General Tensor Structure for Electron Scattering in Terms of Invariant Responses

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Abstract

The use of invariant response functions in treatments of electron scattering from hadronic targets is reviewed. Various classes of reaction are treated, building from the simplest (and best known) case of inclusive scattering from unpolarized targets, to more complicated cases involving polarized electrons and possibly polarized spin-1/2 targets. In particular, the general structure of semi-inclusive polarized electron scattering from polarized spin-1/2 targets is emphasized. A summary is presented of how the leptonic and hadronic tensors that enter in the formalism are constructed in a general covariant way in terms of kinematic factors that are frame dependent but model independent and invariant response functions which contain all of the model-dependent dynamics. In the process of reviewing the general problem the relationships to the conventional responses expressed in terms of the frame-dependent helicity components of the exchanged virtual photon are presented.

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Keywords: Keywords here.

1. Introduction and General Developments

Studies of electron scattering within the Plane-Wave Born Approximation (PWBA) involve the contraction of the relatively simple electron tensor with the tensor that captures the dynamical content of the system from which the electrons are being scattered (see, for instance, [1] for a general overview of the subject). For the latter we have in mind hadronic systems, specifically nucleons or nuclei. Typically this contraction of leptonic and hadronic tensors is decomposed into the individual Lorentz components that are written in a coordinate system oriented along and transverse to the direction of the virtual photon exchanged between the electron and the hadronic system (see [2] and [3] and references therein for the specific conventions employed in the present work). This approach has been employed for many decades and has proven to be useful in that sometimes, for instance at very high energies, the various pieces of the response are not all of similar importance and accordingly approximations may be made; note, however, that this situation is not always the case. A drawback of this approach is that the response functions that enter are specific components of the hadronic tensor and thus are not Lorentz invariant. This means that when one wishes to inter-relate results in different frames of reference, for instance between the target rest frame and a frame in which the electrons and target hadrons are colliding, it becomes necessary to perform a Lorentz transform on the second-rank hadronic tensor.

In contrast, at least within the context of inclusive electron scattering, it is well-known that the tensor contraction may be written in terms of **Lorentz invariant** response functions multiplied by simple kinematic factors. For example, when scattering from unpolarized systems the familiar decomposition into Lorentz scalar response functions $W_{1,2}$ is conventional (see later and [1] for discussion of this simplest of cases). These two response functions may be shown to be functions of two Lorentz scalar quantities, as will be discussed in detail later in this review. These ideas have been extended to more complicated reactions in which some particle or particles in the final state may be detected in coincidence with the scattered electron or where the target is polarized. In the present study we present a review of the general formalism for a selection of these reactions employing invariant hadronic response functions throughout. In the case of so-called semi-inclusive reactions where the incident electron is in general polarized, where the target is assumed to be polarized and where one particle is detected in coincidence with the scattered electron (its polarization is assumed not to be measured) we provide detailed arguments for the structure of the tensor contraction and the resulting cross section, since this situation is likely to be the most relevant in future experimental studies.

The paper is organized in the following way: in the present section some familiar general developments are summarized which involve the contraction of the leptonic and hadronic tensors and include the specific forms for the electron scattering tensors in the Extreme Relativistic Limit (ERL_e). This is followed in Sec. 2 with a review of the steps followed in developing detailed constructions of the general hadronic tensors for several specific classes of electron scattering reactions. This is followed in Sec. 3 with several examples where the hadronic tensors are expressed in terms of frame-dependent response functions, including a discussion of how these are related to their counterparts written in terms of invariant response functions. In Sec. 4 the semi-inclusive cross section is given for a general situation where the polarized spin-1/2 target is moving in some arbitrary direction — this for use in collider physics. For completeness the simpler situation of polarized inclusive electron scattering from a (moving) polarized spin-1/2 target is presented in Sec. 5. To conclude the paper a summary is given in Sec. 6.

We begin with some general developments that are common to all electron scattering formalism at the level of the plane-wave Born approximation.[2]

[2] We use the conventions of [4] in this work. We also employ the conventions previously used by us and others in many previous studies. Namely we denote 4-vectors by capital letters

The general cross section is proportional to the contraction of the leptonic and hadronic tensors $\eta_{\mu\nu}$ and $W^{\mu\nu}$, respectively

$$\eta_{\mu\nu} W^{\mu\nu}. \tag{1}$$

Being composed of bilinear products of the corresponding leptonic and hadronic current matrix elements $\left(j_{fi}\right)_\mu$ and $\left(J_{fi}\right)^\mu$, respectively, in the forms

$$\eta_{\mu\nu} \sim \overline{\sum_{if}} \left(j_{fi}\right)^*_\mu \left(j_{fi}\right)_\nu \tag{2}$$

$$W^{\mu\nu} \sim \overline{\sum_{if}} \left(J_{fi}\right)^{\mu *} \left(J_{fi}\right)^\nu , \tag{3}$$

with appropriate averages over initial and sums over final states, one has immediately that

$$\eta_{\nu\mu} = \left(\eta_{\mu\nu}\right)^* \tag{4}$$

$$W^{\nu\mu} = \left(W^{\mu\nu}\right)^* . \tag{5}$$

Instead of $\eta_{\mu\nu}$ we employ the following convention for the leptonic tensor (see [2])

$$\chi^{\mu\nu} \equiv 4m_e^2 \eta^{\mu\nu} \tag{6}$$

$$= \chi^{\mu\nu}_{unpol} + \chi^{\mu\nu}_{pol}. \tag{7}$$

Also, since the electromagnetic current is conserved,

$$Q^\mu \left(j_{fi}\right)_\mu = Q_\mu \left(J_{fi}\right)^\mu = 0, \tag{8}$$

one has that

$$Q^\mu \chi_{\mu\nu} = \chi_{\mu\nu} Q^\nu = Q_\mu W^{\mu\nu} = W^{\mu\nu} Q_\nu = 0. \tag{9}$$

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and 3-vectors by lower case letters, $A^\mu = (A^0, \mathbf{a})$, $B^\mu = (B^0, \mathbf{b})$, *etc.* The scalar product of two 4-vectors is then $A \cdot B = A^0 B^0 - \mathbf{a} \cdot \mathbf{b}$ and therefore the scalar product of a 4-vector with itself is $A^2 = (A^0)^2 - a^2$ where $a \equiv |\mathbf{a}|$. One potential point of confusion can occur with these conventions, *viz.* for the momentum transfer 4-vector $Q^\mu = (Q^0, \mathbf{q}) = (\omega, \mathbf{q}) = (\nu, \mathbf{q})$ we have $Q^2 = (Q^0)^2 - q^2$ which for electron scattering is spacelike, and accordingly $Q^2 < 0$. One should be careful not to confuse our sign convention for this quantity with the so-called SLAC convention which has the opposite sign $Q^2_{SLAC} = -Q^2 > 0$.

Since one can decompose the tensors into symmetric and anti-symmetric contributions (*i.e.,* under exchange of μ and ν), namely,

$$\chi^s_{\mu\nu} \equiv \frac{1}{2}\left(\chi_{\mu\nu} + \chi_{\nu\mu}\right) \tag{10}$$

$$\chi^a_{\mu\nu} \equiv \frac{1}{2}\left(\chi_{\mu\nu} - \chi_{\nu\mu}\right) \tag{11}$$

$$W^{\mu\nu}_s \equiv \frac{1}{2}\left(W^{\mu\nu} + W^{\nu\mu}\right) \tag{12}$$

$$W^{\mu\nu}_a \equiv \frac{1}{2}\left(W^{\mu\nu} - W^{\nu\mu}\right) \tag{13}$$

with

$$\chi^{\mu\nu} = \chi^{\mu\nu}_s + \chi^{\mu\nu}_a \tag{14}$$

$$W^{\mu\nu} = W^{\mu\nu}_s + W^{\mu\nu}_a. \tag{15}$$

Clearly one has the individual continuity equation relationships

$$Q_\mu W^{\mu\nu}_s = Q_\mu W^{\mu\nu}_a = 0, \tag{16}$$

and also only symmetric (anti-symmetric) leptonic tensors will contract with symmetric (anti-symmetric) hadronic tensors when forming the cross section, the last going as

$$\chi_{\mu\nu} W^{\mu\nu} = \chi^s_{\mu\nu} W^{\mu\nu}_s + \chi^a_{\mu\nu} W^{\mu\nu}_a. \tag{17}$$

We also have from Eqs. (4) and (5) that

$$\chi^{\mu\nu}_s = \mathrm{Re}\chi^{\mu\nu} \tag{18}$$

$$\chi^{\mu\nu}_a = i\mathrm{Im}\chi^{\mu\nu} \tag{19}$$

$$W^{\mu\nu}_s = \mathrm{Re}W^{\mu\nu} \tag{20}$$

$$W^{\mu\nu}_a = i\mathrm{Im}W^{\mu\nu}; \tag{21}$$

we shall make use of this when constructing explicit forms for the tensors by including the factor i in the appropriate places.

Furthermore, one can isolate contributions that contain the target spin from those that do not by forming the unpolarized (spin sum) terms and polarized (spin difference) terms, so that the total becomes

$$\chi^{\mu\nu}_{s,a} = \left(\chi^{\mu\nu}_{s,a}\right)_{unpol} + \left(\chi^{\mu\nu}_{s,a}\right)_{pol} \tag{22}$$

$$W^{\mu\nu}_{s,a} = \left(W^{\mu\nu}_{s,a}\right)_{unpol} + \left(W^{\mu\nu}_{s,a}\right)_{pol} \tag{23}$$

with all four contributions individually satisfying the continuity equation constraint:

$$Q_\mu \left(\chi^{\mu\nu}_{s,a}\right)_{unpol} = Q_\mu \left(\chi^{\mu\nu}_{s,a}\right)_{pol} = 0 \tag{24}$$

$$Q_\mu \left(W^{\mu\nu}_{s,a}\right)_{unpol} = Q_\mu \left(W^{\mu\nu}_{s,a}\right)_{pol} = 0. \tag{25}$$

When only the incident electrons may be polarized but the scattered electron's polarization is assumed not to be measured one can show that the leptonic tensor contributions that do not involve the electron polarization are only symmetric, while those that do involve the electron polarization are only anti-symmetric (see [2]).

The incident electron has 4-momentum $K^\mu = (\epsilon, \mathbf{k})$, the scattered electron has 4-momentum $K'^\mu = (\epsilon', \mathbf{k}')$ and $Q^\mu = K^\mu - K'^\mu$. We shall adopt the convention where $\mathbf{q}$ points along the 3-direction, so that the 4-vector momentum transfer is

$$Q^\mu = (\omega, 0, 0, q) \tag{26}$$

with energy transfer $\omega = \nu$ (the former is commonly employed in nuclear physics while the latter is almost always chosen for use in particle physics; we use the two interchangeably) and 3-momentum transfer $q = |\mathbf{q}|$. One can show that for electron scattering the 4-momentum transfer must be spacelike:

$$Q^2 = \omega^2 - q^2 = -4kk' \sin^2 \theta_e/2 \leq 0; \tag{27}$$

here we invoke the Extreme Relativistic Limit (ERL_e) for the electron, *i.e.,* we take the electron mass m_e to be much smaller than ϵ and ϵ' (the situation where the mass terms are not ignored is discussed in [2]). For convenience later we also define

$$\nu' \equiv \omega/q = \nu/q \tag{28}$$

$$\rho \equiv |Q^2/q^2| = 1 - \nu'^2 \tag{29}$$

with $0 \le \nu' \le 1$ and $0 \le \rho \le 1$.

We quote the standard results for the leptonic coefficients in the ERL$_e$; their derivation can be found *e.g.* in [3]. By convention, upon removing a common factor v_0 from the leptonic tensor given above, where

$$v_0 \equiv (\epsilon + \epsilon')^2 - q^2 = 4kk' \cos^2\theta_e/2, \tag{30}$$

we are left with the following six leptonic ERL$_e$ "Rosenbluth" factors

$$v_L = \rho^2 \equiv \left(\frac{-Q^2}{q^2}\right)^2 \tag{31}$$

$$v_T = \frac{1}{2}\rho + \tan^2\theta_e/2 \tag{32}$$

$$v_{TT} = -\frac{1}{2}\rho \tag{33}$$

$$v_{TL} = -\frac{1}{\sqrt{2}}\rho\sqrt{\rho + \tan^2\theta_e/2} \tag{34}$$

$$v_{T'} = \tan\theta_e/2\sqrt{\rho + \tan^2\theta_e/2} \tag{35}$$

$$v_{TL'} = -\frac{1}{\sqrt{2}}\rho\tan\theta_e/2. \tag{36}$$

The first four, v_L, v_T, v_{TT} and v_{TL}, arise from the symmetric part of the leptonic tensor, while $v_{T'}$ and $v_{TL'}$ stem from its anti-symmetric part. Note that in the general case there are nine such factors; see [3] for discussions of the general case. In the present study we assume that only the incident electron may be polarized and we invoke the ERL$_e$ throughout. Similar labelling conventions prove to be useful for the hadronic tensor in many applications and we shall employ them throughout this study:

$$\begin{aligned}
W^{L} &\equiv \left(W_{fi}^{00}\right)_s & (37)\\
W^{T} &\equiv \left(W_{fi}^{22}\right)_s + \left(W_{fi}^{11}\right)_s & (38)\\
W^{TT} &\equiv \left(W_{fi}^{22}\right)_s - \left(W_{fi}^{11}\right)_s & (39)\\
W^{TL} &\equiv 2\sqrt{2}\left(W_{fi}^{01}\right)_s = 2\sqrt{2}\mathrm{Re}W_{fi}^{01} & (40)\\
W^{T'} &\equiv 2i\left(W_{fi}^{12}\right)_a = -2\mathrm{Im}W_{fi}^{12} & (41)\\
W^{TL'} &\equiv 2\sqrt{2}i\left(W_{fi}^{02}\right)_a = -2\sqrt{2}\mathrm{Im}W_{fi}^{02} & (42)\\
W^{\underline{TT}} &\equiv 2\left(W_{fi}^{12}\right)_s = 2\mathrm{Re}W_{fi}^{12} & (43)\\
W^{\underline{TL}} &\equiv 2\sqrt{2}\left(W_{fi}^{02}\right)_s = 2\sqrt{2}\mathrm{Re}W_{fi}^{02} & (44)\\
W^{\underline{TL}'} &\equiv -2\sqrt{2}i\left(W_{fi}^{01}\right)_a = 2\sqrt{2}\mathrm{Im}W_{fi}^{01}. & (45)
\end{aligned}$$

As in the cited work the notation here is the following: the quantities labelled L refer to contributions involving the $\mu\nu = 00$ parts of the tensors; those labelled T, TT, T' and $\underline{TT}$ involve only transverse components of the tensors; and those labelled TL, TL', $\underline{TL}$ and $\underline{TL}'$ involve interferences having real or imaginary parts of the $\mu\nu = 01$ and 02 components of the tensors. Unprimed quantities arise from symmetric tensors, *viz.,* those that do not involve polarized electrons, whereas those with primes only occur when electron polarizations enter. The underlined quantities labelled $\underline{TT}$ and $\underline{TL}$ occur only when the electron beam is polarized *and* the polarization of the scattered electron is measured (see [2]); since we will not consider this situation in the present study, these contributions are henceforth dropped. Finally, the sector labelled $\underline{TL}'$ does occur when only the electron beam is polarized, although at high energies these can also safely be ignored since they go as $1/\gamma$ where γ is the usual ratio of energy to mass for the electron and thus are also neglected in the present work, leaving 6 classes of response. Accordingly, for the situation of interest in the present study the full contraction of the leptonic and hadronic tensors may then be written in terms of these real quantities, 4 involving symmetric contributions and 2 involving anti-symmetric contributions:

$$\begin{aligned}\mathcal{C} &= v_0\left[\left(v_L W^L + v_T W^T + v_{TL} W^{TL} + v_{TT} W^{TT}\right)\right.\\ &\left. +h\left(v_{T'} W^{T'} + v_{TL'} W^{TL'}\right)\right], \end{aligned} \tag{46}$$

where $\mathcal{C}$ is a Lorentz invariant. Here h is the incident electron's helicity. We note that, while the entire right-hand side of the equation forms a Lorentz invariant, the individual factors are all frame-dependent.

One may also re-write the leptonic factors in a way that involves the so-called photon longitudinal polarization. One begins with the transverse term in Eq. (32)

$$v_T = \frac{1}{2}\rho + \tan^2\theta_e/2 \tag{47}$$

$$= \frac{1}{2}\rho\left[1 + \frac{2}{\rho}\tan^2\theta_e/2\right], \tag{48}$$

thereby defining the photon longitudinal polarization

$$\mathcal{E} \equiv \left[1 + \frac{2}{\rho}\tan^2\theta_e/2\right]^{-1}, \tag{49}$$

which implies that

$$\tan^2\theta_e/2 = \frac{\rho}{2}\left(\mathcal{E}^{-1} - 1\right). \tag{50}$$

If one defines the ratios

$$u_X \equiv \frac{v_X}{v_T} \tag{51}$$

with $X = L, T, TT, TL, T'$ and TL' and substitutes in the above equations for v_X for the factor $\tan\theta_e/2$ one finds that

$$\begin{aligned} u_L &= 2\rho\mathcal{E} \\ u_T &= 1 \\ u_{TT} &= -\mathcal{E} \\ u_{TL} &= -\sqrt{\rho}\sqrt{\mathcal{E}\left(1+\mathcal{E}\right)} \\ u_{T'} &= \sqrt{1-\mathcal{E}^2} \\ u_{TL'} &= -\sqrt{\rho}\sqrt{\mathcal{E}\left(1-\mathcal{E}\right)}. \end{aligned} \tag{52}$$

The invariant in Eq. (46) in this notation in the ERL$_e$ then becomes

$$\begin{aligned} \mathcal{C} &= v_0 v_T \left[\left(2\rho\mathcal{E}W^L + W^T - \mathcal{E}W^{TT} - \sqrt{\rho}\sqrt{\mathcal{E}\,(1+\mathcal{E})}W^{TL} \right) \right. \\ &\quad \left. + h \left(\sqrt{1-\mathcal{E}^2}W^{T'} - \sqrt{\rho}\sqrt{\mathcal{E}\,(1-\mathcal{E})}W^{TL'} \right) \right]. \end{aligned} \tag{53}$$

Equations (46) and (53) for electron scattering in general have the following properties: the entire expressions for $\mathcal{C}$ are Lorentz invariant; however, the factors on the right-hand sides of the equations are not, but are all frame-dependent. That is, v_0, the "Rosenbluth" factors v_X, with $X = L, T, \cdots$, *etc.*, and the response functions W^X all depend on the chosen reference frame — clearly the last, since they involve particular Lorentz components of the hadronic tensor which are frame dependent. If one wants to relate these in one frame to another then they must be Lorentz-transformed (*i.e.,* with respect to both Lorentz indices in $W^{\mu\nu}$). Specific cases exist where this transformation is relatively simple, such as between the target rest frame and the target-virtual photon center-of-momentum frame where the boost is along the **q**-direction (see, for instance, [3]). However, while certainly possible, it is not so simple when, for instance, going between the rest frame and a general collider frame involving crossed beams.

