A DETERMINATION OF THE BLOWUP SOLUTIONS TO THE FOCUSING NLS WITH MASS EQUAL TO THE MASS OF THE SOLITON

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Abstract. In this paper we prove rigidity for blowup solutions to the focusing, mass-critical nonlinear Schrödinger equation in dimensions $2 \leq d \leq 15$ with mass equal to the mass of the soliton. We prove that the only such solutions are the solitons and the pseudoconformal transformation of the solitons. We show that this implies a Liouville result for the nonlinear Schrödinger equation.

1. Introduction

In this paper we continue the study begun in [Dod21a] of the focusing, mass-critical nonlinear Schrödinger equation

$$iu_t + \Delta u + |u|^{\frac{4}{d}} u = 0, \qquad u(0,x) = u_0(x) \in L^2(\mathbb{R}^d). \tag{1.1}$$

In [Dod21a] we proved a rigidity result for solutions to (1.1) in one dimension with mass equal to the mass of the soliton. In this paper we address higher dimensions.

In general, the Hamiltonian equation

$$iu_t + \Delta u + |u|^{p-1} u = 0, \tag{1.2}$$

has the scaling symmetry

$$u(t,x) \mapsto v(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x). \tag{1.3}$$

That is, if u solves (1.2), then v solves (1.2) for any $\lambda > 0$, where v is given by (1.3). In particular, (1.1) is called L^2-critical or mass-critical, since if u solves (1.1), then

$$v(t,x) = \lambda^{d/2} u(\lambda^2 t, \lambda x) \tag{1.4}$$

is also a solution to (1.1) with initial data $v_0 = \lambda^{d/2} u_0(\lambda x)$. A change of variables calculation verifies that $\|v_0\|_{L^2} = \|u_0\|_{L^2}$.

As in one dimension, the scaling symmetry (1.4) completely controls the local well-posedness theory of (1.1). Indeed, [CW90] proved

Theorem 1. *The initial value problem* (1.1) *is locally well-posed for any* $u_0 \in L^2$.

1. *For any* $u_0 \in L^2$ *there exists* $T(u_0) > 0$ *such that* (1.1) *is locally well-posed on the interval* $(-T, T)$.
2. *If* $\|u_0\|_{L^2}$ *is small then* (1.1) *is globally well-posed, and the solution scatters both forward and backward in time. That is, there exist* u_+, $u_- \in L^2(\mathbb{R}^d)$ *such that*

$$\lim_{t \nearrow +\infty} \|u(t) - e^{it\Delta} u_+\|_{L^2} = 0, \qquad \lim_{t \searrow -\infty} \|u(t) - e^{it\Delta} u_-\|_{L^2} = 0. \tag{1.5}$$

(3) *If I is the maximal interval of existence for a solution to* (1.1) *with initial data u_0, u is said to blow up forward in time if*

$$\lim_{T \nearrow \sup(I)} \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}([0,T]\times\mathbb{R}^d)} = +\infty. \tag{1.6}$$

If u does not blow up forward in time, then $\sup(I) = +\infty$ and u scatters forward in time.

(4) *If $\sup(I) < \infty$, then for any $s > 0$,*

$$\lim_{t \nearrow \sup(I)} \|u(t)\|_{H^s} = +\infty. \tag{1.7}$$

(5) *Time reversal symmetry implies that the results corresponding to* (3) *and* (4) *also hold going backward in time.*

Proof. The proof in [CW90] combines Strichartz estimates (see [GV92], [KT98], [Str77b],[Yaj87]) with Picard iteration. See Section 1.3 of [Dod19] or Section 3.3 of [Tao06] for a detailed proof. See also [GV79b], [GV79a], [GV85], and [Kat87]. □

Theorem 1 also holds for the defocusing, nonlinear Schrödinger equation,

$$iu_t + \Delta u - |u|^{4/d}u = 0, \qquad u(0,x) = u_0(x) \in L^2(\mathbb{R}^d), \tag{1.8}$$

and the proof is identical.

However, the global theory for (1.1) with large data differs substantially from the global theory for (1.8) with large data. Observe that the equation

$$iu_t + \Delta u - \mu|u|^{p-1}u = 0, \tag{1.9}$$

is the Hamiltonian equation for the Hamiltonian

$$E(u(t)) = \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 + \frac{\mu}{p+1}\|u(t)\|_{L^{p+1}}^{p+1} = E(u(0)). \tag{1.10}$$

Both (1.1) and (1.8) conserve the mass

$$M(u(t)) = \int |u(t,x)|^2 dx = M(u(0)). \tag{1.11}$$

When $\mu = +1$, as in (1.8), the energy (1.10) is positive definite, so for $u_0 \in H^1(\mathbb{R}^d)$, conservation of energy guarantees a uniform bound on $\|u(t)\|_{H^1(\mathbb{R}^d)}$, which by (1.7) guarantees that the solution u to (1.8) with initial data $u_0 \in H^1$ is global. Global well-posedness and scattering for (1.8) for general $u_0 \in L^2$ was proved in [Dod12], [Dod16b], and [Dod16a]. When $\mu = -1$, the energy is given by

$$E(u(t)) = \frac{1}{2}\int |\nabla u(t,x)|^2 dx - \frac{d}{2(d+2)}\int |u(t,x)|^{\frac{2(d+2)}{d}} dx. \tag{1.12}$$

The most that (1.11) and (1.12) guarantee is a uniform bound on $\|\nabla u(t)\|_{L^2}$ for $\|u_0\|_{L^2}$ below a threshold mass. Indeed, in two dimensions, a straightforward application of the fundamental theorem of calculus and Hölder's inequality implies

(1.13)

$$\int |u(t,x,y)|^4 dxdy \leq \int |u(t,x,y)|^2 (\int |u_y(t,x,s_2)|^2 ds_2)^{1/2} (\int |u(t,x,s_2)|^2 ds_2)^{1/2} dxdy$$

$$\leq \int\int (\int |u_x(t,s_1,y)|^2 ds_1)^{1/2} (\int |u(t,s_1,y)|^2 ds_1)^{1/2} (\int |u_y(t,x,s_2)|^2 ds_2)^{1/2} (\int |u(t,x,s_2)|^2 ds_2)^{1/2} dxdy$$

$$\leq \|\nabla u\|_{L^2}^2 \|u\|_{L^2}^2.$$

Therefore, there exists a threshhold mass M_0 for which, if $\|u_0\|_{L^2} < M_0$,

$$E(u(t)) \gtrsim_{M_0} \|u(t)\|^2_{\dot{H}^1(\mathbb{R}^2)}, \tag{1.14}$$

with implicit constant $\searrow 0$ as $\|u_0\|_{L^2} \nearrow M_0$. In higher dimensions, by the Sobolev embedding theorem,

$$\|u\|^{\frac{2(d+2)}{d}}_{L_x^{\frac{2(d+2)}{d}}(\mathbb{R}^d)} \lesssim \|u\|^{\frac{4}{d}}_{L^2} \|\nabla u\|^2_{L^2}, \tag{1.15}$$

so then (1.15) holds in dimensions $d \geq 3$ for some M_0 that may depend on d.

Remark 1. *See the introduction to* [Dod21a] *and references therein for more information on blowup for mass-subcritical and mass-supercritical results.*

From [Wei83], the optimal constant in (1.13) and (1.15) is given by the Gagliardo–Nirenberg inequality,

$$\|u\|^{\frac{2(d+2)}{d}}_{L^{\frac{2(d+2)}{d}}(\mathbb{R}^d)} \leq \frac{d+2}{d} \left(\frac{\|u\|^{4/d}_{L^2}}{\|Q\|^{4/d}_{L^2}}\right) \|\nabla u\|^2_{L^2}, \tag{1.16}$$

where Q is the unique positive, radial solution to

$$\Delta Q + Q^{1+\frac{4}{d}} = Q. \tag{1.17}$$

Remark 2. *The unique positive, radial solution to* (1.17) *is called the ground state. See* [BL78], [BLP81], [Kwo89], *and* [Str77a] *for the existence and uniqueness of a ground state solution to* (1.17) *in general dimensions.*

Thus, (1.12) implies that (1.1) with initial data $u_0 \in H^1$ and $\|u_0\|_{L^2} < \|Q\|_{L^2}$ is globally well-posed. Global well-posedness and scattering for (1.1) with a general $\|u_0\|_{L^2} < \|Q\|_{L^2}$ was proved in [Dod15].

It is straightforward to see that $u(t,x) = e^{it}Q(x)$ solves (1.1), which gives a solution to (1.1) with mass $\|u_0\|_{L^2} = \|Q\|_{L^2}$ that blows up both forward and backward in time (according to (1.6)). Making a pseudoconformal transformation of $e^{it}Q(x)$,

$$u(t,x) = \frac{1}{t^{d/2}} e^{\frac{i}{t} + \frac{i|x|^2}{4t}} Q\left(\frac{x}{t}\right), \tag{1.18}$$

is a solution to (1.1) that blows up as $t \searrow 0$. Observe that $\|\nabla u(t)\|_{L^2} \nearrow \infty$ as $t \searrow 0$ for (1.18).

The soliton solutions and the pseudoconformal transformations of the solitons are the only solutions to (1.1) with mass $\|u_0\|_{L^2} = \|Q\|_{L^2}$. Previously, in [Dod21a], we proved

Theorem 2. *In dimension $d = 1$, the only solutions to* (1.1) *with mass $\|u_0\|_{L^2} = \|Q\|_{L^2}$ that blow up forward in time are the family of soliton solutions*

$$e^{-i\theta - it\xi_0^2} e^{i\lambda^2 t} e^{ix\xi_0} \lambda^{1/2} Q(\lambda(x - 2t\xi_0) + x_0), \qquad \lambda > 0, \qquad \theta \in \mathbb{R}, \qquad x_0 \in \mathbb{R}, \qquad \xi_0 \in \mathbb{R}, \tag{1.19}$$

and the pseudoconformal transformation of the family of solitons,

$$\frac{\lambda^{1/2}}{(T-t)^{1/2}} e^{i\theta} e^{\frac{i(x-\xi_0)^2}{4(t-T)}} e^{i\frac{\lambda^2}{t-T}} Q\left(\frac{\lambda(x-\xi_0) - (T-t)x_0}{T-t}\right), \tag{1.20}$$

$$\text{where} \qquad \lambda > 0, \qquad \theta \in \mathbb{R}, \qquad x_0 \in \mathbb{R}, \qquad \xi_0 \in \mathbb{R}, \qquad T \in \mathbb{R}, \qquad t < T.$$

In this paper we prove the same result in dimensions $2 \leq d \leq 15$.

Theorem 3. *In dimensions $2 \leq d \leq 15$, the only solutions to (1.1) with mass $\|u_0\|_{L^2} = \|Q\|_{L^2}$ that blow up forward in time are the family of soliton solutions*

(1.21)
$$e^{-i\theta - it|\xi_0|^2} e^{i\lambda^2 t} e^{ix\cdot\xi_0} \lambda^{d/2} Q(\lambda(x - 2t\xi_0) + x_0), \qquad \lambda > 0, \qquad \theta \in \mathbb{R}, \qquad x_0 \in \mathbb{R}^d, \qquad \xi_0 \in \mathbb{R}^d,$$

and the pseudoconformal transformation of the family of solitons,

(1.22)
$$\frac{\lambda^{d/2}}{(T-t)^{d/2}} e^{i\theta} e^{\frac{i|x-\xi_0|^2}{4(t-T)}} e^{i\frac{\lambda^2}{t-T}} Q(\frac{\lambda(x-\xi_0) - (T-t)x_0}{T-t}),$$
$$\text{where} \qquad \lambda > 0, \qquad \theta \in \mathbb{R}, \qquad x_0 \in \mathbb{R}^d, \qquad \xi_0 \in \mathbb{R}^d, \qquad T \in \mathbb{R}, \qquad t < T.$$

Applying time reversal symmetry to (1.1), this theorem completely settles the question of qualitative behavior of solutions to (1.1) for initial data satisfying $\|u_0\|_{L^2} = \|Q\|_{L^2}$.

Remark 3. *The obstruction to proving Theorem 3 in dimensions $d \geq 16$ appears to be a purely technical obstruction. The issue arises only in section ten, and will be discussed in more detail there.*

The proof of Theorem 3 relies heavily on the virial identity

(1.23)
$$\frac{d}{dt} \int x \cdot Im[\bar{u}\nabla u](t,x)dx = 4E(u(t)).$$

Using the Pohozaev identity,

(1.24)
$$E(Q) = \frac{1}{2} \int (Q - \Delta Q - Q^{1+4/d})(\frac{d}{2}Q + x \cdot \nabla Q)dx = 0.$$

Thus, by (1.16), Q is a minimizer of the energy when $\|u\|_{L^2} = \|Q\|_{L^2}$. In fact, up to scaling, Q is the unique minimizer of the energy (see [Wei83]). So when $\|u\|_{L^2} = \|Q\|_{L^2}$, the energy $E(u)$ in (1.12) gives a good measurement for the distance from u to the set

(1.25)
$$\{e^{i\theta}\lambda^{d/2}Q(\lambda x + x_0) : \lambda > 0, \qquad x_0 \in \mathbb{R}^d, \qquad \theta \in \mathbb{R}\}.$$

However, $E(u)$ is not invariant under the scaling symmetry (1.4), so this notion will not be made precise until later. It will also be necessary to account for the Galilean transformation, which does change the energy.

The proof of Theorem 3 will occupy most of the paper, and will follow the argument in [Dod21a]. There are many places where the argument is exactly the same, and in those places the argument will often be abbreviated, and the reader will be referred to [Dod21a] for more details. There are other places where the argument is much more technically difficult, especially in dimensions $d \geq 3$. This is due to the fact that $F(x) = |x|^{4/d}x$ is not a smooth function of x in dimensions $d \geq 3$. Circumventing this difficulty will rely on the tools in [Tay07] and [Vis07], along with bounds on $\frac{\nabla Q(x)}{Q(x)^{1-\alpha}}$ for some $\alpha > 0$, when Q is the ground state solution to (1.17). Theorem 25 is the only obstacle to the proof of Theorem 3 in dimensions $d \geq 16$.

However, before proving Theorem 3, it will be useful to cite two previous results making partial progress in the direction of Theorem 3.

Theorem 4. *If $u_0 \in H^1$, $\|u_0\|_{L^2} = \|Q\|_{L^2}$, and the solution $u(t)$ to (1.1) blows up in finite time $T > 0$, then $u(t,x)$ is in the form of (1.22).*

Proof. This result was proved in [Mer92] and [Mer93], and was proved for the focusing, mass-critical nonlinear Schrödinger equation in every dimension. □

For the mass-critical nonlinear Schrödinger equation in higher dimensions with radially symmetric initial data, [KLVZ09] proved

Theorem 5. *If $\|u_0\|_{L^2} = \|Q\|_{L^2}$ is radially symmetric, and u is the solution to the focusing, mass-critical nonlinear Schrödinger equation with initial data u_0, and u blows up both forward and backward in time, then u is equal to* (1.21) *with $x_0 = \xi_0 = 0$.*

After proving Theorem 3, we will show how Theorem 3 implies a Liouville result for blowup solutions to the mass-critical problem. This result is very similar in nature to the Liouville result in [MM00] for the generalized KdV equation. This result holds in any dimension for which Theorems 2 or 3 hold.

2. Reduction of a blowup solution

As in [Dod21a], the first step is to reduce Theorem 3 to a result for solutions to (1.1) that blow up forward in time and are close to the family of solitons for every positive time.

Theorem 6. *Let $0 < \eta_* \ll 1$ be a small, fixed constant to be defined later. If u is a solution to* (1.1) *on the maximal interval of existence $I \subset \mathbb{R}$, $\|u_0\|_{L^2} = \|Q\|_{L^2}$, u blows up forward in time, and*

$$\sup_{t\in[0,\sup(I))} \inf_{\gamma\in\mathbb{R},\lambda>0,\xi\in\mathbb{R}^d,\tilde{x}\in\mathbb{R}^d} \|e^{i\gamma}e^{ix\cdot\xi}\lambda^{d/2}u(t,\lambda x+\tilde{x}) - Q\|_{L^2} \leq \eta_*, \tag{2.1}$$

then u is a soliton solution of the form (1.17) *or a pseudoconformal transformation of the soliton of the form* (1.18).

Theorem 3 follows from Theorem 6.

