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Global well-posedness for the defocusing, cubic nonlinear Schrödinger equation with initial data in a critical space

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Abstract. In this note we prove global well-posedness for the defocusing, cubic nonlinear Schrödinger equation with initial data lying in a critical Sobolev space.

1. Introduction

In this note, we discuss the defocusing, cubic, nonlinear Schrödinger equation in three dimensions,

$$iu_t + \Delta u = F(u) = |u|^2 u, \qquad u(0,x) = u_0 \in \dot H^{1/2}(\mathbb{R}^3). \tag{1.1}$$

Equation (1.1) has a scaling symmetry. For any $\lambda > 0$, if u solves (1.1), then

$$u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x), \tag{1.2}$$

also solves (1.1). The initial data $u_0(x)$ has $\dot H^{1/2}(\mathbb{R}^3)$ norm that is invariant under the scaling (1.2).

The local theory for initial data lying in $\dot H^{1/2}(\mathbb{R}^3)$ has been completely worked out, and the scaling symmetry has been shown to control the local well-posedness theory.

Theorem 1.1. *Assume $u_0 \in \dot H^{1/2}(\mathbb{R}^3)$, $\|u_0\|_{\dot H^{1/2}(\mathbb{R}^3)} \le A$. Then there exists $\delta = \delta(A)$ such that if $\|e^{it\Delta} u_0\|_{L^5_{t,x}(I\times\mathbb{R}^3)} < \delta$, then there exists a unique solution to (1.1) on $I \times \mathbb{R}^3$ with $u \in C(I; \dot H^{1/2}(\mathbb{R}^3))$, and*

$$\|u\|_{L^5_{t,x}(I\times\mathbb{R}^3)} \le 2\delta.$$

Moreover, if $u_{0,k} \to u_0$ in $\dot H^{1/2}(\mathbb{R}^3)$, then the corresponding solutions $u_k \to u$ in $C(I; \dot H^{1/2}(\mathbb{R}^3))$.

This theorem was proved in [3].

From this, it is straightforward to show that local well-posedness holds for (1.1) for any initial data $u_0 \in \dot H^{1/2}(\mathbb{R}^3)$. Indeed, by the dominated convergence principle combined

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with Strichartz estimates, for any $u_0 \in \dot{H}^{1/2}(\mathbb{R}^3)$,

$$\lim_{T \searrow 0} \|e^{it\Delta}u_0\|_{L^5_{t,x}([-T,T]\times\mathbb{R}^3)} = 0. \tag{1.3}$$

Since $\delta(A)$ is decreasing as $A \nearrow +\infty$, Strichartz estimates imply that there exists $\delta_0 > 0$ such that if $\|u_0\|_{\dot{H}^{1/2}(\mathbb{R}^3)} < \delta_0$, (1.1) has a global solution that scatters. By scattering, we mean that there exist u_0^+, u_0^- so that

$$\lim_{t \to +\infty} \|u(t) - e^{it\Delta}u_0^+\|_{\dot{H}^{1/2}} = 0;$$

and

$$\lim_{t \to -\infty} \|u(t) - e^{it\Delta}u_0^-\|_{\dot{H}^{1/2}} = 0.$$

However, it is important to note that while (1.3) holds for any fixed $u_0 \in \dot{H}^{1/2}(\mathbb{R}^3)$, the convergence is not uniform, even for $\|u_0\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \leq A < \infty$. Thus, one cannot conclude directly from [3] that a uniform bound for $\|u(t)\|_{\dot{H}^{1/2}(\mathbb{R}^3)}$ on the entire time of the existence of the solution to (1.1) implies that the solution is global. This result was instead proved in [9], using concentration compactness methods.

Theorem 1.2. *Suppose that u is a solution of (1.1) with initial data $u_0 \in \dot{H}^{1/2}(\mathbb{R}^3)$ and a maximal interval of existence $I = (-T_-, T_+)$. Also assume that $\sup_{t\in(-T_-,T_+)} \|u(t)\|_{\dot{H}^{1/2}(\mathbb{R}^3)} = A < \infty$. Then $T_+(u_0) = +\infty$, $T_-(u_0) = \infty$, and the solution u scatters.*

It is conjectured that (1.1) is globally well-posed and scattering for any $u_0 \in \dot{H}^{1/2}(\mathbb{R}^3)$, without the a priori assumption of a universal bound on the $\dot{H}^{1/2}$ norm of the solu-tion $u(t)$. Partial progress has been made in this direction.

A solution to (1.1) has the conserved quantities mass,

$$M(u(t)) = \int |u(t,x)|^2\, dx = M(u(0));$$

and energy,

$$E(u(t)) = \frac{1}{2}\int |\nabla u(t,x)|^2\, dx + \frac{1}{4}\int |u(t,x)|^4\, dx. \tag{1.4}$$

This fact implies global well-posedness for (1.1) with $u_0 \in H^1_x(\mathbb{R}^3)$, where $H^1_x(\mathbb{R}^3)$ is the inhomogeneous Sobolev space of order one. In this case, one could also prove bounds on the scattering size directly, using the interaction Morawetz estimate of [5].

Theorem 1.3. *If u is a solution to (1.1), on an interval I, then*

$$\|u\|^4_{L^4_{t,x}(I\times\mathbb{R}^3)} \lesssim \|u\|^2_{L^\infty_t L^2_x(I\times\mathbb{R}^3)} \|u\|^2_{L^\infty_t \dot{H}^{1/2}(I\times\mathbb{R}^3)} \lesssim E(u)^{1/2} M(u)^{3/2}. \tag{1.5}$$

Interpolating (1.4) and (1.5) then implies

$$\|u\|^4_{L^8_t L^4_x(I\times\mathbb{R}^3)} \lesssim M(u)^{3/4} E(u)^{3/4}; \tag{1.6}$$

with bounds independent of $I \subset \mathbb{R}$. Combining Strichartz estimates and local well-posedness theory, a uniform bound on (1.6) for any $I \subset \mathbb{R}$ directly implies a uniform bound on

$$\|u\|_{L^5_{t,x}(I \times \mathbb{R}^3)}.$$

The argument from [3] implies that proving scattering is equivalent to proving

$$\|u\|_{L^5_{t,x}(\mathbb{R} \times \mathbb{R}^3)} < \infty. \tag{1.7}$$

Indeed, assuming that (1.7) is true, the interval $\mathbb{R}$ may be partitioned into finitely many pieces J_k such that

$$\|u\|_{L^5_{t,x}(J_k \times \mathbb{R}^3)} \ll 1.$$

Then iterate the argument over the intervals J_k, which proves scattering.