The goal of the present work is to review an alternative set of developments where, instead of frame-dependent response functions, the results are expressed in terms of **invariant** hadronic response functions multiplied by general kinematic factors which are frame-dependent. This will allow universal invariant response functions from measurement in different frames to be compared. Again we emphasize the fact that the results in Eqs. (46) and (53) are not wrong, since the overall expressions are Lorentz invariant, only that the hadronic responses that would be deduced upon analyzing measurements in two different frames would not be the same, that is, in general $[W^{\mu\nu}]_{frame1} \neq [W^{\mu\nu}]_{frame2}$

2. Hadronic Tensors and Invariant Response Functions

In this section we review the past developments of the use of invariant response functions in semi-leptonic electroweak interactions. The general strategy

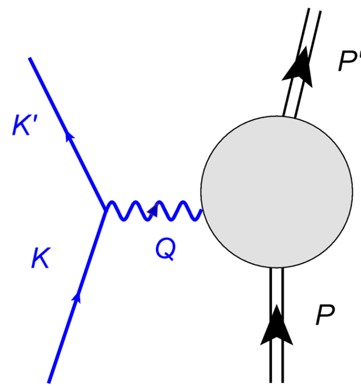

Figure 1: Feynman diagram for inclusive electron scattering. The 4-momenta here are discussed in the text.

followed here is to use the basic 4-vectors that enter for a specific choice of conditions and to write the hadronic tensor involved as a linear combination of all allowable contributions built from those 4-vectors that can be formed multiplied by invariant response functions. We begin with a review of the simplest case of inclusive scattering of unpolarized electrons from unpolarized targets where the procedures are easiest to follow and then successively turn to more complicated situations.

2.1. Unpolarized Inclusive Electron Scattering

Here no final-state particles are assumed to be detected and both the electrons and the target are assumed to be unpolarized. This case was developed in the early 1960s [5], [6], [7] (see also [8]) and provides the prototype for all other cases to follow. The hadronic vertex has 4-momentum Q^μ incoming via the exchanged virtual photon, together with P^μ, the 4-momentum of the target, and P'^μ, the undetected 4-momentum of the final state, corresponding to the reaction being assumed to be inclusive (see Fig. 1). Conservation of 4-momentum allows the last to be eliminated, *viz.*

$$P'^\mu = P^\mu + Q^\mu. \tag{54}$$

The Lorentz scalars for this reaction are those that can be built from the two independent 4-vectors (Q^μ, P^μ), namely, Q^2, $Q \cdot P$ and P^2, and since $P^2 = M^2$, where M is the mass of the target, one has only two independent Lorentz scalars upon which the invariant response functions can depend. These may be chosen to be Q^2 and $Q \cdot P$, or alternatively, one may define a Lorentz invariant version of the Bjorken x-variable via

$$x \equiv \frac{|Q^2|}{2Q \cdot P}, \tag{55}$$

where x reverts to the usual definition in the target rest frame, *viz.*, $x_R = |Q^2|/2M\nu$, and then use as Lorentz scalar variables (Q^2, x).

In building the general form for the hadronic tensor in this case it is convenient to employ instead of P^μ the projected 4-vector

$$U^\mu \equiv \frac{1}{M}\left(P^\mu - \left(\frac{Q \cdot P}{Q^2}\right) Q^\mu\right), \tag{56}$$

where by construction

$$Q \cdot U = 0 \tag{57}$$

$$U^2 = 1 - \frac{(Q \cdot P)^2}{M^2 Q^2}. \tag{58}$$

This strategy will be used in the later more complicated cases and will be seen to greatly simplify the developments there. As we have seen in Sec. 1, when the electrons are unpolarized only symmetric tensors enter and hence for this inclusive unpolarized case one can build the most general symmetric second-rank tensor:

$$\begin{aligned}(W_s^{\mu\nu})^{incl}_{unpol} &= X_1 g^{\mu\nu} + X_2 Q^\mu Q^\nu \\ &\quad + X_3 U^\mu U^\nu + X_4 \left(Q^\mu U^\nu + U^\mu Q^\nu\right),\end{aligned} \tag{59}$$

where general Lorentz invariant response functions $X_{1,2,3,4}$ have been introduced; each is a function of the two Lorentz scalars discussed above, say, (Q^2, $Q \cdot P$). Since the electromagnetic current is conserved one has in momentum space that

$$Q_\mu \, (W_s^{\mu\nu})^{incl}_{unpol} = 0 \tag{60}$$

and hence that

$$\left(X_1 + X_2 Q^2\right) Q^\nu + \left(X_4 Q^2\right) U^\nu = 0. \tag{61}$$

and since Q^ν and U^ν are linearly independent 4-vectors this means that

$$X_1 + X_2 Q^2 = X_4 = 0, \tag{62}$$

leaving

$$(W_s^{\mu\nu})^{incl}_{unpol} = -\left(W_1\right)^{incl}\left(g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}\right) + \left(W_2\right)^{incl} U^\mu U^\nu, \tag{63}$$

where we follow standard convention and define $X_1 \equiv -\left(W_1\right)^{incl}$ and $X_3 \equiv \left(W_2\right)^{incl}$. Here the motivation for including the factor M in Eq. (56) becomes clear: all four of the tensors above have the same dimensions. As long as Lorentz invariant scalars are used for the arguments of the *two* invariant response functions $\left(W_{1,2}\right)^{incl}$ these factors in Eq. (63) are Lorentz invariant, *viz.,* do not depend on the particular frame of reference involved in a specific situation. Hence $\left(W_{1,2}\right)^{incl}$ determined in the rest frame, the CM frame, the Breit frame, or in any specific collider frame are all identical, in contrast to the response functions $W^{L,T,\cdots}$ introduced in Sec. 1 which are frame-dependent.

Note that if one wishes to use as response functions $F_{1,2}$ as is common in the high-energy regime (HER), then to maintain their Lorentz invariant properties one should use the definitions

$$\left(F_1\right)^{incl} \equiv M \left(W_1\right)^{incl} \tag{64}$$

$$\left(F_2\right)^{incl} \equiv \frac{Q \cdot P}{M} \left(W_2\right)^{incl}, \tag{65}$$

and treat these as functions of the Lorentz scalars Q^2 and x in Eq. (55).

While the focus in this work is on parity-conserving electron scattering we note that when studying the full electroweak interaction even for inclusive scattering that additional contributions enter since then one has both polar- and axial-vector currents, the latter not being conserved. For instance, see [9], [10], [11] for discussions of neutrino reactions, and [9], [10], [12], [13] for discussions of parity-violating electron scattering.

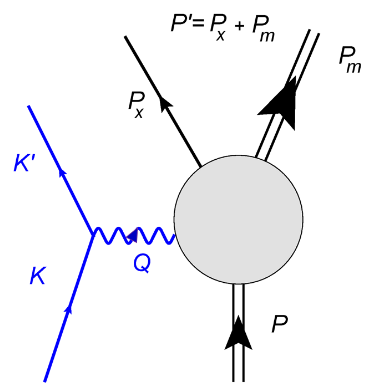

Figure 2: Feynman diagram for semi-inclusive electron scattering. The 4-momenta here are discussed in the text. In particular, particle x is assumed to be detected in coincidence with the scattered electron and thus P_x^μ is assumed to be known. Since the total final-state momentum P'^μ is known (see Fig. 1 for inclusive scattering) this implies that the missing 4-momentum is also known via the relationship $P_m^\mu = P'^\mu - P_x^\mu$.

2.2. Semi-inclusive Scattering from Unpolarized Targets

Semi-inclusive electron scattering entails reactions of the sort $(e, e'x)$, $(\overrightarrow{e}, e'x)$, $(e, e'\overrightarrow{x})$ and $(\overrightarrow{e}, e'\overrightarrow{x})$, where the scattered electron and some other particle x are assumed to be detected in coincidence in the final state (see Fig. 2). The incident electron may be polarized, the detected particle (if it is not spin-0) may have its polarization measured, or both. Let us begin with the cases where the polarization of particle x is assumed not to be measured.

The strategy reviewed in the previous section is easily generalized. We follow the developments presented in [14] and employ the same conventions as in that work. Now one has three 4-vectors upon which to build the hadronic tensor, namely, (Q^μ, P^μ, P_x^μ), where the 4-momentum of particle x is $P_x^\mu = (E_x, \mathbf{p}_x)$. As above the invariant response functions are functions of the available Lorentz scalars, namely, the four quantities (Q^2, $Q \cdot P$, $Q \cdot P_x$, $P \cdot P_x$) having eliminated the other two possibilities via their (assumed known) masses: $P^2 = M^2$ and $P_x^2 = M_x^2$, where M_x is the mass of particle x. Again it proves convenient to use projected 4-vectors: we use U^μ rather than P^μ as above and introduce

$$V^\mu \equiv \frac{1}{M}\left(P_x^\mu - \left(\frac{Q\cdot P_x}{Q^2}\right)Q^\mu\right). \tag{66}$$

By construction, one has

$$Q\cdot U = Q\cdot V = 0 \tag{67}$$

and

$$\begin{aligned}
U^2 &= 1 - \frac{(Q\cdot P)^2}{M^2 Q^2} && (68)\\
V^2 &= \frac{1}{M^2}\left(M_x^2 - \frac{(Q\cdot P_x)^2}{Q^2}\right) && (69)\\
U\cdot V &= \frac{1}{M^2}\left(P\cdot P_x - \frac{(Q\cdot P)(Q\cdot P_x)}{Q^2}\right). && (70)
\end{aligned}$$

Note that we have chosen to use the target mass M above and not the mass of particle x, namely M_x, since we want to allow the latter to be general enough to include the photon. Furthermore, we can replace V^μ with a 4-vector that is orthogonal not only to Q^μ but to U^μ as well:

$$X^\mu \equiv V^\mu - \left(\frac{U\cdot V}{U^2}\right)U^\mu, \tag{71}$$

where then

$$Q\cdot U = Q\cdot X = U\cdot X = 0 \tag{72}$$

and

$$X^2 = V^2 - \frac{(U\cdot V)^2}{U^2}. \tag{73}$$

Given the above 4-vector building blocks, we now proceed to construct second-rank hadronic tensors with the appropriate symmetries. We begin with the symmetric cases where no target polarization is involved.

$$W_{1,s}^{\mu\nu} \equiv g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2} \tag{74}$$

$$W_{2,s}^{\mu\nu} \equiv U^\mu U^\nu \tag{75}$$

$$W_{3,s}^{\mu\nu} \equiv X^\mu X^\nu \tag{76}$$

$$W_{4,s}^{\mu\nu} \equiv U^\mu X^\nu + X^\mu U^\nu. \tag{77}$$

Here the motivation for including the factors M becomes clear: all four of the tensors above have the same dimensions. We have the following upon contracting with Q_μ:

$$Q_\mu W^{\mu\nu}_{m,s} = 0 \tag{78}$$

for $m = 1, 2, 3, 4$. The general tensor of this type is obtained by summing over the 4 contributions, where each is multiplied by a Lorentz scalar, invariant response function, A_m, that in turn depends only on the four Lorentz scalars in the problem, namely

$$(W^{\mu\nu}_s)^{semi-1}_{unpol} = \sum_{m=1}^{4} A_m W^{\mu\nu}_{m,s} \tag{79}$$

and as above one has

$$Q_\mu \, (W^{\mu\nu}_s)^{semi-1}_{unpol} = 0 \tag{80}$$

as required for the overall symmetric, unpolarized tensor by the continuity equation. The notation "semi-1" is used to denote that one particle is assumed to be detected in coincidence with the scattered electron. Thus the symmetric, unpolarized second-rank hadronic tensor may then be written

$$\begin{aligned} (W^{\mu\nu}_s)^{semi-1}_{unpol} &= -\left(W_1\right)^{semi-1}\left(g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}\right) + \left(W_2\right)^{semi-1} U^\mu U^\nu \\ &\quad + \left(W_3\right)^{semi-1} X^\mu X^\nu + \left(W_4\right)^{semi-1} \left(U^\mu X^\nu + X^\mu U^\nu\right), \end{aligned} \tag{81}$$

namely, with *four* contributions involving invariant functions $(W_m)^{semi-1}$, $m = 1, 2, 3, 4$ (here we have shifted from using invariant functions A_m to more familiar notation, including the minus sign in the $(W_1)^{semi-1}$ case, which is conventional).

Additionally in the semi-inclusive case with no hadronic polarizations one can now have an anti-symmetric tensor; there is only one anti-symmetric contribution that uses Q^μ, U^μ and X^ν as a basis [14], namely

$$W^{\mu\nu}_{1,a} \equiv i(U^\mu X^\nu - X^\mu U^\nu), \tag{82}$$

where here and below the factor i has been included following Eq. (21). Contracting the valid anti-symmetric tensor with Q_μ yields zero and we find that the anti-symmetric, unpolarized tensor is constructed from the *single* basis tensor of the correct type with an invariant functions here called $(W_5)^{semi-1}$:

$$(W_a^{\mu\nu})_{unpol}^{semi-1} = i\,(W_5)^{semi-1}\,(U^\mu X^\nu - X^\mu U^\nu)\,, \tag{83}$$

namely the so-called 5th response (see, for instance, [14] and earlier references therein).

This semi-inclusive analysis has been extended to neutrino reactions which entails dealing with both polar- and axial-vector currents [15].

Without providing the details let us note that in past work [14] cases have been studied where two particles x_1 and x_2 are assumed to be detected in coincidence with the scattered electron (denoted "semi-2"), but where the polarizations of those particles are assumed not to be measured, namely, for the reactions $(e, e'x_1x_2)$ and $(\overrightarrow{e}, e'x_1x_2)$. In analogy with the semi-1 situation 4-vectors for the two final-state hadrons are involved, $P^\mu_{x_1}$ and $P^\mu_{x_2}$, and one must work with a set of four 4-vectors, $\left(Q^\mu, P^\mu, P^\mu_{x_1}, P^\mu_{x_2}\right)$. Now the set of dynamical Lorentz scalars is $(Q^2,\, Q\cdot P,\, Q\cdot P_{x_1},\, Q\cdot P_{x_2},\, P\cdot P_{x_1},\, P\cdot P_{x_2},\, P_{x_1}\cdot P_{x_2})$, namely, all invariant response functions are now functions of seven dynamical Lorentz scalars for this semi-2 situation. As above one can form the corresponding projected analogs of Eq. (66) denoted V_1^μ and V_2^μ defined such that $Q\cdot V_{1,2} = 0$, and then orthogonalize the complete set of four 4-vectors following a similar procedure to that outlined above. Finally, one can proceed to write the semi-2 symmetric and anti-symmetric hadronic tensors as above and employ the continuity equation to arrive at the extensions of Eqs. (81) and (83); the details are omitted here and the reader is directed to [14] for the full analysis.

Note also that, if one wishes to proceed to reactions with three or more particles in coincidence with the scattered electron, for example, $(e, e'x_1x_2x_3\cdots)$ and $(\overrightarrow{e}, e'x_1x_2x_3\cdots)$, a change in the logic occurs, namely, at the level of semi-2 one already has four independent 4-vectors $\left(Q^\mu, P^\mu, P^\mu_{x_1}, P^\mu_{x_2}\right)$ and accordingly any additional 4-vectors may be written as linear combinations of these four, for example,

$$P_{x_3}^{\mu} = aQ^{\mu} + bP^{\mu} + cP_{x_1}^{\mu} + dP_{x_2}^{\mu}, \tag{84}$$

where a, b, c, d are Lorentz scalars. That is, any set of four linearly independent 4-vectors spans the 4D space. Hence, the form of the semi-2 cross section is the most general, although each invariant response function must depend on the complete set of Lorentz scalars for each case semi-2, semi-3, *etc.* (see [14] for more detail on this and other issues).

Finally, in reviewing the past developments of the various hadronic tensors in terms of invariant response functions let us note that the cases of $(e, e'\overrightarrow{x})$ and $(\overrightarrow{e}, e'\overrightarrow{x})$ reactions have been presented in [16].

2.3. Semi-inclusive Scattering from Polarized Spin-1/2 Targets

In this section we proceed to summarize the procedures for building the most general tensors for semi-inclusive electron scattering from polarized spin-1/2 targets written in terms of invariant response functions. Later, in the following section, we will connect this approach with the more familiar one where frame-dependent responses are involved. This situation is likely to be one of the most relevant in future experimental studies and accordingly we provide a detailed summary of the procedures involved. In particular, we employ notation that is consistent with that used in previous studies where cross sections for electron scattering from unpolarized targets have been developed, namely, those summarized above. The simpler unpolarized-target cases may then straightforwardly be recovered from the more general results. At the end of the section we briefly discuss alternative schemes.