Proof that Theorem 6 implies Theorem 3. Let u be a blowup solution to (1.1) with mass $\|u_0\|_{L^2} = \|Q\|_{L^2}$. By (1.17),

$$\begin{aligned} &\inf_{\lambda>0,\gamma\in\mathbb{R},\tilde{x}\in\mathbb{R}^d,\xi\in\mathbb{R}^d} \|e^{i\gamma}e^{ix\cdot\xi}\lambda^{d/2}u_0(\lambda x+\tilde{x}) - Q(x)\|_{L^2(\mathbb{R}^d)} \\ &= \inf_{\lambda>0,\gamma\in\mathbb{R},\tilde{x}\in\mathbb{R}^d,\xi\in\mathbb{R}^d} \|u_0(x) - e^{-i\gamma}e^{-ix\cdot\frac{\xi}{\lambda}}\lambda^{-d/2}Q(\frac{x-\tilde{x}}{\lambda})\|_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{2.2}$$

As in the $d = 1$ case, there exist $x_0 \in \mathbb{R}^d$, $\xi_0 \in \mathbb{R}^d$, $\lambda_0 > 0$, $\gamma_0 \in \mathbb{R}$ where this infimum is attained.

Lemma 1. *There exists $\lambda_0 > 0$, $\gamma_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^d$, and $\xi_0 \in \mathbb{R}^d$, such that*

$$\|u_0(x) - e^{-i\gamma_0}e^{-ix\cdot\frac{\xi_0}{\lambda_0}}\lambda_0^{-d/2}Q(\frac{x-x_0}{\lambda_0})\|_{L^2} = \inf_{\lambda>0,\gamma\in\mathbb{R},\tilde{x}\in\mathbb{R}^d,\xi\in\mathbb{R}^d} \|u_0(x) - e^{-i\gamma}e^{-ix\cdot\frac{\xi}{\lambda}}\lambda^{-d/2}Q(\frac{x-\tilde{x}}{\lambda})\|_{L^2}. \tag{2.3}$$

Proof. As in one dimension, the ground state solution Q is smooth and rapidly decreasing.

Theorem 7. *There exists a unique positive radially symmetric solution Q to*

$$\Delta Q + Q^{1+4/d} = Q \tag{2.4}$$

in H^1 which is called the ground state. In addition, $Q \in C^\infty(\mathbb{R}^d)$, $\partial_r Q(r) < 0$ for all $r > 0$, and there exists $\delta > 0$ such that

$$|\partial^\alpha Q(x)| \lesssim_\alpha e^{-\delta|x|}, \qquad \forall x \in \mathbb{R}^d, \qquad \forall \alpha \in \mathbb{Z}_+^d. \tag{2.5}$$

Proof. See page 641 of [CMPS16]. □

Throughout the paper, $(f,g)_{L^2}$ will denote the inner product

$$(f,g)_{L^2} = Re \int \overline{f(x)} g(x) dx. \tag{2.6}$$

Since Q is smooth and rapidly decreasing,

$$\begin{aligned} &(u_0(x) - e^{-i\gamma} e^{-ix\cdot\frac{\xi}{\lambda}} \lambda^{-d/2} Q(\frac{x-\tilde{x}}{\lambda}), u_0(x) - e^{-i\gamma} e^{-ix\cdot\frac{\xi}{\lambda}} \lambda^{-d/2} Q(\frac{x-\tilde{x}}{\lambda}))_{L^2} \\ &\qquad = 2\|Q\|_{L^2}^2 - 2(e^{-i\gamma} e^{-ix\cdot\frac{\xi}{\lambda}} \lambda^{-d/2} Q(\frac{x-\tilde{x}}{\lambda}), u_0(x))_{L^2}, \end{aligned} \tag{2.7}$$

is differentiable in $\gamma \in \mathbb{R}$, $\xi \in \mathbb{R}^d$, $\tilde{x} \in \mathbb{R}^d$, and $\lambda > 0$. Since

$$\int_0^{2\pi} (e^{-i\gamma} e^{-ix\cdot\frac{\xi}{\lambda}} \lambda^{-d/2} Q(\frac{x-\tilde{x}}{\lambda}), u_0(x))_{L^2} d\gamma = 0, \tag{2.8}$$

if $(2.3) = 2\|Q\|_{L^2}^2$, then $(2.7) = 2\|Q\|_{L^2}^2$ for any λ, γ, $\tilde{x}$, and ξ. In this case it is convenient to take $\lambda_0 = 1$, $\gamma_0 = 0$, $\xi_0 = 0$, and $\tilde{x} = 0$. If $(2.3) < 2\|Q\|_{L^2}^2$, then since

$$(e^{-i\gamma} e^{-ix\cdot\frac{\xi}{\lambda}} \lambda^{-d/2} Q(\frac{x-\tilde{x}}{\lambda}), u_0(x))_{L^2} \to 0, \tag{2.9}$$

as $\lambda \nearrow \infty$, $\lambda \searrow 0$, $|\tilde{x}| \to \infty$, or $|\xi| \to \infty$, (2.3) is equal to the infimum over a compact set $[\lambda_1, \lambda_2] \times \{\xi : |\xi| \leq R\} \times \{\tilde{x} : |\tilde{x}| \leq R\} \times [0, 2\pi]$. Indeed, (2.9) holds as $\lambda \searrow 0$ or $\lambda \nearrow \infty$, uniformly over $\tilde{x}$ and ξ, so restrict $\lambda \in [\lambda_1, \lambda_2]$. Then (2.9) holds as $|\tilde{x}| \to \infty$, uniformly over $\lambda \in [\lambda_1, \lambda_2]$ and $\xi \in \mathbb{R}^d$, so restrict $\tilde{x} \in \{x : |x| \leq R\}$ for some large R. Finally, (2.9) holds as $|\xi| \to \infty$ for $|\tilde{x}| \leq R$ and $\lambda \in [\lambda_1, \lambda_2]$. Since (2.7) is continuous in λ, $\tilde{x}$, ξ, and γ, the infimum is attained on this compact set. □

Now recall the sequential convergence results of [Fan21] and [Dod21b].

Theorem 8. *Assume u is a symmetric solution to (1.1) with $\|u\|_{L^2} = \|Q\|_{L^2}$ that does not scatter forward in time. Let $(T_-(u), T_+(u))$ be its lifespan. Then there exists a sequence $t_n \to T_+(u)$ and a family of parameters λ_n, γ_n such that*

$$e^{i\gamma_n} \lambda_n^{1/2} u(t_n, \lambda_n x) \to Q, \qquad in \qquad L^2. \tag{2.10}$$

Proof. This is proved in Theorem 1.3 of [Fan21]. □

Theorem 9. *Assume u is a solution to (1.1) with $\|u_0\|_{L^2} = \|Q\|_{L^2}$ which blows up forward in time. Let $(T_-(u), T_+(u))$ be its lifespan. Then there exists a sequence $t_n \nearrow T_+(u)$ and a family of parameters $\lambda_{*,n}$, $\gamma_{*,n}$, $\xi_{*,n}$, $x_{*,n}$ such that*

$$e^{i\gamma_{*,n}} e^{ix\cdot\xi_{*,n}} \lambda_{*,n}^{d/2} u(t_n, \lambda_{*,n} x + x_{*,n}) \to Q, \qquad in \qquad L^2. \tag{2.11}$$

Proof. This is proved in Theorem 2 of [Dod21b]. □

Then make the same argument as in [Dod21a], where we proved that Theorem 6 implies Theorem 4 and that Theorem 20 implies Theorem 5. □

3. Decomposition of the solution near Q

Under the assumptions of Theorem 6, when $\eta_* \ll 1$ is sufficiently small, it is possible to decompose u into the sum of a soliton and a remainder for which the linearization of (1.17) has good spectral properties. Recall that in one dimension, the linearization of (1.17),

$$\mathcal{L} = -\partial_{xx} + 1 - 5Q^4, \tag{3.1}$$

has one negative eigenvalue and one zero eigenvalue. Choose λ, ξ, $\tilde{x}$, and γ so that the difference ϵ between the soliton and the solution u acted on by the representation of the group element $(\lambda, \xi, \tilde{x}, \gamma)$ is orthogonal to these two eigenvectors. The fact that $\mathcal{L}$ is positive definite on a subspace containing ϵ gives lower bounds on various virial and energy identities as a function of the size of ϵ.

The same can be done in higher dimensions. In two dimensions, the spectral theory for

$$\mathcal{L} = -\Delta + 1 - 3Q^2, \qquad \mathcal{L}_- = -\Delta + 1 - Q^2, \tag{3.2}$$

is found in [CGNT08], [Kwo89], and [Mar02]. From the product rule, for any $j = 1, 2$,

$$\mathcal{L}(Q_{x_j}) = -\Delta Q_{x_j} + Q_{x_j} - 3Q^2 Q_{x_j} = \partial_{x_j}(-\Delta Q + Q - Q^3) = 0. \tag{3.3}$$

Thus Q_{x_j} are eigenvectors of $\mathcal{L}$ with eigenvalue zero. These are the only two, and there is only one eigenvector of $\mathcal{L}$ with negative eigenvalue.

Theorem 10. *The following holds for an operator $\mathcal{L}$ defined in* (3.2).

1. *$\mathcal{L}$ is a self-adjoint operator and $\sigma_{ess}(\mathcal{L}) = [1, +\infty)$.*
2. *$Ker(\mathcal{L}) = span\{Q_{x_1}, Q_{x_2}\}$.*
3. *$\mathcal{L}$ has a unique single negative eigenvalue $-\lambda_0$ associated to a positive, radially symmetric eigenfunction χ_0. Without loss of generality, χ_0 can be chosen such that $\|\chi_0\|_{L^2} = 1$. Moreover, there exists $\delta > 0$ such that $|\chi_0(x)| \lesssim e^{-\delta|x|}$ for all $x \in \mathbb{R}^2$.*

Proof. This is Theorem 3.3 of [FHRY18]. □

In higher dimensions, [CGNT08], [Kwo89], and [Wei85] proved a similar result for the spectral theory of $\mathcal{L}$ and $\mathcal{L}_-$.

$$\mathcal{L} = -\Delta + 1 - \frac{d+4}{d} Q^{\frac{4}{d}}, \qquad \mathcal{L}_- = -\Delta + 1 - Q^{\frac{4}{d}}, \tag{3.4}$$

Theorem 11. *The following holds for an operator $\mathcal{L}$ defined in* (3.4).

1. *$\mathcal{L}$ is a self-adjoint operator and $\sigma_{ess}(\mathcal{L}) = [1, +\infty)$.*
2. *$Ker(\mathcal{L}) = span\{Q_{x_1}, \cdots, Q_{x_d}\}$.*
3. *$\mathcal{L}$ has a unique single negative eigenvalue $-\lambda_d$ associated to a positive, radially symmetric eigenfunction χ_0. Without loss of generality, χ_0 can be chosen such that $\|\chi_0\|_{L^2} = 1$. Moreover, there exists $\delta > 0$ such that $|\chi_0(x)| \lesssim e^{-\delta|x|}$ for all $x \in \mathbb{R}^d$.*

Proof. This theorem is copied from page 642 of [CMPS16]. □

For u_0 sufficiently close to Q, it is possible to choose λ, γ, $\tilde{x}$, and ξ so that the remainder is orthogonal to the kernel of $\mathcal{L}$ and to the negative eigenvector.

Theorem 12. *Take $u \in L^2$. There exists $\alpha > 0$ sufficiently small such that if there exist $\lambda_0 > 0$, $\gamma_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^d$, and $\xi_0 \in \mathbb{R}^d$ that satisfy*

$$\|e^{i\gamma_0} e^{ix\cdot\xi_0} \lambda_0^{d/2} u(\lambda_0 x + x_0) - Q(x)\|_{L^2} \leq \alpha, \tag{3.5}$$

then there exist unique $\lambda > 0$, $\gamma \in \mathbb{R}$, $\tilde{x} \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$ *such that if*

$$\epsilon(x) = e^{i\gamma} e^{ix\cdot\xi} \lambda^{d/2} u(\lambda x + \tilde{x}) - Q(x), \tag{3.6}$$

then for $j = 1, ..., d$,

$$(\epsilon, \chi_0)_{L^2} = (\epsilon, i\chi_0)_{L^2} = (\epsilon, Q_{x_j})_{L^2} = (\epsilon, iQ_{x_j})_{L^2} = 0, \tag{3.7}$$

and

$$\|\epsilon\|_{L^2} + |\frac{\lambda}{\lambda_0} - 1| + |\gamma - \gamma_0 - \xi_0 \cdot (\tilde{x} - x_0)| + |\xi - \frac{\lambda}{\lambda_0}\xi_0| + |\frac{\tilde{x} - x_0}{\lambda_0}| \lesssim \|e^{i\gamma_0} e^{ix\cdot\xi_0} \lambda_0 u(\lambda_0 x + x_0) - Q\|_{L^2}. \tag{3.8}$$

Remark 4. *As usual,* γ *is unique up to translation by* $2\pi n$*, where* n *is an integer.*

Proof. Let f denote an element of the set

$$f \in \{\chi_0, i\chi_0, Q_{x_1}, ..., Q_{x_d}, iQ_{x_1}, ..., iQ_{x_d}\}. \tag{3.9}$$

By Hölder's inequality,

$$|(e^{i\gamma_0} e^{ix\cdot\xi_0} \lambda_0^{d/2} u(t, \lambda_0 x + \tilde{x}) - Q(x), f)_{L^2}| \lesssim \|e^{i\gamma_0} e^{ix\cdot\xi_0} \lambda_0^{d/2} u(\lambda_0 x + \tilde{x}) - Q\|_{L^2}. \tag{3.10}$$

The inner product in (3.10) is C^1 as a function of γ, λ, $\tilde{x}$, and ξ. Indeed,

$$\frac{\partial}{\partial\gamma}(e^{i\gamma} e^{ix\cdot\xi} \lambda^{d/2} u(\lambda x + \tilde{x}) - Q(x), f)_{L^2} = (ie^{i\gamma} e^{ix\cdot\xi} \lambda^{d/2} u(\lambda x + \tilde{x}), f)_{L^2} \lesssim \|u\|_{L^2} \|f\|_{L^2}. \tag{3.11}$$

Next, since all f are smooth with rapidly decreasing derivatives,

$$\frac{\partial}{\partial\xi}(e^{i\gamma} e^{ix\cdot\xi} \lambda^{d/2} u(\lambda x + \tilde{x}) - Q(x), f)_{L^2} = (ixe^{i\gamma} e^{ix\cdot\xi} \lambda^{d/2} u(\lambda x + \tilde{x}), f)_{L^2} \lesssim \|u\|_{L^2} \|xf\|_{L^2}. \tag{3.12}$$

Integrating by parts,

$$\frac{\partial}{\partial\tilde{x}}(e^{i\gamma} e^{ix\cdot\xi} \lambda^{d/2} u(\lambda x + \tilde{x}) - Q(x), f)_{L^2} = (e^{i\gamma} e^{ix\cdot\xi} \lambda^{d/2} \nabla u(\lambda x + \tilde{x}), f)_{L^2} \tag{3.13}$$

$$= -\frac{1}{\lambda}(e^{i\gamma} e^{ix\cdot\xi} \lambda^{d/2} u(\lambda x + \tilde{x}), \nabla f)_{L^2} - \frac{\xi}{\lambda}(ie^{i\gamma} e^{ix\cdot\xi} \lambda^{d/2} u(\lambda x + \tilde{x}), f)_{L^2} \lesssim \frac{1}{\lambda}\|u\|_{L^2}\|\nabla f\|_{L^2} + \frac{|\xi|}{\lambda}\|u\|_{L^2}\|f\|_{L^2}.$$

Finally,

$$\frac{\partial}{\partial\lambda}(e^{i\gamma} e^{ix\cdot\xi} \lambda^{d/2} u(\lambda x + \tilde{x}) - Q(x), f)_{L^2} = (\frac{d}{2}\lambda^{d/2-1} e^{i\gamma} e^{ix\cdot\xi} u(\lambda x + \tilde{x}) + e^{i\gamma} e^{ix\cdot\xi} \lambda^{d/2} x \cdot \nabla u(\lambda x + \tilde{x}), f)_{L^2} \tag{3.14}$$

$$\lesssim \frac{1}{\lambda}\|u\|_{L^2}\|f\|_{L^2} + \frac{|\xi|}{\lambda}\|u\|_{L^2}\|xf\|_{L^2} + \frac{1}{\lambda}\|u\|_{L^2}\|x\nabla f\|_{L^2}.$$

Therefore, the inner product is a C^1 function of γ, ξ, λ, and $\tilde{x}$. Repeating the above calculations would also show that the inner product is C^2.

