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Nina Lebedeva* and Anton Petrunin

5-Point CAT(0) Spaces after Tetsu Toyoda

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Abstract: We give another proof of Toyoda's theorem that describes 5-point subspaces in CAT(0) length spaces.

Keywords: CAT(0); finite metric space; comparison inequality; Alexandrov comparison

MSC: 53C23, 30L15, 51F99

1 Introduction

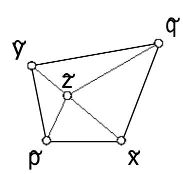
The CAT(0) comparison is a certain inequality for 6 distances between 4 points in a metric space. The following descriptions, the so-called *(2+2)-comparison*, is the most standard, we refer to [3, 4] for other definitions and their equivalences.

Given a quadruple of points p, q, x, y in a metric space X, consider two *model triangles* (that is, a plane triangle with the same sides) $[\tilde p\tilde x\tilde y] = \tilde\triangle(pxy)$ and $[\tilde q\tilde x\tilde y] = \tilde\triangle(qxy)$ with common side $[\tilde x\tilde y]$.

If the inequality

$$|p-q|_X \leqslant |\tilde p-\tilde z| + |\tilde z-\tilde q|$$

holds for any point $\tilde z \in [\tilde x\tilde y]$, then we say that the quadruple p, q, x, y *satisfies* CAT(0) *comparison*; here $|p-q|_X$ denotes the distance from p to q in X.



If CAT(0) comparison holds for any quadruple (and any of its relabeling) in a metric space X, then we say that X *is* CAT(0).

It is not hard to check that if a quadruple of points satisfies CAT(0) comparison for all relabeling, then it admits a distance-preserving inclusion into a length CAT(0) space. The following theorem generalizes this statement to 5-point metric spaces.

1.1. Toyoda's theorem. *Let P be a 5-point metric space that satisfies* CAT(0) *comparison. Then P admits a distance-preserving inclusion into a length* CAT(0) *space X.*

Moreover, X can be chosen to be a subcomplex of a 4-simplex such that (1) each simplex in X has Euclidean metric and (2) the inclusion maps the 5 points on P to the vertexes of the simplex.

A slightly weaker version of this theorem was proved by Tetsu Toyoda [7]. Our proof is shorter; it uses the fact that convex spacelike hypersurfaces in $\mathbb{R}^{3,1}$ equipped with the induced length metrics are CAT(0) spaces

***Corresponding Author: Nina Lebedeva:** Saint Petersburg State University, Saint Petersburg, Russia and Steklov Institute of Mathematics, Saint Petersburg, Russia, E-mail: lebed@pdmi.ras.ru
Anton Petrunin: Mathematics Department, Pennsylvania State University, University Park, USA, E-mail: petrunin@math.psu.edu

[1]. We construct a distance-preserving inclusion ι of P into $\mathbb{R}^4$ or $\mathbb{R}^{3,1}$. In the case of $\mathbb{R}^4$ the convex hull K of $\iota(P)$ can be taken as X; in the case of $\mathbb{R}^{3,1}$ we take as X a spacelike part of the boundary of K.

It is expected that *any 5 point metric space P as in the theorem admits a distance-preserving inclusion in a product of trees*.

An analog of Toyoda's theorem does not hold for 6-point sets. It can be seen by using the so-called (4+2)-comparison introduced in [2]; this comparison holds for any length CAT(0) space, but may not hold for a space with CAT(0) comparison (if it is not a length space).

The (4+2)-comparison is not a sucient condition for 6-point spaces. More precisely, there are 6-point metric spaces that satisfy (4+2) and (2+2)-comparisons but do not admit a distance-preserving embedding into a length CAT(0) space. An example was constructed by the first author; it is described in [2] right aŒer 7.2. See the final section for related questions.

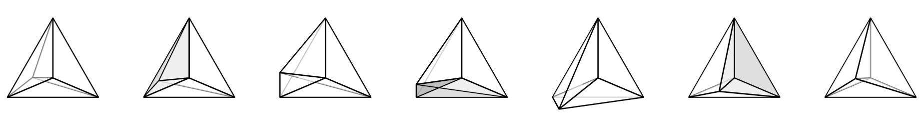
2 5-point arrays in 3-space

Denote by $\mathcal{A}$ the space of all 5 point arrays in $\mathbb{R}^3$ that is nondegenerate in the following sense: (1) all 5 points do not lie on one plane and (2) no three points lie on one line. Note that $\mathcal{A}$ is connected.