The process is shown schematically in Fig. 3 (see also Fig. 2). That is, we consider reactions of the type $\overrightarrow{e} + \overrightarrow{A}(1/2) \to e' + x + B$ where the incident electron may be polarized, the spin-1/2 target A may be polarized and where we assume that, in addition to the scattered electron, some (unpolarized) particle x is detected in coincidence. The sum of all open channels that make up the final state is denoted B and is assumed not to be detected. Employing notation commonly used in nuclear physics the reaction may be written $\overrightarrow{A}(1/2)(\overrightarrow{e}, e'x)B$. We shall

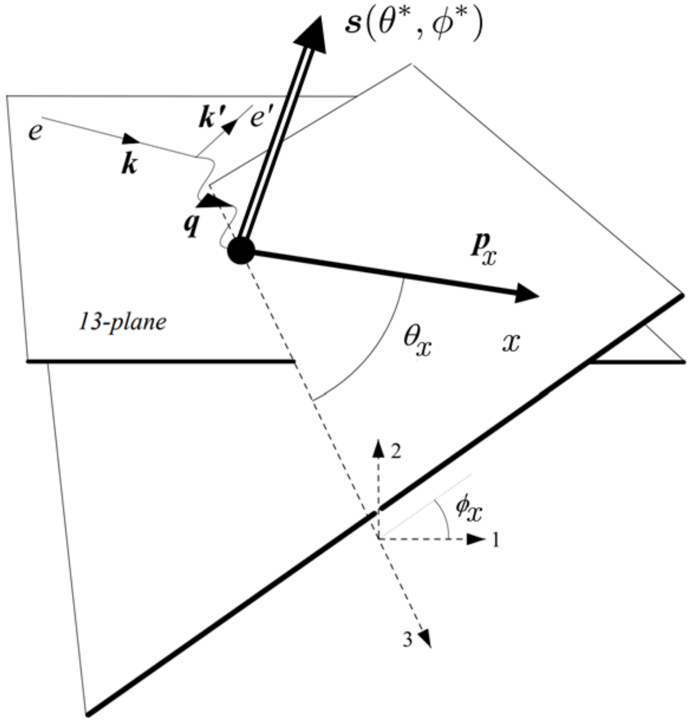

Figure 3: Schematic representation of semi-inclusive electron scattering. The coordinate system is chosen such that the electron scattering occurs in the 13-plane and has the 3-momentum transfer along the 3-axis. The particle x detected in coincidence with the scattered electron has 3-momentum $\mathbf{p}_x$ which lies in a plane in general inclined at an azimuthal angle ϕ_x with respect to the electron scattering plane and has polar angle θ_x with respect to $\mathbf{q}$. The polarization of the spin-1/2 target involves the spin 3-vector $\mathbf{s}$ with polar and azimuthal angles θ^* and ϕ^*, respectively, in the chosen coordinate system.

discuss how such semi-inclusive reactions are related to the inclusive cross section, *i.e.,* for reactions of the type $\overrightarrow{e} + \overrightarrow{A}(1/2) \rightarrow e' + X$ or $\overrightarrow{A}(1/2)(\overrightarrow{e}, e')X$, where X denotes the complete (undetected) final state. The formalism is developed in a general coordinate system as one wishes to be able to relate the response in different frames of reference, in particular, in the target rest frame and in a frame where the incident electrons and the spin-1/2 target are both moving and colliding.

The developments summarized here are general and intended for use at any energy scale and for studies both of particle and nuclear physics. For instance, past and ongoing studies involve fixed-target (target rest frame) measurements say at SLAC or JLab or other fixed-target facilities as well as at colliders. The energies involved in the former are typically 10s of GeV or lower, while the latter range up to quite high energies — the high-energy regime (HER). In the future one anticipates studies at the EIC collider facility where electrons at 10s of GeV will be collided with hadronic targets at 100s of GeV. In fact, where polarized electrons are to be scattered from polarized spin-1/2 targets in that facility it is anticipated that both polarized protons and polarized ^{3}He nuclei will be employed. Accordingly one is motivated to develop the formalism for general semi-inclusive scattering of polarized electrons from polarized spin-1/2 targets in a covariant way through the use of general invariant hadronic response functions for use in all situations at all energies.

Two examples where the ideas are relevant, one from particle physics and one from nuclear physics are the following. For the former consider charged pion production from a polarized proton target (see, for instance, [17] or [3] and references therein). For single-pion production one then has the (exclusive) reaction $\overrightarrow{e} + \overrightarrow{p} \rightarrow e' + n + \pi^+$ with a neutron and a positive pion in the final state. As a semi-inclusive reaction one then has either $\overrightarrow{p}(\overrightarrow{e}, e'n)\pi^+$ where particle x is a neutron and the pion is undetected or $\overrightarrow{p}(\overrightarrow{e}, e'\pi^+)n$ where particle x is a π^+ and the neutron is undetected. In fact these are the same reaction and accordingly they constitute a single channel. Clearly there are experimental considerations involved in which particle is the one detected in coincidence;

however, theoretically they are indistinguishable. For higher-energy kinematics one reaches a threshold where additional channels open. For instance, once the relevant threshold is reached, two-pion production becomes possible, $\overrightarrow{e} + \overrightarrow{p} \rightarrow e' + n + \pi^+ + \pi^0$ and then $\overrightarrow{e} + \overrightarrow{p} \rightarrow e' + p + \pi^- + \pi^+$, and so on, with more and more particles in the final state. Of those a given semi-inclusive reaction is to be taken as having some given particle detected in coincidence with the scattered electron and all other particles undetected.

A second example, taken from nuclear physics, is where the polarized electron is scattered from a polarized ^{3}He target. Let us focus on the reaction $^3\overrightarrow{\mathrm{He}}(\overrightarrow{e}, e'p)$ where a proton is assumed to be detected in coincidence with the scattered electron. The unobserved part of the final state depends on the specific kinematics of the reaction. At threshold one has the (exclusive) two-body reaction $\overrightarrow{e} +^3 \overrightarrow{\mathrm{He}} \rightarrow e' + p + d$ and then for slightly higher missing energies the three-body breakup reaction $\overrightarrow{e} +^3 \overrightarrow{\mathrm{He}} \rightarrow e' + p + p + n$. Alternatively one could have a neutron as the particle detected in coincidence with the scattered electron, $^3\overrightarrow{\mathrm{He}}(\overrightarrow{e}, e'n)$. In this case the two-body channel does not occur, although the three-body breakup channel does. In fact, for the latter the final state is the same and this will have consequences later when we discuss the issue of avoiding double counting. As in the particle physics example above, when the energy increases a threshold is reached where pion production can occur and the final state becomes even more complicated. Nevertheless, the semi-inclusive reaction is well defined: the point is that a specific particle is assumed to be the one called "x", namely, the one that is detected, whereas all other particles in the final state must be summed while avoiding double counting.

Before providing a review of the detailed formalism involved with this class of reactions let us anticipate a few of the salient features that will emerge.

- We shall see that there are four sectors which may be separated by employing the polarizations. When unpolarized electrons are involved only symmetric tensors enter, whereas when the incident electrons are longitudinally polarized only anti-symmetric tensors occur.

- The four types of polarization (electrons polarized or not with target polarized or not) may be separated using those polarizations. We shall see that there are four symmetric invariant responses for the fully unpolarized case, one anti-symmetric invariant response when the electron is polarized but the target is not, eight symmetric invariant responses when the electron is unpolarized but the target is polarized, and five anti-symmetric invariant responses when both electrons and target are polarized.

- These 18 invariant response functions will be shown to be functions of four Lorentz scalar invariants. The 18 responses may be sub-divided into two sets of nine according to their properties under parity and time-reversal; these two sets typically behave quite differently.

- We also detail how the hadronic response may be characterized using the helicity decomposition of the virtual photon to label the various contributions. We shall detail how this representation relates to the decomposition in terms of invariant response functions.

- A prime motivation for such studies is to have the semi-inclusive cross section written in a completely general frame of reference. This then allows one to relate the results in (say) the collider frame to the target rest frame, or to relate the results in the rest frame to those in the photon-target center-of-momentum frame. This can prove to be essential when models are being developed for the hadronic physics that are non-relativistic and hence cannot be boosted — polarized ^{3}He would be one such example — since only in the target rest frame will such models make sense.

- Finally, we review how inclusive (polarized) scattering emerges via specific integrals over semi-inclusive cross sections with appropriate sums over all open channels.

We begin by extending the analysis outlined in the previous sections, providing more detail here since this case is the main focus of the present study.

As shown in Fig. 3, the laboratory system involves the choice where the 3-momentum transfer lies along the 3-direction and the electrons lie in the 13-plane, as discussed in Sec. 1; accordingly, we have the 123-system shown in the figure. Since we want to retain the usual meaning for the leptonic and hadronic factors discussed in Sec. 1, it is important to employ this system for the developments above. In this system we have the following 4-vectors:

$$\begin{aligned} Q^\mu &= (\omega, \mathbf{q}) && (85) \\ P^\mu &= (E_p, \mathbf{p}) && (86) \\ P_x^\mu &= (E_x, \mathbf{p}_x) && (87) \\ S^\mu &= \left(S^0, \mathbf{s}\right) && (88) \end{aligned}$$

with 3-vectors

$$\begin{aligned} \mathbf{q} &= q\mathbf{u}_3 && (89) \\ \mathbf{p} &= p\left(\sin\theta\cos\phi\mathbf{u}_1 + \sin\theta\sin\phi\mathbf{u}_2 + \cos\theta\mathbf{u}_3\right) && (90) \\ \mathbf{p}_x &= p_x\left(\sin\theta_x\cos\phi_x\mathbf{u}_1 + \sin\theta_x\sin\phi_x\mathbf{u}_2 + \cos\theta_x\mathbf{u}_3\right) && (91) \\ \mathbf{s} &= s\left(\sin\theta^*\cos\phi^*\mathbf{u}_1 + \sin\theta^*\sin\phi^*\mathbf{u}_2 + \cos\theta^*\mathbf{u}_3\right). && (92) \end{aligned}$$

The target (mass M) and particle detected in coincidence with the scattered electron (mass M_x) are both on-shell and thus $E_p = \sqrt{p^2 + M^2}$ and $E_x = \sqrt{p_x^2 + M_x^2}$. Note that in Eq. (92) the magnitude s and the angles (θ^*, ϕ^*) are assumed to be in the general frame; they can be related to rest-frame variables by employing rotations and a boost. The target spin 4-vector S^μ may be developed further by exploiting the two conditions it must satisfy, namely

$$P \cdot S = 0 \tag{93}$$

and

$$S^2 = \left(S^0\right)^2 - s^2 = -1, \tag{94}$$

which may be verified by going to the target rest frame. We shall not pursue these developments in the present work, leaving that for another time.

Thus we see that as building blocks we may employ the 4-momentum transfer Q^μ, the target 4-momentum P^μ, the 4-momentum of some particle detected in the final state P_x^μ, and the 4-vector that characterizes the target spin, S^μ. As usual, it is convenient to replace the last three with projected 4-vectors, *i.e.,* vectors that are by construction orthogonal to Q^μ. When the spin is not involved the analysis is the one presented in the previous section involving the 4-vectors (Q^μ, P^μ, P_x^μ) and therefore (Q^μ, U^μ, X^μ) with the constraints $Q \cdot U = Q \cdot V = Q \cdot X = U \cdot X = 0$. Note that we can also define a fourth 4-vector via

$$D^\mu \equiv \frac{1}{M}\varepsilon^{\mu\alpha\beta\gamma}Q_\alpha U_\beta X_\gamma = \frac{1}{M^3}\varepsilon^{\mu\alpha\beta\gamma}Q_\alpha P_\beta P_{x\gamma} \tag{95}$$

which is dual to the above set, behaves as an axial-vector and satisfies $Q \cdot D = U \cdot D = X \cdot D = 0$. As above we have the four dynamical Lorentz scalars $(Q^2,\, Q \cdot P,\, Q \cdot P_x,\, P \cdot P_x)$ as arguments of the invariant response functions, or, equivalently we may define the following dimensionless invariants

$$I_p \equiv \frac{Q \cdot P}{Q^2} \tag{96}$$

$$I_x \equiv \frac{Q \cdot P_x}{Q^2} \tag{97}$$

$$I_{pp_x} \equiv \frac{P \cdot P_x}{Q^2}, \tag{98}$$

and alternatively employ the following four Lorentz scalars as arguments of the invariant response functions, $(Q^2, I_p, I_x, I_{pp_x})$.

When the spin is involved we then have the 4-vector Σ^μ

$$\Sigma^\mu \equiv S^\mu - I_s Q^\mu, \tag{99}$$

where

$$I_s \equiv \frac{Q \cdot S}{Q^2}\cdot \tag{100}$$

satisfying the constraint $Q \cdot \Sigma = 0$. Note that the spin 4-vector does not enter as a dynamical Lorentz scalar since it occurs as part of the projection operator

$$\mathcal{P}_{spin} \equiv \frac{1}{2}\left(1 + \gamma_5 \gamma_\mu S^\mu\right) \tag{101}$$

and either does not enter (unpolarized) or occurs explicitly (polarized) where, being part of the projection operator, it only enters linearly. Since $P \cdot S = 0$ and $S^2 = -1$, we have

$$U \cdot \Sigma = -\frac{(Q \cdot P)\,(Q \cdot S)}{MQ^2} \tag{102}$$

and

$$\Sigma^2 = -\left[1 + \frac{(Q \cdot S)^2}{Q^2}\right]. \tag{103}$$

We can also define two 4-vectors that contain the spin 4-vector linearly and are dual to specific combinations of the others, namely,

$$\overline{X}^\mu \equiv \frac{1}{M}\varepsilon^{\mu\alpha\beta\gamma} S_\alpha Q_\beta U_\gamma = \frac{1}{M^2}\varepsilon^{\mu\alpha\beta\gamma} S_\alpha Q_\beta P_\gamma \tag{104}$$

$$\overline{U}^\mu \equiv \frac{1}{M}\varepsilon^{\mu\alpha\beta\gamma} S_\alpha Q_\beta X_\gamma = \frac{1}{M}\varepsilon^{\mu\alpha\beta\gamma} S_\alpha Q_\beta V_\gamma - \left(\frac{U \cdot V}{U^2}\right)\overline{X}^\mu. \tag{105}$$

One has that

$$Q \cdot \overline{U} \quad = \quad X \cdot \overline{U} = \Sigma \cdot \overline{U} = 0 \tag{106}$$

$$Q \cdot \overline{X} \quad = \quad U \cdot \overline{X} = \Sigma \cdot \overline{X} = 0 \tag{107}$$

and additionally that

$$I_0 \quad \equiv \quad U \cdot \overline{U} = -X \cdot \overline{X} \tag{108}$$

$$= \quad \frac{1}{M}\varepsilon^{\alpha\beta\gamma\delta}\Sigma_\alpha Q_\beta U_\gamma X_\delta \tag{109}$$

$$= \quad \frac{1}{M^3}\varepsilon^{\alpha\beta\gamma\delta} S_\alpha Q_\beta P_\gamma P_{x\delta}, \tag{110}$$

an invariant that depends linearly on the target spin. Note that a tensor of the form

$$\overline{Q}^\mu \equiv \varepsilon^{\mu\alpha\beta\gamma}\Sigma_\alpha U_\beta X_\gamma \tag{111}$$

is redundant, since it can be shown that

$$\overline{Q}^\mu = -\frac{MI_0}{Q^2} Q^\mu \tag{112}$$

where Q^μ will be used instead as a building block.

Next, let us consider the 4-vector $\overline{X}^{\mu}$ defined in Eq. (104). In contracting with $\epsilon^{\mu\alpha\beta\gamma}$ the contributions in Σ_{α} and U_{γ} containing Q_{α} and Q_{γ}, respectively, may be ignored due to the explicit factor Q_{β}, and hence we can write

$$\overline{X}^{\mu} = \frac{1}{M^2}\epsilon^{\mu\alpha\beta\gamma}S_{\alpha}Q_{\beta}P_{\gamma} \tag{113}$$

$$= -\frac{1}{M^2}\left[\omega\epsilon^{\mu 0\alpha\gamma} - q\epsilon^{\mu 3\alpha\gamma}\right]S_{\alpha}P_{\gamma}, \tag{114}$$

the latter expression in the 123-system. If we define the following anti-symmetric tensor

$$F^{\mu\nu} \equiv \frac{1}{M}\left(P^{\mu}S^{\nu} - S^{\mu}P^{\nu}\right) \tag{115}$$

and evaluate the result in Eq. (114) explicitly we find that

$$\overline{X}^{0} = \frac{q}{M}F^{12} \tag{116}$$

$$\overline{X}^{1} = \frac{1}{M}\left(\omega F^{23} + qF^{02}\right) \tag{117}$$

$$\overline{X}^{2} = \frac{1}{M}\left(\omega F^{31} - qF^{01}\right) \tag{118}$$

$$\overline{X}^{3} = \nu'\overline{X}^{0}. \tag{119}$$

Again these may be developed, leading to the following expressions: one can show that one obtains

$$\overline{X}^{i} = \frac{1}{M^2}\left(\left[(\omega\mathbf{p} - E_p\mathbf{q})\times\mathbf{s}\right] + S^0\left(\mathbf{q}\times\mathbf{p}\right)\right)^{i},\ i = 1,2,3 \tag{120}$$

$$\overline{X}^{0} = \frac{1}{\nu'}\overline{X}^{3}. \tag{121}$$

And finally we have the 4-vector $\overline{U}^{\mu}$ defined in Eq. (105)

$$\overline{U}^{\mu} = \frac{1}{M}\epsilon^{\mu\alpha\beta\gamma}S_{\alpha}Q_{\beta}X_{\gamma} \tag{122}$$

$$= \frac{1}{M}\epsilon^{\mu\alpha\beta\gamma}S_{\alpha}Q_{\beta}V_{\gamma} - \left(\frac{U\cdot V}{U^2}\right)\overline{X}^{\mu} \tag{123}$$

$$= \overline{T}^{\mu} - \left(\frac{U\cdot V}{U^2}\right)\overline{X}^{\mu}, \tag{124}$$

where

$$\overline{T}^{\mu} \equiv \frac{1}{M^2}\epsilon^{\mu\alpha\beta\gamma}S_{\alpha}Q_{\beta}P_{x\gamma} \tag{125}$$

$$= -\frac{1}{M^2}\left[\omega\epsilon^{\mu 0\alpha\gamma} - q\epsilon^{\mu 3\alpha\gamma}\right]S_{\alpha}P_{x\gamma}. \tag{126}$$

As above we can define

$$G^{\mu\nu} \equiv X^{\mu}S^{\nu} - S^{\mu}X^{\nu} \tag{127}$$

and find that

$$\begin{aligned}
\overline{U}^0 &= \frac{q}{M}G^{12} && (128)\\
\overline{U}^1 &= \frac{1}{M}\left(\omega G^{23} + qG^{02}\right) && (129)\\
\overline{U}^2 &= \frac{1}{M}\left(\omega G^{31} - qG^{01}\right) && (130)\\
\overline{U}^3 &= \frac{\omega}{M}G^{12} = \nu'\overline{U}^0. && (131)
\end{aligned}$$

We can also develop $\overline{T}^{\mu}$; define

$$F_x^{\mu\nu} \equiv \frac{1}{M}\left(P_x^{\mu}S^{\nu} - S^{\mu}P_x^{\nu}\right) \tag{132}$$

and find that

$$\begin{aligned}
\overline{T}^0 &= \frac{q}{M}F_x^{12} && (133)\\
\overline{T}^1 &= \frac{1}{M}\left(\omega F_x^{23} + qF_x^{02}\right) && (134)\\
\overline{T}^2 &= \frac{1}{M}\left(\omega F_x^{31} - qF_x^{01}\right) && (135)\\
\overline{T}^3 &= \frac{\omega}{M}F_x^{12} = \nu'\overline{T}^0. && (136)
\end{aligned}$$

Again these may be developed leading to the following expressions:

$$\begin{aligned}
\overline{T}^i &= \frac{1}{M^2}\left(\left[(\omega\mathbf{p}_x - E_x\mathbf{q})\times\mathbf{s}\right] + S^0\left(\mathbf{q}\times\mathbf{p}_x\right)\right)^i,\ i = 1,2,3 && (137)\\
\overline{T}^3 &= \nu\overline{T}^0. && (138)
\end{aligned}$$

2.3.1. Second-Rank Tensors: Symmetric, polarized

Let us begin the symmetric polarized developments by starting with a set of symmetric second-rank tensors that starts with the set of 4 symmetric tensors obtained in the unpolarized case, $W_{m,s}^{\mu\nu}$, with $m = 1\cdots 4$ as in Eqs. (74–77), multiplied by I_0, namely

$$\begin{aligned} W'^{\mu\nu}_{1,s} &\equiv \left(g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2} \right) I_0 \\ W'^{\mu\nu}_{2,s} &\equiv (U^\mu U^\nu)\, I_0 \\ W'^{\mu\nu}_{3,s} &\equiv (X^\mu X^\nu)\, I_0 \\ W'^{\mu\nu}_{4,s} &\equiv (U^\mu X^\nu + X^\mu U^\nu)\, I_0. \end{aligned} \tag{139}$$

Here and below the prime is included to denote the fact that the target spin is involved. These all have the desired properties, namely, they behave as vector/vector and are linear in the spin; they all have the same dimensions. Contractions with Q^μ yield zero as above. To these we can add another set built from $\bar{U}^\mu$ and $\overline{X}^\mu$ together with the 4-vectors Q^μ, U^μ and X^μ.