This argument also shows that a solution to (1.1) blowing up at a finite time $T_0 < \infty$ is equivalent to

$$\|u\|_{L^5_{t,x}([0,T_0) \times \mathbb{R}^3)} = \infty.$$

Remark. Prior to [5], [8] and [10] proved scattering using the standard Morawetz estimate. See [12] for more details on Strichartz estimates.

Many have attempted to lower the regularity needed in order to prove global well-posedness. For any $s > 1/2$, the inhomogeneous Sobolev space $H^s_x(\mathbb{R}^3) \subset \dot{H}^{1/2}(\mathbb{R}^3)$. Therefore, if $u_0 \in H^s_x(\mathbb{R}^3)$, then it would be conjectured that the solution to (1.1) with initial data u_0 is global and scatters.

Proving a uniform bound on the $H^s_x(\mathbb{R}^3)$ norm would be enough, since by interpolation this would guarantee a uniform bound on the $\dot{H}^{1/2}_x(\mathbb{R}^3)$ norm. The difficulty is that there does not exist a conserved quantity at regularity s that controls the $\dot{H}^s$ norm for $1/2 < s < 1$.

Instead, [2] used the Fourier truncation method (see also [1] for the cubic problem in two dimensions). Decompose the initial data

$$u_0 = P_{\leq N} u_0 + P_{>N} u_0 = v_0 + w_0.$$

Then $v_0 \in H^1(\mathbb{R}^3)$, and $\|w_0\|_{\dot{H}^{1/2}(\mathbb{R}^3)}$ is small. Thus, (1.1) has a global solution for initial data v_0 or w_0, call them v and w. Since (1.1) is a nonlinear equation, it is necessary to also estimate the interaction between v and w in the nonlinearity of (1.1). Then, [2] proved global well-posedness for (1.1) with initial data $u_0 \in H^s_x(\mathbb{R}^3)$ when $s > 11/13$. Moreover, [2] proved that the solution is of the form

$$e^{it\Delta} u_0 + v(t); \qquad \text{where } v(t) \in H^1_x(\mathbb{R}^3).$$

The results from the Fourier truncation method for (1.1) were improved using the I-method. First, [4] improved the regularity necessary for global well-posedness to $s > 5/6$. Then, [5] improved the necessary regularity to $s > 4/5$. To the author's best knowledge, the best known regularity result is the result of [11], proving global well-posedness and scattering for regularity $s > 5/7$. For radial initial data, [6] proved global well-posedness and scattering for any $s > 1/2$. This result is almost sharp at high frequencies.

In this paper, we study the cubic nonlinear Schrödinger equation (1.1) with initial data lying in the Sobolev space $W_x^{7/6,11/7}(\mathbb{R}^3)$. That is,

$$\| |\nabla|^{11/7} u_0 \|_{L^{7/6}(\mathbb{R}^3)} < \infty.$$

Remark. This norm is well-defined using the Littlewood–Paley decomposition. See for example [13].

This norm is preserved under the scaling (1.2), and is therefore a critical Sobolev norm. Moreover, $W_x^{7/6,11/7}(\mathbb{R}^3) \subset \dot{H}^{1/2}(\mathbb{R}^3)$, so (1.1) has a local solution for this initial data. We prove global well-posedness for (1.1) with this initial data.

Theorem 1.4. The cubic nonlinear Schrödinger equation is globally well-posed for initial data $u_0 \in W_x^{7/6,11/7}(\mathbb{R}^3)$.

The proof of this theorem will heavily utilize dispersive estimates. Interpolating between the fact that $e^{it\Delta}$ is a unitary operator,

$$\| e^{it\Delta} u_0 \|_{L^2(\mathbb{R}^3)} = \| u_0 \|_{L^2(\mathbb{R}^3)},$$

and the dispersive estimate,

$$\| e^{it\Delta} u_0 \|_{L^\infty(\mathbb{R}^3)} \lesssim \frac{1}{t^{3/2}} \| u_0 \|_{L^1(\mathbb{R}^3)},$$

gives the estimate

$$\| e^{it\Delta} u_0 \|_{L^7(\mathbb{R}^3)} \lesssim \frac{1}{t^{15/14}} \| u_0 \|_{L^{7/6}(\mathbb{R}^3)}. \tag{1.8}$$

This implies that the linear solution $e^{it\Delta} u_0$ has very good behavior when $t > 1$, in fact it is integrable in time. We then rescale so that u_0 has a local solution on an interval $[-1, 1]$. We prove that this solution may be decomposed into

$$u(t) = e^{it\Delta} u_0 + v(t) + w(t).$$

In particular,

$$u(1) = e^{i\Delta} u_0 + v(1) + w(1).$$

The term

$$e^{i(t-1)\Delta} e^{i\Delta} u_0 = e^{it\Delta} u_0$$

has good properties when $t > 1$. We can also show that

$$\| \nabla e^{i(t-1)\Delta} v(1) \|_{L^\infty} \lesssim \frac{1}{t^{3/2}},$$

which also has good properties when $t > 1$. Finally, $w(1) \in \dot{H}_x^1$ and has finite energy. Making a Gronwall argument shows that

$$\| u(t) - e^{it\Delta} u_0 - e^{i(t-1)\Delta} v(1) \|_{\dot{H}^1},$$

is uniformly bounded on $[1, \infty)$. This is enough to give global well-posedness, but not scattering.