Computing (3.11)–(3.14) at $u = Q$, $\xi = \tilde{x} = \gamma = 0$, $\lambda = 1$,

$$\frac{\partial}{\partial\gamma}(e^{i\gamma} e^{ix\cdot\xi} \lambda^{d/2} u(\lambda x + \tilde{x}) - Q(x), f)_{L^2}|_{u=Q,\lambda=1,\gamma=\tilde{x}=\xi=0} = (iQ, f)_{L^2} = 0 \qquad \text{if} \qquad f \in \{\chi_0, Q_{x_j}, iQ_{x_j}\}, \tag{3.15}$$

$$= (iQ, i\chi_0)_{L^2} = (Q, \chi_0)_{L^2} > 0, \qquad \text{if} \qquad f = i\chi_0.$$

The fact that $(Q,\chi_0)_{L^2}>0$ follows from the fact that $\chi_0>0$ and $Q>0$. Next,

(3.16)
$$\frac{\partial}{\partial\xi_k}(e^{i\gamma}e^{ix\cdot\xi}\lambda^{d/2}u(\lambda x+\tilde{x})-Q(x),f)_{L^2}|_{u=Q,\lambda=1,\gamma=\tilde{x}=\xi=0}=(ix_kQ,f)_{L^2}=0,\qquad \text{if}\qquad f\in\{\chi_0,Q_{x_j},i\chi_0\},$$
$$=(ix_kQ,iQ_{x_j})_{L^2}=(x_kQ,Q_{x_j})_{L^2}=-\frac{\delta_{jk}}{2}\|Q\|_{L^2}^2\qquad \text{if}\qquad f=iQ_{x_j},\qquad j=1,...,d.$$

Next,

$$\frac{\partial}{\partial\tilde{x}_k}(e^{i\gamma}e^{ix\cdot\xi}\lambda^{d/2}u(\lambda x+\tilde{x})-Q(x),f)_{L^2}|_{u=Q,\lambda=1,\gamma=\tilde{x}=\xi=0}=(Q_{x_k},f)_{L^2}=0,$$
(3.17)
$$\text{if}\qquad f\in\{i\chi_0,iQ_{x_j},\chi_0\}$$
$$=(Q_{x_k},Q_{x_j})_{L^2}=\frac{\delta_{jk}}{d}\|\nabla Q\|_{L^2}^2,\qquad \text{if}\qquad f=Q_{x_j},\qquad j=1,...,d.$$

Finally,

(3.18)
$$\frac{\partial}{\partial\lambda}(e^{i\gamma}e^{ix\cdot\xi}\lambda^{d/2}u(\lambda x+\tilde{x})-Q(x),f)_{L^2}|_{u=Q,\lambda=1,\gamma=\tilde{x}=\xi=0}=(\frac{d}{2}Q+x\cdot\nabla Q,f)_{L^2}=0$$
$$\text{if}\qquad f\in\{i\chi_0,iQ_{x_j},Q_{x_j}\},$$
$$=(\frac{d}{2}Q+x\cdot\nabla Q,\chi_0)_{L^2}=\frac{-1}{\lambda_d}(\frac{d}{2}Q+x\cdot\nabla Q,\mathcal{L}\chi_0)_{L^2}=-\frac{1}{\lambda_d}(\mathcal{L}(\frac{d}{2}Q+x\cdot\nabla Q),\chi_0)_{L^2}=\frac{2}{\lambda_d}(Q,\chi_0)>0,$$
$$\text{if}\qquad f=\chi_0.$$

Therefore, by the inverse function theorem, there exists a unique $\lambda>0$, $\tilde{x}\in\mathbb{R}^2$, $\xi\in\mathbb{R}^2$ and $\gamma\in\mathbb{R}/2\pi\mathbb{Z}$ in a neighborhood of λ_0, ξ_0, x_0, and γ_0 such that (3.7) holds.

The proof of (3.8) follows from acting on u by symmetries to map to $\lambda_0=1$, $\tilde{x}=\xi=\gamma=0$, applying the inverse function theorem, and then mapping back to the original u. The proof of uniqueness in $\mathbb{R}_{>0}\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}/2\pi\mathbb{Z}$ is identical to the proof of uniqueness in [Dod21a]. □

Therefore, in Theorem 6, there exist functions

$$\lambda(t):[0,\sup(I))\to(0,\infty),\qquad \xi(t):[0,\sup(I))\to\mathbb{R}^d,$$
(3.19)
$$x(t):[0,\sup(I))\to\mathbb{R}^d,\qquad \gamma(t):[0,\sup(I))\to\mathbb{R},$$

such that (3.7) holds for all $t\in[0,\sup(I))$.

Furthermore, define a monotone function,

$$s(t):[0,\sup(I))\to[0,\infty),\qquad s(t)=\int_0^t\lambda(\tau)^{-2}d\tau.\tag{3.20}$$

As in Theorem 10 of [Dod21a], the functions in (3.19) are differentiable in time for almost every $t\in[0,\sup(I))$, and furthermore, taking $\epsilon=\epsilon_1+i\epsilon_2$,

(3.21)
$$\epsilon_s=i(\gamma_s+1)(Q+\epsilon)+i\xi_s\cdot x(Q+\epsilon)+\frac{\lambda_s}{\lambda}(\frac{d}{2}(Q+\epsilon)+x\cdot\nabla(Q+\epsilon))-i\frac{\lambda_s}{\lambda}\xi(s)\cdot x(Q+\epsilon)$$
$$+\frac{x_s}{\lambda}\cdot\nabla(Q+\epsilon)-i\frac{x_s}{\lambda}\cdot\xi(s)(Q+\epsilon)+i\mathcal{L}\epsilon_1-\mathcal{L}_-\epsilon_2+2\xi(s)\cdot\nabla(Q+\epsilon)-i|\xi(s)|^2(Q+\epsilon)$$
$$+iO(|Q|^{4/d-1}|\epsilon|^2+|\epsilon|^{1+4/d}),\qquad \text{when}\qquad 2\le d\le 4,\qquad +iO(|\epsilon|^{1+4/d})\qquad \text{when}\qquad d>5.$$

For f in (3.9),

$$\frac{d}{ds}(\epsilon, f)_{L^2} = (\epsilon_s, f)_{L^2} = 0. \tag{3.22}$$

Plugging (3.21) into (3.22) and following the analysis in Section 10.2 of [Dod21a], for any $a \in \mathbb{Z}_{\geq 0}$,

$$\int_a^{a+1} |\gamma_s + 1 - \frac{x_s}{\lambda} \cdot \xi(s) - |\xi(s)|^2| ds \lesssim \int_a^{a+1} \|\epsilon(s)\|_{L^2}^2 ds, \tag{3.23}$$

$$\int_a^{a+1} |\xi_s - \frac{\lambda_s}{\lambda} \xi(s)| ds \lesssim \int_a^{a+1} \|\epsilon(s)\|_{L^2}^2 ds, \tag{3.24}$$

$$\int_a^{a+1} |\frac{\lambda_s}{\lambda}| ds \lesssim \int_a^{a+1} \|\epsilon(s)\|_{L^2} ds, \tag{3.25}$$

and

$$\int_a^{a+1} |\frac{x_s}{\lambda} + 2\xi| ds \lesssim \int_a^{a+1} \|\epsilon(s)\|_{L^2} ds. \tag{3.26}$$

As in [Dod21a], it is also true that under the conditions of Theorem 6, for any $a \geq 0$,

$$\sup_{s \in [a,a+1]} \|\epsilon(s)\|_{L^2} \sim \inf_{s \in [a,a+1]} \|\epsilon(s)\|_{L^2}. \tag{3.27}$$

4. A long time Strichartz estimate when $d = 2$

As in the one dimensional case, the next step is to obtain a long time Strichartz estimate. Roughly speaking, if I is an interval of length $|I| = T$, and $\lambda(t) = 1$ on I, the goal is to obtain good Strichartz estimates at frequencies greater than or equal to T^{α} for some $\alpha < \frac{1}{2}$. In two dimensions, as in one dimension, $\alpha = \frac{1}{3}$ will do. In higher dimensions, we will take T^{α_d}, where $\alpha_d \to \frac{1}{2}$ as d goes to infinity.

The main new technical difficulty in two dimensions is utilizing the interaction Morawetz bilinear estimate of [PV09]. Setting

$$M(t) = \int |u(t,y)|^2 \frac{(x-y)_j}{|(x-y)_j|} Im[\bar{u} \partial_j u](t,x) dx dy, \tag{4.1}$$

if u solves (1.1),

$$\begin{aligned} \frac{d}{dt} M(t) = &-2 \int \partial_k Im[\bar{u} \partial_k u](t,y) \frac{(x-y)_j}{|(x-y)_j|} Im[\bar{u} \partial_j u](t,x) dx dy + \frac{1}{2} \int |u(t,x)|^2 \frac{(x-y)_j}{|(x-y)_j|} \partial_j \Delta(|u|^2) dx dy \\ &-2 \int |u(t,y)|^2 \frac{(x-y)_j}{|(x-y)_j|} \partial_k Re[\partial_j \bar{u} \partial_k](t,x) dx dy + \frac{2}{d+2} \int |u(t,y)|^2 \frac{(x-y)_j}{|(x-y)_j|} \partial_j (|u(t,x)|^{\frac{2(d+2)}{d}}) dx dy. \end{aligned} \tag{4.2}$$

In one dimension, integrating by parts,

$$\frac{d}{dt} M(t) = \int |\partial_x (|u|^2)(t,x)|^2 dx - \frac{2}{3} \int |u(t,x)|^8 dx. \tag{4.3}$$

If the u's in the interaction Morawetz estimate are replaced by Fourier truncations of u, this gives good bilinear estimates, as was used in [Dod21a]. In two dimensions, fixing j, say $j = 1$,

$$\frac{d}{dt} M(t) = \int |\partial_{x_1} (u(t,x_1,x_2) \overline{u(t,x_1,y_2)})|^2 dx_1 dx_2 dy_2 - \frac{1}{2} \int |u(t,x_1,x_2)|^2 |u(t,x_1,y_2)|^4 dx_1 dx_2 dy_2. \tag{4.4}$$

In order to turn (4.4) into a bilinear estimate of the form (4.3), we utilize the Fourier support of u. Suppose again that the u's are replaced by a u at frequencies $\geq k$ and a u at frequencies $\leq k$. By Hölder's inequality and the product rule, if $\tilde{\psi}$ is a rapidly decreasing function obtained from the Littlewood–Paley kernel,

(4.5)
$$\begin{aligned}
&\int |\partial_{x_1}(P_{\leq k}u(t,x_1,x_2)\overline{P_{\geq k}u(t,x_1,x_2)})|^2 dx_1 dx_2 \\
&\lesssim \int |\partial_{x_1}(P_{\geq k}u(t,x_1,x_2)|^2 |P_{\leq k}u(t,x_1,x_2)|^2 dx_1 dx_2 + \|\partial_{x_1}P_{\leq k}u\|_{L^p}^2 \|P_{\geq k}u\|_{L^q}^2 \\
&\lesssim 2^k \int \tilde{\psi}(2^k(x_2-y_2))|\partial_{x_1}(P_{\leq k}u(t,x_1,x_2)\overline{P_{\geq k}u(t,x_1,y_2)})|^2 dx_1 dx_2 dy_2 \\
&+2^k \int \tilde{\psi}(2^k(x_2-y_2))|\partial_{x_1}P_{\leq k}u(t,x_1,x_2)|^2 |P_{\geq k}u(t,x_1,y_2)|^2 dx_1 dx_2 dy_2 + \|\partial_{x_1}P_{\leq k}u\|_{L^p}^2 \|P_{\geq k}u\|_{L^q}^2 \\
&\lesssim 2^k \int |\partial_{x_1}(P_{\leq k}u(t,x_1,x_2)\overline{P_{\geq k}u(t,x_1,y_2)})|^2 dx_1 dx_2 dy_2 + \|\partial_{x_1}P_{\leq k}u\|_{L^p}^2 \|P_{\geq k}u\|_{L^q}^2.
\end{aligned}$$

This calculation appeared previously in [Dod16a] and [Dod19]. Here $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$.

Let $J=[a,b]$ be an interval. As in the one dimensional case, choose

$$0<\eta_1 \ll \eta_0 \ll 1, \tag{4.6}$$

such that

$$\sup_{t\in J}\|\epsilon(t,x)\|_{L^2}^2 \leq \eta_0^2, \tag{4.7}$$

and that

$$\int_{|\xi|\geq \eta_1^{-1/2}} |\hat{Q}(\xi)|^2 \leq \eta_0^2, \tag{4.8}$$

where Q is the soliton solution (1.17). Furthermore, suppose that

$$\frac{1}{\eta_1} \leq \lambda(t) \leq \frac{1}{\eta_1}T^{1/100}, \qquad \text{and} \qquad \frac{|\xi(t)|}{\lambda(t)} \leq \eta_0, \qquad \text{for all} \qquad t\in J, \tag{4.9}$$

and that there exists $k\in\mathbb{Z}_{\geq 0}$ such that

$$\int_J \lambda(t)^{-2}dt = T, \qquad \text{and} \qquad \eta_1^{-2}T = 2^{3k}. \tag{4.10}$$

When $i\in\mathbb{Z}$, $i>0$, let P_i denote the standard Littlewood-Paley projection operator. When $i=0$, let P_i denote the projection operator $P_{\leq 0}$, and when $i<0$, let P_i denote the zero operator. It is convenient to calculate for $\lambda(t)=\frac{1}{\eta_1}$ first and then generalize to (4.9). It is also convenient to assume without loss of generality that $a=0$.

Theorem 13. *Suppose $J=[0,\eta_1^{-2}T]$ is an interval for which*

(4.11)
$$\lambda(t)=\frac{1}{\eta_1}, \qquad \textit{and} \qquad \frac{|\xi(t)|}{\lambda(t)}\leq\eta_0, \qquad \textit{for all} \qquad t\in J, \qquad \int_J \lambda(t)^{-2}dt = T, \qquad \textit{and} \qquad \eta_1^{-2}T=2^{3k}.$$

Then for some $p > 2$, $p = 5/2$ will do, define the norm

(4.12)
$$\|u\|^2_{X([0,\eta_1^{-2}T]\times\mathbb{R}^2)} = \sup_{0\leq i\leq k}\sup_{1\leq a\leq 2^{3k-3i}} \|P_{\geq i}u\|^2_{U^p_\Delta([(a-1)2^{3i},a2^{3i}]\times\mathbb{R}^2)} + \frac{1}{\eta_0^{1/10}}\|(P_{\geq i}u)(P_{\leq i-3}u)\|^2_{L^2_{t,x}([(a-1)2^{3i},a2^{3i}]\times\mathbb{R}^2)}.$$

Also, for any $0 \leq j \leq k$, let

(4.13)
$$\begin{aligned}\|u\|^2_{X_j([0,\eta_1^{-2}T]\times\mathbb{R}^2)} &= \sup_{0\leq i\leq j}\sup_{1\leq a\leq 2^{3k-3i}} \|P_{\geq i}u\|^2_{U^p_\Delta([(a-1)2^{3i},a2^{3i}]\times\mathbb{R}^2)} \\ &+ \sup_{0\leq i\leq j}\sup_{1\leq a\leq 2^{3k-3i}} \frac{1}{\eta_0^{1/10}}\|(P_{\geq i}u)(P_{\leq i-3}u)\|^2_{L^2_{t,x}([(a-1)2^{3i},a2^{3i}]\times\mathbb{R}^2)}.\end{aligned}$$

Then the long time Strichartz estimate,

$$\|u\|_{X([0,\eta_1^{-2}T]\times\mathbb{R}^2)} \lesssim 1, \tag{4.14}$$

holds with implicit constant independent of T.

Proof. This estimate is proved by induction on j. Local well-posedness arguments combined with the fact that $\lambda(t) = \frac{1}{\eta_1}$ for any $t \in [0,T]$ imply that

$$\|u\|_{U^2_\Delta([a,a+1]\times\mathbb{R}^2)} \lesssim 1, \tag{4.15}$$

and when $i = 0$,

$$(P_{\geq i}u)(P_{\leq i-3}u) = 0. \tag{4.16}$$

Therefore,

$$\|u\|_{X_j([0,\eta_1^{-2}T]\times\mathbb{R}^2)} \lesssim 1, \tag{4.17}$$

when $j = 0$. This is the base case.

To prove the inductive step, recall that by Duhamel's principle, if $J_a^{(k)} = [(a-1)2^{3k-3i}, a2^{3k-3i}]$, then for any $t_0 \in J_a^{(k)}$,

$$u(t) = e^{i(t-t_0)\Delta}u(t_0) + i\int_{t_0}^t e^{i(t-\tau)\Delta}(|u|^2u)d\tau, \tag{4.18}$$

and

$$\|P_{\geq i}u\|_{U^p_\Delta(J_a^{(k)}\times\mathbb{R}^2)} \lesssim \|P_{\geq i}(u(t_0))\|_{L^2} + \|\int_{t_0}^t e^{i(t-\tau)\Delta}P_{\geq i}(|u|^2u)d\tau\|_{U^p_\Delta(J_a^{(k)}\times\mathbb{R}^2)}. \tag{4.19}$$

Since $\|u(t_0)\|_{L^2} \lesssim 1$, turn to the Duhamel term. Choose $v \in V_\Delta^{p'}(J\times\mathbb{R})$ such that $\|v\|_{V_\Delta^{p'}(J\times\mathbb{R})} = 1$ and $\hat{v}(t,\xi)$ is supported on the Fourier support of P_i. It is a well-known fact that

$$\|\int_{t_0}^t e^{i(t-\tau)\Delta}P_{\geq i}(|u|^2u)d\tau\|_{U^p_\Delta(J\times\mathbb{R})} \lesssim \sup_v \|vP_{\geq i}(|u|^2u)\|_{L^1_{t,x}}, \tag{4.20}$$

where $\sup_v$ is the supremum over all such v supported on P_i satisfying $\|v\|_{V_\Delta^{p'}(J\times\mathbb{R})} = 1$. See [HHK09] for a proof.