For the remaining building blocks constructed from tensors containing the spin we use

$$W'^{\mu\nu}_{5,s} \equiv U^\mu \bar{U}^\nu + U^\nu \bar{U}^\mu \tag{140}$$

$$W'^{\mu\nu}_{6,s} \equiv U^\mu \overline{X}^\nu + U^\nu \overline{X}^\mu \tag{141}$$

$$W'^{\mu\nu}_{7,s} \equiv X^\mu \bar{U}^\nu + X^\nu \bar{U}^\mu \tag{142}$$

$$W'^{\mu\nu}_{8,s} \equiv X^\mu \overline{X}^\nu + X^\nu \overline{X}^\mu, \tag{143}$$

again with no contributions that are proportional to Q^μ or Q^ν as these would yield zero when contracted with the electron tensor. Again these behave as vector/vector and are linear in the spin and all yield zero when contracted with Q^μ. Accordingly, if we expand the symmetric polarized tensor in this set of basis tensors,

$$(W^{\mu\nu}_s)_{pol} = \sum_{m=1}^{8} A'_m W'^{\mu\nu}_{m,s} \tag{144}$$

with general invariant response functions A'_m, and impose the continuity equation constraint $Q_\mu \left(W_s^{\mu\nu}\right)_{pol} = 0$ we obtain the following:

$$\begin{aligned}
\left(W_s^{\mu\nu}\right)_{pol} &= \Bigg[-W_1'\left(g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}\right) + W_2' U^\mu U^\nu \\
&\quad + W_3' X^\mu X^\nu + W_4'\left(U^\mu X^\nu + X^\mu U^\nu\right)\Big] I_0 \\
&\quad + W_5'\left(U^\mu \bar{U}^\nu + U^\nu \bar{U}^\mu\right) + W_6'\left(U^\mu \overline{X}^\nu + U^\nu \overline{X}^\mu\right) \\
&\quad + W_7'\left(X^\mu \bar{U}^\nu + X^\nu \bar{U}^\mu\right) + W_8'\left(X^\mu \overline{X}^\nu + X^\nu \overline{X}^\mu\right), \qquad (145)
\end{aligned}$$

again shifting from generic invariant functions A_m' to the more conventional notation involving invariant W_m'. Thus, for the symmetric, polarized case we are left with *eight* contributions. All tensors here have the same dimensions and consequently all invariant functions have the same dimensions.

2.3.2. Second-Rank Tensors: Anti-symmetric, polarized

In this sector we begin with a basis tensor that involves the Levi-Civita symbol and is linear in spin:

$$W_{1,a}'^{\mu\nu} \equiv \frac{i}{M}\varepsilon^{\mu\nu\alpha\beta}\Sigma_\alpha Q_\beta. \qquad (146)$$

Note that one has the following identities,

$$Q^2\varepsilon^{\mu\nu\alpha\beta}\Sigma_\alpha X_\beta = M\left(Q^\mu \bar{U}^\nu - Q^\nu \bar{U}^\mu\right) \qquad (147)$$

$$Q^2\varepsilon^{\mu\nu\alpha\beta}\Sigma_\alpha U_\beta = M\left(Q^\mu \overline{X}^\nu - Q^\nu \overline{X}^\mu\right) \qquad (148)$$

and hence no terms having the Levi-Civita symbol as here are needed, since they also yield zero upon contraction with the electron tensor. Since we want tensors that are linear in spin and of vector/vector form we can also have the following tensors:

$$W_{2,a}'^{\mu\nu} \equiv i(U^\mu \bar{U}^\nu - U^\nu \bar{U}^\mu) \qquad (149)$$

$$W_{3,a}'^{\mu\nu} \equiv i(U^\mu \overline{X}^\nu - U^\nu \overline{X}^\mu) \qquad (150)$$

$$W_{4,a}'^{\mu\nu} \equiv i(X^\mu \bar{U}^\nu - X^\nu \bar{U}^\mu) \qquad (151)$$

$$W_{5,a}'^{\mu\nu} \equiv i(X^\mu \overline{X}^\nu - X^\nu \overline{X}^\mu) \qquad (152)$$

with no terms of the form $Q^\mu \bar{U}^\nu - Q^\nu \bar{U}^\mu$ or $Q^\mu \overline{X}^\nu - Q^\nu \overline{X}^\mu$, since, as above, these yield zero when contracted with the lepton tensor. Finally, as in the symmetric case we can use the anti-symmetric contribution above (Eq. (82)) multiplied by the invariant I_0:

$$W'^{\mu\nu}_{6,a} \equiv i(U^\mu X^\nu - X^\mu U^\nu) I_0. \tag{153}$$

however, one can prove the following identity

$$\begin{aligned} -I_0\left(U^\mu X^\nu - U^\nu X^\mu\right) &= \frac{1}{M} U^2 X^2 \epsilon^{\mu\nu\alpha\beta} \Sigma_\alpha Q_\beta + X^2 \left(U^\mu \overline{X}^\nu - U^\nu \overline{X}^\mu\right) \\ &\quad + U^2 \left(X^\mu \overline{U}^\nu - X^\nu \overline{U}^\mu\right) \end{aligned} \tag{154}$$

and hence the tensor $W'^{\mu\nu}_{6,a}$ is redundant. The remaining five tensors all yield zero when contracted with Q^μ. Accordingly we have the following *five* independent contributions:

$$\begin{aligned} (W^{\mu\nu}_a)_{pol} &= i\left[\frac{1}{M} W'_9 \varepsilon^{\mu\nu\alpha\beta} \Sigma_\alpha Q_\beta \right. \\ &\quad + W'_{10}(U^\mu \bar{U}^\nu - U^\nu \bar{U}^\mu) + W'_{11}(U^\mu \overline{X}^\nu - U^\nu \overline{X}^\mu) \\ &\quad \left. + W'_{12}(X^\mu \bar{U}^\nu - X^\nu \bar{U}^\mu) + W'_{13}(X^\mu \overline{X}^\nu - X^\nu \overline{X}^\mu)\right]. \end{aligned} \tag{155}$$

As above, we have shifted notation to make this sector coherent with the previous ones; all tensors have the same dimensions, implying that the invariant functions all have the same dimensions. As an alternative it is also possible to expand the contraction of the leptonic and hadronic tensors in terms of Lorentz scalars rather than employing the 4-vectors as we have here. The resulting form is documented in [18] [3].

Let us end this section with a brief discussion of how the use of time-reversal invariance allows one to separate the four types of contributions into two classes.

[3] An extended version of this study is available in the cited reference: there explicit results are given in the target rest frame and six appendices are included detailing the conventions used, expressing the contraction of the tensors entirely in terms of invariants, inverting the invariant response representations in terms of photon helicity projections, detailing the nature of the cross section as the available phase-space increases and more channels become open, including some connections with conventional kinematic variables and discussing inclusive scattering in more detail to make connections with (more) familiar material. For brevity these addenda have been omitted in this shorter version.

The basic requirement for the time-reversal operator is to relate a given matrix element to one that describes the process running in the opposite direction, that is to a matrix element where the incoming state now contains all of the particles from the original final state and the final state contains the particles from the original initial state. If the original matrix element has a final state with two or more interacting particles this requires that the boundary condition for this state be changed from the incoming boundary condition to the outgoing boundary condition.

The effects of time-reversal on the hadronic tensor have been studied in great detail in the context of multipole expansions for arbitrary target spin (see, for instance, [3] and references therein). The result is that the matrix elements must fall into two classes: one where the transition multipole moment is real and another where it is imaginary. These two classes result in response functions that are either even or odd under time-reversal, TRE or TRO, respectively. Note that time-reversal invariance is assumed throughout this work; being TRE or TRO does not imply violation of this symmetry.

For the case of a spin-1/2 particle in the initial or final state the effects of time-reversal can be greatly simplified by the simultaneous application of both time-reversal and parity [16]. This is particularly useful in the case where the hadronic tensor is written as a linear combination of invariant functions of inner products of the available 4-momenta and second-rank tensors constructed from these four-momenta and the spin vector, such as we have done above. For the purpose of this discussion let

$$W^{\mu\nu}(Q,P,P_x,P_m,S,(-)) = \langle P,S|\, J^{\mu\dagger}(Q)\, |P_x,P_m,S,(-)\rangle \times \langle P_x,P_m,S,(-)|\, J^{\mu}(Q)(-)\, |P,S\rangle\,, \qquad (156)$$

where $(-)$ denotes the incoming boundary conditions for the final scattering state. This trivially implies that

$$W^{*\mu\nu}(Q,P,P_x,P_m,S,(-)) = W^{\nu\mu}(Q,P,P_x,P_m,S,(-)). \qquad (157)$$

Equations (81,83,145,155) are constructed such that $W_i,\ i=1,\ldots,5$ and $W_i',\ i=$

$1, \ldots, 13$ are real.

The components of the hadronic tensor in Eqs. (81,83,145,155) are parameterized in terms of Lorentz 4-vectors. The result of combining time-reversal and parity causes no change to the momentum 4-vectors while causing the spin 4-vector to change sign. Most importantly, time-reversal causes a change in the boundary condition of the scattering state from incoming ((−)) to outgoing ((+)). This gives

$$\begin{aligned} W^{\mu\nu}(Q, P, P_x, P_m, S, (-)) \xrightarrow{\mathcal{TP}} & W^{\nu\mu}(Q, P, P_x, P_m, -S, (+)) \\ & = -W^{*\mu\nu}(Q, P, P_x, P_m, S, (+)) \,. \end{aligned} \tag{158}$$

Since Q^μ, U^μ and X^μ depend only on the momentum 4-vectors one has

$$\begin{aligned} Q^\mu &\xrightarrow{\mathcal{TP}} Q^\mu \\ U^\mu &\xrightarrow{\mathcal{TP}} U^\mu \\ X^\mu &\xrightarrow{\mathcal{TP}} X^\mu \,. \end{aligned} \tag{159}$$

The vectors Σ^μ, $\overline{X}^\mu$ and $\overline{U}^\mu$ are linear in S^μ and thus

$$\begin{aligned} \Sigma^\mu &\xrightarrow{\mathcal{TP}} -\Sigma^\mu \\ \overline{X}^\mu &\xrightarrow{\mathcal{TP}} -\overline{X}^\mu \\ \overline{U}^\mu &\xrightarrow{\mathcal{TP}} -\overline{U}^\mu . \end{aligned} \tag{160}$$

The scalar I_0 is also linear in S^μ and accordingly

$$I_0 \xrightarrow{\mathcal{TP}} -I_0 \,. \tag{161}$$

The invariant functions W_i and W'_i are real and the complex conjugation changes the sign of all factors of i.

Applying these rules to Eqs. (81,83,145,155) yields

$$W_i(-) \xrightarrow{\mathcal{TP}} W_i(+), \qquad i = 1, \ldots, 4 \tag{162}$$

$$W_5(-) \xrightarrow{\mathcal{TP}} -W_5(+) \tag{163}$$

$$W'_i(-) \xrightarrow{\mathcal{TP}} -W'_i(+), \qquad i = 1, \ldots, 8 \tag{164}$$

$$W'_i(-) \xrightarrow{\mathcal{TP}} W'_i(+), \qquad i = 9, \ldots, 13 \,. \tag{165}$$

		Number	Time-Reversal
Unpolarized	Symmetric	4	Even
	Anti-symmetric	1	Odd
Polarized	Symmetric	8	Odd
	Anti-symmetric	5	Even

Table 1: This table shows the number of invariant functions falling into the four sectors according to polarization and symmetry indicating the time-reversal properties of each sector.

Under conditions where the boundary condition has no effect, such as the plane-wave impulse approximation, factorization approximations or where the final state is obtained through a single resonance at the energy where only the real part contributes, the invariant functions in Eqs. (163) and (164) must be zero. In such a special case this reduces the number of invariant functions from 18 to 9 with a similar reduction in the number of reponse functions. Generally speaking, however, all 18 play a role. This is the same as would be obtained by applying the multipole analysis with time-reversal only [3]. Indeed, this is a very old result which goes back at least to early studies of pion electroproduction using polarized electrons and scattering from polarized proton targets (see [3] and references therein); our point here is simply to note that the present analysis in terms of invariant hadronic response functions has, as it should, the same structure.

In summary we have 18 invariant response functions falling into the four sectors categorized in Table 1, with the symmetric contributions entering when the incident electrons are unpolarized and the anti-symmetric contributions when they are polarized, in fact, longitudinally polarized when in the ERL_e. The sectors are otherwise specified by whether or not the spin-1/2 target is unpolarized or polarized.

3. Frame-dependent Forms for the Response

3.1. Frame-dependent Hadronic Responses in the Present Work

We next proceed to write explicit forms for the hadronic tensors defined in Sec. 1. Clearly, since these involve specific Lorentz components of the general hadronic tensor, these quantities are **frame-dependent**. We begin with the **symmetric, unpolarized** case given in Eq. (81) which immediately yields the following for the minimal set of components:

$$\left(W_s^{00}\right)_{unpol} = -\frac{1}{\rho}W_1 + \left(U^0\right)^2 W_2 + \left(X^0\right)^2 W_3 + \left(2U^0X^0\right) W_4 \quad (166)$$

$$\left(W_s^{01}\right)_{unpol} = \left(U^0U^1\right) W_2 + \left(X^0X^1\right) W_3 + \left(U^0X^1 + X^0U^1\right) W_4 \quad (167)$$

$$\left(W_s^{11}\right)_{unpol} = W_1 + \left(U^1\right)^2 W_2 + \left(X^1\right)^2 W_3 + \left(2U^1X^1\right) W_4 \quad (168)$$

$$\left(W_s^{22}\right)_{unpol} = W_1 + \left(U^2\right)^2 W_2 + \left(X^2\right)^2 W_3 + \left(2U^2X^2\right) W_4 \quad (169)$$

$$\left(W_s^{02}\right)_{unpol} = \left(U^0U^2\right) W_2 + \left(X^0X^2\right) W_3 + \left(U^0X^2 + X^0U^2\right) W_4 \quad (170)$$

$$\left(W_s^{12}\right)_{unpol} = \left(U^1U^2\right) W_2 + \left(X^1X^2\right) W_3 + \left(U^1X^2 + X^1U^2\right) W_4. \quad (171)$$

Note that, since the symmetric leptonic tensor in this work may be shown to have no $\mu\nu = 02$ or 12 components, the last two hadronic contributions (Eqs. (170-171)) do not enter when the tensors are contracted, leaving a total of four terms, as expected for the situation where only the incident electrons may be polarized and the ERL_e is invoked [2]. In the situation where the incident electrons are polarized and where the scattered electron's polarization is assumed to be measured these last two contributions do enter. Following the nomenclature in

[2] we have

$$
\begin{aligned}
W^{L}_{unpol} &\equiv \left(W^{00}_{s}\right)_{unpol} = -\frac{1}{\rho}W_1 + \left(U^0\right)^2 W_2 + \left(X^0\right)^2 W_3 \\
&\quad +2U^0X^0W_4 && (172)\\
W^{T}_{unpol} &\equiv \left(W^{22+11}_{s}\right)_{unpol} = 2W_1 + \left[\left(U^1\right)^2 + \left(U^2\right)^2\right] W_2 \\
&\quad + \left[\left(X^1\right)^2 + \left(X^2\right)^2\right] W_3 + 2\left[U^1X^1 + U^2X^2\right] W_4 && (173)\\
W^{TT}_{unpol} &\equiv \left(W^{22-11}_{s}\right)_{unpol} = \left[-\left(U^1\right)^2 + \left(U^2\right)^2\right] W_2 \\
&\quad + \left[-\left(X^1\right)^2 + \left(X^2\right)^2\right] W_3 + 2\left[-U^1X^1 + U^2X^2\right] W_4 && (174)\\
W^{TL}_{unpol} &\equiv 2\sqrt{2}\left(W^{01}_{s}\right)_{unpol} = 2\sqrt{2}\left[U^0U^1W_2 + X^0X^1W_3\right. \\
&\quad \left.+\left(U^0X^1 + X^0U^1\right) W_4\right]. && (175)
\end{aligned}
$$

Next, for the **anti-symmetric, unpolarized** case we have the following from Eq. (83):

$$
\begin{aligned}
\left(W^{02}_{a}\right)_{unpol} &= iW_5\left(U^0X^2 - X^0U^2\right) && (176)\\
\left(W^{12}_{a}\right)_{unpol} &= iW_5\left(U^1X^2 - X^1U^2\right), && (177)
\end{aligned}
$$

yielding

$$
\begin{aligned}
W^{TL'}_{unpol} &\equiv 2\sqrt{2}\left(iW^{02}_{a}\right)_{unpol} = -2\sqrt{2}W_5\left(U^0X^2 - X^0U^2\right) && (178)\\
W^{T'}_{unpol} &\equiv 2\left(iW^{12}_{a}\right)_{unpol} = -2W_5\left(U^1X^2 - X^1U^2\right). && (179)
\end{aligned}
$$

These can all contribute in a situation where the incident electron is polarized. However, note the following: if mass terms in the electron tensor are retained (even in the PWBA) then one finds that the TL' and T' contributions are of leading order whereas the $\underline{TL}'$ contributions go as $1/\gamma_e$ or $1/\gamma'_e$ where $\gamma_e = \epsilon/m_e$ and $\gamma'_e = \epsilon'/m_e$ and hence may usually be neglected at high energies, leaving only the TL' and T' contributions.