This result could be compared to the result in [7] for the nonlinear wave equation. There, the author proved global well-posedness and scattering for the cubic wave equation with initial radial data in the Besov space $B^2_{1,1} \times B^1_{1,1}$. Here, we do not require radial symmetry, however, we only prove global well-posedness. We are unable to prove scattering at this time due to the lack of a scale invariant conformal symmetry.

We prove a local well-posedness result in section two, and a global result in sec-tion three. This argument could be generalized to many intercritical, defocusing nonlinear Schrödinger equations.

2. Local well-posedness

The Sobolev embedding theorem implies that $W_x^{7/6,11/7}(\mathbb{R}^3)$ is embedded into $\dot{H}^{1/2}(\mathbb{R}^3)$. Therefore, (1.1) is locally well-posed, and there exists some $T(u_0) > 0$ such that (1.1) has a solution on $[-T, T]$ and $\|u\|_{L^5_{t,x}([-T,T]\times\mathbb{R}^3)} = \epsilon_0$, for some $\epsilon_0(\|u_0\|_{\dot{H}^{1/2}})$ small. After rescaling using (1.2), suppose

$$\|u\|_{L^5_{t,x}([-1,1]\times\mathbb{R}^3)} = \epsilon_0. \tag{2.1}$$

Since $(3, 18/5)$ is an admissible pair, Strichartz estimates imply

$$\begin{aligned} &\||\nabla|^{1/2} u\|_{L^\infty_t L^2_x \cap L^2_t L^6_x([-1,1]\times\mathbb{R}^3)} \\ &\quad\lesssim \||\nabla|^{1/2} u_0\|_{L^2_x(\mathbb{R}^3)} + \||\nabla|^{1/2} u\|_{L^3_t L^{18/5}_x([-1,1]\times\mathbb{R}^3)} \|u\|^2_{L^5_{t,x}([-1,1]\times\mathbb{R}^3)}. \end{aligned} \tag{2.2}$$

Therefore,

$$\||\nabla|^{1/2} u\|_{L^\infty_t L^2_x \cap L^2_t L^6_x([-1,1]\times\mathbb{R}^3)} \lesssim \|u_0\|_{\dot{H}^{1/2}}. \tag{2.3}$$

Also, by Duhamel's principle, for any $t \in [-1, 1]$,

$$u(t) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-\tau)\Delta} F(u(\tau))\, d\tau = u_l(t) + u_{nl}(t). \tag{2.4}$$

Remark. Recall from (1.1) that $F(u) = |u|^2 u$.

We begin with a technical lemma. This lemma allows us to make a Littlewood–Paley decomposition of u_{nl}, treat each $P_j u_{nl}$ separately, and then sum up. It also implies that u_{nl} retains all the properties of a solution to the linear Schrödinger equation with initial data in a Besov space.

Remark. In this section, all implicit constants depend on the norm $\|u_0\|_{W^{7/6,11/7}}$.

Remark. Throughout this section we rely very heavily on the bilinear Strichartz estimate

$$\|(e^{it\Delta} P_j u_0)(e^{it\Delta} P_k v_0)\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \lesssim 2^{-j/2} 2^k \|P_j u_0\|_{L^2} \|P_k v_0\|_{L^2}.$$

See [1] for a proof.

Lemma 2.1. Let P_j be the customary Littlewood–Paley projection operator. Also suppose that u is a solution to (1.1) satisfying (2.1). Then

$$\sum_j 2^{j/2}\|P_jF(u)\|_{L_t^1L_x^2([-1,1]\times\mathbb{R}^3)} \lesssim 1. \tag{2.5}$$

Proof. Decompose the nonlinearity,

$$P_jF(u) = P_jF(P_{\geq j-3}u) + 3P_j((P_{\leq j-3}u)^2(P_{\geq j-3}u)) + 3P_j((P_{j-3\leq\cdot\leq j+3}u)(P_{\leq j-3}u)^2).$$

By Bernstein's inequality, and (2.2),

$$\begin{aligned} 2^{j/2}\|P_jF(P_{\geq j-3}u)\|_{L_t^1L_x^2([-1,1]\times\mathbb{R}^3)} &\lesssim 2^{j/2}\|P_{\geq j-3}u\|^3_{L_t^3L_x^6([-1,1]\times\mathbb{R}^3)} \\ &\lesssim 2^{j/2}\Big(\sum_{l\geq j-3}2^{-l/6}\||\nabla|^{1/6}P_lu\|_{L_t^3L_x^6}\Big)^3. \end{aligned} \tag{2.6}$$

Next,

$$\begin{aligned} 2^{j/2}\|P_j((P_{\leq j-3}u)^2(P_{\geq j-3}u))\|_{L_t^1L_x^2([-1,1]\times\mathbb{R}^3)} \\ \lesssim 2^{j/2}\sum_{l\geq j-3}2^{-l/4}\||\nabla|^{1/4}P_lu\|_{L_t^3L_x^{36/7}}\|u\|^2_{L_t^3L_x^9}. \end{aligned} \tag{2.7}$$