Throughout this section it is not so important to distinguish between u and $\bar{u}$. Since

$$P_{\geq i}(|u_{\leq i-3}|^2u_{\leq i-3}) = 0, \tag{4.21}$$

decompose

$$P_{\geq i}(|u|^2u) = P_{\geq i}O((P_{\geq i-3}u)^3) + P_{\geq i}O((P_{\geq i-3}u)^2(P_{\leq i-3}u)) + P_{\geq i}O((P_{\geq i-3}u)(P_{\leq i-3}u)^2). \tag{4.22}$$

By Hölder's inequality,

(4.23)
$$\|v(u_{\geq i-3})^2(u_{\leq i-3})\|_{L^1_{t,x}} + \|v(u_{\geq i-3})^3\|_{L^1_{t,x}} \lesssim \|v\|_{L^\infty_t L^2_x}\|u_{\geq i-3}\|^3_{L^3_t L^6_x} + \|v\|_{L^3_t L^6_x}\|u_{\geq i-3}\|^2_{L^3_t L^6_x}\|u_{\leq i-3}\|_{L^\infty_t L^2_x}.$$

Since $V_\Delta^{p'} \subset U_\Delta^2$ for any $p > 2$, again see [HHK09],

$$\|v\|_{L^\infty_t L^2_x} + \|v\|_{L^3_t L^6_x} \lesssim \|v\|_{U^2_\Delta} \lesssim \|v\|^{p'}_{V_\Delta} = 1. \tag{4.24}$$

Therefore, when $i > 4$, by (4.7), (4.8), (4.11), and interpolation,

$$\|u_{\geq i-3}\|^3_{L^3_t L^6_x} \lesssim \|u_{\geq i-3}\|^{5/2}_{L^{5/2}_t L^{10}_x}\|u_{\geq i-3}\|^{1/2}_{L^\infty_t L^2_x} \lesssim \eta_0^{1/2}\|u_{\geq i-3}\|^{5/2}_{L^{5/2}_t L^{10}_x} \lesssim \eta_0^{1/2}\|u\|^{5/2}_{X_{i-3}([0,T]\times\mathbb{R})}. \tag{4.25}$$

When $i \leq 4$, simply use (3.6), which implies

$$u(t,x) = \frac{1}{\lambda(t)}e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}Q(\frac{x-x(t)}{\lambda(t)}) + \frac{1}{\lambda(t)}e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}\epsilon(t,\frac{x-x(t)}{\lambda(t)}). \tag{4.26}$$

By local well-posedness arguments and $\lambda(t) = \frac{1}{\eta_1}$, for any $a \in \mathbb{Z}_{\geq 0}$,

$$\|u\|_{L^{5/2}_t L^{10}_x([a,a+1]\times\mathbb{R}^2)} \lesssim 1, \qquad \|\epsilon\|_{L^{5/2}_t L^{10}_x([a,a+1]\times\mathbb{R}^2)} \lesssim \eta_0, \tag{4.27}$$

and therefore by (4.7) and (4.8),

$$\|u_{\geq i-3}\|^3_{L^3_t L^6_x([a,a+1]\times\mathbb{R}^2)} \lesssim \eta_0. \tag{4.28}$$

Therefore,

(4.29)
$$\|u_{\geq i-3}\|^3_{L^3_t L^6_x} + \|u_{\geq i-3}\|^2_{L^3_t L^6_x}\|u_{\leq i-3}\|_{L^\infty_t L^2_x} \lesssim \eta_0^{1/2}\|u\|^3_{X_{i-3}([0,T]\times\mathbb{R})} + \eta_0^{1/2}\|u\|^{5/2}_{X_{i-3}([0,T]\times\mathbb{R})} + \eta_0^{1/2}.$$

Now compute

$$\|v((P_{\geq i-3}u)(P_{\leq i-3}u)^2)\|_{L^1_{t,x}} \lesssim \|(P_{\geq i-3}u)(P_{\leq i-3}u)\|_{L^2_{t,x}}\|v(P_{\leq i-3}u)\|_{L^2_{t,x}}. \tag{4.30}$$

By (4.13),

$$\|(P_{\geq i-3}u)(P_{\leq i-3}u)\|_{L^2_{t,x}([(a-1)2^{3i},a2^{3i}]\times\mathbb{R}^2)} \lesssim \eta_0^{1/20}\|u\|_{X_{i-3}}. \tag{4.31}$$

Next, suppose that it is true that for any $\|v_0\|_{L^2} = 1$, where $\hat{v_0}$ is supported on $|\xi| \geq 2^i$,

$$\sup_{v_0}\|(e^{it\Delta}v_0)(u_{\leq i-3})\|_{L^2_{t,x}} \lesssim 1 + \eta_0\|u\|^3_{X_{i-3}}. \tag{4.32}$$

Then (4.32) combined with $V_\Delta^{p'} \subset U_\Delta^2$ implies

$$\|v(P_{\leq i-3}u)\|_{L^2_{t,x}} \lesssim 1 + \eta_0\|u\|^3_{X_{i-3}}, \tag{4.33}$$

and therefore,

$$\sup_{1\leq a\leq 2^{3(k-i)}}\|P_{\geq i}u\|_{U^p_\Delta([(a-1)2^{3i},a2^{3i}]\times\mathbb{R}^2)} \lesssim 1 + \eta_0^{1/20}\|u\|_{X_{i-3}}(1+\eta_0\|u\|^3_{X_{i-3}}). \tag{4.34}$$

This bound is fine for the induction on frequency argument.

The bilinear estimate (4.32) is proved using the interaction Morawetz estimate described at the beginning of this section. To simplify notation, let

$$v(t,x) = e^{it\Delta}v_0, \tag{4.35}$$

where $\|v_0\|_{L^2} = 1$ and $\hat{v}_0$ is supported on P_j for some $j \geq i$. The function v may be split into a piece supported in frequency near the $\hat{e}_1$ axis and a piece supported in frequency near the $\hat{e}_2$ axis. Suppose without loss of generality that $\hat{v}$ is supported near the $\hat{e}_1$ axis. Then set

(4.36)
$$M(t) = \int |v(t,y)|^2 \frac{(x-y)_1}{|(x-y)_1|} Im[\bar{u}_{\leq i-3} \partial_{x_1} u_{\leq i-3}] dxdy + \int |u_{\leq i-3}|^2 \frac{(x-y)_1}{|(x-y)_1|} Im[\bar{v} \partial_{x_1} v] dxdy.$$

Let $F(u) = |u|^2 u$. Then $u_{\leq i-3}$ solves the equation

$$i\partial_t u_{\leq i-3} + \Delta u_{\leq i-3} + F(u_{\leq i-3}) = F(u_{\leq i-3}) - P_{\leq i-3} F(u) = -\mathcal{N}_{i-3}. \tag{4.37}$$

Making a direct computation,

$$\begin{aligned} \frac{d}{dt} M(t) = 2 \int \int_{x_1 = y_1} |\partial_{x_1} (\overline{v(t,y)} u_{\leq i-3}(t,x))|^2 dxdy - \int \int_{x_1 = y_1} |v(t,y)|^2 |u_{\leq i-3}(t,x)|^4 dxdy \\ + \int |v(t,y)|^2 \frac{(x-y)_1}{|(x-y)_1|} \cdot Re[\overline{u_{\leq i-3}} \partial_{x_1} \mathcal{N}_{i-3}](t,x) dxdy \\ - \int |v(t,y)|^2 \frac{(x-y)_1}{|(x-y)_1|} \cdot Re[\overline{\mathcal{N}_{i-3}} \partial_{x_1} u_{\leq i-3}](t,x) dxdy \\ + 2 \int Im[\overline{u_{\leq i-3}} \mathcal{N}_{i-3}](t,y) \frac{(x-y)_1}{|(x-y)_1|} \cdot Im[\bar{v} \partial_{x_1} v](t,x) dxdy. \end{aligned} \tag{4.38}$$

Then by the fundamental theorem of calculus, Bernstein's inequality, the Fourier support of $\bar{v} u_{\leq i-3}$, $\|v_0\|_{L^2} = 1$, the fact that $\|u\|_{L^2} = \|Q\|_{L^2}$, and (4.5),

(4.39)
$$\begin{aligned} 2^{2j} \|\bar{v} u_{\leq i-3}\|^2_{L^2_{t,x}(J \times \mathbb{R}^2)} \lesssim 2^i \int_{x_1 = y_1} \int_{x_1 = y_1} |\partial_{x_1} (\overline{v(t,y)} u_{\leq i-3}(t,x))|^2 dxdydt + \int_J \|\nabla u_{\leq i-3}\|^2_{L^p} \|v\|^2_{L^q} dt \\ \lesssim 2^{j+i} + 2^i \int \int_{x_1 = y_1} |v(t,y)|^2 |u_{\leq i-3}(t,x)|^4 dxdydt \\ -2^i \int \int \int |v(t,y)|^2 \frac{(x-y)_1}{|(x-y)_1|} \cdot Re[\overline{\mathcal{N}_{i-3}} \partial_{x_1} u_{\leq i-3}](t,x) dxdydt \\ +2^i \int \int \int |v(t,y)|^2 \frac{(x-y)_1}{|(x-y)_1|} Re[\overline{u_{\leq i-3}} \partial_{x_1} \mathcal{N}_{i-3}](t,x) dxdydt \\ +2^{i+1} \int \int \int Im[\overline{u_{\leq i-3}} \mathcal{N}_{i-3}](t,y) \frac{(x-y)_1}{|(x-y)_1|} Im[\bar{v} \partial_{x_1} v](t,x) dxdydt + \int \|\nabla u_{\leq i-3}\|^2_{L^p} \|v\|^2_{L^q} dt. \end{aligned}$$

Also note that

$$\|\bar{v} u_{\leq i-3}\|^2_{L^2_{t,x}} = \|\bar{v} v \bar{u}_{\leq i-3} u_{\leq i-3}\|_{L^1_{t,x}} = \|v u_{\leq i-3}\|^2_{L^2_{t,x}}, \tag{4.40}$$

so it is not too important to pay attention to complex conjugates in the proceeding calculations.

First, by (4.7), Bernstein's inequality, and the fact that $\lambda(t) = \frac{1}{\eta_1}$,

$$\begin{aligned} \int_{x_1 = y_1} |v(t,y)|^2 |u_{\leq i-3}(t,x)|^4 dxdy \lesssim 2^{-2j} \|u_{\leq i-3}\|^2_{L^\infty} \int_{x_1 = y_1} |\partial_{x_1} (\overline{v(t,y)} u_{\leq i-3}(t,x))|^2 dxdy \\ \lesssim \eta_0^2 2^{2i-2j} \int_{x_1 = y_1} |\partial_{x_1} (\overline{v(t,y)} u_{\leq i-3}(t,x))|^2 dxdy. \end{aligned} \tag{4.41}$$

Since $j \geq i$ and $\eta_0 \ll 1$ is small,

$$\sum_{j\geq i}(4.41) \ll \int_{x_1=y_1} |\partial_{x_1}(\bar{v}(t,y)u_{\leq i-3}(t,x))|^2 dxdy, \tag{4.42}$$

which is more than sufficient for our purposes. Next, Strichartz estimates, (4.7), (4.8), and (4.11) imply

(4.43)
$$\int \|v\|_{L^{20}}^2 \|\nabla P_{\leq i-3}u\|_{L^{5/2}} \|\nabla P_{\leq i-3}u\|_{L^2} dt \lesssim \|v\|^2_{L_t^{20/9}L_x^{20}} \|\nabla P_{\leq i-3}u\|_{L_t^{10}L_x^{5/2}} \|\nabla P_{\leq i-3}u\|_{L_t^\infty L_x^2} \lesssim \eta_0 2^{2i}\|u\|_{X_{i-3}}.$$

This calculation is also sufficient since $\sum_{j\geq i} 2^{2i-2j}\eta_0\|u\|_{X_{i-3}}$ is bounded by the right hand side of (4.33).

Now consider the term,

$$\mathcal{N}_{i-3} = P_{\leq i-3}F(u) - F(u_{\leq i-3}). \tag{4.44}$$

Since by Fourier support arguments

$$P_{\leq i-3}F(u_{\leq i-6}) - F(u_{\leq i-6}) = 0, \tag{4.45}$$

(4.46)
$$\begin{aligned}\mathcal{N}_i = P_{\leq i-3}(2|u_{\leq i-6}|^2 u_{\geq i-6} + (u_{\leq i-6})^2\overline{u_{\geq i-6}}) - (2|u_{\leq i-6}|^2 u_{i-6\leq\cdot\leq i-3} + (u_{\leq i-6})^2\overline{u_{i-6\leq\cdot\leq i-3}})\\ + P_{\leq i-3}O((u_{\geq i-6})^2 u) + O((u_{i-6\leq\cdot\leq i-3})^2 u) = \mathcal{N}_{i-3}^{(1)} + \mathcal{N}_{i-3}^{(2)}.\end{aligned}$$

Following (4.23)–(4.31),

(4.47)
$$\begin{aligned}\||\mathcal{N}_{i-3}^{(2)}||u_{\leq i-3}|\|_{L^1_{t,x}} \lesssim \||u_{\geq i-6}|^2|u_{\leq i-9}|^2\|_{L^1_{t,x}} + \||u_{\geq i-6}|^2|u_{\geq i-9}|^2\|_{L^1_{t,x}}\\ \lesssim \|(u_{\geq i-6})(u_{\leq i-9})\|^2_{L^2_{t,x}} + \|u_{\geq i-6}\|^2_{L^4_{t,x}}\|u_{\geq i-9}\|^2_{L^4_{t,x}} \lesssim \eta_0^{1/10}\|u\|^2_{X_{i-3}(J\times\mathbb{R}^2)}(1+\|u\|^6_{X_{i-3}(J\times\mathbb{R}^2)}).\end{aligned}$$

Therefore, since $\|v_0\|_{L^2} = 1$,

$$\begin{aligned}&-\int\int\int |v(t,y)|^2 \frac{(x-y)_1}{|(x-y)_1|} Re[\bar{\mathcal{N}}_{i-3}^{(2)}\partial_{x_1}u_{\leq i-3}](t,x)dxdydt\\ &+\int\int\int |v(t,y)|^2 \frac{(x-y)_1}{|(x-y)_1|} Re[\bar{u}_{\leq i-3}\partial_{x_1}\mathcal{N}_{i-3}^{(2)}](t,x)dxdydt\\ &+2\int\int\int Im[\bar{u}_{\leq i-3}\mathcal{N}_{i-3}^{(2)}](t,y)\frac{(x-y)_1}{|(x-y)_1|} Im[\bar{v}\partial_{x_1}v](t,x)dxdydt\\ &\lesssim \eta_0^{1/10}2^j\|u\|^2_{X_{i-3}([0,T]\times\mathbb{R}^2)}(1+\|u\|^6_{X_{i-3}([0,T]\times\mathbb{R}^2)}).\end{aligned} \tag{4.48}$$

Next, observe that

(4.49)
$$2P_{\leq i-3}(|u_{\leq i-6}|^2u_{\geq i-6}) - 2(|u_{\leq i-6}|^2u_{i-6\leq\cdot\leq i-3}) = 2P_{>i-3}(|u_{\leq i-6}|^2u_{i-6\leq\cdot\leq i-3}) + 2P_{\leq i-3}(|u_{\leq i-6}|^2u_{>i-3}).$$

The computation for

$$P_{\leq i-3}((u_{\leq i-6})^2\overline{u_{\geq i-6}}) - (u_{\leq i-6})^2(\overline{u_{i-6\leq\cdot\leq i-3}}) \tag{4.50}$$

is similar.