Next we consider the contributions that arise from contractions of the symmetric leptonic tensor with the symmetric hadronic tensor for the case where the target is polarized – the **symmetric, polarized** case. From the developments in the last section we find that the following contributions enter in this sector:

$$
\begin{aligned}
W^L_{pol} &\equiv \left(W^{00}_s\right)_{pol} \\
&= \left\{-W'_1/\rho + \left(U^0\right)^2 W'_2 + \left(X^0\right)^2 W'_3 + 2U^0X^0W'_4\right\} I_0 \\
&\quad +2\left\{U^0\bar{U}^0W'_5 + U^0\overline{X}^0W'_6 + X^0\bar{U}^0W'_7 + X^0\overline{X}^0W'_8\right\} \qquad (180)
\end{aligned}
$$

$$
\begin{aligned}
W^T_{pol} &\equiv \left(W^{22}_s + W^{11}_s\right)_{pol} = \Big\{2W'_1 + \left(\left(U^2\right)^2 + \left(U^1\right)^2\right) W'_2 \\
&\quad + \left(\left(X^2\right)^2 + \left(X^1\right)^2\right) W'_3 + 2\left(U^2X^2 + U^1X^1\right) W'_4\Big\} I_0 \\
&\quad +2\Big\{\left(U^2\bar{U}^2 + U^1\bar{U}^1\right) W'_5 + \left(U^2\overline{X}^2 + U^1\overline{X}^1\right) W'_6 \qquad (181) \\
&\quad + \left(X^2\bar{U}^2 + X^1\bar{U}^1\right) W'_7 + \left(X^2\overline{X}^2 + X^1\overline{X}^1\right) W'_8\Big\}
\end{aligned}
$$

$$
\begin{aligned}
W^{TT}_{pol} &\equiv \left(W^{22}_s - W^{11}_s\right)_{pol} = \Big\{\left(\left(U^2\right)^2 - \left(U^1\right)^2\right) W'_2 \\
&\quad + \left(\left(X^2\right)^2 - \left(X^1\right)^2\right) W'_3 + 2\left(U^2X^2 - U^1X^1\right) W'_4\Big\} I_0 \qquad (182) \\
&\quad +2\Big\{\left(U^2\bar{U}^2 - U^1\bar{U}^1\right) W'_5 + \left(U^2\overline{X}^2 - U^1\overline{X}^1\right) W'_6 \\
&\quad + \left(X^2\bar{U}^2 - X^1\bar{U}^1\right) W'_7 + \left(X^2\overline{X}^2 - X^1\overline{X}^1\right) W'_8\Big\}
\end{aligned}
$$

$$
\begin{aligned}
W^{TL}_{pol} &\equiv 2\sqrt{2}\left(W^{01}_s\right)_{pol} \\
&= 2\sqrt{2}\Big[\left\{U^0U^1W'_2 + X^0X^1W'_3 + \left(U^0X^1 + U^1X^0\right) W'_4\right\} I_0 \\
&\quad + \left(U^0\bar{U}^1 + U^1\bar{U}^0\right) W'_5 + \left(U^0\overline{X}^1 + U^1\overline{X}^0\right) W'_6 \qquad (183) \\
&\quad + \left(X^0\bar{U}^1 + X^1\bar{U}^0\right) W'_7 + \left(X^0\overline{X}^1 + X^1\overline{X}^0\right) W'_8\Big]
\end{aligned}
$$

following conventional notation.

Finally, we need to develop the **anti-symmetric, polarized** case. From

Eq. (155) we have that

$$\begin{aligned}
\left(W_a^{02}\right)_{pol} &= i\left[\frac{1}{M}W_9'\varepsilon^{02\alpha\beta}\Sigma_\alpha Q_\beta \right.\\
&+W_{10}'(U^0\bar{U}^2-U^2\bar{U}^0)+W_{11}'(U^0\overline{X}^2-U^2\overline{X}^0)\\
&\left.+\,W_{12}'(X^0\bar{U}^2-X^2\bar{U}^0)+W_{13}'(X^0\overline{X}^2-X^2\overline{X}^0)\right] \qquad (184)\\
\left(W_a^{12}\right)_{pol} &= i\left[\frac{1}{M}W_9'\varepsilon^{12\alpha\beta}\Sigma_\alpha Q_\beta \right.\\
&+W_{10}'(U^1\bar{U}^2-U^2\bar{U}^1)+W_{11}'(U^1\overline{X}^2-U^2\overline{X}^1)\\
&\left.+\,W_{12}'(X^1\bar{U}^2-X^2\bar{U}^1)+W_{13}'(X^1\overline{X}^2-X^2\overline{X}^1)\right], \qquad (185)
\end{aligned}$$

where no cases with components $\mu\nu = 03$, 13 or 23 are needed, since they can be eliminated using the continuity equation. These yield three possible responses, of which we further develop only the two below, as $\underline{TL'}$ is typically suppressed, see the discussion after eq. (179).

$$\begin{aligned}
W_{pol}^{T'} &\equiv 2\left(iW_a^{12}\right)_{pol} \qquad (186)\\
&= -2\left[\frac{1}{M}W_9'\varepsilon^{12\alpha\beta}\Sigma_\alpha Q_\beta \right.\\
&+W_{10}'(U^1\bar{U}^2-U^2\bar{U}^1)+W_{11}'(U^1\overline{X}^2-U^2\overline{X}^1)\\
&\left.+\,W_{12}'(X^1\bar{U}^2-X^2\bar{U}^1)+W_{13}'(X^1\overline{X}^2-X^2\overline{X}^1)\right] \qquad (187)\\
W_{pol}^{TL'} &\equiv 2\sqrt{2}\left(iW_a^{02}\right)_{pol} \qquad (188)\\
&= -2\sqrt{2}\left[\frac{1}{M}W_9'\varepsilon^{02\alpha\beta}\Sigma_\alpha Q_\beta \right.\\
&+W_{10}'(U^0\bar{U}^2-U^2\bar{U}^0)+W_{11}'(U^0\overline{X}^2-U^2\overline{X}^0)\\
&\left.+\,W_{12}'(X^0\bar{U}^2-X^2\bar{U}^0)+W_{13}'(X^0\overline{X}^2-X^2\overline{X}^0)\right]. \qquad (189)
\end{aligned}$$

This completes the general structure of both the leptonic and hadronic tensors in a general frame where the spin-1/2 target is polarized and moving with some general 4-momentum P^μ. Once we know the detailed descriptions of the basic 4-vectors involved (see the next section) the tensors involved in forming the semi-inclusive cross section are then completely specified, requiring only the invariant response functions W_m and W_m', these, of course, being functions of the four basic Lorentz scalar invariants in the problem (again, see the next section). Our strategy below is first to express the basic 4-vectors involved in a rotated frame of reference, and then to boost our results to the target rest frame. In this last frame we will then be in a position to evaluate the tensors involved and hence to extract specific results for the invariant response functions required.

3.2. Relationships to other Conventions

Above we have summarized the relationships between the frame-dependent and Lorentz invariant formulations of the hadronic response following the conventions that have been adopted by many in studies of electron scattering for over half a century. Other conventions have also been employed and, while attempting to review how all of these are inter-related would go beyond the scope of the present work, in this section we do provide connections to two other ways of expressing the response for the present class of reactions.

One formulation of the problem is quite old, coming from studies of exclusive pion electroproduction from polarized protons [17] (see also [3] and references therein). While this involves an exclusive final-state channel, nevertheless the structure of the cross section falls within the class of reactions being summarized here and, for instance, is well-known to involve 18 individual (frame dependent) response functions. Typically, for this reaction, one adopts the virtual photon/target center-of-momentum frame, since the exclusive final state is conveniently handled in that frame. This frame may be related relatively easily to the target rest frame, since the Lorentz transformation involved is rather straightforward (see, for example, [3]).

From [17] one has the following:

$$\frac{d\sigma}{d\Omega} = \sigma_0 + \sigma_e + \sigma_t + \sigma_{et}, \tag{190}$$

where σ_0 arises when neither the electron nor the target are polarized, σ_e arises when the electron is longitudinally polarized but the target is unpolarized, σ_t arises when the electron is unpolarized but the target is polarized, and σ_{et} arises when both the electron and the target are polarized. Beginning with the fully unpolarized case from [17] one has

$$\sigma_0 = \sigma_U + \mathcal{E}\sigma_L + \mathcal{E}\sigma_T \cos 2\phi + \sqrt{\frac{1}{2}\mathcal{E}\,(1+\mathcal{E})}\sigma_I \cos\phi \tag{191}$$

and accordingly one can show that the four contributions $\sigma_{U,L,T,I}$ are related to four specific response functions used in the present work, namely

$$\begin{aligned}
\sigma_U &= \mathcal{N}_1 \cdot \left(W^{T}_{unpol}\right) &(192)\\
\sigma_L &= \mathcal{N}_1 \cdot \left(2\rho\mathcal{E}W^{L}_{unpol}\right) &(193)\\
\sigma_T \cos 2\phi &= \mathcal{N}_1 \cdot \left(-W^{TT}_{unpol}\right) &(194)\\
\sigma_I \cos\phi &= \mathcal{N}_1 \cdot \left(-\sqrt{2\rho}W^{TL}_{unpol}\right). &(195)
\end{aligned}$$

Here, as usual, $\rho = \left|Q^2/q^2\right|$ and $\mathcal{N}_1$ is a factor that relates the normalization conventions employed in [17] to those employed in this work — we postpone the discussion of the overall normalization of the response functions until Sec. 4 where the semi-inclusive cross section is discussed in more detail. It is then straightforward to relate these expressions directly to expressions containing the invariant response functions. For example, using Eqs. (172) and (173) one has

$$\begin{aligned}
\sigma_L &= \mathcal{N}_1 \cdot \left[2\rho\mathcal{E}\left(W^{00}_s\right)_{unpol}\right] &(196)\\
&= \mathcal{N}_1 \cdot \left[2\rho\mathcal{E}\left\{-\frac{1}{\rho}W_1 + \left(U^0\right)^2 W_2 + \left(X^0\right)^2 W_3\right\}\right] &(197)\\
\sigma_U &= \mathcal{N}_1 \cdot \left(W^{22+11}_s\right)_{unpol} &(198)\\
&= \mathcal{N}_1 \cdot \Big[2W_1 + \left[\left(U^1\right)^2 + \left(U^2\right)^2\right] W_2 \\
&\quad + \left[\left(X^1\right)^2 + \left(X^2\right)^2\right] W_3 + 2\left[U^1X^1 + U^2X^2\right] W_4\Big] &(199)
\end{aligned}$$

and likewise for the other responses here and below. Of course, the kinematic factors ρ, U^μ and X^μ must be evaluated in the appropriate frame of reference.

When only the electron is polarized (that is, longitudinally polarized and in the ERL_e) the result from [17] may be written

$$\sigma_e \equiv \sqrt{\frac{1}{2}\mathcal{E}\,(1-\mathcal{E})}\sigma'_I \sin\phi, \tag{200}$$

where σ_I' is the imaginary part of the same expression for which σ_I is the real part (see [17] Eqs. (22a) and (23); we have condensed the notation here for clarity). This is similarly related to a fifth contribution found in the present work, namely

$$\sigma_I' \sin\phi = \mathcal{N}_1 \cdot \left(-\sqrt{2\rho} W_{unpol}^{TL'}\right), \tag{201}$$

involving the so-called 5th response function.

When the target is polarized but not the incident electron analogously to Eq. (191) one has

$$\begin{aligned}\sigma_t &= \widetilde{\sigma}_U + \mathcal{E}\widetilde{\sigma}_L + \mathcal{E}\left[\widetilde{\sigma}_{T1}\cos 2\phi + \widetilde{\sigma}_{T2}\sin 2\phi\right] \\ &\quad + \sqrt{\frac{1}{2}\mathcal{E}\left(1+\mathcal{E}\right)}\left[\widetilde{\sigma}_{I1}\cos\phi + \widetilde{\sigma}_{I2}\sin\phi\right],\end{aligned} \tag{202}$$

now containing both sine and cosine contributions, where

$$\widetilde{\sigma}_U = \mathcal{N}_1 \cdot \left(W_{pol}^{T}\right) \tag{203}$$

$$\widetilde{\sigma}_L = \mathcal{N}_1 \cdot \left(2\rho\mathcal{E} W_{pol}^{L}\right) \tag{204}$$

$$\widetilde{\sigma}_{T1}\cos 2\phi + \widetilde{\sigma}_{T2}\sin 2\phi = \mathcal{N}_1 \cdot \left(-W_{pol}^{TT}\right) \tag{205}$$

$$\widetilde{\sigma}_{I1}\cos\phi + \widetilde{\sigma}_{I2}\sin\phi = \mathcal{N}_1 \cdot \left(-\sqrt{2\rho} W_{pol}^{TL}\right). \tag{206}$$

Here we have suppressed any explicit dependence on the target polarization. In [17] and later in the present review if projections of the target spin along $\mathbf{q}$ the momentum transfer direction (unit vector $\mathbf{u}_{L'}$), in the direction $\mathbf{q}\times\mathbf{p}_x$ (unit vector $\mathbf{u}_{N'}$) and in the direction $\mathbf{u}_{S'} = \mathbf{u}_{N'} \times \mathbf{u}_{L'}$ one can show that $\widetilde{\sigma}_U$, $\widetilde{\sigma}_L$, $\widetilde{\sigma}_{T1}$ and $\widetilde{\sigma}_{I1}$ involve four distinct responses containing the N' projection, $\widetilde{\sigma}_{T2}$ and $\widetilde{\sigma}_{I2}$, involve two distinct responses containing the L' projection, and two more distinct responses containing the N' projection. Namely, in this sector one has eight distinct types of response.

And then when both the target and the incident electron are polarized one can write the results from [17] in the form

$$\sigma_{et} = \sqrt{1-\mathcal{E}^2}\,\widetilde{\sigma}_T' + \sqrt{\frac{1}{2}\mathcal{E}\left(1-\mathcal{E}\right)}\left[\widetilde{\sigma}_{I1}'\sin\phi + \widetilde{\sigma}_{I2}'\cos\phi\right] \tag{207}$$

with

$$\widetilde{\sigma}'_T = \mathcal{N}_1 \cdot \left(W_{pol}^{T'}\right) \tag{208}$$

$$\widetilde{\sigma}'_{I1} \sin\phi + \widetilde{\sigma}'_{I2} \cos\phi = \mathcal{N}_1 \cdot \left(-\sqrt{2\rho} W_{pol}^{TL'}\right). \tag{209}$$

In this case it can be shown (see [17]) that $\widetilde{\sigma}'_T$ has two distinct responses involving the L' and S'projections, $\widetilde{\sigma}'_{I1}$ has one response involving the N' projection, and $\widetilde{\sigma}'_{I2}$ has two distinct responses involving the L' and S'projections, for a total of five. In total there are 18 distinct responses. As stated above, all of these results may be written in terms of invariant response functions using the expressions given in Sec. 3.

Other conventions have been employed for several decades (see, for example, [19], [20]): for instance, in the high-energy regime one has a set of conventions that have been adopted in several studies and, while the notation varies from study to study, these are all essentially the same.

For example, in [21], semi-inclusive electron scattering is discussed from the point of view of SIDIS. Just like in the earlier paper by [3], the 18 response functions (as they are called in a nuclear context) or structure functions (as they are called in a DIS context), are discussed in two specific laboratory frames of reference, the laboratory gamma-lepton frame and the laboratory gamma-hadron frame. The author mentions how to obtain 18 independent functions using constraints following [5], but does not relate the invariant functions with the functions in the specific frames he discusses.

In other work Diehl and Sapeta [22] focus on how the polarization is specified, with respect to the lepton beam or the virtual photon direction. They discuss the transformation between these two polarization coordinate systems for the target polarization (longitudinal or transverse target polarization) and work in the target rest frame, stating that for a collider everything can be boosted along the lepton beam momentum. In this review, we use a completely general axis for the target polarization, characterized by the starred angles. The relationships we provide here include these two specific choices of polarization axes, and the use of invariant functions eliminates the need for a potentially cumbersome boost.