A 5 point array $x_1, \dots, x_5 \in \mathbb{R}^3$ defines an ane map from a 4-simplex to $\mathbb{R}^3$. Fix an orientation of the 4-simplex and consider the induced orientations on its 5 facets. Each facet may be mapped in an orientation-preserving, degenerate, or orientation-reversing way. For each array consider the triple of integers (n_+, n_0, n_-), where n_+, n_0, and n_- denote the number of orientation-preserving, degenerate, or orientation-reversing facets respectively.

Clearly $n_+ + n_0 + n_- = 5$ and since all 5 points cannot lie in one plane, we have that $n_+ \geq 1$, $n_- \geq 1$, and $n_0 \leq 1$. Therefore, the value $m = n_- - n_+$ can take an integer value between -3 and 3; in this case, we say that an array belongs to $\mathcal{A}_m$.

It defines a subdivision of $\mathcal{A}$ into 7 subsets $\mathcal{A}_{-3}, \dots, \mathcal{A}_3$ with combinatorial configuration as on the diagram; quadruples in one plane are marked in gray and the triple (n_+, n_0, n_-) is written below.



$\mathcal{A}_{-3}$ $(1, 0, 4)$ $\mathcal{A}_{-2}$ $(1, 1, 3)$ $\mathcal{A}_{-1}$ $(2, 0, 3)$ $\mathcal{A}_0$ $(2, 1, 2)$ $\mathcal{A}_1$ $(3, 0, 2)$ $\mathcal{A}_2$ $(3, 1, 1)$ $\mathcal{A}_3$ $(4, 0, 1)$

Every two quadrilaterals in the array have 3 common points that define a plane. If the remaining two points lie on opposite sides from the plane, then the corresponding facets have the same orientation; if they lie on one side, then the orientations are opposite. Therefore, the 7 subsets $\mathcal{A}_{-3}, \dots \mathcal{A}_3$ can be described in the following way:

$\mathcal{A}_{-3}$ — a tetrahedron with preserved orientation and one point inside.
$\mathcal{A}_{-2}$ — a tetrahedron with preserved orientation and one point on a facet.
$\mathcal{A}_{-1}$ — a double triangular pyramid formed by two tetrahedrons with preserved orientation.
$\mathcal{A}_0$ — a pyramid over a convex quadrilateral
$\mathcal{A}_1$ — a double triangular pyramid formed by two tetrahedrons with reversed orientation.
$\mathcal{A}_2$ — a tetrahedron with reversed orientation and one point on a facet.
$\mathcal{A}_3$ — a tetrahedron with reversed orientation and one point inside.

Note that the complement $\mathcal{A} \setminus \mathcal{A}_0$ has two connected components formed by $\mathcal{A}_- = \mathcal{A}_{-3} \cup \mathcal{A}_{-2} \cup \mathcal{A}_{-1}$ and $\mathcal{A}_+ = \mathcal{A}_3 \cup \mathcal{A}_2 \cup \mathcal{A}_1$. Observe that each array in $\mathcal{A}_-$ has at least 3 positively oriented facets and each array in $\mathcal{A}_+$ has at least 3 negatively oriented facets.

2.1. Observation. *Let Q be a connected subset of $\mathcal{A}$ that does not intersect $\mathcal{A}_0$. Then either $Q \subset \mathcal{A}_+$ or $Q \subset \mathcal{A}_-$.*

3 Associated form

In this section we recall some facts about the so-called *associated* form introduced in [6]; it is a quadratic form $W_{\boldsymbol{x}}$ on R^{n-1} associated to a given n-point array $\boldsymbol{x} = (x_1, \dots, x_n)$ in a metric space X.

Construction. Let 4 be the standard simplex 4 in R^{n-1}; that is, the first $(n-1)$ of its vertices $v_1, \dots, v_n$ form the standard basis on R^{n-1}, and $v_n = 0$.

Recall that j$a - b$j$_X$ denotes the distance between points a and b in the metric space X. Set

$$W_{\boldsymbol{x}}(v_i - v_j) = \mathrm{j}x_i - x_j\mathrm{j}_X^2$$

for all i and j. Note that this identity defines $W_{\boldsymbol{x}}$ uniquely.

The constructed quadratic form $W_{\boldsymbol{x}}$ will be called the *form associated to the point array* $\boldsymbol{x}$.

Note that an array $\boldsymbol{x} = (x_1, \dots, x_n)$ in a metric space X is isometric to an array in Euclidean space if and only if $W_{\boldsymbol{x}}(v) > 0$ for any v 2 R^{n-1}.

