Self-Crossing Geodesics

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This column is a place for those bits of contagious mathematics that travel from person to person in the community, because they are so elegant, surprising, or appealing that one has an urge to pass them on. Contributions are most welcome.

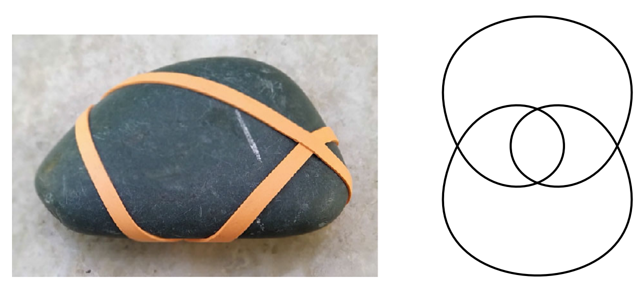
A rubber band is stretched around a pebble, and it crosses itself at several points (see Figure 1). The combinatorics of self-crossings can be described by a closed plane curve—it is the rubber band in a parameterization of the surface with one point removed. For example, if you could turn the pebble around, you would see that the self-crossings are described by the plane curve in Figure 1(b).

We assume that the surface of the pebble is strongly convex, smooth, and frictionless; in this case, the rubber band models a closed geodesic. Suppose that we are interested in possible patterns of self-crossings. More precisely, what are the possible combinatoric types of self-crossings of a closed geodesic on a strongly convex smooth closed surface?

Consider the six possible patterns with three double crossings, shown in Figure 2. Configurations 1, 2, 3, and 4 can be realized as mirror-symmetric geodesics on mirror-symmetric surfaces; the projections on the plane of symmetry are sketched in Figure 3.

Further, we will discuss forbidden configurations, that is, configurations that cannot appear for a closed geodesic. These are configurations 5 and 6 in Figure 2.

Determining the forbidden configurations is a good exercise—it can be explained to anyone, but an answer



(a) (b)

Figure 1. (a) A rubber band stretched around a pebble is described (b) by a closed plane curve.

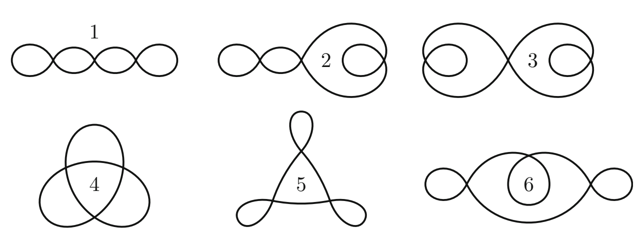


Figure 2. The six possible patterns with three double crossings.

➢ Submissions should be uploaded to http://tmin.edmgr.com or sent directly to Sophie Morier-Genoud (sophie.morier-genoud@imj-prg.fr) or Valentin Ovsienko (valentin.ovsienko@univ-reims.fr).

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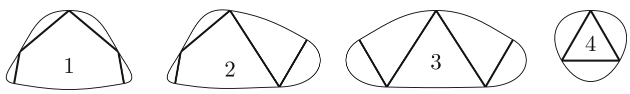

Figure 3. The projections on the plane of symmetry of configurations 1–4 in Figure 2.

requires a considerable part of the theory. The reader is welcome to check that there are no forbidden patterns with fewer than three double crossings and to try the cases with more self-crossings (see [2, Figures 15–17]). By the way, is there an algorithm for solving the general case? Our question is closely related to the so-called flat knot types of geodesics; see [1] and the references therein.

In what follows, we discuss the Gauss–Bonnet formula as well as the Alexandrov–Toponogov theorem and apply them to forbidden configurations 5 and 6. These theorems are covered in the textbook [3], which I like, but they are treated in plenty of other places as well.

Gauss–Bonnet and Configuration 5

Suppose that Δ is an n-gon with geodesic sides in a surface Σ. Recall that by the Gauss–Bonnet formula, the sum of the external angles of Δ equals

$$2 \cdot \pi \cdot \chi(\Delta) - \int_{\Delta} K,$$

where $\chi(\Delta)$ denotes the Euler characteristic of Δ, and K is the Gaussian curvature of Σ.

Further, we assume that Σ is a closed strongly convex surface. In this case:

- ◇ Σ has strictly positive Gaussian curvature;
- ◇ Σ is homeomorphic to the sphere, and therefore $\chi(\Sigma) = 2$;
- ◇ Δ is homeomorphic to the disk, and therefore $\chi(\Delta) = 1$.

It follows that the sum of the internal angles of Δ is greater than $(n-2) \cdot \pi$. In particular, if Δ is a triangle with angles α, β, and γ, then

$$(*) \qquad \alpha + \beta + \gamma > \pi.$$

The Gauss–Bonnet formula can be applied to the whole surface; it implies that the integral of the Gaussian curvature along Σ is exactly $4 \cdot \pi$.

Configuration 5 Is Forbidden

Suppose there is a geodesic with self-crossings as in Figure 4. It divides the surface Σ into one triangle, say Δ, one hexagon, and three monogons. Denote by α, β, and γ the internal angles of Δ.

Note that the three monogons have internal angles α, β, and γ. By Gauss–Bonnet, the integrals of the Gaussian curvature along these monogons are respectively $\pi + \alpha$, $\pi + \beta$, and $\pi + \gamma$. By $(*)$, the integral of the Gaussian curvature along the three monogons exceeds $4 \cdot \pi$. But $4 \cdot \pi$ is the integral of the Gaussian curvature along the whole surface—a contradiction.

