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Moon in a Puddle and the Four-Vertex Theorem

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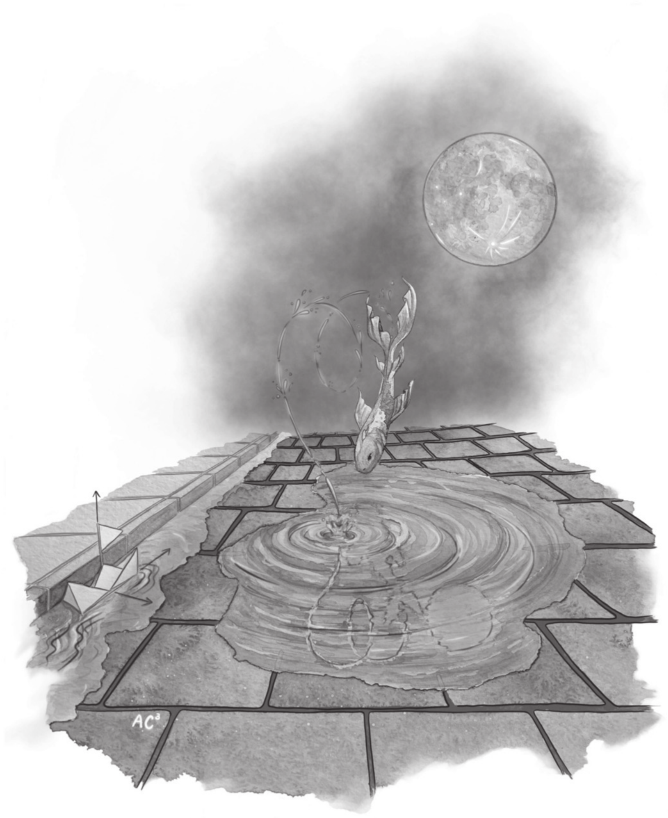
with artwork by Ana Cristina Chávez Cáliz

Abstract. We present a proof of the Moon in a puddle theorem, and use its key lemma to prove a generalization of the four-vertex theorem.



INTRODUCTION. The theorem about the Moon in a puddle provides the simplest meaningful example of a local-to-global theorem which is mainly what differential geometry is about. Yet, the theorem is surprisingly not well-known. This paper aims to redress this omission by calling attention to the result and applying it to a well-known theorem.

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MOON IN A PUDDLE. The following question was initially asked by Abram Fet and solved by Vladimir Ionin and German Pestov [10].

Theorem 1. *Assume γ is a simple closed smooth regular plane curve with curvature bounded in absolute value by 1. Then the region surrounded by γ contains a unit disc.*

We present the proof from our textbook [12] which is a slight improvement of the original proof. Both proofs work under the weaker assumption that the signed curvature is at most one, assuming that the sign is chosen suitably. A more general statement for a barrier-type bound on the curvature was given by Anders Aamand, Mikkel Abrahamsen, and Mikkel Thorup [1]. There are other proofs. One is based on the curve-shortening flow; it is given by Konstantin Pankrashkin [8]. Another one uses cut locus; it is sketched by Victor Toponogov [13, Problem 1.7.19]; see also [9, 11].

Let us mention that an analogous statement for surfaces does not hold—there is a solid body V in the Euclidean space bounded by a smooth surface whose principal curvatures are bounded in absolute value by 1 such that V does not contain a unit ball; moreover one can assume that V is homeomorphic to the 3-ball. Such an example can be obtained by inflating a nontrivial contractible 2-complex in $\mathbb{R}^3$ (Bing's house constructed in [3] would do the job). This problem is discussed by Abram Fet and Vladimir Lagunov [5, 6]; see also [12].

A path $\gamma : [0, 1] \to \mathbb{R}^2$ such that $\gamma(0) = \gamma(1)$ will be called a *loop*; the point $\gamma(0)$ is called the *base* of the loop. A loop is *smooth*, *regular*, and *simple* if it is smooth and regular in $[0, 1]$, and injective in the open interval $(0, 1)$.