In particular, the condition $W_{\boldsymbol{x}} > 0$ for a triple $\boldsymbol{x} = (x_1, x_2, x_3)$ means that all three triangle inequalities for the distances between x_1, x_2, and x_3 hold. For an n-point array, it implies that $W_{\boldsymbol{x}}(v) > 0$ for any vector v in a plane spanned by a triple x_i, x_j, x_k. In particular, we get the following:

3.1. Observation. *Let $W_{\boldsymbol{x}}$ be a form on* R^{n-1} *associated with a point array* $\boldsymbol{x} = (x_1, \dots, x_n)$. *Suppose that L is a subspace of* R^{n-1} *such that* $W_{\boldsymbol{x}}(v) < 0$ *for any nonzero vector v* 2 *L. Then the projections of any 3 vertices of* 4 *to the quotient space* R^{n-1}/L *are not collinear.*

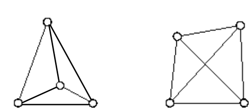
CAT(0) condition. Consider a point array $\boldsymbol{x}$ with 4 points. From 3.1, it follows that $W_{\boldsymbol{x}}$ is nonnegative on every plane parallel to a face of the tetrahedron 4 . In particular, $W_{\boldsymbol{x}}$ can have at most one negative eigenvalue.

Assume $W_{\boldsymbol{x}}(w) < 0$ for some w 2 R^3. From 3.1, the line L_w spanned by w is transversal to each of 4 planes parallel to a face of 4 .

Consider the projection of 4 along L_w to a transversal plane. The projection of the 4 vertices of 4 lie in general position; that is, no three of them lie on one line. Therefore, we can see one of two combinatorial pictures shown on the diagram. Since the set of lines L_w with $W_{\boldsymbol{x}}(w) < 0$ is connected, the combinatorics of the picture does not depend on the choice of w.

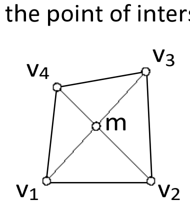
3.2. Claim. *If* CAT(0) *comparison holds in X, then the diagram on the right cannot appear.*

(The converse holds as well, but we will not need it.)



Proof. Suppose we see the picture on the right.

Let $[v_1, v_3]$ and $[v_2, v_4]$ be the line segments of 4 that correspond to the diagonals on the picture. Denote by m the point of $[v_1, v_3]$ that corresponds to the point of intersection.



In the plane spanned by $[v_2, v_4]$ and w, the vector w is timelike. Therefore we have the following reversed triangle inequality:

$$\mathrm{j}v_2 - m\mathrm{j} + \mathrm{j}v_4 - m\mathrm{j} < \mathrm{j}v_2 - v_4\mathrm{j};$$

here we use shortcut $|a-b| = \sqrt{W(a-b)}$.

Note that the triangles $[v_1v_2v_3]$ and $[v_1v_3v_4]$ with metric induced by W are isometric to model triangles of $[x_1x_2x_3]$ and $[x_1x_3x_4]$. Whence (2+2)-point comparison does not hold. □

The claim implies the following:

3.3. Observation. *Suppose a metric on* $\boldsymbol{x} = (x_1, \ldots, x_n)$ *satisfies* CAT(0) *comparison and* $W_{\boldsymbol{x}}$ *is its associated form on* $\mathbb{R}^{n-1}$. *Assume that* L *is a subspace of* $\mathbb{R}^{n-1}$ *such that* $W_{\boldsymbol{x}}(v) < 0$ *for any nonzero vector* $v \in L$. *Then if the projections of 4 vertices of* Δ *to the quotient space* $\mathbb{R}^{n-1}/L$ *lies in one plane, then its projection looks like the picture on the left, that is, one of the points lies in the triangle formed by the remaining three points.*

3.4. Corollary. *Suppose a metric on* $\boldsymbol{x} = (x_1, \ldots, x_5)$ *satisfies* CAT(0) *comparison and* $W_{\boldsymbol{x}}$ *is its associated form on* $\mathbb{R}^4$. *Assume that* L *is a subspace of* $\mathbb{R}^4$ *such that* $W_{\boldsymbol{x}}(v) < 0$ *for any nonzero vector* $v \in L$. *Then* $\dim L \leq 1$.