Finally, by the bilinear Strichartz estimate

$$\|(e^{it\Delta}P_ju_0)(e^{it\Delta}P_{l_1}u_0)\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \lesssim 2^{-j/2}2^{l_1}\|P_ju_0\|_{L^2}\|P_{l_1}u_0\|_{L^2}, \tag{2.8}$$

combined with the principle of superposition and (2.4),

$$\begin{aligned} &\|(P_ju)(P_{l_1}u)\|_{L^2_{t,x}} \\ &\quad\lesssim 2^{-j/2}2^{l_1}(\|P_ju_0\|_{L^2}+\|P_jF(u)\|_{L_t^1L_x^2})(\|P_{l_1}u_0\|_{L^2}+\|P_{l_1}F(u)\|_{L_t^1L_x^2}); \end{aligned} \tag{2.9}$$

and the Sobolev embedding properties of Littlewood–Paley projections,

$$\begin{aligned} &2^{j/2}\|(P_{j-3\leq\cdot\leq j+3}u)(P_{\leq j-3}u)^2\|_{L_t^1L_x^2([-1,1]\times\mathbb{R}^3)} \\ &\lesssim 2^{j/2}\sum_{l_1\leq j-3}\|(P_{l_1}u)(P_{j-3\leq\cdot\leq j+3}u)\|_{L^2_{t,x}}\sum_{l_1\leq l_2\leq j-3}\|P_{l_2}u\|_{L_t^2L_x^\infty} \\ &\lesssim \||\nabla|^{1/2}u\|_{L_t^2L_x^6}\sum_{l_1\leq j-3}2^{l_1}(j-l_1)\big(\|P_{j-3\leq\cdot\leq j+3}u_0\|_{L^2} \\ &\qquad + \|P_{j-3\leq\cdot\leq j+3}F(u)\|_{L_t^1L_x^2}\big)\big(\|P_{l_1}u_0\|_{L^2}+\|P_{l_1}F(u)\|_{L_t^1L_x^2}\big). \end{aligned} \tag{2.10}$$

By Strichartz estimates, (2.3), Plancherel's theorem, and the fractional product rule,

$$\begin{aligned} \sum_j 2^j\|P_ju_0\|^2_{L^2} + \sum_j 2^j\|P_jF(u)\|^2_{L_t^1L_x^2([-1,1]\times\mathbb{R}^3)} &\lesssim \|u_0\|^2_{\dot H^{1/2}} + \||\nabla|^{1/2}F(u)\|^2_{L_t^1L_x^2} \\ &\lesssim \|u_0\|^2_{\dot H^{1/2}} + \||\nabla|^{1/2}u\|^2_{L_t^3L_x^{18/5}}\|u\|^4_{L_t^3L_x^9} \lesssim 1. \end{aligned}$$

Combining (2.6)–(2.10) with the Cauchy–Schwarz inequality implies

$$\sum_j 2^{j/2}\|P_j F(u)\|_{L^1_t L^2_x([0,1]\times\mathbb{R}^3)} \lesssim 1, \tag{2.11}$$

which proves the lemma. ■

Next, decompose u_{nl} in the following manner:

$$u_{nl}(t) = -i\int_0^{(1-\epsilon)t} e^{i(t-\tau)\Delta}F(u(\tau))\,d\tau - i\int_{(1-\epsilon)t}^{t} e^{i(t-\tau)\Delta}F(u(\tau))\,d\tau = v(t) + w(t),$$

for some $\epsilon > 0$ sufficiently small, to be specified later.

Lemma 2.2. For any $t \in [0,1]$,

$$\|v(t)\|_{L^\infty} \lesssim \frac{1}{\epsilon^{1/2} t^{1/2}}, \tag{2.12}$$

and

$$\|\nabla v(t)\|_{L^\infty} \lesssim \frac{1}{\epsilon t}. \tag{2.13}$$

Proof. By the dispersive estimate, since $\|u\|_{L^3} \lesssim \|u\|_{\dot H^{1/2}}$ is uniformly bounded on $[0,1]$,

$$\|v(t)\|_{L^\infty} \lesssim \left\|\int_0^{(1-\epsilon)t} e^{i(t-\tau)\Delta}F(u)\,d\tau\right\|_{L^\infty} \lesssim \int_0^{(1-\epsilon)t}\frac{1}{|t-\tau|^{3/2}}\|u\|^3_{L^3}\,d\tau \lesssim \frac{1}{\epsilon^{1/2}t^{1/2}}.$$

To prove (2.13), observe that by the product rule,

$$\nabla F(u) = 2|u|^2\nabla u + u^2\nabla\bar{u}.$$

Interpolating,

$$\||\nabla|^{1/2}u_l\|_{L^2} \lesssim \||\nabla|^{1/2}u_0\|_{L^2} \lesssim 1, \tag{2.14}$$

with

$$t^{15/14}\||\nabla|^{11/7}u_l\|_{L^7} \lesssim \||\nabla|^{11/7}u_0\|_{L^{7/6}} \lesssim 1, \tag{2.15}$$

we have

$$t^{1/2}\|\nabla u_l\|_{L^3} \lesssim 1. \tag{2.16}$$

Making a dispersive estimate and using (2.16),

$$\begin{aligned}\left\|\int_0^{(1-\epsilon)t} e^{i(t-\tau)\Delta}|u|^2\nabla u_l(\tau)\,d\tau\right\|_{L^\infty} &\lesssim \int_0^{(1-\epsilon)t}\frac{1}{|t-\tau|^{3/2}}\|\nabla u_l(\tau)\|_{L^3}\|u\|^2_{L^3}\,d\tau\\ &\lesssim \int_0^{(1-\epsilon)t}\frac{1}{|t-\tau|^{3/2}}\frac{1}{|\tau|^{1/2}}\,d\tau \lesssim \frac{1}{\epsilon t}.\end{aligned}$$

The same computation may also be made for $u^2\nabla\bar{u}_l$.