Again following (4.23)–(4.32),

$$\begin{aligned}(4.51)\qquad &\|P_{\leq i-3}(|u_{\leq i-6}|^2 u_{>i-3})(u_{i-6\leq\cdot\leq i-3})\|_{L^1_{t,x}} \\ &+\|P_{\leq i-3}(|u_{\leq i-6}|^2 u_{>i-3})(u_{i-6\leq\cdot\leq i-3})\|_{L^1_{t,x}} \lesssim \eta_0^{1/10}\|u\|^2_{X_{i-3}(J\times\mathbb{R}^2)}(1+\|u\|^6_{X_{i-3}(J\times\mathbb{R}^2)}).\end{aligned}$$

Finally, observe that the Fourier support of

$$(4.52)\qquad 2P_{>i-3}(|u_{\leq i-6}|^2 u_{i-6\leq\cdot\leq i-3})(u_{\leq i-6}) + 2P_{\leq i-3}(|u_{\leq i-6}|^2 u_{>i-3})(u_{\leq i-6})$$

is on frequencies $|\xi| \geq 2^{i-6}$. Therefore, integrating by parts,

$$\begin{aligned}(4.53)\qquad &\int\int\int Im[\bar{u}_{\leq i-6}P_{>i-3}(|u_{\leq i-6}|^2 u_{i-6\leq\cdot\leq i-3})](t,y)\frac{(x-y)_1}{|(x-y)_1|}Im[\bar{v}\partial_{x_1}v](t,x)dxdydt \\ &= \int\int\int_{x_1=y_1} Im[\bar{v}\partial_{x_1}v](t,y)\cdot\frac{\partial_{x_1}}{\partial_{x_1}^2}Im[\bar{u}_{\leq i-6}P_{>i-3}(|u_{\leq i-6}|^2 u_{i-6\leq\cdot\leq i-3})](t,x)dxdydt \\ &\lesssim 2^{-i}(\int\int\int_{x_1=y_1}|\bar{v}(u_{\leq i-6})|^2 dxdydt)^{1/2}(\int\int\int_{x_1=y_1}|\partial_{x_1}\bar{v}(u_{\leq i-6})|^2 dxdydt)^{1/2}\|u_{\leq i-3}\|^2_{L^\infty_{t,x}} \\ &\lesssim 2^{j+i}\eta_0^2(\int\int\int_{x_1=y_1}|\bar{v}(u_{\leq i-6})|^2 dxdydt).\end{aligned}$$

A similar calculation gives the estimate

$$\begin{aligned}(4.54)\qquad &\int\int\int Im[\bar{u}_{\leq i-6}P_{\leq i-3}(|u_{\leq i-6}|^2 u_{>i-3})](t,y)\frac{(x-y)_1}{|(x-y)_1|}Im[\bar{v}\partial_{x_1}v](t,x)dxdydt \\ &\lesssim 2^{j+i}\eta_0^2(\int\int\int_{x_1=y_1}|\bar{v}(u_{\leq i-6})|^2 dxdydt).\end{aligned}$$

We can analyze the terms

$$(4.55)\qquad \int\int\int |v(t,y)|^2\frac{(x-y)_1}{|(x-y)_1|}Re[\bar{\mathcal{N}}^{(1)}_{i-3}\partial_{x_1}u_{\leq i-3}](t,x)dxdydt,$$

and

$$(4.56)\qquad \int\int\int |v(t,y)|^2\frac{(x-y)_1}{|(x-y)_1|}Re[\bar{u}_{\leq i-3}\partial_x\mathcal{N}^{(1)}_{i-3}](t,x)dxdydt,$$

in a similar manner.

Plugging (4.40)–(4.56) into (4.39) gives

$$(4.57)\qquad 2^{2j}\|\bar{v}u_{\leq i-3}\|^2_{L^2_{t,x}} + 2^{2j}\|vu_{\leq i-3}\|^2_{L^2_{t,x}} \lesssim 2^{i+j} + \eta_0^{1/10}2^{i+j}(1+\|u\|^8_{X_{i-3}}).$$

Summing up over $j \geq i$ implies (4.32).

The estimate of

$$(4.58)\qquad \|(P_{\geq i}u)(P_{\leq i-3}u)\|_{L^2_{t,x}},$$

also uses an interaction Morawetz estimate. This time, for a fixed $j \geq i$, define the Morawetz potential,

$$(4.59)\qquad M_j(t) = \int |P_j u(t,y)|^2\frac{(x-y)_1}{|(x-y)_1|}Im[\overline{u_{\leq i-3}}\partial_{x_1}u_{\leq i-3}]dxdy + \int |u_{\leq i-3}|^2\frac{(x-y)_1}{|(x-y)_1|}Im[\overline{P_j u}\partial_{x_1}P_j u]dxdy.$$

Making a direct computation, since $P_j u$ is not a solution to the linear Schrödinger equation, we have three additional terms,

(4.60)
$$\begin{aligned}\frac{d}{dt}M_j(t) = 2\int\int_{x_1=y_1} |\partial_{x_1}(\bar{P_j}u(t,y)u_{\leq i-3}(t,x))|^2 dxdy - \int\int_{x_1=y_1} |P_j u(t,y)|^2 |u_{\leq i-3}(t,x)|^4 dxdy\\ + \int\int |P_j u(t,y)|^2 \frac{(x-y)_1}{|(x-y)_1|}\cdot Re[\overline{u_{\leq i-3}}\partial_{x_1}\mathcal{N}_{i-3}](t,x)dxdy\\ - \int\int |P_j u(t,y)|^2 \frac{(x-y)_1}{|(x-y)_1|}\cdot Re[\overline{\mathcal{N}_{i-3}}\partial_{x_1}u_{\leq i-3}](t,x)dxdy\\ +2\int\int Im[\overline{u_{\leq i-3}}\mathcal{N}_{i-3}](t,y)\frac{(x-y)_1}{|(x-y)_1|}\cdot Im[\bar{P_j}u\partial_{x_1}P_j u](t,x)dxdy\\ + \int\int |P_{\leq i-3}u|^2 \frac{(x-y)_1}{|(x-y)_1|} Re[\overline{P_j u}\partial_{x_1}P_j(|u|^2 u)]dxdy\\ - \int\int |P_{\leq i-3}u|^2 \frac{(x-y)_1}{|(x-y)_1|} Re[\overline{P_j(|u|^2 u)}\partial_{x_1}P_j u]dxdy\\ +2\int\int Im[\overline{P_j u}P_j(|u|^2 u)]\frac{(x-y)_1}{|(x-y)_1|} Im[\overline{u_{\leq i-3}}\partial_{x_1}u_{\leq i-3}]dxdy.\end{aligned}$$

Now then,

$$\sum_{j\geq i} 2^{i-2j}\sup_{t\in J}|M_j(t)| \lesssim \|P_{\geq i}u\|^2_{L_t^\infty L_x^2} \lesssim \eta_0^2. \tag{4.61}$$

Following (4.40)–(4.56), (4.6) and (4.7) imply

(4.62)
$$\begin{aligned}-\sum_{j\geq i} 2^{i-2j}\int\int\int_{x_1=y_1} |P_j u(t,y)|^2 |u_{\leq i-3}(t,x)|^4 dxdydt\\ +\sum_{j\geq i} 2^{i-2j}\int\int\int |P_j u(t,y)|^2 \frac{(x-y)_1}{|(x-y)_1|}\cdot Re[\overline{u_{\leq i-3}}\partial_{x_1}\mathcal{N}_{i-3}](t,x)dxdydt\\ -\sum_{j\geq i} 2^{i-2j}\int\int\int |P_j u(t,y)|^2 \frac{(x-y)_1}{|(x-y)_1|}\cdot Re[\overline{\mathcal{N}_{i-3}}\partial_{x_1}u_{\leq i-3}](t,x)dxdydt\\ +2\sum_{j\geq i} 2^{i-2j}\int\int\int Im[\overline{u_{\leq i-3}}\mathcal{N}_{i-3}](t,y)\frac{(x-y)_1}{|(x-y)_1|}\cdot Im[\overline{P_j u}\partial_{x_1}P_j u](t,x)dxdydt \lesssim \eta_0^2(1+\|u\|^8_{X_{i-3}}).\end{aligned}$$

To analyze the new terms,

$$\begin{aligned}2\sum_{j\geq i} 2^{i-2j}\int\int\int Im[\overline{P_j u}P_j(|u|^2 u)]\frac{(x-y)_1}{|(x-y)_1|} Im[\overline{u_{\leq i-3}}\partial_{x_1}u_{\leq i-3}]dxdydt\\ +\sum_{j\geq i} 2^{i-2j}\int\int\int |P_{\leq i-3}u|^2\frac{(x-y)_1}{|(x-y)_1|} Re[\overline{P_j u}\partial_{x_1}P_j(|u|^2 u)]dxdydt\\ -\sum_{j\geq i} 2^{i-2j}\int\int\int |P_{\leq i-3}u|^2\frac{(x-y)_1}{|(x-y)_1|} Re[\overline{P_j(|u|^2 u)}\partial_{x_1}P_j u]dxdydt,\end{aligned} \tag{4.63}$$

first observe that by (4.7), (4.8), (4.11), (4.26), and (4.47),

$$\begin{aligned} &2\sum_{j\geq i} 2^{i-2j}\int\int\int Im[\overline{P_j u}P_j(|u|^2u)]\frac{(x-y)_1}{|(x-y)_1|}Im[\overline{u_{\leq i-3}}\partial_{x_1}u_{\leq i-3}]dxdydt \\ &\lesssim 2^{-i}\|\nabla P_{\leq i-3}u\|_{L^2}\|P_{\leq i-3}u\|_{L^2}(\|P_{\geq i-6}u\|_{L^4_{t,x}}^4+\|(P_{\geq i-3}u)(P_{\leq i-6}u)\|_{L^2_{t,x}}^2) \\ &\lesssim \eta_0(\eta_0^{1/10}\|u\|_{X_{i-3}}^2(1+\|u\|_{X_{i-3}}^6)). \end{aligned} \tag{4.64}$$

For the other two terms in (4.63), decompose

$$\begin{aligned} &P_j(|u|^2u)=P_j(O((P_{\geq i-3}u)^3))+P_j(O((P_{\geq i-3}u)^2(P_{\leq i-3}u))) \\ &+P_j(2|P_{\leq i-3}u|^2(P_{>i-3}u)+(P_{\leq i-3}u)^2(P_{>i-3}\bar{u}))=\mathcal{N}_1+\mathcal{N}_2. \end{aligned} \tag{4.65}$$

By (4.25),

$$\begin{aligned} &\sum_{j\geq i}2^{i-2j}\int\int\int |P_{\leq i-3}u|^2\frac{(x-y)_1}{|(x-y)_1|}Re[\overline{P_j u}\partial_{x_1}P_j\mathcal{N}_1]dxdydt \\ &-\sum_{j\geq i}2^{i-2j}\int\int\int |P_{\leq i-3}u|^2\frac{(x-y)_1}{|(x-y)_1|}Re[\mathcal{N}_1\partial_{x_1}P_j u]dxdydt\lesssim \|P_{\geq i-3}u\|_{L^3_tL^6_x}^3\lesssim \eta_0^{1/2}\|u\|_{X_{i-3}}^{5/2}. \end{aligned} \tag{4.66}$$

Next, if m is a smooth Fourier multiplier satisfying $\nabla m(\xi)\lesssim \frac{1}{|\xi|}$, $|m(\xi+\xi_1)-m(\xi)|\lesssim \frac{|\xi_1|}{|\xi|}$ for $|\xi_1|\ll|\xi|$, and therefore, as in (4.25),

$$\begin{aligned} &\|(P_j u)\cdot[P_j(|P_{\leq i-3}u|^2(P_{>i-3}u))-|P_{\leq i-3}u|^2(P_j u)]\|_{L^1_{t,x}} \\ &+\|(P_j u)\cdot[P_j((P_{\leq i-3}u)^2(\overline{P_{>i-3}u}))-(P_{\leq i-3}u)^2(\overline{P_j u})]\|_{L^1_{t,x}} \\ &\lesssim \|\nabla P_{\leq i-3}u\|_{L_t^5L_x^{10/3}}\|u_{\leq i-3}\|_{L_t^\infty L_x^2}\|P_{\geq i-3}u\|_{L_t^{5/2}L_x^{10}}^2\lesssim \eta_0^{1/2}\|u\|_{X_{i-3}}^{5/2}. \end{aligned} \tag{4.67}$$

Also by the product rule,

$$\begin{aligned} &2Re[\overline{P_j u}\partial_{x_1}(|P_{\leq i-3}u|^2(P_j u))]-2Re[|P_{\leq i-3}u|^2(\overline{P_j u})\partial_{x_1}P_j u] \\ &+Re[\overline{P_j u}\partial_{x_1}((P_{\leq i-3}u)^2(P_j\bar{u}))]-Re[(\overline{P_{\leq i-3}u})^2(P_j u)\partial_{x_1}(P_j u)]=O((\nabla P_{\leq i-3}u)(P_{\leq i-3}u)(P_j u)^2). \end{aligned} \tag{4.68}$$

Using the analysis in (4.67), (4.68) $\lesssim \eta_0^{1/2}\|u\|_{X_{i-3}}^{5/2}$. Finally, integrating by parts and using (4.41),

$$\begin{aligned} &\sum_{j\geq i}2^{i-2j}\int\int\int |P_{\leq i-3}u|^2\frac{(x-y)_1}{|(x-y)_1|}Re[\partial_{x_1}((P_j u)^2(P_{\leq i-3}u)^2]dxdydt \\ &\ll \int_{x_1=y_1}|\partial_{x_1}(\bar{v}(t,y)u_{\leq i-3}(t,x))|^2dxdy. \end{aligned} \tag{4.69}$$

Therefore, by the fundamental theorem of calculus,

$$\begin{aligned} &\sum_{j\geq i}2^{-2j}\|\partial_{x_1}(P_j u\bar{u}_{\leq i-3})\|_{L^2_{t,x}}^2 \\ &\lesssim \sum_{j\geq i}2^{i-2j}\int\int_{x_1=y_1}|\partial_{x_1}(\overline{P_j u}(t,y)u_{\leq i-3}(t,x))|^2dxdydt\lesssim \eta_0^2+\eta_0^{1/2}(1+\|u\|_{X_{i-3}}^8). \end{aligned} \tag{4.70}$$

It is possible to perform the same analysis with ∂_{x_1} replaced by ∂_{x_2}, so

$$\sum_{j\geq i} 2^{-2j}\|\nabla(P_j u\bar{u}_{\leq i-3})\|^2_{L^2_{t,x}} \lesssim \eta_0^2 + \eta_0^{1/2}\|u\|^4_{X_{i-3}}. \tag{4.71}$$

Using the Fourier support properties of $P_j u(\overline{u_{\leq i-3}})$ implies

$$\|(P_{\geq i}u)(P_{\leq i-3}u)\|^2_{L^2_{t,x}} \lesssim \eta_0^2 + \eta_0^{1/2}\|u\|^4_{X_{i-3}} \tag{4.72}$$

Combining (4.34) with (4.72) and (4.17), and arguing by induction on i implies Theorem 13. □

Theorem 13 may be upgraded when u is close to a soliton, as in Theorem 6.