Moreover, if $\dim L = 1$, *then the projections of the vertices of* Δ *to the quotient space* $\mathbb{R}^3 = \mathbb{R}^4/L$ *belong to* $\mathfrak{A}\setminus\mathfrak{A}_0$ *(defined in the previous section).*

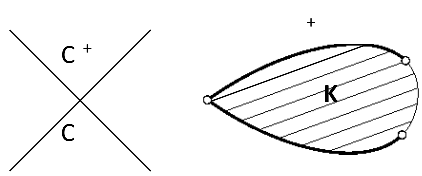
Proof. If $\dim L \geq 2$, then $\dim(\mathbb{R}^4/L) \leq 2$. By 3.1, these 5 projections lie in a general position; that is, no three of these projections lie on one line. Therefore, $\mathbb{R}^4/L = \mathbb{R}^2$ is the plane.

Any 5 points in a general position on the plane include 4 vertices of a convex quadrangle. The latter contradicts 3.3. □

4 Convex spacelike surfaces

Let W be a quadratic form on $\mathbb{R}^4$. Suppose that W has exactly one negative eigenvalue. Choose future and past cones C^+ and C^- for W; that is, C^+ and C^- are connected components of the set $\{v \in \mathbb{R}^4 \mid W(v) < 0\}$. A subset S in $\mathbb{R}^4$ will be called *spacelike* if $W(x-y) \geq 0$ for any $x, y \in S$.

Let K be a convex body in $\mathbb{R}^4$; denote by Σ the surface of K. A point p lies on the *upper side* of Σ (briefly $p \in \Sigma^+$) if there is a spacelike hyperplane in $\mathbb{R}^4$ that supports Σ at p from above; more precisely if the Minkowski sum $\{p\} + C^+$ does not intersect K.



Similarly, we define the *lower side* of Σ denoted by Σ^-. Note that Σ^+ and Σ^- might have common points. The subsets Σ^+ and Σ^- are spacelike; in particular, the length of any Lipschitz curve in these subsets can be defined and it leads to induced intrinsic pseudometrics on Σ^+ and Σ^-. Abusing notation, we will not distinguish a pseudometric space and the corresponding metric space.

4.1. Lemma. *Let* Σ *be the surface of a convex set* K *in* $\mathbb{R}^4$ *and* $C^\pm$ *be the future and past cones for a quadratic form* W. *Then the upper and lower sides* Σ^+ *and* Σ^- *of* Σ *equipped with the induced intrinsic metric are* CAT(0) *length spaces.*

Moreover, if a line segment $[pq]$ *in* $\mathbb{R}^4$ *lies on* $\Sigma^\pm$, *then* $[pq]$ *is a minimizing geodesic in* $\Sigma^\pm$; *that is,*

$$|p-q|^2_{\Sigma^\pm} = W(p-q).$$

This lemma is essentially stated by Anatolii Milka [1, Theorem 4]; we give a sketch of alternative proof based on smooth approximation.

Sketch. We can assume that W is nondegenerate; that is, after a linear change of coordinates it is the standard form on $\mathbb{R}^{3,1}$. If not, then there is a W-preserving projection of $\mathbb{R}^4$ to a W-nondegenerate subspace; apply this projection and note that this subspace is isometric a subspace of $\mathbb{R}^{3,1}$.

Assume S is a smooth strictly spacelike hypersurface in $\mathbb{R}^{3,1}$ with convex epigraph. By Gauss formula, S has nonpositive sectional curvature.

Suppose a strictly spacelike hyperplane Π cuts from S a disc D. Recall that Liberman's lemma [1, Theorem 3] implies that time coordinate is convex on any geodesic in S. We may assume that time is vanishing on Π; therefore, by the lemma, D has a convex set in S. Therefore the Cartan–Hadamard theorem [4] implies that that D is CAT(0).

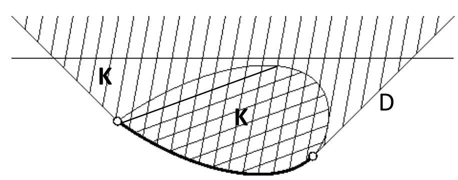
Now suppose D_n is a sequence of smooth discs of the described type that converges to a (possibly nonsmooth) disc D. Note that the metric on D_n converges to the induced pseudometric on D. It follows that the metric space D^0 that corresponds to D is CAT(0).

The disc D might contain lightlike segments which have zero length. Note that every maximal lightlike segment in D starts at its interior point and goes to the boundary. Consider the map $\iota\colon D \to D$ that sends each maximal lightlike segment to its starting point. Note that the sublemma below implies that ι is length-nonincreasing. Since $|x - \iota(x)|_D = 0$, we get that the D^0 is isometric to the image of ι with the induced metric.