Let us consider one of these other treatments of the problem in a little more detail. Specifically we relate the present conventions to those employed in the target rest frame by [23]. The aim of their paper is to model the cross section at tree level in terms of transverse-momentum dependent parton distribution and fragmentation functions. For the target-unpolarized terms one has

$$\begin{aligned}
F_{UU,T} &= \mathcal{N}_2 \cdot \left(W^{T}_{unpol}\right) && (210)\\
F_{UU,L} &= \mathcal{N}_2 \cdot \left(2\rho W^{L}_{unpol}\right) && (211)\\
\cos\phi_h F_{UU}^{\cos\phi_h} &= \mathcal{N}_2 \cdot \left(-\sqrt{\rho/2}W^{TL}_{unpol}\right) && (212)\\
\cos 2\phi_h F_{UU}^{\cos 2\phi_h} &= \mathcal{N}_2 \cdot \left(-W^{TT}_{unpol}\right) && (213)\\
\sin\phi_h F_{LU}^{\sin\phi_h} &= \mathcal{N}_2 \cdot \left(-\sqrt{\rho/2}W^{TL'}_{unpol}\right), && (214)
\end{aligned}$$

where again $\mathcal{N}_2$ is an overall normalization factor (as above, we postpone the discussion of the overall normalization of the response functions until Sec. 4 where the semi-inclusive cross section is discussed in more detail), and when the target is polarized one has

$$\begin{aligned}
|S_\perp|\sin(\phi_h-\phi_s)F_{UT,T}^{\sin(\phi_h-\phi_s)} &= \mathcal{N}_2 \cdot \left(W^{T}_{pol}\right) && (215)\\
|S_\perp|\sin(\phi_h-\phi_s)F_{UT,L}^{\sin(\phi_h-\phi_s)} &= \mathcal{N}_2 \cdot \left(2\rho W^{L}_{pol}\right) && (216)\\
S_\parallel \sin\phi_h F_{UL}^{\sin\phi_h} + |S_\perp| \Big[\sin\phi_s F_{UT}^{\sin\phi_s} & \\
+\sin(2\phi_h-\phi_s)F_{UT}^{\sin(2\phi_h-\phi_s)}\Big] &= \mathcal{N}_2 \cdot \left(-\sqrt{\rho/2}W^{TL}_{pol}\right) && (217)\\
S_\parallel \sin 2\phi_h F_{UL}^{\sin 2\phi_h} + |S_\perp| \Big[\sin(\phi_h+\phi_s) F_{UT}^{\sin(\phi_h+\phi_s)} & \\
+\sin(3\phi_h-\phi_s)F_{UT}^{\sin(3\phi_h-\phi_s)}\Big] &= \mathcal{N}_2 \cdot \left(-W^{TT}_{pol}\right) && (218)\\
S_\parallel F_{LL} + |S_\perp|\cos(\phi_h-\phi_s)F_{LT}^{\cos(\phi_h-\phi_s)} &= \mathcal{N}_2 \cdot \left(W^{T'}_{pol}\right) && (219)\\
S_\parallel \cos\phi_h F_{LL}^{\cos\phi_h} + |S_\perp| \Big[\cos\phi_s F_{LT}^{\cos\phi_s} & \\
+\cos(2\phi_h-\phi_s)F_{LT}^{\cos(2\phi_h-\phi_s)}\Big] &= \mathcal{N}_2 \cdot \left(-\sqrt{\rho/2}W^{TL'}_{pol}\right). && (220)
\end{aligned}$$

The target polarizations are related by the following:

$$S_{\parallel} = \mathcal{P}_{L'} \tag{221}$$

$$|S_{\perp}|\cos(\phi_h - \phi_s) = \mathcal{P}_{S'} \tag{222}$$

$$|S_{\perp}|\sin(\phi_h - \phi_s) = -\mathcal{P}_{N'}, \tag{223}$$

where the polarizations on the right-hand sides of the equations are discussed in more detail in Sec. 4.3. As for the previous example, all of these results may be written in terms of invariant response functions using the expressions given in Sec. 3.

Finally, we note that other reactions have been studied using similar approaches to those employed here. For instance, in the dilepton production paper of [24] the hadronic side of the reaction involves two (polarized) hadrons which collide, exchanging a virtual photon which then has a lepton pair plus X at its other end. However, whatever specific choices of colliding hadrons are made, they are not as in the present paper, namely, a hadron (the spin-1/2 target, perhaps polarized) for one and a state typically having a hadron plus unobserved "missing" particles ("breakup"). Similar reasoning applies to the case of polarized hadron pair production in electron-positron annihilation discussed in [25], leading to a different state than discussed here. For the present case one may make the following arguments (see the end of Sec. 2.3): applying time-reversal interchanges initial and final hadronic states and applying complex conjugation restore the states to their original order, albeit while now requiring the complex conjugate of the matrix element involved. Two types of contributions occur: TRE corresponding to real parts of matrix elements and TRO corresponding to imaginary parts. We again emphasize that within the present context P and T are assumed to be good symmetries; the TRE/TRO characterization, of course, does not imply violation of time-reversal invariance. When discrete states are involved initial- and final-state phases cancel and thus TRO contributions are absent. For example, for elastic scattering (see, for instance, [26]) where initial and final hadronic states are the same and where any chosen phase simply cancels this means that only real parts can survive and thus that TRO contributions are absent. This is well-known for elastic electron scattering where only even Coulomb, odd magnetic and no electric multipoles occur when P and T are taken to be good symmetries (see, for instance [1], Chapters 7 and 15).

However, for inelastic scattering involving breakup (as in the present work) the initial and final states are not in general relatively real, the imaginary parts do occur and therefore TRO contributions are in general non-zero (they may be small, however not always). Such contributions have been measured in specific cases. Typically the reason some feel they are zero stems from the model assumptions made. For instance, in high-energy physics if the "handbag" assumption is invoked, then the TRO contributions are zero. However, if higher-order diagrams are added (for instance by adding gluon lines between the quark lines) then one is faced with loop integrals, which means complexity and means both TRE and TRO contributions can be present. In hadronic physics discussions of pion electroproduction or nuclear applications such as $(e, e'p)$ reactions with nuclei one naturally sees such contributions as coming from "final-state interactions". And in the latter case, when final-state interactions are assumed to be absent, then again TRO contributions are as well.

The case of pp or $p\bar{p}$ dilepton production [24] provides an example where both frame-dependent and Lorentz invariant responses have been discussed; however, that situation is akin to the situation outlined above for elastic scattering [26] where a reduced set of response functions enters. In this special case those two reactions may be related; however, the "breakup" class of reactions being reviewed in this work is more general and involves a larger number of (different) response functions.

4. Semi-inclusive Cross Section for Electron Scattering from a Polarized Spin-1/2 Target

The full semi-inclusive electron scattering cross section in a general frame of reference may be written in terms of the Mott cross section, some kinematic factors that arise from using the Feynman rules [4], together with a general response function $\mathcal{F}^{semi}$. We begin the discussion in this section by introducing useful notation for the kinematic variables involved in semi-inclusive scattering.

4.1. Kinematics for Semi-inclusive Scattering

Referring to Figs. 3 and 2, as discussed above, we are assuming that the initial state has two particles of masses m_e and M with 4-momenta $K^\mu = (\epsilon, \mathbf{k})$ and $P^\mu = (E_p, \mathbf{p})$, where $\epsilon = \sqrt{k^2 + m_e^2}$ and $E_p = \sqrt{p^2 + M^2}$, respectively, which collide, leaving a particle of mass m_e with 4-momentum $K'^\mu = (\epsilon', \mathbf{k}')$ where $\epsilon' = \sqrt{k'^2 + m_e^2}$ and producing a final state with 4-momentum $P'^\mu = (E_{p'}, \mathbf{p}')$ and hence invariant mass $W = \sqrt{E_{p'}^2 - p'^2}$. In turn, the final state is assumed to be divided into two pieces, one the specific particle "x" that is assumed to be detected, having 4-momentum $P_x^\mu = (E_x, \mathbf{p}_x)$, where $E_x = \sqrt{p_x^2 + M_x^2}$, together with the undetected ("missing") parts of the final state having 4-momentum $P_m^\mu = (E_m^{tot}, \mathbf{p}_m)$ with missing energy E_m^{tot}, missing momentum $\mathbf{p}_m$, and invariant mass $W_m = \sqrt{\left(E_m^{tot}\right)^2 - p_m^2}$. Note: for the *total* missing energy we use E_m^{tot}, since we reserve the notation E_m to denote a different, but related quantity (see below). See Fig. 2 where conservation of 4-momentum requires that $Q^\mu + P^\mu = P'^\mu = P_x^\mu + P_m^\mu$ and thus

$$E_m^{tot} = E_{p'} - E_x \tag{224}$$

$$\mathbf{p}_m = \mathbf{p}' - \mathbf{p}_x. \tag{225}$$

From above we have that

$$P_m^\mu = Q^\mu + P^\mu - P_x^\mu \tag{226}$$

and therefore that

$$E_m^{tot} = \omega + E_p - E_x \tag{227}$$

$$\mathbf{p}_m = \mathbf{p}' - \mathbf{p}_x. \tag{228}$$

Following the procedures adopted in studies of scaling [27] let us employ as independent kinematic variables the missing momentum $\mathbf{p}_m$ and, rather than the missing energy E_m, the following energy

$$\begin{aligned} \mathcal{E}_m(p_m) &\equiv E_m^{tot} - \left(E_m^{tot}\right)_T \geq 0 \qquad (229) \\ &= \sqrt{W_m^2 + p_m^2} - \sqrt{\left(W_m^T\right)^2 + p_m^2}, \qquad (230) \end{aligned}$$

where the threshold value of the invariant mass of the missing momentum is denoted W_m^T; examples of this are given later. This quantity has the merit of taking on the value $\mathcal{E}_m = 0$ at threshold. When used in the context of nuclear physics the missing 3-momentum is typically much smaller than the invariant masses of either the daughter threshold value (often the daughter ground-state mass) or any higher-energy daughter state and thus Eq. (230) may be written

$$\begin{aligned} \mathcal{E}_m(p_m) &= W_m\sqrt{1 + \left(\frac{p_m}{W_m}\right)^2} - W_m^T\sqrt{1 + \left(\frac{p_m}{W_m^T}\right)^2} \qquad (231) \\ &= W_m\left(1 + \frac{p_m^2}{2W_m^2} + \cdots\right) - W_m^T\left(1 + \frac{p_m^2}{2\left(W_m^T\right)^2} + \cdots\right) \qquad (232) \\ &= \left(W_m - W_m^T\right)\left[1 - \delta_m + \cdots\right] \qquad (233) \end{aligned}$$

where

$$\delta_m \equiv \frac{p_m^2}{2W_m W_m^T} \ll 1 \qquad (234)$$

typically. Often setting δ_m to zero is an excellent approximation; this correction involves only the difference between the kinetic energy of recoil when the daughter system is at threshold and when it is in some excited state. However, it is not necessary ever to make these approximations and the exact expressions can alway be employed.

In studies of nuclear physics it is common to define a different quantity (confusingly also called the missing energy) where kinetic energies are employed, E_m. Defining the kinetic energies

$$\begin{aligned} T &\equiv E_p - M \qquad (235) \\ T_x &\equiv E_x - M_x \qquad (236) \\ T_m &\equiv E_m^{tot} - W_m, \qquad (237) \end{aligned}$$

one has

$$E_m \equiv \omega - (T_x + T_m) \tag{238}$$

$$= \left(W_m - W_m^T\right) + E_s - T \tag{239}$$

$$\simeq \mathcal{E}_m(p_m) + E_s - T, \tag{240}$$

where the so-called separation energy

$$E_s \equiv M_x + W_m^T - M \geq 0 \tag{241}$$

has been introduced and the approximation in the third equation above corresponds to neglecting the correction involving δ_m discussed above.

Using the energy conservation condition in Eq. (227) we have

$$\mathcal{E}_m(p_m) = (E_p + \omega) - \left(E_m^{tot}\right)_T - \sqrt{M_x^2 + p'^2 + p_m^2 - 2p_m p' \cos\theta_m}, \tag{242}$$

where θ_m is the angle between $\mathbf{p}'$ and $\mathbf{p}_m$ and $p_m = |\mathbf{p}_m|$. By setting $\mathcal{E}_m$ to zero and solving the above equation for p_m under the limiting conditions where $\cos\theta_m = \pm 1$ it is straightforward to show that the above equation at $\mathcal{E}_m = 0$ has two solutions

$$p_m^+ \equiv Y = \frac{1}{W^2}\left[(E_p + \omega)\sqrt{\Lambda^2 - W^2\left(W_m^T\right)^2} + p'\Lambda\right] \tag{243}$$

$$-p_m^- \equiv y = \frac{1}{W^2}\left[(E_p + \omega)\sqrt{\Lambda^2 - W^2\left(W_m^T\right)^2} - p'\Lambda\right], \tag{244}$$

where, following the notation of [27] we have introduced the quantity

$$\Lambda \equiv \frac{1}{2}\left[W^2 + \left(W_m^T\right)^2 - M_x^2\right]. \tag{245}$$

Note that the quantity in the square root may be written

$$\Lambda^2 - W^2\left(W_m^T\right)^2 = \frac{1}{4}\left[W^2 - \left(W_m^T + M_x\right)^2\right]\left[W^2 - \left(W_m^T - M_x\right)^2\right] \tag{246}$$

and, since the argument of the square root must be non-negative, that

$$W \geq W^T = W_m^T + M_x. \tag{247}$$

Upon setting $y = 0$ one finds that

$$\omega = \omega_0 \equiv \sqrt{M_x^2 + q^2} + W_m^T - M. \tag{248}$$

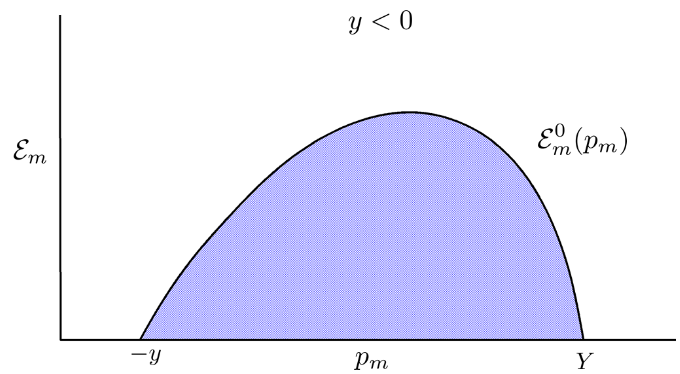

Figure 4: Physically allowed region for the situation where $y < 0$. The variables employed here are discussed in the text.

Given these relationships it is then straightforward to determine the physically allowed regions in the $\mathcal{E}_m$-p_m plane: for $y \geq 0$ corresponding to $\omega \geq \omega_0$ one has

$$\begin{aligned} &\mathcal{E}^0_m(-p_m) \leq \mathcal{E}(p_m) \leq \mathcal{E}^0_m(p_m) \quad && \text{for } 0 \leq p_m \leq y \\ &0 \leq \mathcal{E}(p_m) \leq \mathcal{E}^0_m(p_m) && \text{for } y \leq p_m \leq Y, \end{aligned} \tag{249}$$

while for $y \leq 0$ corresponding to $\omega \leq \omega_0$ one has

$$0 \leq \mathcal{E}(p_m) \leq \mathcal{E}^0_m(p_m) \quad \text{for } -y \leq p_m \leq Y, \tag{250}$$

where

$$\mathcal{E}^0_m(p_m) \equiv (E_p + \omega) - \left(E^{tot}_m\right)_T - \sqrt{M_x^2 + (p' - p_m)^2}, \tag{251}$$

namely, the value of $\mathcal{E}_m(p_m)$ when $\cos\theta_m = +1$. These regions are shown in Figs. 4 and 5. The region in Fig. 5 is seen to be bounded from below by the curve $\mathcal{E}^0_m(-p_m)$ which occurs when $\theta_m = \pi$ and above by the curve $\mathcal{E}^0_m(p_m)$ which occurs when $\theta_m = 0$ for $0 \leq p_m \leq y$, while the other regions are all bounded by zero from below and by the curve $\mathcal{E}^0_m(p_m)$ from above. When $\mathcal{E}_m(p_m) = 0$ one has from Eq. (242) that

$$\cos\theta_m = \frac{1}{2p_m p'}\left\{M_x^2 + p'^2 + p_m^2 - \left[(E_p + \omega) - \left(E^{tot}_m\right)_T\right]^2\right\}, \tag{252}$$

which determines θ_m for this boundary.

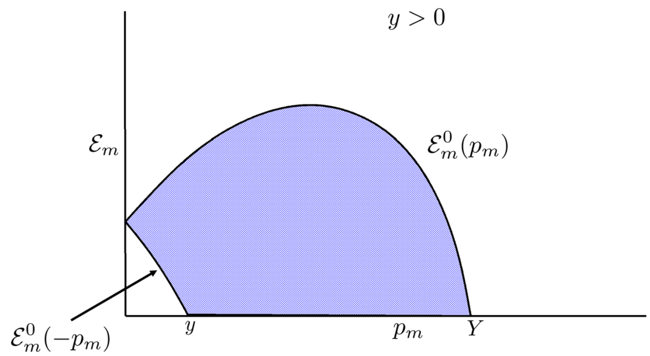

Figure 5: Physically allowed region for the situation where $y > 0$. The variables employed here are discussed in the text.

Thus we have the allowed regions of kinematics in the $\mathcal{E}_m$-p_m plane for given values of q and ω or, equivalently, of Q^2 and $\omega = \nu$ or q and y, where $y = y\,(q, \omega)$ given above is often used to replace ω in scaling analyses [27]. In turn these impose limits on the allowed values of the energy, 3-momentum and polar angle for the detected particle x: first, taking the scalar and cross product of $\mathbf{p}'$ with $\mathbf{p}_x = \mathbf{p}' - \mathbf{p}_m$ yields

$$p_x \cos\theta_x = p' - p_m \cos\theta_m \quad (253)$$

$$p_x \sin\theta_x = p_m \sin\theta_m \quad (254)$$

and thus

$$E_x = (E_p + \omega) - \left(\left(E_m^{tot} \right)_T + \mathcal{E}_m(p_m) \right) \quad (255)$$

$$p_x = \sqrt{p'^2 + p_m^2 - 2 p_m p' \cos\theta_m} \quad (256)$$

$$\tan\theta_x = \frac{p_m \sin\theta_m}{p' - p_m \cos\theta_m}. \quad (257)$$

By evaluating these expressions on the above boundaries one can then determine the physically allowed regions for P_x^μ. Let us denote the allowed region for the variables p_x (and hence E_x) and the polar angle θ_x by Γ_x. The above equations define the kinematic boundaries within which all values of (p_x, θ_x) are allowed and outside of which no physically allowed values exist. Later we discuss the roles played by the azimuthal angle ϕ_x where all values $(0, 2\pi)$ are allowed.

These results may be specialized from the general frame to the rest frame where $p = 0$ and thus $T = 0$ by making the following replacements: the energy E and the 3-momentum $\mathbf{p}'$ are replaced by M and $\mathbf{q}$, respectively, and θ_m becomes the angle between $\mathbf{q}$ and $\mathbf{p}_m$; W and Λ are Lorentz invariants and so do not change. The results one then obtains are the ones that are familiar from analyses of scaling [27].