Next, consider the contribution of $|u|^2 \nabla u_{nl}$. By (2.5), we can, without loss of generality, consider only one P_j Littlewood–Paley multiplier, provided the estimate is uniform in $2^{j/2}\|P_j F(u)\|_{L_t^1 L_x^2}$:

$$|u|^2(\nabla P_j u_{nl}) = |P_{\le j} u|^2 (\nabla P_j u_{nl}) + 2\,\mathrm{Re}\big((P_{>j}\bar{u})(P_{\le j}\bar{u})\big)(\nabla P_j u_{nl}) + |P_{>j} u|^2 (\nabla P_j u_{nl}).$$

Using the bilinear Strichartz estimate in (2.9), as well as (2.11) and the Cauchy–Schwartz inequality,

$$\| |u_{\le j}|^2 (\nabla P_j u_{nl})\|_{L_t^2 L_x^1([0,1]\times\mathbb{R}^3)} \lesssim \sum_{j_1 \le j_2 \le j} \|(P_{j_1} u)(P_j \nabla u_{nl})\|_{L^2_{t,x}} \|P_{j_2} u\|_{L_t^\infty L_x^2} \tag{2.17}$$

$$\lesssim \sum_{j_1\le j_2\le j} 2^{j_1/2}\, 2^{-j_2/2}\, 2^{j/2}\|P_j F(u)\|_{L_t^1 L_x^2} \big(\||\nabla|^{1/2} P_{j_1} u_0\|_{L^2} + \||\nabla|^{1/2} P_{j_1} F(u)\|_{L_t^1 L_x^2}\big) \big(\||\nabla|^{1/2} P_{j_2} u_0\|_{L^2} + \||\nabla|^{1/2} P_{j_2} F(u)\|_{L_t^1 L_x^2}\big) \lesssim 1.$$

Also, by Bernstein's inequality and Lemma 2.1,

$$\| |\nabla P_j u_{nl}|\, |P_{>j} u| (|P_{\le j} u| + |P_{>j} u|)\|_{L_t^2 L_x^1} \lesssim \||\nabla|^{1/2} P_j u_{nl}\|_{L_t^2 L_x^6} \||\nabla|^{1/2} P_{>j} u\|_{L_t^\infty L_x^2} \|u\|_{L_t^\infty L_x^3} \lesssim 1.$$

Therefore,

$$\Big\| \int_0^{(1-\delta)t} e^{i(t-\tau)\Delta} |u|^2 \nabla u_{nl}(\tau)\, d\tau \Big\|_{L^\infty} \lesssim \int_0^{(1-\delta)t} \frac{1}{|t-\tau|^{3/2}} \||u|^2 \nabla u_{nl}\|_{L^1}\, d\tau \lesssim \frac{1}{\delta t} \||u|^2 \nabla u_{nl}\|_{L_t^2 L_x^1} \lesssim \frac{1}{\delta t}. \tag{2.18}$$

The same computation can be also be made for $u^2 \nabla \bar{u}_{nl}$. This completes the proof of Lemma 2.2. ∎

Lemma 2.3. For any $t \in [0,1]$,

$$\||\nabla|^{1/2} w(t)\|_{L^3} \lesssim \frac{1}{\delta^{1/4} t^{1/4}}. \tag{2.19}$$

Proof. First observe that by interpolation, Bernstein's inequality, and (2.16),

$$\||\nabla|^{1/2} e^{it\Delta} u_0\|_{L^3} \lesssim t^{1/4}\|\nabla e^{it\Delta} P_{\le t^{-1/2}} u_0\|_{L^3} + t^{-1/4}\|P_{\ge t^{-1/2}} u_0\|_{\dot{H}^{1/2}} \lesssim t^{-1/4}. \tag{2.20}$$

Also since $e^{it\Delta}$ is unitary in L^2, by (2.1) and (2.2),

$$\|v(t)\|_{\dot{H}^{1/2}} = \|u_{nl}((1-\delta)t)\|_{\dot{H}^{1/2}} \lesssim \||\nabla|^{1/2} u\|_{L_t^3 L_x^{18/5}} \|u\|^2_{L^5_{t,x}} \lesssim \epsilon_0^2, \tag{2.21}$$

so interpolating (2.12), (2.13), and (2.21),

$$\||\nabla|^{1/2} v\|_{L^3} \lesssim \||\nabla|^{1/2} v\|_{L^\infty}^{1/3} \||\nabla|^{1/2} v\|_{L^2}^{2/3} \lesssim \frac{\epsilon_0^{4/3}}{\delta^{1/4} t^{1/4}}. \tag{2.22}$$

Finally, making a dispersive estimate, for any $t \in [0,1]$ by (2.20) and (2.22), if $\eta^{1/4} \ll \epsilon_0$,

$$\eta^{1/4}t^{1/4}\left\|\int_{(1-\eta)t}^{t} e^{i(t-\tau)\Delta}|\nabla|^{1/2}F(u)\,d\tau\right\|_{L^3} \tag{2.23}$$
$$\lesssim \eta^{1/4}t^{1/4}\int_{(1-\eta)t}^{t}\frac{1}{|t-\tau|^{1/2}}\||\nabla|^{1/2}u(\tau)\|_{L^3}\|u(\tau)\|_{L^6}^2\,d\tau$$
$$\lesssim \sup_{t\in[0,1]}\eta^{1/4}t^{1/4}\||\nabla|^{1/2}u\|_{L^3}^3 \lesssim \epsilon_0^4 + \sup_{t\in[0,1]}\eta^{1/4}t^{1/4}\||\nabla|^{1/2}w\|_{L^3}^3.$$

Thus, absorbing the second term on the right-hand side into the left-hand side of (2.23) proves (2.19):

$$\||\nabla|^{1/2}w(t)\|_{L^3} \lesssim \frac{\epsilon_0^4}{\eta^{1/4}t^{1/4}}. \qquad \blacksquare$$

Remark. To make the proofs of Lemmas 2.2 and 2.3 completely rigorous, truncate u_0 in frequency. Then the bounds (2.12), (2.13), and (2.19) all hold on some open subset of $[0,1]$ that contains 0. Making the bootstrap argument using the proof of Lemma 2.3 gives bounds on all of $[0,1]$ that do not depend on the frequency truncation of u_0. Standard perturbation arguments then give the lemmas.

Lemma 2.3 can be strengthened to an estimate on the $\dot H^1$ norm of w.