Let us use the term *circline* as a shorthand for a *circle or line*. Note that the osculat-ing circle of a smooth regular curve is defined at each of its points—there is no need to assume that the curvature does not vanish.

Suppose that γ is a closed simple smooth plane loop. We say that a circline σ *supports* γ at a point p if the point p lies on both σ and γ, and the cicrline σ lies in one of the closed regions that γ cuts from the plane. If furthermore this region is bounded, then we say that σ *supports* γ *from inside*. Otherwise, we say that σ *supports* γ *from the outside*.

Key lemma. *Assume γ is a simple smooth regular plane loop. Then at one point of γ (distinct from its base), its osculating circle σ supports γ from inside.*

Spherical and hyperbolic versions of this lemma were given in [9, Lemma 8.2] and [2, Proposition 7.1] respectively.

Proof of the theorem modulo the key lemma. Since γ has absolute curvature of at most 1, each osculating circle has radius of at least 1.

According to the key lemma, one of the osculating circles σ supports γ from inside. In this case, σ lies inside γ, whence the result. ■

Proof of the key lemma. Denote by F the closed region surrounded by γ. Arguing by contradiction, assume that the osculating circle at each point $p = p_0$ on γ does not lie in F. Given such a point p, let us consider the maximal circle σ that lies entirely in F and is tangent to γ at p. The circle σ will be called the *incircle* of F at p.

Note that the curvature of the incircle σ has to be strictly larger than the curvature of γ at p, hence there is a neighborhood of p in γ that intersects σ only at p. Further note that the circle σ has to touch γ at another point at least; otherwise, we could increase σ slightly while keeping it inside F.

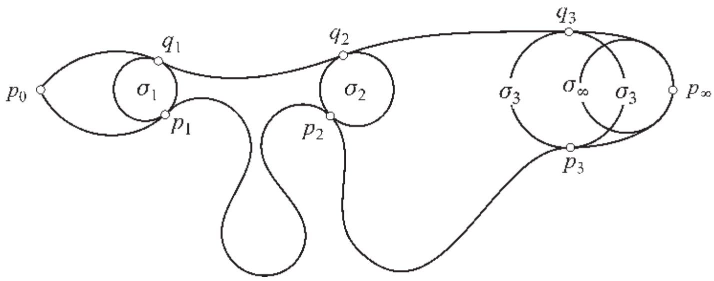

Figure 1. The sequences p_n and q_n converge to a point p_∞.

Choose a point $p_1 \neq p_0$ on γ, and let σ_1 be the incircle at p_1. Choose an arc γ_1 of γ from p_1 to a first point q_1 on σ_1. Denote by $\hat\sigma_1$ and $\check\sigma_1$ the two arcs of σ_1 from p_1 to q_1 such that the cyclic concatenation of $\hat\sigma_1$ and γ_1 surrounds $\check\sigma_1$.

Let p_2 be the midpoint of γ_1, and σ_2 be the incircle at p_2.

Note that σ_2 cannot intersect $\hat\sigma_1$. Otherwise, if s is a point of the intersection, then σ_2 must have two more common points with $\check\sigma_1$, say x and y—one for each arc of σ_2 from p_2 to s. Therefore $\sigma_1 = \sigma_2$ since these two circles have three common points: s, x, and y. On the other hand, by construction, $p_2 \in \sigma_2$ and $p_2 \notin \sigma_1$—a contradiction.

Recall that σ_2 has to touch γ at another point. From above it follows that it cannot touch $\gamma \setminus \gamma_1$, and therefore we can choose an arc γ_2 in γ_1 that runs from p_2 to a first point q_2 on σ_2. Since p_2 is the midpoint of γ_1, we have that

$$(*) \qquad \operatorname{length}\gamma_2 < \tfrac{1}{2} \cdot \operatorname{length}\gamma_1.$$

Repeating this construction recursively, we obtain an infinite sequence of arcs $\gamma_1 \supset \gamma_2 \supset \dots$; by $(*)$, we also get that

$$\operatorname{length}\gamma_n \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore the intersection $\gamma_1 \cap \gamma_2 \cap \dots$ contains a single point; denote it by p_∞.