That said, it should be noted that all of these developments are also valid for studies of particle physics at high energies.

4.2. Semi-inclusive Cross Section

Having established the allowed regions for the kinematics in semi-inclusive reactions we may now proceed to a discussion of the cross section. The Feynman rules followed in this work are those of [4]: we provide details in an appendix of [18] for how the general expression for the six-fold semi-inclusive cross section is obtained. That general answer may be re-written in the following form to connect with the above development of the leptonic and hadronic tensors

$$\left[\frac{d^6\sigma}{d\Omega dk' dp_x d\cos\theta_x d\phi_x}\right]_x = \frac{1}{2\pi}\sigma_{\text{Mott}} f \frac{M}{E_p}\frac{p_x^2}{E_x}\left[\mathcal{F}^{semi}\right]_x \tag{258}$$

where

$$\frac{\alpha^2 v_0 k'}{Q^4 k} = \sigma_{\text{Mott}} = \left(\frac{\alpha\cos\theta_e/2}{2\epsilon\sin^2\theta_e/2}\right)^2 \tag{259}$$

is the Mott cross section and $\left[\mathcal{F}^{semi}\right]_x$ is the invariant called $\mathcal{C} = \chi_{\mu\nu}W^{\mu\nu}$ divided by the factor v_0, namely

$$\left[\mathcal{F}^{semi}\right]_x = \chi_{\mu\nu}W_x^{\mu\nu}/v_0 \tag{260}$$

$$= v_L\left[W_x^L\right]^{semi} + v_T\left[W_x^T\right]^{semi} + \cdots \tag{261}$$

as discussed below and where the subscript "x" has been added to remind us that this forms the semi-inclusive cross section where particle x is assumed to be detected. The factor M/E_p arises from applying the Feynman rules in a general frame where the target is moving; this factor becomes unity in the target rest frame. Furthermore, the factor [28, 29]

$$f = \left[(\boldsymbol{\beta}_e - \boldsymbol{\beta}_p)^2 - (\boldsymbol{\beta}_e \times \boldsymbol{\beta}_p)^2\right]^{-1/2}, \tag{262}$$

with $\boldsymbol{\beta}_e = \boldsymbol{k}/\epsilon$ and $\boldsymbol{\beta}_p = \mathbf{p}/E_p$ as usual, accounts for the flux of the (in general colliding) beams. In the rest frame one has $\beta_p = 0$ and thus $f^R = 1/\beta_e$ which equals unity in the ERL$_e$.

In Eq. (258) a specific choice has been made for the normalization. In particular, while any constants or Lorentz invariants could be absorbed into the definitions of the invariant functions we choose to fix the conventions so that upon integrating the semi-inclusive cross section over the detected particle's 3-momentum and summing over all open channels, *i.e.,* all particles x while taking care not to double-count, one should recover the inclusive cross section with its conventional normalization. That is, to obtain the contribution of the channel having particle x to the inclusive cross section one should perform the integral over p_x, $\cos\theta_x$ and ϕ_x over the allowed physical region for the semi-inclusive reaction $(e, e'x)$ (see above for detailed discussion concerning the allowed region)

$$\left[\frac{d^2\sigma}{d\Omega dk'}\right]_x = \left\{\int dp_x \int d\cos\theta_x \int_0^{2\pi} d\phi_x \left[\frac{d^6\sigma}{d\Omega dk' dp_x d\cos\theta_x d\phi_x}\right]_x\right\}_{\text{allowed}} \tag{263}$$

$$= \frac{1}{2\pi}\sigma_{\text{Mott}} f \frac{M}{E_p}\left\{\int dp_x \frac{p_x^2}{E_x}\int d\cos\theta_x \left[\mathcal{G}^{semi}\right]_x\right\}_{\text{allowed}}, \tag{264}$$

where

$$\left[\mathcal{G}^{semi}\right]_x \equiv \int_0^{2\pi} d\phi_x \left[\mathcal{F}^{semi}\right]_x . \tag{265}$$

Then the full inclusive cross section is obtained by summing over all open channels, taking care not to double-count:

$$\frac{d^2\sigma}{d\Omega dk'} = \widehat{\sum_x}\left[\frac{d^2\sigma}{d\Omega dk'}\right]_x, \tag{266}$$

where the requirement not to double-count is indicated by the hat over the summation. In the next section the full inclusive cross section is also written in the form

$$\frac{d^2\sigma}{d\Omega dk'} = \sigma_{\text{Mott}} f \frac{M}{E_p}\mathcal{R}^{incl}, \tag{267}$$

where

$$\mathcal{R}^{incl} = R_1^{incl} + \cdots \tag{268}$$

and

$$R_1^{incl} = \left[v_L R_{unpol}^L\right]^{incl} + \cdots \tag{269}$$

Clearly the integral over ϕ_x for contributions that have no explicit ϕ_x-dependence simply accounts for the factor 2π put in the denominator above.

One may now change variables in the following ways. Since from Eq. (228) $\mathbf{p}_m = \mathbf{p}' - \mathbf{p}_x$ and we are keeping $\mathbf{q}$ and $\mathbf{p}$ constant and hence also $\mathbf{p}' = \mathbf{p} + \mathbf{q}$ constant, one has

$$p_x^2 dp_x d\cos\theta_x = p_m^2 dp_m d\cos\theta_m \tag{270}$$

and thus the semi-inclusive cross section may be written as differential in the missing-momentum plus changing p_x^2 to p_m^2. Since we also have from Eq. (242) that

$$\mathcal{E}_m(p_m) = (E_p + \omega) - \left(E_m^{tot}\right)_T - \sqrt{M_x^2 + p'^2 + p_m^2 - 2p_m p' \cos\theta_m}, \tag{271}$$

we can change variables from $\cos\theta_m$ to $\mathcal{E}$:

$$\left[\frac{\partial \mathcal{E}_m}{\partial \cos\theta_m}\right]_{p_m} = \frac{p_m p'}{E_x} \tag{272}$$

and so

$$\left[\frac{d^6\sigma}{d\Omega dk' dp_m d\mathcal{E}_m d\phi_x}\right]_x = \frac{1}{2\pi p'} \sigma_{\mathrm{Mott}} f \frac{M}{E_p} p_m \left[\mathcal{F}^{semi}\right]_x . \tag{273}$$

To form the inclusive cross section one may then proceed to integrate over p_m, $\mathcal{E}_m$ and ϕ_x (which is unchanged from the previous treatment), where now the physical region defining the boundaries in the $(p_m, \mathcal{E}_m)$-plane is that discussed above.

The above has been developed in a general frame; if one wishes to have the results in the target rest frame all that is necessary is to set p to zero, in which case $\mathbf{p}' \to \mathbf{q}$, θ_m becomes the angle between $\mathbf{q}$ and $\mathbf{p}_m$ and $E_p \to M$.

As discussed in detail above where the invariant response functions have been developed, the overall response can be decomposed into the four sectors that are classified by the types of polarization they involve

$$\mathcal{F}^{semi} = \mathcal{F}_1^{semi} + h\mathcal{F}_2^{semi} + h^*\mathcal{F}_3^{semi} + hh^*\mathcal{F}_4^{semi}. \tag{274}$$

In the semi-inclusive case, as we have seen earlier, the responses here depend on four scalar invariants, $(Q^2, I_p, I_x, I_{pp_x})$, together with the kinematic variables that enter through the lepton tensor. Clearly again the four sectors can be separated by flipping the electron helicity h and the direction of the target spin via the factor h^*. Explicitly we have

$$\mathcal{F}_1^{semi} = v_L \left[W_{unpol}^{L}\right]^{semi} + v_T \left[W_{unpol}^{T}\right]^{semi} + v_{TT} \left[W_{unpol}^{TT}\right]^{semi} + v_{TL} \left[W_{unpol}^{TL}\right]^{semi} \tag{275}$$

$$h\mathcal{F}_2^{semi} = v_{T'} \left[W_{unpol}^{T'}\right]^{semi} + v_{TL'} \left[W_{unpol}^{TL'}\right]^{semi} \tag{276}$$

$$h^*\mathcal{F}_3^{semi} = v_L \left[W_{pol}^{L}\right]^{semi} + v_T \left[W_{pol}^{T}\right]^{semi} + v_{TT} \left[W_{pol}^{TT}\right]^{semi} + v_{TL} \left[W_{pol}^{TL}\right]^{semi} \tag{277}$$

$$hh^*\mathcal{F}_4^{semi} = v_{T'} \left[W_{pol}^{T'}\right]^{semi} + v_{TL'} \left[W_{pol}^{TL'}\right]^{semi} . \tag{278}$$

Here the responses $\left[W_{unpol}^{K}\right]^{semi}$ and $\left[W_{pol}^{K}\right]^{semi}$ with $K = L,\ T,\ TL,\ TT,\ T'$ and TL' are the semi-inclusive quantities developed earlier, now with the label *semi* appended to distinguish them from the inclusive responses discussed above. As we found earlier, $\mathcal{F}_{1,4}^{semi}$ are TRE while $\mathcal{F}_{2,3}^{semi}$ are TRO. In turn, the individual responses are built from the 18 invariant response functions W_m, $m = 1, \ldots, 5$ and W'_m, $m = 1, \ldots, 13$. Note: the invariant responses here are for *semi-inclusive* scattering and depend on the four chosen scalar invariants; these quantities should not be confused with the *inclusive* invariant response functions discussed below.

4.3. Two Coordinate Systems for the Target Spin

We will have occasion to use two different coordinate system to specify the axis of quantization for the target spin. In the discussions above we chose the lepton-plane oriented coordinate system where $\mathbf{q}$ is along the 3-axis and the 2-axis is normal to the electron scattering plane (see Fig. 3). It proves to be convenient to introduce a rotated (around the 3-direction) coordinate system which we denote with primes, namely one with $3'$-axis along $\mathbf{q}$ and $2'$-axis normal to the plane formed by $\mathbf{q}$ and $\mathbf{p}_x$ (see Fig. 6). The reason for this choice of rotated system will become apparent in due course. The unit vectors

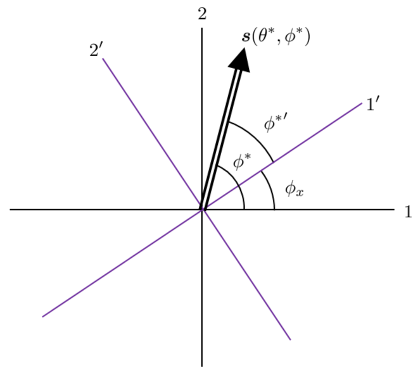

Figure 6: Two coordinate systems for the target spin. The original coordinate system is shown in Fig. 3 and here one can see how the primed system is related via a rotation around the 3-direction (the direction of the 3-momentum transfer $\mathbf{q}$) by the azimuthal angle ϕ_x. Hence in the 123-system the azimuthal angle of the target spin is ϕ^*, while in the $1'2'3'$-system it is $\phi^{*\prime} = \phi^* - \phi_x$.

in these two systems are related by

$$\mathbf{u}_{1'} = \cos\phi_x \mathbf{u}_1 + \sin\phi_x \mathbf{u}_2 \tag{279}$$

$$\mathbf{u}_{2'} = -\sin\phi_x \mathbf{u}_1 + \cos\phi_x \mathbf{u}_2 \tag{280}$$

$$\mathbf{u}_{3'} = \mathbf{u}_3 \tag{281}$$

and the inverse

$$\mathbf{u}_1 = \cos\phi_x \mathbf{u}_{1'} - \sin\phi_x \mathbf{u}_{2'} \tag{282}$$

$$\mathbf{u}_2 = \sin\phi_x \mathbf{u}_{1'} + \cos\phi_x \mathbf{u}_{2'} \tag{283}$$

$$\mathbf{u}_3 = \mathbf{u}_{3'}. \tag{284}$$

One has that

$$\mathbf{q} = q\mathbf{u}_3 = q\mathbf{u}_{3'} \tag{285}$$

while

$$\mathbf{p}_x = p_x \left[\sin\theta_x \mathbf{u}_{1'} + \cos\theta_x \mathbf{u}_{3'}\right] \tag{286}$$

with no $2'$ component, by construction. A simple result (which we use below) is accordingly

$$\mathbf{q} \times \mathbf{p}_x = q p_x \sin\theta_x \left(-\sin\phi_x \mathbf{u}_1 + \cos\phi_x \mathbf{u}_2\right) \tag{287}$$

$$= q p_x \sin\theta_x \mathbf{u}_{2'}, \tag{288}$$

namely having only a $2'$ component. The spin 4-vector may then be written in either the 123 system or the $1'2'3'$ system. One may define projections of the spin 3-vector in the two systems in the following way: the L, S and N directions are obtained by setting $\theta^* = 0$ (for L), $\theta^* = \pi/2$ with $\phi^* = 0$ (for S) and $\phi^* = \pi/2$ (for N), namely, making projections along the 123 system unit vectors

$$\mathcal{P}_L \equiv \mathbf{u}_3 \cdot \mathbf{s} = h^* s \cos\theta^* \tag{289}$$

$$\mathcal{P}_S \equiv \mathbf{u}_1 \cdot \mathbf{s} = h^* s \sin\theta^* \cos\phi^* \tag{290}$$

$$\mathcal{P}_N \equiv \mathbf{u}_2 \cdot \mathbf{s} = h^* s \sin\theta^* \sin\phi^* \tag{291}$$

or doing the same, but for the unit vectors in the $1'2'3'$ system

$$\mathcal{P}_{L'} \equiv \mathbf{u}_{3'} \cdot \mathbf{s} = h^* s \cos\theta^* \tag{292}$$

$$\mathcal{P}_{S'} \equiv \mathbf{u}_{1'} \cdot \mathbf{s} = h^* s \sin\theta^* \cos\phi^{*'} \tag{293}$$

$$\mathcal{P}_{N'} \equiv \mathbf{u}_{2'} \cdot \mathbf{s} = h^* s \sin\theta^* \sin\phi^{*'}. \tag{294}$$

Using the relationships amongst the unit vectors above one has that

$$\mathcal{P}_L = \mathcal{P}_{L'} \tag{295}$$

$$\mathcal{P}_S = \cos\phi_x \mathcal{P}_{S'} - \sin\phi_x \mathcal{P}_{N'} \tag{296}$$

$$\mathcal{P}_N = \sin\phi_x \mathcal{P}_{S'} + \cos\phi_x \mathcal{P}_{N'} \tag{297}$$

$$\mathcal{P}_{S'} = \cos\phi_x \mathcal{P}_S + \sin\phi_x \mathcal{P}_N \tag{298}$$

$$\mathcal{P}_{N'} = -\sin\phi_x \mathcal{P}_S + \cos\phi_x \mathcal{P}_N. \tag{299}$$

Note that $\mathcal{P}_{L'} = \mathcal{P}_L$ contains no dependence on ϕ_x.

Recalling the conventions employed in some analyses of frame-dependent formulations in Sec. 3.2, we see how the approach using the spin azimuthal angle

$\phi^{*\prime}$ referenced to the hadron plane (which is commonly used in treatments of pion electroproduction; see, for example, the discussion in [3]), together with the azimuthal angle between the electron and hadron planes ϕ_h leads to a simple pattern. Here we have $\phi^{*\prime} = \phi^* - \phi_x = \phi_s - \phi_h$, with $\phi_h = \phi_x$ and $\phi_s = \phi^*$; accordingly the dependences in Sec. 3.2 are the following:

$$\sin(\phi_h - \phi_s) = -\sin\phi^{*\prime} \tag{300}$$

$$\cos(\phi_h - \phi_s) = \cos\phi^{*\prime} \tag{301}$$

$$\sin\phi_s = \sin\phi_h\cos\phi^{*\prime} + \cos\phi_h\sin\phi^{*\prime} \tag{302}$$

$$\cos\phi_s = \cos\phi_h\cos\phi^{*\prime} - \sin\phi_h\sin\phi^{*\prime} \tag{303}$$

$$\sin(\phi_h + \phi_s) = \sin 2\phi_h\cos\phi^{*\prime} + \cos 2\phi_h\sin\phi^{*\prime} \tag{304}$$

$$\sin(2\phi_h - \phi_s) = \sin\phi_h\cos\phi^{*\prime} - \cos\phi_h\sin\phi^{*\prime} \tag{305}$$

$$\cos(2\phi_h - \phi_s) = \cos\phi_h\cos\phi^{*\prime} + \sin\phi_h\sin\phi^{*\prime} \tag{306}$$

$$\sin(3\phi_h - \phi_s) = \sin 2\phi_h\cos\phi^{*\prime} - \cos 2\phi_h\sin\phi^{*\prime} \tag{307}$$

together with $\sin\phi_h$, $\cos\phi_h$, $\sin 2\phi_h$ and $\cos 2\phi_h$. We see that for the spin-dependent sector the dependences on the spin azimuthal angle enter not at all (L') or through factors $\cos\phi^{*\prime}$ (S') or $\sin\phi^{*\prime}$ (N'). The dependences on the azimuthal angle ϕ_h are the following: no dependence for L, T and T' responses, via factors $\sin\phi_h$ and $\cos\phi_h$ for TL and TL' responses, and via factors $\sin 2\phi_h$ and $\cos 2\phi_h$ for TT responses. This simple pattern is well-known in studies of pion electroproduction (see, for example, Sec. 3 of [3] and Table 2 of [18]).