Lemma 2.4. For any $t \in [0,1]$,

$$\|\nabla w(t)\|_{L^2} \lesssim \frac{1}{\eta^{1/4}t^{1/4}}.$$

Proof. Once again make use of the bilinear Strichartz estimate. Again by the product rule,

$$\nabla F(u) = 2|u|^2\nabla u + u^2\nabla\bar u.$$

First, by Strichartz estimates, (2.16), Lemma 2.3, and the Sobolev embedding theorem,

$$\left\|\int_{(1-\eta)t}^{t} e^{i(t-\tau)\Delta}[2|u|^2\nabla u_l + u^2\nabla\bar u_l]\,d\tau\right\|_{L^2} \lesssim \|2|u|^2\nabla u_l + u^2\nabla\bar u_l\|_{L_t^2L_x^{6/5}}$$
$$\lesssim \eta^{1/2}t^{1/2}\|\nabla u_l\|_{L_t^\infty L_x^3([(1-\eta)t,t]\times\mathbb{R}^3)}\|u\|_{L_t^\infty L_x^3([(1-\eta)t,t]\times\mathbb{R}^3)}\||\nabla|^{1/2}u\|_{L_t^\infty L_x^3([(1-\eta)t,t]\times\mathbb{R}^3)}$$
$$\lesssim \frac{\eta^{1/4}}{t^{1/4}}.$$

Next, by (2.19), bilinear Strichartz estimates in (2.9), and the Littlewood–Paley theorem,

$$\|2|u_{\le j}|^2(\nabla P_j u_{nl}) + (u_{\le j})^2(\nabla P_j\bar u_{nl})\|_{L_t^2L_x^{6/5}}$$
$$\lesssim \sum_{k\ge 0}2^{-k/2}\left\|\Big(\sum_{j_1\le j}2^{j_1+k}|P_{j_1+k}u|^2\Big)^{1/2}\Big(\sum_{j_1\le j}2^{-j_1}2^{2j}|P_{j_1}u|^2|P_j u_{nl}|^2\Big)^{1/2}\right\|_{L_t^2L_x^{6/5}}$$
$$\lesssim \sum_{k\ge 0}2^{-k/2}\||\nabla|^{1/2}u(t)\|_{L_t^\infty L_x^3([(1-\eta)t,t]\times\mathbb{R}^3)}$$
$$\times\sum\Big(\|P_{j_1}u_0\|_{\dot H^{1/2}}^2 + \|P_{j_1}F(u)\|_{L_t^1L_x^2}^2\Big)^{1/2}2^{j/2}\|P_j F(u)\|_{L_t^1L_x^2}$$
$$\lesssim \frac{1}{\eta^{1/4}t^{1/4}}\||\nabla|^{1/2}P_j F(u)\|_{L_t^1L_x^2}.$$

Next, by Bernstein’s inequality and (2.19)–(2.21),

$$
\begin{aligned}
\|(\nabla P_j u_{nl})|u_l||u|\|_{L_t^2 L_x^{6/5}} &\lesssim \iota^{1/4} t^{1/4} \||\nabla|^{1/2} u\|^2_{L_t^\infty L_x^3([(1-\iota)t, t] \times \mathbb{R}^3)} \||\nabla|^{1/2} P_j u_{nl}\|_{L_t^4 L_x^3([(1-\iota)t, t] \times \mathbb{R}^3)} \\
&\lesssim \frac{1}{\iota^{1/4} t^{1/4}} \||\nabla|^{1/2} P_j F(u)\|_{L_t^1 L_x^2([0,\infty) \times \mathbb{R}^3)}.
\end{aligned}
$$

Summing up in j using Lemma 2.1 completes the proof. ■

Remark. The above arguments would work equally well in the time interval $[-\infty, 0]$.

3. Global well-posedness

We are ready to prove Theorem 1.4. The proof will use conservation of the energy (1.4). Decompose

$$u(1) = \tilde{v}(1) + w(1),$$

where

$$\tilde{v}(1) = u_l(1) + v(1), \tag{3.1}$$

and $w(1)$ is the w in the previous section. Let $T_0 > 1$ be a time value for which we know that (1.1) has a solution on $[0, T_0)$. By standard local well-posedness arguments and we know that such a T_0 exists. Then on $[1, T_0)$, decompose

$$u(t) = \tilde{v}(t) + w(t),$$

where $\tilde{v}(t)$ is the solution to

$$(i\partial_t + \Delta)\tilde{v}(t) = 0, \qquad \tilde{v}(1) = \tilde{v}(1, x), \tag{3.2}$$

and $w(t)$ is the solution to

$$(i\partial_t + \Delta) w = |u|^2 u, \qquad w(1) = w(1, x). \tag{3.3}$$

Let $E(t)$ denote the energy of w,

$$E(t) = \frac{1}{2} \int |\nabla w|^2 + \frac{1}{4} \int |w|^4.$$

First observe that Lemma 2.4 and $\|w(1)\|_{\dot{H}^{1/2}} \lesssim 1$ implies that $E(1) < \infty$. The estimate $\|w(1)\|_{\dot{H}^{1/2}}$ is a consequence of Lemma 2.1 and the definition of w. To prove Theorem 1.4, it suffices to prove that for any $T_0 > 1$ such that (1.1) has a solution on $[0, T_0)$,

$$\sup_{t \in [1, T_0)} E(t) < \infty. \tag{3.4}$$

Indeed, by interpolation and the Sobolev embedding theorem, $E(t) < \infty$ implies that $\|w(t)\|_{L^5} < \infty$. Meanwhile, by (2.14)–(2.16), (2.12), and (2.21), $\|\tilde{v}(t)\|_{L^5}$ is uniformly bounded on $\mathbb{R}$. Therefore, (3.4) implies

$$\|u\|_{L^5_{t,x}([0,T_0)\times\mathbb{R}^3)} < \infty.$$

To estimate the growth of $E(t)$, compute the derivative in time of the energy. By (3.3),