Theorem 14. *When $J = [0, \eta_1^{-2}T]$ is an interval that satisfies* (4.11)*, and u satisfies Theorem* 6*,*

$$\|P_{\geq k}u\|_{U^p_\Delta([0,\eta_1^{-2}T]\times\mathbb{R}^2)} + \|(P_{\geq k}u)(P_{\leq k-3}u)\|_{L^2_{t,x}([0,\eta_1^{-2}T]\times\mathbb{R}^2)} \lesssim (\frac{\eta_1^2}{T}\int_0^{\eta_1^{-2}T}\|\epsilon(t)\|^2_{L^2}dt)^{1/2} + \frac{1}{T^{10}}. \tag{4.73}$$

Proof. Make another induction on frequency argument starting at level $\frac{k}{2}$. First, observe that since Theorem 13 is invariant under translation in time, for any $a \in \mathbb{Z}$,

$$\|P_{\geq \frac{k}{2}}u\|_{U^2_\Delta([a\eta_1^{-1}T^{1/2},(a+1)\eta_1^{-1}T^{1/2}]\times\mathbb{R}^2)} \lesssim 1. \tag{4.74}$$

Next, following Theorem 13,

$$\begin{aligned}\|P_{\geq \frac{k}{2}+3}u\|_{U^2_\Delta([512a\eta_1^{-1}T^{1/2},512(a+1)\eta_1^{-1}T^{1/2}]\times\mathbb{R}^2)} \lesssim & \inf_{t\in[512a\eta_1^{-1}T^{1/2},512(a+1)\eta_1^{-1}T^{1/2}]}\|P_{\geq\frac{k}{2}+3}u(t)\|_{L^2}\\ &+\eta_0\|P_{\geq\frac{k}{2}}u\|_{U^2_\Delta([512a\eta_1^{-1}T^{1/2},512(a+1)\eta_1^{-1}T^{1/2}]\times\mathbb{R}^2)}.\end{aligned} \tag{4.75}$$

Now then, by (4.26), (4.11), and the fact that Q is smooth and all its derivatives are rapidly decreasing,

$$\|P_{\geq\frac{k}{2}+3}u(t)\|_{L^2} \leq \|\epsilon(t)\|_{L^2} + \|P_{\geq\frac{k}{2}+3}(\lambda(t)^{-1}e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}Q(\frac{x-x(t)}{\lambda(t)})\|_{L^2} \lesssim \|\epsilon(t)\|_{L^2} + T^{-10}. \tag{4.76}$$

Plugging (4.76) back into (4.75),

$$\begin{aligned}\|P_{\geq\frac{k}{2}+3}u\|_{U^2_\Delta([512a\eta_1^{-1}T^{1/2},512(a+1)\eta_1^{-1}T^{1/2}]\times\mathbb{R}^2)} \lesssim & (\frac{\eta_1}{512T^{1/2}}\int_{512a\eta_1^{-1}T^{1/2}}^{512(a+1)\eta_1^{-1}T^{1/2}}\|\epsilon(t,x)\|^2_{L^2}dt)^{1/2}\\ &+T^{-10}+\eta_0(\sum_{j=1}^{512}\|P_{\geq\frac{k}{2}}u\|^2_{U^2_\Delta([(512a+(j-1))\eta_1^{-1}T^{1/2},(512a+j)\eta_1^{-1}T^{1/2}]\times\mathbb{R}^2)})^{1/2}.\end{aligned} \tag{4.77}$$

Arguing by induction in k, taking $\lfloor\frac{k}{6}\rfloor$ steps in all, for η_0 sufficiently small,

$$\begin{aligned}\|P_{\geq k}u\|_{U^2_\Delta([0,T]\times\mathbb{R}^2)} &\lesssim T^{-10} + 2^{k/2}\eta_0^{-\frac{k}{6}} + (\frac{1}{T}\int_0^T \inf_{\lambda,\gamma}\|e^{i\gamma}\lambda^{1/2}u(t,\lambda x) - Q(x)\|^2_{L^2}dt)^{1/2}\\ &\lesssim T^{-10} + (\frac{1}{T}\int_0^T \inf_{\lambda,\gamma}\|e^{i\gamma}\lambda^{1/2}u(t,\lambda x) - Q(x)\|^2_{L^2}dt)^{1/2}.\end{aligned} \tag{4.78}$$

Indeed, if C is the implicit constant in (4.77) then for $\eta_0 \ll 1$ sufficiently small,

$$(C\eta_0)^{\lfloor\frac{k}{6}\rfloor} \leq T^{-10}. \tag{4.79}$$

□

Throughout the proof, we assumed that $\lambda(t) = \frac{1}{\eta_1}$, although most of the time we only needed $\lambda(t) \geq \frac{1}{\eta_1}$. For the general case when $\lambda(t) \geq \frac{1}{\eta_1}$, replace the intervals in (4.15) with intervals on which

$$\int_{J_a} \lambda(t)^{-2} dt = \eta_1^2, \tag{4.80}$$

and then argue by induction in exactly the same manner.

Corollary 1. *When J is an interval that satisfies*

$$\frac{1}{\eta_1} \leq \lambda(t), \qquad \text{and} \qquad \frac{|\xi(t)|}{\lambda(t)} \leq \eta_0, \qquad \text{for all} \qquad t \in J, \tag{4.81}$$

and (4.10)*, and u satisfies the conditions of Theorem* 6*,*

$$\|P_{\geq k} u\|_{U_\Delta^p(J \times \mathbb{R}^2)} + \|(P_{\geq k} u)(P_{\leq k-3} u)\|_{L_{t,x}^2(J \times \mathbb{R}^2)} \lesssim (\frac{1}{T} \int_J \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2} dt)^{1/2} + \frac{1}{T^{10}}. \tag{4.82}$$

5. A long time Strichartz estimate when $d \geq 3$

In dimensions $d \geq 3$, the proof of long time Strichartz estimates is complicated by the fact that $F(x) = |x|^{4/d} x$ is not a smooth function. To circumvent this difficulty, we will utilize a bound on $\frac{\nabla Q}{Q^{1-\alpha}}$ for $\alpha > 0$.

Theorem 15. *If Q is the unique positive solution to $\Delta Q + Q^{1+4/d} = Q$, $\frac{\nabla Q}{Q^{1-\alpha}}$ is uniformly bounded for any $\alpha > 0$.*

Proof. To see this, first observe that since Q is smooth and positive, $\frac{\nabla Q}{Q}$ is uniformly bounded on the set $\{x : |x| \leq 1\}$.

Next, since Q is radially symmetric,

$$Q_{rr} + \frac{d-1}{r} Q_r = r^{-(d-1)} \partial_r (r^{d-1} Q_r) = Q - Q^{1+4/d}. \tag{5.1}$$

By the fundamental theorem of calculus, since Q and all its derivatives are rapidly decreasing, and Q is strictly decreasing, for any $\alpha > 0$,

$$r^{d-1} Q_r \leq \int_r^\infty s^{d-1} Q(s) ds \lesssim_\alpha Q(r)^{1-\alpha}. \tag{5.2}$$

This gives a bound on $\frac{\nabla Q}{Q^{1-\alpha}}$ when $r > 1$. □

As before, let $J = [a, b]$ be an interval, and choose

$$0 < \eta_1 \ll \eta_0 \ll 1, \tag{5.3}$$

such that

$$\sup_{t \in J} \|\epsilon(t, x)\|_{L^2}^2 \leq \eta_0^2, \tag{5.4}$$

and that

$$\int_{|\xi| \geq \eta_1^{-1/2}} |\hat{Q}(\xi)|^2 \leq \eta_0^2, \tag{5.5}$$

where Q is the soliton solution (1.17). Suppose

$$\frac{1}{\eta_1} \leq \lambda(t) \leq \frac{1}{\eta_1} T^{1/50d}, \qquad \text{and} \qquad \frac{|\xi(t)|}{\lambda(t)} \leq \eta_0, \qquad \text{for all} \qquad t \in J, \tag{5.6}$$

and that there exists $k \in \mathbb{Z}_{\geq 0}$ such that

$$\int_J \lambda(t)^{-2} dt = T, \qquad \text{and} \qquad \eta_1^{-2} T = 2^{\alpha_d k}, \qquad \text{where} \qquad \alpha_d = 3 - \frac{1}{5d} \qquad \text{when} \qquad 3 \leq d \leq 8,$$
$$\alpha_d = 2(1 + \frac{4}{d}) - \frac{1}{5d}, \qquad \text{when} \qquad d \geq 9. \tag{5.7}$$

Once again, when $i \in \mathbb{Z}$, $i > 0$, let P_i denote the standard Littlewood-Paley projection operator. When $i = 0$, let P_i denote the projection operator $P_{\leq 0}$, and when $i < 0$, let P_i denote the zero operator.

Theorem 16. *If J is an interval on which $|x(t)| \leq T^{\frac{1}{2000d^2}}$, and (5.6) and (5.7) hold, letting p satisfy $\frac{1}{p} = \frac{1}{2} - \frac{1}{4000d^2}$, for $\frac{k}{10d} \leq i \leq k(1 + \frac{1}{10d})$,*

$$\|P_{\geq i} u\|_{U^p_\Delta(J \times \mathbb{R}^d)} + \|P_{\geq i} u\|_{L^2_t L^{\frac{2d}{d-2}}_x (J \times \mathbb{R}^d)} \lesssim 2^{\frac{\alpha_d}{2}(k(1+\frac{1}{10d})-i)} (\frac{1}{T} \int_J \|\epsilon(t)\|^2_{L^2} \lambda(t)^{-2} dt)^{1/2} + T^{-10}. \tag{5.8}$$

Proof. This theorem is also proved using induction on frequency. Make the decomposition

$$u(t,x) = e^{-i\gamma(t)} e^{-ix \cdot \frac{\xi(t)}{\lambda(t)}} \lambda(t)^{-d/2} Q(\frac{x - x(t)}{\lambda(t)}) + e^{-i\gamma(t)} e^{-ix \cdot \frac{\xi(t)}{\lambda(t)}} \lambda(t)^{-d/2} Q(\frac{x - x(t)}{\lambda(t)}) = \tilde{\epsilon}(t,x) + \tilde{Q}. \tag{5.9}$$

By (5.6), (5.7), the fact that Q is smooth and all its derivatives are rapidly decreasing, and local well-posedness theory,

$$\|P_{\geq \frac{k}{10d}} u\|_{L^2_t L^{\frac{2d}{d-2}}_x (J \times \mathbb{R}^d)} \lesssim (\int_J \|\epsilon(t)\|^2_{L^2} \lambda(t)^{-2} dt)^{1/2} + T^{-10} = 2^{\frac{\alpha_d k}{2}} (\frac{1}{T} \int_J \|\epsilon(t)\|^2_{L^2} \lambda(t)^{-2} dt)^{1/2} + T^{-10}. \tag{5.10}$$

For $i \geq \frac{k}{10d}$, by Duhamel's principle, for $t_0, t \in J$,

$$u(t) = e^{i(t-t_0)\Delta} u(t_0) + i \int_{t_0}^t e^{i(t-\tau)\Delta} F(u) d\tau. \tag{5.11}$$

Also, by (5.6), (5.7), and the fact that Q is smooth and all its derivatives are rapidly decreasing, for $i \geq \frac{k}{10d}$, choosing $t_0 \in J$ such that $\|\tilde{\epsilon}(t_0)\|_{L^2} = \inf_{t \in J} \|\tilde{\epsilon}(t)\|_{L^2}$,

$$\|P_{\geq i} e^{it\Delta} u(t_0)\|_{L^2_t L^{\frac{2d}{d-2}}_x (J \times \mathbb{R}^d)} + \|P_{\geq i} e^{it\Delta} u(t_0)\|_{U^p_\Delta(J \times \mathbb{R}^d)} \lesssim (\frac{1}{T} \int_J \|\epsilon(t)\|^2_{L^2} \lambda(t)^{-2} dt)^{1/2} + T^{-10}. \tag{5.12}$$

For the Duhamel term, observe that by the endpoint Strichartz estimates of [KT98] and [HHK09] (compare to (4.20)),

$$\| \int_{t_0}^t e^{i(t-\tau)\Delta} F(\tau) d\tau \|_{L^2_t L^{\frac{2d}{d-2}}_x \cap U^p_\Delta(J \times \mathbb{R}^d)} \lesssim \|F\|_{L^2_t L^{\frac{2d}{d+2}}_x (J \times \mathbb{R}^d)}. \tag{5.13}$$

We will split $F = F_1 + F_2$, where $F_1 \in L_t^2 L_x^{\frac{2d}{d+2}}$ and F_2 will be estimated in a different function space. Use Taylor's formula to expand the nonlinearity,

$$F(u) = F(u_{\leq i}) + F(u) - F(u_{\leq i}) = F(u_{\leq i}) + \int_0^1 F'(u_{\leq i} + s u_{>i}) \cdot (u_{>i}) ds. \tag{5.14}$$

Remark 5. *When proving Theorem* 16*, it is not so important to distinguish between u and $\bar{u}$.*

Split

$$|F'(u_{\leq i} + s u_{>i})| \lesssim |e^{-i\gamma(t)} e^{-ix \cdot \frac{\xi(t)}{\lambda(t)}} \frac{1}{\lambda(t)^{d/2}} Q(\frac{x - x(t)}{\lambda(t)})|^{4/d} + |e^{-i\gamma(t)} e^{-ix \cdot \xi(t)} \frac{1}{\lambda(t)^{d/2}} \epsilon(t, \frac{x - x(t)}{\lambda(t)})|^{4/d}. \tag{5.15}$$

By (5.4),

$$\| |P_{>i} u| |e^{-i\gamma(t)} e^{-ix \cdot \frac{\xi(t)}{\lambda(t)}} \frac{1}{\lambda(t)^{d/2}} \epsilon(t, \frac{x - x(t)}{\lambda(t)})|^{4/d} \|_{L_t^2 L_x^{\frac{2d}{d+2}}} \lesssim \eta_0^{4/d} \| P_{>i} u \|_{L_t^2 L_x^{\frac{2d}{d-2}}}. \tag{5.16}$$

Next, using local smoothing,

Lemma 2. *Under the assumptions of Theorem* 16*,*

$$\| |\frac{1}{\lambda(t)^{d/2}} Q(\frac{x - x(t)}{\lambda(t)})|^{4/d} |P_{\geq i} u| \|_{L_t^2 L_x^{\frac{2d}{d+2}}(J \times \mathbb{R}^d)} \lesssim \eta_1 T^{\frac{1}{2000 d^2}} 2^{-\frac{i}{p}} \| P_{\geq i} u \|_{U_\Delta^p(J \times \mathbb{R}^d)}. \tag{5.17}$$

Proof of Lemma 2. If v is a solution to $(i\partial_t + \Delta) v = 0$ and $\hat{v}_0$ is supported on $|\xi| \geq 2^i$, then for any $R > 0$,

$$\| v \|_{L_{t,x}^2(\mathbb{R} \times \{x : |x| \leq R\})} \lesssim R 2^{-\frac{i}{2}} \| v_0 \|_{L^2}. \tag{5.18}$$

Then, by (5.18), $|x(t)| \leq T^{\frac{1}{2000 d^2}}$, (5.6), and the fact that Q is rapidly decreasing,

$$\| v | \frac{1}{\lambda(t)^{d/2}} Q(\frac{x - x(t)}{\lambda(t)})|^{4/d} \|_{L_t^2 L_x^{\frac{2d}{d+2}}} \lesssim \| v | \frac{1}{\lambda(t)^{d/2}} Q(\frac{x - x(t)}{\lambda(t)})|^{2/d} \|_{L_{t,x}^2} \| Q \|_{L^2}^{2/d} \lesssim \eta_1^{1/2} 2^{-\frac{i}{2}} T^{\frac{1}{4000 d^2}} \| v_0 \|_{L^2}. \tag{5.19}$$

Also,

$$\begin{aligned} &\| v | \frac{1}{\lambda(t)^{d/2}} Q(\frac{x - x(t)}{\lambda(t)})|^{4/d} \|_{L_t^2 L_x^{\frac{2d}{d+2}}(J \times \mathbb{R}^d)} \\ &\lesssim \| v | \frac{1}{\lambda(t)^{d/2}} Q(\frac{x - x(t)}{\lambda(t)})|^{2/d} \|_{L_{t,x}^2}^{2/p} \| v \|_{L_t^\infty L_x^2}^{\frac{p-2}{p}} \| Q \|_{L^2}^{2/d} \cdot (\int_J \lambda(t)^{-2} dt)^{\frac{1}{4000 d^2}} \lesssim \eta_1^{\frac{1}{p}} 2^{-\frac{i}{p}} T^{\frac{1}{2000 d^2}} \| v_0 \|_{L^2}. \end{aligned} \tag{5.20}$$

Lemma 2 follows by replacing v with a U_Δ^p atom and summing up. □

Since $\eta_1^{\frac{1}{p}} 2^{-\frac{i}{p}} T^{\frac{1}{2000 d^2}} \ll 1$ when $i \geq \frac{k}{10d}$, the contributions of (5.16) and (5.17) may be absorbed into the left hand side of (5.8). Therefore, by (5.11) and (5.14), for any $\frac{k}{10d} \leq i \leq k(1 + \frac{1}{10d})$,

$$\begin{aligned} &\| P_{\geq i} u \|_{U_\Delta^p(J \times \mathbb{R}^d)} + \| P_{\geq i} u \|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbb{R}^d)} \\ &\lesssim \| \int_{t_0}^t e^{i(t-\tau)\Delta} P_{\geq i} F(u_{\leq i}) d\tau \|_{U_\Delta^p \cap L_t^2 L_x^{\frac{2d}{d-2}}} + (\int_J \| \epsilon(t) \|_{L^2}^2 \lambda(t)^{-2} dt)^{1/2} + T^{-10}. \end{aligned} \tag{5.21}$$

By (1.17), (5.6), and (5.7), for $i \geq \frac{k}{10d}$,

$$\| P_{\geq i} F(\tilde{Q}) \|_{L_t^1 L_x^2(J \times \mathbb{R}^d)} \lesssim T^{-10}, \tag{5.22}$$

so it only remains to compute

$$\| \int_{t_0}^{t} e^{i(t-\tau)\Delta} P_{\geq i}[F(u_{\leq i}) - F(\tilde{Q})] d\tau \|_{U^p_\Delta \cap L^2_t L^{\frac{2d}{d-2}}_x (J\times\mathbb{R}^d)}. \tag{5.23}$$

Once again from [HHK09]

$$\| \int_{t_0}^{t} e^{i(t-\tau)\Delta} F d\tau \|_{L^2_t L^{\frac{2d}{d-2}}_x \cap U^p_\Delta} \lesssim \sup_v \int_J (v, F)_{L^2} dt, \tag{5.24}$$

where $\|v\|_{U^p_\Delta(J\times\mathbb{R}^d)} \lesssim \|v\|_{V^2_\Delta(J\times\mathbb{R}^d)} = 1$. Therefore, suppose $\|v\|_{V^2_\Delta(J\times\mathbb{R}^d)} = 1$ and $\hat{v}(t,\xi)$ is supported on $|\xi| \geq 2^i$. By Bernstein's inequality,

$$\begin{aligned} \int_J (v, F(u_{\leq i}) - F(\tilde{Q}))_{L^2} dt &\lesssim 2^{-i} \int_J (v, (\nabla u_{\leq i}) F'(u_{\leq i}) - (\nabla \tilde{Q}) F'(\tilde{Q}))_{L^2} dt \\ &= 2^{-i} \int_J (v, \nabla(u_{\leq i} - \tilde{Q}) F'(u_{\leq i}) + \nabla \tilde{Q}(F'(u_{\leq i}) - F'(\tilde{Q})))_{L^2} dt. \end{aligned} \tag{5.25}$$

