, h) \geq 0.$$

Therefore, $\mathrm{MI}^{\phi_1,\dots\phi_a,\varphi_1,\dots\varphi_b}(S) = 0$ and the proof is finished.

□

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