$$\frac{d}{dt}E(t) = \langle -\Delta w, w_t\rangle + \langle |w|^2 w, w_t\rangle = \langle |w|^2 w - |u|^2 u, w_t\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the inner product

$$\langle f, g\rangle = \operatorname{Re}\int f(x)\bar{g}(x)\,dx.$$

By the product rule,

$$\begin{aligned}\langle w_t, |u|^2 u - |w|^2 w\rangle &= \frac{d}{dt}\langle |w|^2 w, \tilde{v}\rangle + \frac{d}{dt}\langle |\tilde{v}|^2, |w|^2\rangle \\ &\quad + \frac{1}{2}\frac{d}{dt}\operatorname{Re}\int \bar{w}^2\tilde{v}^2 + \frac{d}{dt}\langle w, |\tilde{v}|^2\tilde{v}\rangle - 2\langle \tilde{v}_t\bar{\tilde{v}}, |w|^2\rangle \\ &\quad - \langle |w|^2 w, \tilde{v}_t\rangle - \operatorname{Re}\int w^2\bar{\tilde{v}}\bar{\tilde{v}}_t - 2\langle w, |\tilde{v}|^2\tilde{v}_t\rangle - \langle w, \tilde{v}^2\bar{\tilde{v}}_t\rangle.\end{aligned} \tag{3.5}$$

Then define the modified energy,

$$\mathcal{E}(t) = E(t) - \langle |w|^2 w, \tilde{v}\rangle - \langle |\tilde{v}|^2, |w|^2\rangle - \frac{1}{2}\operatorname{Re}\int w^2\bar{\tilde{v}}^2 - \langle w, |\tilde{v}|^2\tilde{v}\rangle.$$

By Hölder's inequality, and the fact that $\|\tilde{v}\|_{L^4} \lesssim_\epsilon 1$ for all $t \in [1,\infty)$ (again using (2.14)–(2.16), (2.12), and (2.21)),

$$\langle |w|^2 w, \tilde{v}\rangle + \langle |\tilde{v}|^2, |w|^2\rangle + \frac{1}{2}\operatorname{Re}\int w^2\bar{\tilde{v}}^2 + \langle w, |\tilde{v}|^2\tilde{v}\rangle \lesssim E(t)^{3/4} + E(t)^{1/4}.$$

Therefore, when $E(t)$ is large, $\mathcal{E}(t) \sim E(t)$. Since we are attempting to prove a uniform bound for $E(t)$, it is enough to uniformly bound $\mathcal{E}(t)$.

Also, by (3.5),

$$\frac{d}{dt}\mathcal{E}(t) = -\langle |w|^2 w, \tilde{v}_t\rangle - 2\langle \tilde{v}_t\bar{\tilde{v}}, |w|^2\rangle - \operatorname{Re}\int w^2\bar{\tilde{v}}\bar{\tilde{v}}_t - 2\langle w, |\tilde{v}|^2\tilde{v}_t\rangle - \langle w, \tilde{v}^2\bar{\tilde{v}}_t\rangle.$$

Since $\tilde{v}$ solves (3.2), $\tilde{v}_t = i\Delta\tilde{v} = i\Delta u_l + i\Delta v$.

Lemma 2.2 implies that for any $t > 1$,

$$\|v(t)\|_{L^\infty} + \|\nabla v(t)\|_{L^\infty} = \left\|\int_0^{(1-\epsilon)} e^{i(t-\tau)\Delta}\langle\nabla\rangle F(u)\,d\tau\right\|_{L^\infty} \lesssim \frac{1}{\epsilon^{3/2}t^{3/2}}. \tag{3.6}$$

Therefore,

$$\langle |w|^2 w, i\Delta v\rangle = -\langle \nabla(|w|^2 w), i\nabla v\rangle \lesssim \|\nabla v\|_{L^\infty}\|\nabla w\|_{L^2}\|w\|^2_{L^4} \lesssim_\epsilon \frac{1}{t^{3/2}}E(t).$$

Remark. Since $\epsilon > 0$ is fixed, we will ignore it from now on.

Also, by Hölder's inequality and (1.8),

$$\langle i\Delta(e^{it\Delta}u_0), |w|^2 w\rangle \lesssim \||\nabla|^{11/7} u_l\|_{L^7} \|\nabla w\|^{3/7}_{L^2} \|w\|^{18/7}_{L^4} \lesssim \frac{1}{t^{15/14}} E(t)^{6/7}.$$

This takes care of the contribution of $\langle \tilde{v}_t, |w|^2 w\rangle$.

Next, integrating by parts,

$$(3.7)\quad 2\langle i(\Delta\tilde{v})\bar{\tilde{v}}, |w|^2\rangle = -2\langle i|\nabla\tilde{v}|^2, |w|^2\rangle - 2\langle i(\nabla\tilde{v})\bar{\tilde{v}}, \nabla|w|^2\rangle = -2\langle i(\nabla\tilde{v})\bar{\tilde{v}}, \nabla|w|^2\rangle.$$

Then by Hölder's inequality and (3.6), since $\|\tilde{v}\|_{L^4} \lesssim 1$,

$$\langle i(\nabla v)\bar{\tilde{v}}, \nabla|w|^2\rangle \lesssim \|\nabla v\|_{L^\infty} \|\tilde{v}\|_{L^4} \|w\|_{L^4} \|\nabla w\|_{L^2} \lesssim \frac{1}{t^{3/2}}$$

$E(t)^{3/4}$. Also, by Hölder's inequality and interpolation,

$$(3.8)\quad \langle i(\nabla u_l)(u_l), \nabla|w|^2\rangle \lesssim \|\nabla u_l\|_{L^\infty_x} \|u_l\|_{L^4} \|\nabla w\|_{L^2} \|w\|_{L^4} \lesssim \frac{1}{t}\frac{1}{t^{1/8}} E(t)^{3/4}.$$