Consider the Minkowski sum

$$K^- = K + C^+;$$

it has a convex spacelike boundary ∂K^-. Choose a strictly spacelike hyperplane Π that lies above K. Denote by D the subset of ∂K^- below Π. Let us equip D with induced intrinsic pseudometric. By construction Σ^- is isometric to $\iota(D)$. It follows that Σ^- is CAT(0).



Now suppose a line segment $[pq]$ in $\mathbb{R}^4$ lies on Σ^-. Choose a supporting hyperplane Π at the midpoint of $[pq]$. Choose time coordinate that vanish on Π; by Liberman's lemma, every shortest path in Σ^- between p and q has to lie on Π; that is, the intersection $\Sigma^- \setminus \Pi$ is a convex subset of Σ^-. Therefore $[pq]$ is convex in Σ^- which implies the second statement. □

4.2. Sublemma. *Let u and v be two lightlike vectors in $\mathbb{R}^{3,1}$. Suppose that the union of two half-lines $s \mapsto p+su$ and $t \mapsto q+tv$ for $s,t>0$ is a spacelike set. Then the function $(s,t)\mapsto |(p+su)-(q+tv)|$ is nondecreasing in both arguments, where $|w| := \sqrt{\langle w,w\rangle}$ for a spacelike vector w.*

Proof. Since u and v are lightlike, $\langle u,u\rangle = \langle v,v\rangle = 0$. Since the union of two half-lines is spacelike, $(p+su)-(q+tv)$ is spacelike for any $s,t>0$. It follows that

$$\begin{aligned}0 \leqslant |(p+su)-(q+tv)|^2 &= \\ &= |p-q|^2 - 2s\langle u, q-p\rangle - 2t\langle v, p-q\rangle - 2st\langle u, v\rangle\end{aligned}$$

for any $s,t>0$. Therefore

$$\langle u, q-p\rangle \leqslant 0, \qquad \langle v, p-q\rangle \leqslant 0 \qquad \langle u, v\rangle \leqslant 0.$$

Whence the result. □

Assume v is a nonzero vector in $\mathbb{R}^4$ and $p \in \Sigma$. We say that p lies on the *upper side of Σ with respect to v* (briefly $p \in \Sigma^+(v)$) if $p + tv \in K$ for any $t > 0$. Correspondingly, p lies on the *lower side of Σ with respect to v* (briefly $p \in \Sigma^-(v)$) if $p + tv \in K$ for any $t < 0$.

4.3. Observation. *Let K be a compact convex set in $\mathbb{R}^4$ and $C^\pm$ be the future and past cones for a quadratic form W. Then the upper (lower) side of the boundary surface Σ of K can be described as the intersection of the upper (respectively lower) sides of Σ with respect to all vectors $v \in C^+$; that is,*

$$\Sigma^\pm = \bigcap_{v \in C^+} \Sigma^\pm(v).$$

5 Proof assembling

Proof of Toyoda's theorem. Let $\{x_1, \dots, x_5\}$ be the points in P. Choose a 5-simplex Δ in $\mathbb{R}^4$; denote by W the form associated with the point array $(x_1, \dots, x_5)$.

If $W > 0$, then P admits a distance preserving embedding into Euclidean 4-space, so one can take the convex hull of its image as X.

Suppose $W(v) < 0$ for some $v \in \mathbb{R}^4$. Since P is CAT(0), 3.4 implies that W has exactly one negative eigenvalue. Moreover, if a line L is spanned by a vector v such that $W(v) < 0$, then the projection of the vertices of the simplex to $\mathbb{R}^3 = \mathbb{R}^4/L$ belongs to $\mathrm{A}\setminus\mathrm{A}_0$.

The space of such lines L is connected. By 2.1, we can assume that all the projections belong to A_-. That is, we can choose timelike orientation such that for any $v \in C^+$ the lower part $\Sigma^-(v)$ of $\Sigma = \partial\Delta$ has at least 3 facets of Δ.

In particular, $\Sigma^-(v)$ contains all edges of Δ for any $v \in C^+$. By 4.3, Σ^- contains all edges of Δ. By 4.1, Σ^- with induced (pseudo)metric is a length CAT(0) space.

Since all edges of Δ lie in Σ^-, the inclusion $P \hookrightarrow \Sigma^-$ is distance preserving. Whence we can take $X = \Sigma^-$.

Finally, observe that in each case X is a subcomplex of Δ that includes all edges and has a model metric on each simplex. □

6 Remarks

Let us recall the definition of *graph comparison* given by Vladimir Zolotov and the authors [5] and use it to formulate a few related questions.