To esimate (5.25), it is useful to split into regions where $u_{\leq i} - \tilde{Q} \ll \tilde{Q}$ and where $\tilde{Q} \lesssim u_{\leq i} - \tilde{Q}$. Let ψ be a smooth, cutoff function, $\psi(x) = 1$ for $|x| \geq \frac{1}{2}$, $\psi(x) = 0$ for $|x| \leq \frac{1}{4}$. Now abuse notation and let $\psi(x)$ denote $\psi(x) = \psi(\frac{P_{\leq i} u - \tilde{Q}}{\tilde{Q}})$. There exists a sequence of constants c_j that are uniformly bounded such that

$$\begin{aligned} &(1 - \psi^2(x))[\nabla(u_{\leq i} - \tilde{Q}) F'(u_{\leq i}) + \nabla \tilde{Q}(F'(u_{\leq i}) - F'(\tilde{Q}))] \\ &= (1 - \psi^2(x))[\nabla(u_{\leq i} - \tilde{Q}) \sum_{j\geq 0} c_j \frac{(u_{\leq i} - \tilde{Q})^j}{\tilde{Q}^{j-4/d}} + \nabla \tilde{Q} \sum_{j\geq 1} \frac{(u_{\leq i} - \tilde{Q})^j}{\tilde{Q}^{j-4/d}}. \end{aligned} \tag{5.26}$$

Following the computations in the proof of Lemma 2, since $\|v\|_{U^p_\Delta} \lesssim 1$,

$$\begin{aligned} &2^{-i} \int_J (v, (1 - \psi^2(x)) \nabla(u_{\leq i} - \tilde{Q}) \sum_{j\geq 0} c_j \frac{(u_{\leq i} - \tilde{Q})^j}{\tilde{Q}^{j-4/d}})_{L^2} dt \\ &\lesssim 2^{-i-\frac{i}{p}} \int_J (v, \nabla(u_{\leq i} - \tilde{Q}) |\tilde{Q}|^{4/d})_{L^2} dt \lesssim \eta_1^{1/p} 2^{-\frac{i}{p} - i} T^{\frac{1}{2000 d^2}} \|\nabla(u_{\leq i} - \tilde{Q})\|_{L^2_t L^{\frac{2d}{d-2}}_x}. \end{aligned} \tag{5.27}$$

Next, on the support of $1 - \psi^2(x)$, by the definition of $\tilde{Q}$, (5.6), (5.7), and Theorem 15,

$$\begin{aligned} \nabla \tilde{Q} \sum_{j\geq 1} \frac{(u_{\leq i} - \tilde{Q})^j}{\tilde{Q}^{j-4/d}} &\lesssim \frac{|\xi(t)|}{\lambda(t)} \tilde{Q} \sum_{j\geq 1} \frac{(u_{\leq i} - \tilde{Q})^j}{\tilde{Q}^{j-4/d}} + \lambda(t)^2 \frac{\nabla Q(\frac{x - x(t)}{\lambda(t)})}{Q(\frac{x-x(t)}{\lambda(t)})^{1-\frac{2}{d}}} \tilde{Q}^{1-\frac{2}{d}} \sum_{j\geq 1} \frac{(u_{\leq i} - \tilde{Q})^j}{\tilde{Q}^{j-4/d}} \\ &\lesssim \frac{|\xi(t)|}{\lambda(t)} \tilde{Q}^{4/d} |u_{\leq i} - \tilde{Q}| + \lambda(t)^2 \frac{\nabla Q(\frac{x-x(t)}{\lambda(t)})}{Q(\frac{x-x(t)}{\lambda(t)})^{1-\frac{2}{d}}} \tilde{Q}^{2/d} |u_{\leq i} - \tilde{Q}|. \end{aligned} \tag{5.28}$$

Again following the computations in the proof of Lemma 2, since $\|v\|_{U^p_\Delta} \lesssim 1$, by (5.6) and (5.7),

$$2^{-i} \int_J (v, (1 - \psi^2(x)) \nabla \tilde{Q} \sum_{j\geq 1} \frac{(u_{\leq i} - \tilde{Q})^j}{\tilde{Q}^{j-4/d}})_{L^2} dt \lesssim \eta_1^{1/p} \eta_0 2^{-\frac{i}{p} - i} T^{\frac{1}{2000 d^2}} \|(u_{\leq i} - \tilde{Q})\|_{L^2_t L^{\frac{2d}{d-2}}_x}. \tag{5.29}$$

Similarly, since $|\tilde{Q}| \lesssim |u_{\leq i} - \tilde{Q}|$ on the support of $\psi(x)$, by the definition of $\tilde{Q}$, (5.6), (5.7), and Theorem 15,

$$\begin{aligned} \psi^2(x)(\nabla \tilde{Q}(F'(u_{\leq i}) - F'(\tilde{Q})) &\lesssim \psi^2(x) \nabla \tilde{Q} \cdot |u_{\leq i} - \tilde{Q}|^{4/d} \\ \lesssim \psi^2(x) \frac{|\xi(t)|}{\lambda(t)} \tilde{Q} |u_{\leq i} - \tilde{Q}|^{4/d} + \psi(x)^2 \lambda(t)^{-2} &\frac{\nabla Q(\frac{x-x(t)}{\lambda(t)})}{Q(\frac{x-x(t)}{\lambda(t)})^{1-2/d}} |u_{\leq i} - \tilde{Q}|^{4/d} \tilde{Q}^{1-2/d} \\ \lesssim \eta_0 |u_{\leq i} - \tilde{Q}| |\tilde{Q}|^{2/d} |\tilde{\epsilon}|^{2/d} + \eta_1 \lambda(t)^{-1} &\frac{\nabla Q(\frac{x-x(t)}{\lambda(t)})}{Q(\frac{x-x(t)}{\lambda(t)})^{1-2/d}} |u_{\leq i} - \tilde{Q}| |\tilde{\epsilon}|^{2/d}. \end{aligned} \tag{5.30}$$

Therefore,

$$2^{-i} \int_J (v, \psi^2(x) \nabla \tilde{Q}[F'(u_{\leq i}) - F'(\tilde{Q})])_{L^2} dt \lesssim \eta_1^{1/p} \eta_0 2^{-\frac{i}{p} - i} T^{\frac{1}{2000 d^2}} \eta_0 \| (u_{\leq i} - \tilde{Q}) \|_{L_t^2 L_x^{\frac{2d}{d-2}}}. \tag{5.31}$$

Now decompose,

$$\nabla(u_{\leq i} - \tilde{Q}) F'(u_{\leq i}) = \nabla(u_{\leq i} - \tilde{Q}) F'(u_{\leq i} - \tilde{Q}) + \nabla(u_{\leq i} - \tilde{Q})[F'(u_{\leq i}) - F'(u_{\leq i} - \tilde{Q})]. \tag{5.32}$$

Since

$$\nabla(u_{\leq i} - \tilde{Q})[F'(u_{\leq i}) - F'(u_{\leq i} - \tilde{Q})] \lesssim |\nabla(u_{\leq i} - \tilde{Q})| |\tilde{Q}|^{4/d}, \tag{5.33}$$

then as in (5.27),

$$2^{-i} \int_J (v, \psi^2(x) |\nabla(u_{\leq i} - \tilde{Q}| [F'(u_{\leq i}) - F'(u_{\leq i} - \tilde{Q})])_{L^2} dt \lesssim \eta_1^{1/p} 2^{-\frac{i}{p} - i} T^{\frac{1}{2000 d^2}} \| \nabla(u_{\leq i} - \tilde{Q}) \|_{L_t^2 L_x^{\frac{2d}{d-2}}}. \tag{5.34}$$

Also, since

$$(1 - \psi^2(x)) |\nabla(u_{\leq i} - \tilde{Q})| |F'(u_{\leq i}) - F'(\tilde{Q})| \lesssim |\nabla(u_{\leq i} - \tilde{Q})| |\tilde{Q}|^{4/d}, \tag{5.35}$$

by (5.27),

$$2^{-i} \int_J (v, (1 - \psi^2(x)) |\nabla(u_{\leq i} - \tilde{Q})| [F'(u_{\leq i} - \tilde{Q})])_{L^2} dt \lesssim \eta_1^{1/p} 2^{-\frac{i}{p} - i} T^{\frac{1}{2000 d^2}} \| \nabla(u_{\leq i} - \tilde{Q}) \|_{L_t^2 L_x^{\frac{2d}{d-2}}}. \tag{5.36}$$

Therefore, it only remains to estimate

$$\begin{aligned} 2^{-i} \| \int_{t_0}^t e^{i(t-\tau)\Delta} P_{\geq i} (\nabla(u_{\leq i} - \tilde{Q}) F'(u_{\leq i} - \tilde{Q})) d\tau \|_{L_t^2 L_x^{\frac{2d}{d-2}} \cap U_\Delta^p(J \times \mathbb{R}^d)} \\ \lesssim 2^{-i} \| P_{\geq i} (\nabla(u_{\leq i} - \tilde{Q}) F'(u_{\leq i} - \tilde{Q})) \|_{L_t^2 L_x^{\frac{2d}{d+2}}(J \times \mathbb{R}^d)}. \end{aligned} \tag{5.37}$$

For this term, use fractional derivative chain rule (Proposition $A.1$ of [Vis07]),

Proposition 1. *Let F be a Hölder continuous function of order $0 < \alpha < 1$. Then for every $0 < \sigma < \alpha$, $1 < p < \infty$, and $\frac{\sigma}{\alpha} < s < 1$, we have*

$$\| |\nabla|^\sigma F(u) \|_{L^p} \lesssim \| |u|^{\alpha - \frac{\sigma}{s}} \|_{L^{p_1}} \| |\nabla|^s u \|_{L^{\frac{\sigma}{s} p_2}}^{\frac{\sigma}{s}}, \tag{5.38}$$

provided $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $(1 - \frac{\sigma}{\alpha s}) p_1 > 1$.

Then for any $0 \le \sigma < 1 + \frac{4}{d}$,

$$\begin{aligned} &2^{-i}\int_J (v, \nabla(u_{\le i} - \tilde{Q})F'(u_{\le i} - \tilde{Q}))_{L^2} dt \\ \lesssim_\sigma\ & 2^{-\sigma i}\||\nabla|^\sigma (u_{\le i} - \tilde{Q})\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \|u_{\le i} - \tilde{Q}\|^{4/d}_{L_t^\infty L_x^2} \lesssim \eta_0^{4/d} 2^{-\sigma i}\||\nabla|^\sigma (u_{\le i} - \tilde{Q})\|_{L_t^2 L_x^{\frac{2d}{d-2}}}. \end{aligned} \tag{5.39}$$

Therefore, arguing by induction on frequency, if (5.8) holds for all $\frac{k}{10} \le i \le j_0$ for some $\frac{k}{10} \le j_0 \le k$, then for $i = j_0 + 1$, (5.8) holds, which proves the Theorem 16 by induction on frequency. Indeed, choosing σ such that $\sigma > \frac{\alpha_d}{2}$, the contributions of (5.23)–(5.39) may be bounded by

$$\begin{aligned} &\eta_1^{1/p} 2^{-\frac{i}{p} - i} T^{\frac{1}{2000 d^2}} \sum_{0 \le j \le i} 2^j \|P_j(u - \tilde{Q})\|_{L_t^2 L_x^{\frac{2d}{d-2}}} + \eta_0^{4/d} \sum_{0 \le j \le i} 2^{\sigma j} \|P_j(u - \tilde{Q})\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbb{R}^d)} \\ &\lesssim (\eta_1^{1/p} 2^{-\frac{i}{11d}} T^{\frac{1}{2000 d^2}} + \eta_0^{4/d})(2^{\frac{\alpha_d}{2}((1+\frac{1}{10d})k - i)} (\frac{1}{T}\int_J \|\epsilon(t)\|_{L^2}^2)^{1/2} + T^{-10}). \end{aligned} \tag{5.40}$$

When $i \ge \frac{k}{10d}$,

$$\eta_1^{1/p} 2^{-\frac{i}{11d}} T^{\frac{1}{2000 d^2}} + \eta_0^{4/d} \ll 1, \tag{5.41}$$

so the induction on frequency step is complete. □

6. Almost conservation of energy

In two dimensions, the almost conservation of energy computation is exactly the same as in one dimension in [Dod21a]. The computations in higher dimensions are more difficult since $F(x) = |x|^{4/d}x$ is not smooth. The good news is that most of the difficult computations have already been done in the previous section.

6.1. Almost conservation of energy in dimension $d = 2$.

Theorem 17. *Let $J = [a, b]$ be an interval for which (4.11) holds. Then,*

$$\sup_{t \in J} E(P_{\le k+9} u(t)) \lesssim \frac{2^{2k}}{T}\int_J \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2} dt + \sup_{t \in J} \frac{|\xi(t)|^2}{\lambda(t)^2} + 2^{2k} T^{-10}. \tag{6.1}$$

Proof. Since $\|\epsilon(t)\|_{L^2}$ is continuous as a function of time, the mean value theorem implies that under the conditions of Theorem 13, there exists $t_0 \in [a, b] = J$ such that

$$\|\epsilon(t_0)\|_{L^2}^2 = \frac{1}{T}\int_a^b \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2} dt. \tag{6.2}$$

Then by (4.26), the fact that Q is a smooth real valued function and all its derivatives are rapidly decreasing, and the Sobolev embedding theorem, taking $\epsilon = \epsilon_1 + i\epsilon_2$,

(6.3)

$$\begin{aligned} E(P_{\le k+9} u) = \frac{1}{2}\|\nabla P_{\le k+9} u\|_{L^2}^2 - \frac{1}{4}\|P_{\le k+9} u\|_{L^4}^4 = \frac{1}{2\lambda(t)^2}\|\nabla Q(x)\|_{L^2}^2 + \frac{1}{2}\frac{|\xi(t)|^2}{\lambda(t)^2}\|Q\|_{L^2}^2 - \frac{1}{4\lambda(t)^2}\|Q\|_{L^4}^4 \\ + \frac{1}{\lambda(t)^2}(\nabla Q, \nabla \epsilon_1)_{L^2} - \frac{\xi(t)}{\lambda(t)^2} \cdot (Q, \nabla \epsilon_2) + \frac{\xi(t)}{\lambda(t)^2} \cdot (\nabla Q, \epsilon_2)_{L^2} + \frac{|\xi(t)|^2}{\lambda(t)^2}(Q, \epsilon_1)_{L^2} - \frac{1}{\lambda(t)^2}(Q^3, \epsilon_1)_{L^2} \\ + O(2^{2k}\|\epsilon\|_{L^2}^2) + O(2^{2k} T^{-10}). \end{aligned}$$

Since $(\epsilon_2, \nabla Q)_{L^2} = (\nabla \epsilon_2, Q)_{L^2} = 0$, $(\nabla Q, \nabla \epsilon_1)_{L^2} - (Q^3, \epsilon_1)_{L^2} = -(Q, \epsilon_1)_{L^2} = \frac{1}{2}\|\epsilon\|_{L^2}^2$, and $E(Q) = 0$,

$$E(P_{\leq k+9}u) = \frac{1}{2}\frac{|\xi(t)|^2}{\lambda(t)^2}\|Q\|_{L^2}^2 + \frac{1}{2\lambda(t)^2}\|\epsilon\|_{L^2}^2 - \frac{|\xi(t)|^2}{2\lambda(t)^2}\|\epsilon\|_{L^2}^2 + O(2^{2k}\|\epsilon\|_{L^2}^2) + O(2^{2k}T^{-10}), \tag{6.4}$$

so (6.1) holds at t_0.