Finally, by (3.6), and Lemma 2.1, which by the Sobolev embedding theorem and the definition of v implies $\|v\|_{L^3} \lesssim 1$

$$(3.9)\quad \langle i(\nabla u_l)v, \nabla|w|^2\rangle \lesssim \|\nabla u_l\|_{L^\infty_x} \|v\|^{3/4}_{L^3} \|v\|^{1/4}_{L^\infty} \|\nabla w\|_{L^2} \|w\|_{L^4} \lesssim \frac{1}{t}\frac{1}{t^{3/8}} E(t)^{3/4}.$$

In (3.8) and (3.9) we used:

Lemma 3.1. For any $t \geq 0$,

$$(3.10)\qquad \|u_l\|_{L^4} \lesssim \frac{1}{t^{1/8}};$$

and

$$(3.11)\qquad \|\nabla u_l\|_{L^\infty} \lesssim \frac{1}{t}.$$

Proof. This is proved by interpolating (2.14)–(2.16). By Bernstein's inequality, (2.15), (2.16), and the Sobolev embedding theorem,

$$(3.12)\qquad \|\nabla P_{\leq t^{-1/2}} u_l\|_{L^\infty} + \|\nabla P_{\geq t^{-1/2}} u_l\|_{L^\infty} \lesssim \frac{1}{t}.$$

Also by the Bernstein inequality and the Sobolev embedding theorem, along with (2.16) and $u_l \in \dot{H}^{1/2}$,

$$(3.13)\qquad \|P_{\leq t^{-1/2}} u_l\|_{L^4} + \|P_{\geq t^{-1/2}} u_l\|_{L^4} \lesssim \frac{1}{t^{1/8}}.$$

This proves the lemma. ∎

The contribution of $2\operatorname{Re}\int w^2 \bar{\tilde v}\,\bar{\tilde v}_t$ may be estimated in a similar manner as the contribution of (3.7), except that there is an additional term to consider,

$$2\operatorname{Re}\int i w^2 (\nabla \bar{\tilde v})^2.$$

Interpolating (3.11) with (2.16),

$$2\operatorname{Re}\int i w^2 (\nabla \bar u_l)^2 \lesssim \|\nabla u_l\|_{L^4}^2 \|w\|_{L^4}^2 \lesssim \frac{1}{t^{5/4}} E(t)^{1/2}.$$

Meanwhile, following (2.17) and using Strichartz estimates,

$$\| |u_l|^2 (\nabla P_{\geq} u_{nl}) \|_{L^1_t L^{3/2}_x([0,\infty)\times\mathbb{R}^3)} \lesssim \sum_{j_1\leq j_2\leq j} \|(P_{j_1} u)(P_j \nabla u_{nl})\|_{L^2_{t,x}} \|P_{j_2} u\|_{L^2_t L^6_x}$$

$$\lesssim \sum_{j_1\leq j_2\leq j} 2^{j_1/2} 2^{-j_2/2} 2^{j/2} \|P_j F(u)\|_{L^1_t L^2_x} \big(\||\nabla|^{1/2} P_{j_1} u_0\|_{L^2} + \||\nabla|^{1/2} P_{j_1} F(u)\|_{L^1_t L^2_x}\big)$$

$$\cdot \big(\||\nabla|^{1/2} P_{j_2} u_0\|_{L^2} + \||\nabla|^{1/2} P_{j_2} F(u)\|_{L^1_t L^2_x}\big) \lesssim 1.$$

Plugging this estimate into (2.18) implies that for $t > 1$,

$$\Big\|\nabla \int_0^{(1-\eta)} e^{i(t-\tau)\Delta} F(u)\Big\|_{L^3_x} \lesssim \frac{1}{t^{1/2}}.$$

Interpolating (3.6) with (3.10),

$$2\operatorname{Re}\int i w^2 (\nabla \bar{\tilde v})^2 \lesssim \|\nabla \tilde v\|_{L^4}^2 \|w\|_{L^4}^2 \lesssim \frac{1}{t^{3/2}} E(t)^{1/2}.$$

Now treat

$$2\langle w, |\tilde v|^2 \tilde v_t\rangle + \langle w, \tilde v^2 \bar{\tilde v}_t\rangle = 2\langle w, |\tilde v|^2 (i\Delta \tilde v)\rangle + \langle w, \tilde v^2 \overline{(i\Delta \tilde v)}\rangle. \tag{3.14}$$

After integrating by parts, by (2.13) and (3.11),

$$(3.14) \lesssim \langle |\nabla \tilde v|^2, |v||w|\rangle + \langle |\nabla \tilde v||\nabla w|, |v|^2\rangle$$

$$\lesssim \|\nabla \tilde v\|_{L^4}^2 \|\tilde v\|_{L^4}\|w\|_{L^4} + \|\nabla w\|_{L^2}\|\nabla \tilde v\|_{L^\infty}\|\tilde v\|_{L^4}^2 \lesssim \frac{1}{t^{5/4}} E(t)^{1/4} + \frac{1}{t} E(t)^{1/2}\|\tilde v(t)\|_{L^4}.$$

Interpolating (3.6) with $\|v\|_{L^3} \lesssim 1$ implies $\|v\|_{L^4} \lesssim t^{-3/8}$. Meanwhile, (3.10) implies $\|u_l\|_{L^4} \lesssim t^{-1/8}$, so therefore, by (3.1), $\|\tilde v\|_{L^4} \lesssim 1/t^{1/8}$. Therefore, we have proved

$$\frac{d}{dt}\mathcal{E}(t) \lesssim \frac{1}{t^{15/14}}(1 + \mathcal{E}(t)). \tag{3.15}$$

By Gronwall's inequality, (3.15) implies a uniform bound on $\mathcal{E}(t)$. This implies a uniform bound on $E(t)$, since $E(t) \sim \mathcal{E}(t)$ when $E(t)$ is large, which proves Theorem 1.4.

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