Let Γ be a graph with vertices $v_1, \dots, v_n$. A metric space X is said to meet the Γ-comparison if for any set of points in X labeled by vertices of Γ there is a model configuration $\tilde v_1, \dots, \tilde v_n$ in the Hilbert space $\mathbb{H}$ such that if v_i is adjacent to v_j, then

$$|\tilde v_i - \tilde v_j|_{\mathbb{H}} \le |v_i - v_j|_X$$

and if v_i is nonadjacent to v_j, then

$$|\tilde v_i - \tilde v_j|_{\mathbb{H}} \ge |v_i - v_j|_X.$$

The C_4-comparison (for the 4-cycle C_4 on the diagram) defines CAT(0) comparison. Tetsu Toyoda have shown that C_4-comparison imlies graph comparisons for all cycles C_n [8]; remakably, the metric space is *not* assumed to be intrinsic. The O_3-comparison (for the octahedron graph O_3 on the diagram) defines another

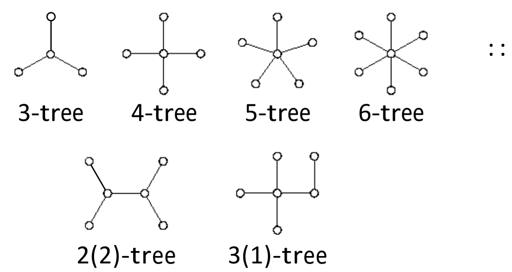
C_4 O_3

comparison. Since O_3 contains C_4 as an induced subgraph, we get that O_3-comparison is stronger than C_4-comparison.

6.1. Open question. *Is it true that octahedron-comparison holds in any 6 points in a length* CAT(0) *space?*

And, assuming the answer is armative, what about the converse: is it true that any 6-point metric space that satisfies octahedron-comparison admits a distance preserving embedding in a length CAT(0) *space?*

The analogous questions for spaces with nonnegative curvature in the sense of Alexandrov (briefly CBB(0)) are open as well. The CBB(0) comparison is equivalent to the 3-tree comparison (for the tripod-tree shown first on the following diagram). It turns out that any length CBB(0) space satisfies the comparison for



the other trees on the diagram; it is formed by an infinite family of star-shaped trees and two trees with 6 vertices [2, 5]. (The 4-tree comparison (the second tree on the diagram) is equivalent to the so-called (4+1)-point comparison in the terminology of [2].)

We expect that this comparison provides a necessary and sucient condition for 5-point sets. Namely, we expect an armative answer to the following stronger question.

6.2. Question. *Suppose a 5-point metric space P satisfies the* 4*-tree comparison. Is it true that P admits a distance preserving embedding into a length* CBB(0) *space?*

Finally, let us mention a related question about a 6-point condition.

6.3. Question. *Suppose a 6-point metric space P satisfies the 5-tree, 2(2)-tree, and 3(1)-tree comparisons. Is it true that P admits a distance preserving embedding into a length* CBB(0) *space?*

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References

[1] À. ˜. ÌŁºŒà. ´ßïóŒºßå ªŁïåðïîâåðıíîæòŁ â ïæåâàïåâŒºŁäîâîì ïðîæòðàíæòâå. *˜îŒº. À˝ ÑÑÑ—*, 284(6):1314–1316, 1985.

[2] S. Alexander, V. Kapovitch, and A. Petrunin. Alexandrov meets Kirszbraun. In *Proceedings of the Gökova Geometry-Topology Conference 2010*, pages 88–109. Int. Press, Somerville, MA, 2011.

[3] S. Alexander, V. Kapovitch, and A. Petrunin. *An invitation to Alexandrov geometry*. SpringerBriefs in Mathematics. CAT(0) spaces.

[4] S. Alexander, V. Kapovitch, and A. Petrunin. Alexandrov geometry: preliminary version no. 1, 2019.

[5] N. Lebedeva, A. Petrunin, and V. Zolotov. Bipolar comparison. *Geom. Funct. Anal.*, 29(1):258–282, 2019.

[6] A. Petrunin. In search of a five-point alexandrov type condition. *St. Petersburg Math. J.*, 29(1):223–225, 2018.

[7] T. Toyoda. An intrinsic characterization of five points in a CAT(0) space. *Anal. Geom. Metr. Spaces*, 8(1):114–165, 2020.

[8] T. Toyoda. A non-geodesic analogue of reshetnyak's majorization theorem, 2020.