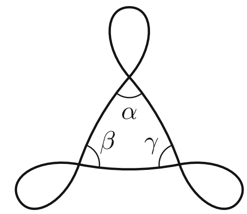

Figure 4. Configuration 5.

Alexandrov–Toponogov and Configuration 6

Let Δ be a geodesic triangle with angles α, β, and γ on the surface Σ. Assume that the sides of Δ are length-minimizing among the curves in Δ with the same endpoints. Then the inequality $(*)$ can be made more exact.

Namely, consider the model triangle $\tilde{\Delta}$ of Δ; that is, $\tilde{\Delta}$ is a plane triangle with equal corresponding sides. Since the sides are length-minimizing, they satisfy the triangle inequality; therefore, the model triangle is well defined.

Denote by $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\gamma}$ the angles of $\tilde{\Delta}$. Then

$$(**) \qquad \alpha > \tilde{\alpha}, \quad \beta > \tilde{\beta}, \quad \gamma > \tilde{\gamma}.$$

Since $\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} = \pi$, this inequality implies $(*)$.

The inequality $(**)$ easily follows from the proof of the Alexandrov–Toponogov theorem, which implies that $(**)$ holds for triangles with length-minimizing sides in the whole surface. The proof is left as an exercise for those familiar with the Alexandrov–Toponogov theorem; others may simply accept it as true.

Configuration 6 Is Forbidden

Suppose that such a geodesic ξ exists; assume that its arcs and angles are labeled as in Figure 5(a). Applying the Gauss–Bonnet formula to the quadrangle and pentagon that ξ cuts from the surface, we get that

$$(\overset{*}{**}) \qquad 2\cdot\alpha < \beta + \gamma, \quad 2\cdot\beta + 2\cdot\gamma < \pi + \alpha,$$

and therefore $\alpha < \frac{\pi}{3}$.

Consider the part of ξ without the arc labeled by a. It cuts from the surface a pentagon Δ with sides and angles as shown in Figure 5(b).

Let us add additional vertices on the sides of Δ so that each side becomes length-minimizing in Δ, as shown in Figure 5(c). Choose a vertex and join it by shortest paths in Δ to every other vertex. Consider a model triangle for each triangle in the obtained subdivision of Δ. The model triangles lie in the plane, and we suppose that they share sides as in Δ. By the comparison inequality $(**)$, the angles of the model triangles do not exceed the corresponding

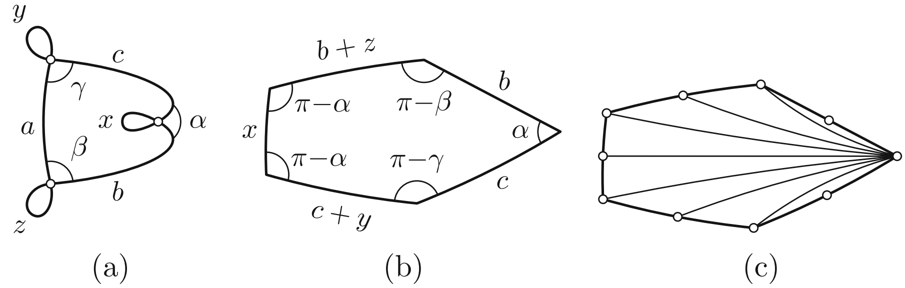

Figure 5. Configuration 6.

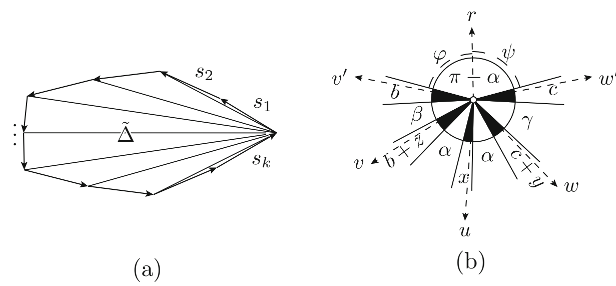

Figure 6. The sides of $\tilde{\Delta}$ oriented counterclockwise.

angles of the original triangle. Therefore, the model triangles form a convex plane polygon, say $\tilde{\Delta}$, such that:

- ◇ The five angles of $\tilde{\Delta}$ that correspond to the angles of Δ do not exceed those.
- ◇ Each side of $\tilde{\Delta}$ equals the corresponding small side of Δ.

It remains to show that no convex plane polygon meets these two conditions.

Let us orient the sides of an alleged polygon $\tilde{\Delta}$ counterclockwise. Denote the obtained vectors by $s_1, \cdots, s_k$; Figure 6(a). Note that the vectors s_i point toward the complement of the white sectors with their angles marked, as shown in Figure 6(b). The sum of the magnitudes of the vectors in each black sector is also marked (each black sector corresponds to a side of Δ).

By $({}^{\,*}_{**})$, we can choose a vector r, as shown in the figure, such that $\varphi > \frac{\pi-\beta}{2}$ and $\psi > \frac{\pi-\gamma}{2}$. Note that $({}^{\,*}_{**})$ implies the inequalities on inner products

$$\langle r,u\rangle < 0,\ \ \langle r,v\rangle < 0,\ \ \langle r,w\rangle < 0,$$
$$\langle r,v\rangle + \langle r,v'\rangle < 0,\ \ \langle r,w\rangle + \langle r,w'\rangle < 0.$$

for any unit vectors v, v', w, and w' in the marked black sectors. It follows that

$$\langle r,s_1\rangle + \cdots + \langle r,s_k\rangle < 0.$$

On the other hand, the vectors s_i circumambulate $\tilde{\Delta}$; so the sum has to vanish—a contradiction.

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