5. Inclusive Scattering of Polarized Electrons from Polarized Spin-1/2 Targets

For inclusive scattering one simply needs to eliminate all contributions that contain the 4-vectors V^μ or X^μ, as well as the invariant I_0 as they involve the 4-vector P_x^μ which does not enter in the inclusive case. All invariant response functions depend only on two scalar quantities, for example, Q^2 and $Q \cdot P =$

$Q^2 I_p$. Accordingly one obtains the following:

$$
\begin{aligned}
\left(W_s^{\mu\nu}\right)_{unpol}^{incl} &= -\left(W_1\right)^{incl}\left(g^{\mu\nu}-\frac{Q^\mu Q^\nu}{Q^2}\right)+\left(W_2\right)^{incl}U^\mu U^\nu && (308)\\
\left(W_a^{\mu\nu}\right)_{unpol}^{incl} &= 0 && (309)\\
\left(W_s^{\mu\nu}\right)_{pol}^{incl} &= \left(W_6'\right)^{incl}\left(U^\mu \overline{X}^\nu + U^\nu \overline{X}^\mu\right) && (310)\\
-i\left(W_a^{\mu\nu}\right)_{pol}^{incl} &= \frac{1}{M}\left(W_9'\right)^{incl}\epsilon^{\mu\nu\alpha\beta}\Sigma_\alpha Q_\beta \\
&\quad +\left(W_{11}'\right)^{incl}\left(U^\mu \overline{X}^\nu - U^\nu \overline{X}^\mu\right) && (311)
\end{aligned}
$$

with 5 inclusive invariant functions $\left(W_m\right)^{incl}$, $m = 1, 2$ and $\left(W_m'\right)^{incl}$, $m = 6, 9, 11$. Using our previous results for semi-inclusive scattering but now dropping all contributions containing V^μ or X^μ we obtain the following: for the **symmetric, unpolarized** cases (now not continuing to develop the $\underline{TL}$ and $\underline{TT}$ cases)

$$
\begin{aligned}
\left[W_{unpol}^{L}\right]^{incl} &= -\frac{1}{\rho}\left(W_1\right)^{incl}+\left(U^0\right)^2\left(W_2\right)^{incl} && (312)\\
\left[W_{unpol}^{T}\right]^{incl} &= 2\left(W_1\right)^{incl}+\left[\left(U^1\right)^2+\left(U^2\right)^2\right]\left(W_2\right)^{incl} && (313)\\
\left[W_{unpol}^{TT}\right]^{incl} &= \left[-\left(U^1\right)^2+\left(U^2\right)^2\right]\left(W_2\right)^{incl} && (314)\\
\left[W_{unpol}^{TL}\right]^{incl} &= 2\sqrt{2}U^0U^1\left(W_2\right)^{incl}, && (315)
\end{aligned}
$$

no results for the **anti-symmetric, unpolarized** case

$$\left(W_a^{\mu\nu}\right)_{unpol}^{incl} = 0, \qquad (316)$$

as can be seen above in discussing the semi-inclusive responses. All contributions there contained explicit factors involving V^μ or X^μ; in fact, potential contributions of this type are parity-violating when electrons are polarized longitudinal or sideways. For the **symmetric, polarized** cases (now not continuing to develop the $\underline{TL}$ and $\underline{TT}$ cases) we have

$$
\begin{aligned}
\left[W_{pol}^{L}\right]^{incl} &= U^0\overline{X}^0 W_6' && (317)\\
\left[W_{pol}^{T}\right]^{incl} &= \left(U^2\overline{X}^2+U^1\overline{X}^1\right)W_6' && (318)\\
\left[W_{pol}^{TT}\right]^{incl} &= \left(U^2\overline{X}^2-U^1\overline{X}^1\right)W_6' && (319)\\
\left[W_{pol}^{TL}\right]^{incl} &= 2\sqrt{2}\left(U^0\overline{X}^1+U^1\overline{X}^0\right)W_6', && (320)
\end{aligned}
$$

all of which are proportional to the same invariant response function W_6'. And, finally, for the **anti-symmetric, polarized** situation (now not continuing to develop the $\underline{TL}'$ case, although it is very similar to the TL' case below, simply having 2 replaced by 1; as noted earlier, this term can occur when only the incident electron is polarized but when the scattered electron's polarization is not measured although the leptonic factor goes as $1/\gamma$ and hence this contribution may be safely neglected at high energies – we do so in the following) we have

$$\begin{aligned}\left[W_{pol}^{T'}\right]^{incl} &= -2\left[\frac{1}{M}\left(W_9'\right)^{incl}\epsilon^{12\alpha\beta}\Sigma_\alpha Q_\beta \right.\\ &\quad \left. +\left(W_{11}'\right)^{incl}\left(U^1\overline{X}^2-\overline{X}^1U^2\right)\right] \qquad (321)\\ \left[W_{pol}^{TL'}\right]^{incl} &= -2\sqrt{2}\left[\frac{1}{M}\left(W_9'\right)^{incl}\epsilon^{02\alpha\beta}\Sigma_\alpha Q_\beta \right.\\ &\quad \left. +\left(W_{11}'\right)^{incl}\left(U^0\overline{X}^2-\overline{X}^0U^2\right)\right]. \qquad (322)\end{aligned}$$

In total we find that 5 invariant response functions enter, $W_{1,2}$ and $W_{9,11}'$ in contributions that are TRE, plus the contributions that involve the invariant response function W_6' and are TRO.

The general inclusive cross section may then be written in the following form:

$$\frac{d^2\sigma}{d\Omega_e dk'} \equiv \sigma_{Mott} f \frac{M}{E_p}\mathcal{R}^{incl} \qquad (323)$$

where σ_{Mott} is the Mott cross section given in Eq. (259) and the full inclusive response is given by

$$\mathcal{R}^{incl} = \mathcal{R}_1^{incl} + h\mathcal{R}_2^{incl} + h^*\mathcal{R}_3^{incl} + hh^*\mathcal{R}_4^{incl}, \qquad (324)$$

in which the four contributions correspond to completely unpolarized, electron polarization only, target polarization only, and double polarization, respectively. As above all responses here depend on two scalar invariants such as Q^2 and $Q \cdot P$ together with the electron scattering angle θ_e which enters via the leptonic factors. Clearly the four sectors can be separated by flipping the electron helicity

h and the direction of the target spin via the factor h^*. Explicitly we have

$$\begin{aligned}
\mathcal{R}_1^{incl} &= v_L \left[W_{unpol}^L\right]^{incl} + v_T \left[W_{unpol}^T\right]^{incl} \\
&\quad + v_{TL} \left[W_{unpol}^{TL}\right]^{incl} + v_{TT} \left[W_{unpol}^{TT}\right]^{incl} \qquad (325) \\
h\mathcal{R}_2^{incl} &= 0 \qquad (326) \\
h^*\mathcal{R}_3^{incl} &= v_L \left[W_{pol}^L\right]^{incl} + v_T \left[W_{pol}^T\right]^{incl} \\
&\quad + v_{TL} \left[W_{pol}^{TL}\right]^{incl} + v_{TT} \left[W_{pol}^{TT}\right]^{incl} \qquad (327) \\
hh^*\mathcal{R}_4^{incl} &= v_{TL'} \left[W_{pol}^{TL'}\right]^{incl} + v_{T'} \left[W_{pol}^{T'}\right]^{incl}, \qquad (328)
\end{aligned}$$

where, as above, we have dropped the small $\underline{TL}'$ contribution. The leptonic factors are given in eqs. (31) - (36) while the inclusive hadronic response functions are given above.

5.1. The Transition from Semi-Inclusive to Inclusive Scattering

While the above developments yield the structure of the general inclusive cross section directly, it is also instructive to follow a different strategy and proceed from the semi-inclusive cross section for a given channel (*i.e.,* for a specific particle x detected in coincidence with the scattered electron), integrating over the allowed kinematics of the 4-momentum that goes with that particle, and then summing over all open channels, of course, paying close attention to issues of double-counting.

We start with the general forms for the semi-inclusive cross section for the specific channel where particle x is assumed to be detected given above in Secs. 4.2 and 4.3. The dependence on the azimuthal angle ϕ_x occurs in the explicit factors $\cos\phi_x$, $\cos 2\phi_x$ and $\sin\phi_x$ in the rest frame for the cases where the target is unpolarized. Clearly, upon performing the integrals over ϕ_x over the range $(0, 2\pi)$ yields zero for the TT, TL and TL' cases, verifying the above inclusive structure. The L and T cases simply pick up a factor 2π when the azimuthal integral is performed. In summary, for the target unpolarized situation one finds that each channel yields only L and T responses.

The situation where the target is polarized is a little more complicated. There one finds that as well as explicit factors $\cos\phi_x$, $\cos 2\phi_x$, $\sin\phi_x$ and $\sin 2\phi_x$, one has implicit dependence on ϕ_x via the factors $\mathcal{P}_{S'}$ and $\mathcal{P}_{N'}$. In this scenario it, of course, makes no sense to use the primed spin-projection variables, since the plane in which the momentum of particle x lies is being integrated over and accordingly we must go back to the original unprimed spin projections which are referred to the electron scattering frame. Two of the symmetric, polarized cases are simple: the L and T results depend on the azimuthal angle solely through the factor $\mathcal{P}_{N'}$, which, by Eq. (303) only has dependences $\sin\phi_x$ and $\cos\phi_x$ and accordingly upon integrations over ϕ_x yield zero. The remaining cases require somewhat more work. The symmetric TL response has three contributions

$$x_1 \quad \sim \quad \cos\phi_x \mathcal{P}_{N'} = \frac{1}{2}\left[-\sin 2\phi_x \mathcal{P}_S + (1+\cos 2\phi_x)\,\mathcal{P}_N\right] \tag{329}$$

$$x_2 \quad \sim \quad \sin\phi_x \mathcal{P}_{L'} = \sin\phi_x \mathcal{P}_L \tag{330}$$

$$x_3 \quad \sim \quad \sin\phi_x \mathcal{P}_{S'} = \frac{1}{2}\left[\sin 2\phi_x \mathcal{P}_S + (1-\cos 2\phi_x)\,\mathcal{P}_N\right]. \tag{331}$$

Upon integrating over ϕ_x one then finds that the x_1 and x_3 cases yield $\pi\mathcal{P}_N$, while the x_2 case yields zero, namely, a nonzero result that goes as $\mathcal{P}_N$. Similarly, the symmetric TT response also has three contributions

$$\begin{aligned} y_1 \quad \sim \quad & \cos 2\phi_x \mathcal{P}_{N'} = \frac{1}{2}\left[-(\sin 3\phi_x - \sin\phi_x)\,\mathcal{P}_S \right. \\ & \left. +(\cos 3\phi_x + \cos\phi_x)\,\mathcal{P}_N\right] \end{aligned} \tag{332}$$

$$y_2 \quad \sim \quad \sin 2\phi_x \mathcal{P}_{L'} = \sin 2\phi_x \mathcal{P}_L \tag{333}$$

$$\begin{aligned} y_3 \quad \sim \quad & \sin 2\phi_x \mathcal{P}_{S'} = \frac{1}{2}\left[(\sin 3\phi_x + \sin\phi_x)\,\mathcal{P}_S \right. \\ & \left. +(-\cos 3\phi_x + \cos\phi_x)\,\mathcal{P}_N\right], \end{aligned} \tag{334}$$

all of which integrate to zero and yield no contribution for the TT term. Next, the anti-symmetric polarized cases are handled similarly: for the T' response the contribution that involves $\mathcal{P}_{S'}$ yields zero upon integration over ϕ_x while the contribution that involves $\mathcal{P}_{L'}$ and hence no dependence on ϕ_x yields a nonzero result arising from the factor 2π coming from the integral. Thus the T' response yields a nonzero result that is proportional to $\mathcal{P}_L$. Finally, the TL' response involves three contributions

$$z_1 \sim \cos\phi_x \mathcal{P}_{L'} = \cos\phi_x \mathcal{P}_L \tag{335}$$

$$z_2 \sim \cos\phi_x \mathcal{P}_{S'} = \frac{1}{2}\left[\left(1+\cos 2\phi_x\right)\mathcal{P}_S + \sin 2\phi_x \mathcal{P}_N\right] \tag{336}$$

$$z_3 \sim \sin\phi_x \mathcal{P}_{N'} = \frac{1}{2}\left[-\left(1-\cos 2\phi_x\right)\mathcal{P}_S + \sin 2\phi_x \mathcal{P}_N\right]. \tag{337}$$

As above, the term involving z_1 integrates to zero, while the z_2 and z_3 terms yields factors of π and $-\pi$, respectively, and involve the spin projection $\mathcal{P}_S$. Thus exactly the structure found above when proceeding to the inclusive cross section directly is found by integrating the semi-inclusive responses over ϕ_x.

Again, the strategy in the present work is the following: given some model for the polarized semi-inclusive cross section in the rest system one can deduce what are the invariant response functions for that model. With these the expressions in a general system immediately yield results for any choice of kinematics. The key feature is having everything written in terms of kinematic factors and invariant responses, since the latter are independent of the choice of frame. So, for example, while the earlier studies referred to above are completely general, they must be re-cast in terms of invariant response functions if one wishes to relate the results in different frames of reference.

6. Summary

The present study has focused on a review of the general formalism for representing electron scattering in terms of Lorentz invariant hadronic response functions. The formalism is very general and meant to be applicable both for low-energy reactions and in the high-energy regime (HER). Together with the well-known leptonic tensor that arises from products of the electron EM current matrix elements in past studies the EM hadronic tensors has been constructed using specific general basis sets of 4-vectors. Several cases are summarized, from the simplest involving unpolarized electrons being inclusively scattered from unpolarized targets to much more complicated cases where, in addition to the scattered electron other particles may be assumed to be detected or where hadronic polarizations enter. After reviewing the well-known cases in the present study we have focused on the specific case involving scattering of polarized electrons from polarized spin-1/2 targets in situations where the scattered electron and some (unpolarized) particle x are detected in coincidence, *viz.,* semi-inclusive scattering. The other simpler reactions may then be recovered as special sub-cases of this general reaction. For the polarized semi-inclusive reaction in total one finds that there are 18 basis tensors, four symmetric ones when both the electron and target are unpolarized, a single anti-symmetric one when the electron is longitudinally polarized while the target is unpolarized, eight symmetric ones when the electron is unpolarized but the target is polarized, and five anti-symmetric ones when the electron and the target are both polarized. The contraction of the leptonic and hadronic tensors that enters when applying the Feynman rules, which is a Lorentz invariant, is then formed as a linear combination involving these 18 hadronic tensors weighted with 18 invariant response functions, W_i, $i = 1, 5$ when the target is unpolarized and W_i', $i = 1, 13$ when the target is polarized. Each of these invariant responses is a function of four Lorentz scalars $(Q^2, I_p, I_x, I_{pp_x})$ (see Eqs. (96–98)). Thus one has the kinematics of the reaction and the target spin dependence expressed in terms of the basis 4-vectors while the dynamics are contained in the 18 invariant response functions. Clearly the former are frame-dependent while the latter are not.

Given the Lorentz invariant contraction of the leptonic and hadronic tensors one can proceed using the Feynman rules to obtain the semi-inclusive cross section in a general frame where both the incident electron and the target are assumed to be moving, the latter with momentum $\mathbf{p}$. All of the kinematic factors summarized above must then be evaluated in this specific frame. Specifically, we require the functions F in Eqs. (279–282) to be combined as in Eq.(278) and inserted into Eq. (277). To accomplish this we require the response functions $\left[W^K_{unpol}\right]^{semi}$ and $\left[W^K_{pol}\right]^{semi}$ with $K = L$, T, TL, TT, T' and TL' developed earlier; see Eqs. (172–175), (178–179), (180–183) and (187–189) in Sect. 3.1 where these response functions are written explicitly in terms of invariant response functions which contain the dynamics of the problem and simple kinematic variables. The kinematic variables of course differ in different frames whereas the invariant response functions do not. These expressions may be employed in any frame simply by evaluating the kinematic factors in the particular frame of interest. One may then obtain the corresponding results in a different frame where the target has a different value for its momentum simply by choosing the appropriate value for $\mathbf{p}$; all other kinematic variables are then to be evaluated in that different frame. Specifically, one can express the semi-inclusive cross section in the target rest frame by setting $p = 0$ (see [18] for details). Importantly, the dynamical content in the problem, which is encapsulated in the invariant response functions summarized above does not change when changing frames. Also, the 18 invariant response functions are functions only of the four Lorentz scalars listed above; these are also invariant.

The semi-inclusive cross section separates into four sectors according to the electron and target polarizations, namely, (I) both unpolarized, (II) electron polarized, target unpolarized, (III) target polarized, electron unpolarized, and (IV) both polarized. Having control of these polarizations then immediately allows the four sectors to be isolated. Furthermore, the cross section has explicit dependence on several kinematic variables that may be evaluated in principle to obtain enough linear equations in the 18 unknowns — the 18 invariant response functions — to invert and thereby determine those response functions. Specifically, the dependences on the electron scattering angle θ_e, on the azimuthal angle for the 3-momentum of the detected particle, ϕ_x, and on the angles θ^* and ϕ^* that specify the axis of quantization of the target spin can be used to isolate the required linear equations (in [18] an appendix is provided with the details).

Hence several strategies are available. In one approach where measurements are made in two different types of experiments the experimental results could be used in principle to isolate the 18 invariant response functions for the kinematical situation involved in the two experiments. Specifically, one could envision one experiment being performed in the target rest frame (fixed-target experiments) and from those measurements the 18 invariant response functions or some subset thereof being determined. One might then have a different experiment where the electron and target are both in motion (collider experiments): nevertheless, the same strategy could be followed and the 18 invariant response functions determined, albeit, perhaps for non-overlapping kinematics. The two sets of invariant responses could then be analyzed in a universal way.

A similar strategy occurs when using theory to make predictions of the semi-inclusive cross section. For instance, one may be forced to work in the target rest frame when modeling the dynamics using ingredients that are not "boostable", which is almost always the case in nuclear physics for nuclei other than the deuteron. However, one could deduce the corresponding invariant response functions working in the target rest frame and then employ them in, say, the collider frame. Specific modeling of this sort will be undertaken by the authors in the future.

To make contact with other approaches, in the process of developing the semi-inclusive cross section we have chosen to express the results in terms of specific Lorentz components of the general hadronic tensor which are governed by the helicity projections of the exchanged virtual photon. In [18] we have included an appendix where this step is skipped and the contraction of leptonic and hadronic tensors is expressed directly in terms of invariant quantities. The two approaches are completely equivalent, but each may have advantages in particular applications.

Finally, we have shown how the inclusive scattering of polarized electrons from polarized spin-1/2 targets is related to integrations of the semi-inclusive cross sections plus sums over all open channels. Again we note that in [18] one may find another appendix containing a few more details on inclusive scattering to help the reader find more familiar ground to aid in navigating the much more intricate problem of semi-inclusive scattering.

Acknowledgements

This work was supported in part by the Office of Nuclear Physics of the U.S. Department of Energy under Grant Contract DE-FG02-94ER40818 (T. W. D.), in part by funds provided by the National Science Foundation under grants No. PHY-1913261 and PHY-2208237 (S. J.), and by Jefferson Science Associates, LLC under U.S. DOE Contract DE-AC05-06OR23177 and U.S. DOE Grant DE-FG02-97ER41028 (J. W. V. O.).

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