Let σ_∞ be the incircle at p_∞; it has to touch γ at another point, say q_∞. The same argument as above shows that $q_\infty \in \gamma_n$ for any n. It follows that $q_\infty = p_\infty$—a contradiction. ∎

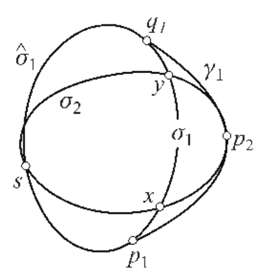

Figure 2. Two ovals pretend to be circles.

Exercise. *Assume that a closed smooth regular curve (possibly with self-intersections) γ lies in a figure F bounded by a closed simple plane curve. Suppose that R is the maximal radius of a disc contained in F. Show that the absolute curvature of γ is at least $\frac{1}{R}$ at some parameter value.*

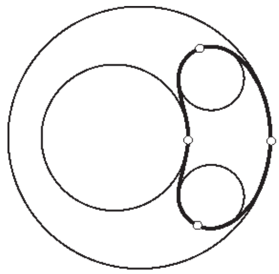

Figure 3. A closed curve with 4 supporting osculating circles.

FOUR-VERTEX THEOREM. Recall that a *vertex* of a smooth regular curve is defined as a critical point of its signed curvature; in particular, any local minimum (or maximum) of the signed curvature is a vertex. For example, every point of a circle is a vertex.

The classical four-vertex theorem says that *any closed smooth regular plane curve without self-intersections has at least four vertices*. It has many different proofs and generalizations. A very transparent proof was given by Robert Osserman [7]; his paper contains a short account of the history of the theorem.

Note that if an osculating circline σ at a point p supports γ, then p is a vertex. The latter can be checked by direct computation, but it also follows from the Tait–Kneser spiral theorem [4]. It states that the *osculating circlines of a curve with monotonic curvature are disjoint and nested*; in particular, none of these circlines can support the curve. Therefore the following theorem is indeed a generalization of the four-vertex theorem:

Theorem 2. *Any smooth regular simple plane curve is supported by its osculating circlines at 4 distinct points; two from inside and two from outside.*

Proof. According to the key lemma, there is a point $p \in \gamma$ such that its osculating circle supports γ from inside.

The curve γ can be considered as a loop with p as its base. Therefore the key lemma implies the existence of another point q with the same property.

This shows the existence of two osculating circles that support γ from inside; it remains to show the existence of two osculating circles that support γ from outside.

Let us apply to γ an inversion with respect to a circle whose center lies inside γ. Then the obtained curve γ_1 also has two osculating circles that support γ_1 from inside.

Note that these osculating circlines are inverses of the osculating circlines of γ. Indeed, the osculating circline at a point x can be defined as the unique circline that has second order of contact with γ at x. It remains to note that inversion, being a local diffeomorphism away from the center of inversion, does not change the order of contact between curves.

Note that the region lying inside γ is mapped to the region outside γ_1 and the other way around. Therefore these two new circlines correspond to the osculating circlines supporting γ from outside. ■

Advanced exercise. *Suppose γ is a closed simple smooth regular plane curve and σ is a circle. Assume γ crosses σ at the points $p_1, \dots, p_{2\cdot n}$ and these points appear in the same cycle order on γ and on σ. Show that γ has at least $2\cdot n$ vertices.*

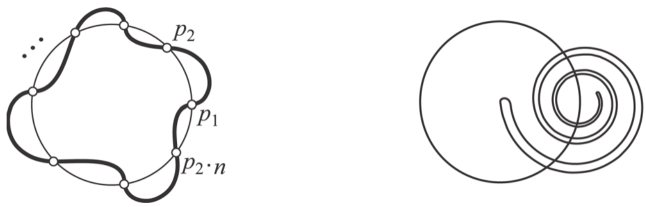

Figure 4.

The order of the intersection points is important. An example with only four vertices and arbitrarily many intersection points can be guessed from the diagram on the right.

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