Next compute the change of energy. This computation utilizes the computations in the Fourier truncation method of [Bou98]. See also the I-method in [CKS+02].

$$\frac{d}{dt}E(P_{\leq k+9}u) = -(P_{\leq k+9}u_t, \Delta P_{\leq k+9}u)_{L^2} - (P_{\leq k+9}u_t, |P_{\leq k+9}u|^2 P_{\leq k+9}u)_{L^2} \tag{6.5}$$

$$= -(P_{\leq k+9}u_t, P_{\leq k+9}F(u) - F(P_{\leq k+9}u))_{L^2} = (i\Delta P_{\leq k+9}u + iP_{\leq k+9}(|u|^2u), P_{\leq k+9}F(u) - F(P_{\leq k+9}u))_{L^2}.$$

First compute

$$\int_{t_0}^{t'} (i\Delta P_{\leq k+9}u, P_{\leq k+9}F(u) - F(P_{\leq k+9}u))_{L^2}dt, \tag{6.6}$$

for some $t' \in J$. Making a Littlewood–Paley decomposition,

$$\int_{t_0}^{t'} (i\Delta P_{\leq k+9}u, P_{\leq k+9}F(u) - F(P_{\leq k+9}u))_{L^2}dt \tag{6.7}$$

$$\sim \sum_{0 \leq k_3 \leq k_2 \leq k_1} \sum_{0 \leq k_4 \leq k+9} \int_{t_0}^{t'} (i\Delta P_{k_4}u, P_{\leq k+9}(P_{k_1}u \cdots P_{k_3}u) - (P_{\leq k+9}P_{k_1}u) \cdots (P_{\leq k+9}P_{k_3}u))_{L^2}dt.$$

Remark 6. *For these computations, it is not so important to distinguish between u and $\bar{u}$.*

Case 1, $k_1 \leq k+6$: In this case $P_{\leq k+9}P_{k_1} = P_{k_1}$ and $P_{\leq k+9}(P_{k_1}u \cdots P_{k_3}u) = P_{k_1}u \cdots P_{k_3}u$, so the contribution of these terms is zero. That is, for $k_1, ..., k_3 \leq k+6$,

$$\int_{t_0}^{t'} (i\Delta P_{k_6}u, P_{\leq k+9}(P_{k_1}u \cdots P_{k_3}u) - (P_{\leq k+9}P_{k_1}u) \cdots (P_{\leq k+9}P_{k_3}u))_{L^2}dt = 0. \tag{6.8}$$

Case 2, $k_1 \geq k+6$ and $k_2 \leq k$: In this case, Fourier support properties imply that $k_4 \geq k+3$. Then by Theorem 14,

$$\int_{t_0}^{t'} (i\Delta P_{k+3\leq \cdot \leq k+9}u, P_{\leq k+9}((P_{\leq k}u)^2(P_{\geq k+6}u)) - (P_{\leq k}u)^2(P_{k+6\leq \cdot \leq k+9}u))_{L^2}dt \tag{6.9}$$

$$\lesssim 2^{2k}\|(P_{\geq k+3}u)(P_{\leq k}u)\|_{L^2_{t,x}}\|(P_{\geq k+6}u)(P_{\leq k}u)\|_{L^2_{t,x}} \lesssim \frac{2^{2k}}{T}\int_J \|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}dt + 2^{2k}T^{-10}. \tag{6.10}$$

Case 3, $k_1 \geq k+6$, $k_2 \geq k$ $k_3 \leq k$: If $k_4 \leq k$, then by Fourier support properties, $k_2 \geq k+3$. In that case,

$$\int_{t_0}^{t'} (i\Delta P_{\leq k}u, P_{\leq k+9}((P_{\geq k+6}u)(P_{\geq k+3}u)(P_{\leq k}u)) - (P_{k+6\leq \cdot \leq k+9}u)(P_{k+3\leq \cdot \leq k+9}u)(P_{\leq k}u))_{L^2}dt \tag{6.11}$$

$$\lesssim 2^{2k}\|(P_{\geq k+6}u)(P_{\leq k}u)\|_{L^2_{t,x}}\|(P_{\geq k+3}u)(P_{\leq k}u)\|_{L^2_{t,x}} \lesssim \frac{2^{2k}}{T}\int_J \|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}dt + 2^{2k}T^{-10}.$$

In the case when $k_4 \geq k$,

(6.12)
$$\int_{t_0}^{t'} (i\Delta P_{k\leq\cdot\leq k+9}u, P_{\leq k+9}((P_{\geq k+6}u)(P_{\geq k}u)(P_{\leq k}u)) - (P_{k+6\leq\cdot\leq k+9}u)(P_{k\leq\cdot\leq k+9}u)(P_{\leq k}u))_{L^2}dt$$
$$\lesssim 2^{2k}\|(P_{\geq k+6}u)(P_{\leq k}u)\|_{L^2_{t,x}}\|P_{\geq k}u\|^2_{L^4_{t,x}} \lesssim \frac{2^{2k}}{T}\int_J \|\epsilon(t)\|^2_{L^2}\lambda(t)^{-2}dt + 2^{2k}T^{-10}.$$

Case 4, $k_1 \geq k+6$ and $k_2, k_3 \geq k$: In this case,

(6.13)
$$\int_{t_0}^{t'} (i\Delta P_{\leq k+9}u, P_{\leq k+9}((P_{\geq k+6}u)(P_{\geq k}u)^2) - (P_{k+6\leq\cdot\leq k+9}u)(P_{k\leq\cdot\leq k+9}u)^2)_{L^2}dt$$
$$\lesssim 2^{2k}\|(P_{\geq k+6}u)(P_{\leq k}u)\|_{L^2_{t,x}}\|P_{\geq k}u\|^2_{L^4_{t,x}} + 2^{2k}\|P_{\geq k}u\|^4_{L^4_{t,x}} \lesssim \frac{2^{2k}}{T}\int_J \|\epsilon(t)\|^2_{L^2}\lambda(t)^{-2}dt + 2^{2k}T^{-10}.$$

The contribution of the nonlinear terms is similar, using the fact that

$$(iP_{\leq k+9}F(u), P_{\leq k+9}F(u) - F(P_{\leq k+9}u))_{L^2} = (iP_{\leq k+9}F(u), F(P_{\leq k+9}u))_{L^2}. \tag{6.14}$$

Then make a Littlewood–Paley decomposition,

$$(iP_{\leq k+9}F(u), F(P_{\leq k+9}u))_{L^2}$$
$$= \sum_{0\leq k_3\leq k_2\leq k_1} \sum_{0\leq k_3'\leq k_2'\leq k_1'} (iP_{\leq k+9}(u_{k_1}\cdots u_{k_3}), (P_{\leq k+9}P_{k_1}u)\cdots(P_{\leq k+9}P_{k_3}u))_{L^2}. \tag{6.15}$$

Case 1: $k_1, k_1' \leq k+6$: Once again, if $k_1, k_1' \leq k+6$, then the right hand side of (6.15) is zero, since $P_{\leq k+9}(u_{k_1}u_{k_2}u_{k_3}) = u_{k_1}u_{k_2}u_{k_3}$ and $P_{\leq k+9}u_{k_j} = u_{k_j}$ for $j = 1, 2, 3$.

Case 2: k_1 or $k_1' \geq k+6$, four terms are $\leq k$: In the case that k_1 or $k_1' \geq k+6$, and four of the terms in (6.15) are at frequency $\leq k$, then by Fourier support properties the final term should be at frequency $\geq k+3$. The contribution in this case is bounded by

$$\|(P_{\geq k+6}u)(P_{\leq k}u)\|_{L^2_{t,x}}\|(P_{\geq k+3}u)(P_{\leq k}u)\|_{L^2_{t,x}}\|P_{\leq k}u\|^2_{L^\infty_{t,x}} \lesssim 2^{2k}\frac{1}{T}\int_J \|\epsilon(t)\|^2_{L^2}dt + 2^{2k}T^{-10}. \tag{6.16}$$

Case 3: k_1 or $k_1' \geq k+6$, two additional terms are $\geq k$: The contribution of the case that k_1 or $k_1' \geq k+6$, two additional terms in (6.15) are at frequency $\geq k$, and the other three terms are at frequency $\leq k$ is bounded by

$$\|(P_{\geq k+6}u)(P_{\leq k}u)\|_{L^2_{t,x}}\|P_{\geq k}u\|^2_{L^4_{t,x}}\|P_{\leq k}u\|^2_{L^\infty_{t,x}} \lesssim 2^{2k}\frac{1}{T}\int_J \|\epsilon(t)\|^2_{L^2}dt + 2^{2k}T^{-10}. \tag{6.17}$$

Case 4: k_1 or $k_1' \geq k+6$ and at least three additional terms in (6.15) **are at frequencies $\geq k$.**

This case may always be reduced to the estimate

$$2^{2k}\|P_{\geq k}u\|^4_{L^4_{t,x}}\|u\|^2_{L^\infty_t L^2_x} \lesssim \frac{2^{2k}}{T}\int_J \|\epsilon(t)\|^2_{L^2}\lambda(t)^{-2}dt + 2^{2k}T^{-10}. \tag{6.18}$$

Indeed, if both $P_{\leq k+9}(u_{k_1}u_{k_2}u_{k_3})$ and $P_{\leq k+9}u_{k_1}\cdot P_{\leq k+9}u_{k_2}\cdot P_{\leq k+9}u_{k_3}$ have two terms at frequency $\leq k$, then place each term in $L^2_tL^1_x$ and then make a Sobolev embedding at frequencies $\leq k+9$ to place each term in $L^2_{t,x}$.

If $P_{\leq k+9}(u_{k_1}u_{k_2}u_{k_3})$ has three terms at frequency $\geq k$, then estimate this triple product in $L^{4/3}_{t,x}$, and place the term in $P_{\leq k+9}u_{k_1} \cdot P_{\leq k+9}u_{k_2} \cdot P_{\leq k+9}u_{k_3}$ at frequency $\geq k$ in $L^4_{t,x}$ and using the Sobolev embedding theorem, place the other two in $L^\infty_{t,x}$.

If $P_{\leq k+9}(u_{k_1}u_{k_2}u_{k_3})$ has only one term at frequency $\geq k$, then estimate this triple product in $L^4_{t,x}$ and place the term $P_{k\leq \cdot \leq k+9}u_{k_1} \cdot P_{k\leq \cdot \leq k+9}u_{k_2} \cdot P_{k\leq \cdot \leq k+9}u_{k_3}$ in $L^{4/3}_{t,x}$.

This completes the proof of Theorem 17. □

This bound on $E(P_{\leq k+9}u)$ gives good bounds on the L^2 norm and $\dot{H}^1$ norms of ϵ.

Theorem 18. *If*

$$\frac{1}{\eta_1} \leq \lambda(t) \leq \frac{1}{\eta_1}T^{1/100}, \qquad \text{and} \qquad \frac{|\xi(t)|}{\lambda(t)} \leq \eta_0, \qquad \text{for all} \qquad t \in J, \tag{6.19}$$

and

$$\int_J \lambda(t)^{-2}dt = T, \qquad \text{and} \qquad \eta_1^{-2}T = 2^{3k}, \tag{6.20}$$

then

$$\sup_{t\in J} \|P_{\leq k+9}(\frac{e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}}{\lambda(t)^{1/2}}\epsilon(t, \frac{x-x(t)}{\lambda(t)}))\|^2_{\dot{H}^1} \lesssim \frac{2^{2k}}{T}\int_J \|\epsilon(t)\|^2_{L^2}\lambda(t)^{-2}dt + \sup_{t\in J}\frac{|\xi(t)|^2}{\lambda(t)^2} + 2^{2k}T^{-10}, \tag{6.21}$$

and

$$\sup_{t\in J} \|\epsilon(t)\|^2_{L^2} \lesssim \frac{2^{2k}T^{1/50}}{\eta_1^2 T}\int_J \|\epsilon(t)\|^2_{L^2}\lambda(t)^{-2}dt + \frac{T^{1/50}}{\eta_1^2}\sup_{t\in J}\frac{|\xi(t)|^2}{\lambda(t)^2} + 2^{2k}\frac{T^{1/50}}{\eta_1^2}T^{-10}. \tag{6.22}$$

Proof. This theorem is a direct consequence of Theorem 17 and an expansion of the energy. Recall from (6.4) that

$$E(P_{\leq k+9}u) = \frac{1}{2}\frac{|\xi(t)|^2}{\lambda(t)^2}\|Q\|^2_{L^2} + \frac{1}{2\lambda(t)^2}\|\epsilon\|^2_{L^2} - \frac{|\xi(t)|^2}{2\lambda(t)^2}\|\epsilon\|^2_{L^2} + \mathcal{P}_2 + \mathcal{P}_3 + \mathcal{P}_4 + O(2^{2k}T^{-10}), \tag{6.23}$$

where $\mathcal{P}_j$, $j = 2, 3, 4$ refers to terms in the expansion of $E(P_{\leq k+9}u)$ with j ϵ's in the product. Split

$$\begin{aligned} P_{\leq k+9}\tilde{\epsilon}(t,x) &= P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}\lambda(t)^{-1}\epsilon(t, \frac{x-x(t)}{\lambda(t)})) \\ &= e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}P_{\leq k+9}(\lambda(t)^{-1}\epsilon(t, \frac{x-x(t)}{\lambda(t)})) + \mathcal{R} = \tilde{\tilde{\epsilon}}(t,x) + \mathcal{R}. \end{aligned} \tag{6.24}$$

Using the fact that $|\frac{|\xi(t)|}{\lambda(t)}| \leq \eta_0$ the Fourier support properties of $P_{\leq k+9}$, and the discussion of Fourier multipliers prior to (4.67),

$$\|\mathcal{R}\|_{L^2} \lesssim 2^{-k}\frac{|\xi(t)|}{\lambda(t)}\|\epsilon\|_{L^2} \lesssim 2^{-k}\eta_0\|\epsilon\|_{L^2}, \qquad \text{and} \qquad \|\nabla\mathcal{R}\|_{L^2} \lesssim \frac{|\xi(t)|}{\lambda(t)}\|\epsilon\|_{L^2} \lesssim \eta_0\|\epsilon\|_{L^2}. \tag{6.25}$$

Since Q is real valued, smooth, and has derivatives that are rapidly decreasing, using (6.24) and (6.25),

$$\begin{aligned} \mathcal{P}_2 = \frac{1}{2}\|\nabla\tilde{\tilde{\epsilon}}\|^2_{L^2} - \frac{1}{\lambda(t)^4}\int Q(\frac{x-x(t)}{\lambda(t)})^2|P_{\leq k+9}\epsilon(t, \frac{x-x(t)}{\lambda(t)})|^2dx \\ -\frac{1}{2\lambda(t)^4}Re\int Q(\frac{x-x(t)}{\lambda(t)})^2(P_{\leq k+9}\epsilon(t, \frac{x-x(t)}{\lambda(t)}))^2dx + O(2^{2k}T^{-10}) + O(\eta_0^2\|\epsilon\|^2_{L^2}) + O(\eta_0\|\epsilon\|_{L^2}\|\nabla\tilde{\tilde{\epsilon}}\|_{L^2}). \end{aligned} \tag{6.26}$$

By the product rule,

(6.27)
$$\frac{1}{2}\|\nabla\tilde{\tilde{\epsilon}}\|_{L^2}^2 = \frac{|\xi(t)|^2}{2\lambda(t)^4}\|P_{\leq k+9}\epsilon(t,\frac{x-x(t)}{\lambda(t)})\|_{L^2}^2 + \frac{\xi(t)}{\lambda(t)^3}\cdot(P_{\leq k+9}\epsilon(t,\frac{x-x(t)}{\lambda(t)}), iP_{\leq k+9}\nabla\epsilon(t,\frac{x-x(t)}{\lambda(t)}))_{L^2} + \frac{1}{2\lambda(t)^2}\|P_{\leq k+9}\epsilon(t,\frac{x-x(t)}{\lambda(t)})\|_{\dot{H}^1}^2.$$

Rescaling, if $2^{n(t)} = \lambda(t)$,

$$\begin{aligned}
&\frac{1}{2\lambda(t)^2}\|P_{\leq k+9}\epsilon(t,\frac{x-x(t)}{\lambda(t)})\|_{\dot{H}^1}^2 - \frac{1}{\lambda(t)^4}\int Q(\frac{x-x(t)}{\lambda(t)})^2|P_{\leq k+9}\epsilon(t,\frac{x-x(t)}{\lambda(t)})|^2dx\\
&\qquad -\frac{1}{2\lambda(t)^4}Re\int Q(\frac{x-x(t)}{\lambda(t)})^2(P_{\leq k+9}\epsilon(t,\frac{x-x(t)}{\lambda(t)}))^2dx\\
&= \frac{1}{2\lambda(t)^2}\|P_{\leq k+9+n(t)}\epsilon(t,x)\|_{\dot{H}^1}^2 - \frac{1}{\lambda(t)^2}\int Q(x)^2|P_{\leq k+9+n(t)}\epsilon(t,x)|^2dx\\
&\qquad -\frac{1}{2\lambda(t)^2}Re\int Q(x)^2(P_{\leq k+9+n(t)}\epsilon(t,x)^2dx.
\end{aligned} \tag{6.28}$$

Then using the spectral theory of $\mathcal{L}$ in Theorem 10, provided

$$g \perp span\{\chi_0, i\chi_0, Q_{x_j}, iQ_{x_j}\}, \tag{6.29}$$

there exists a fixed constant $\lambda_1 > 0$ such that

$$\frac{1}{2}\|\nabla g\|_{L^2}^2 + \frac{1}{2}\|g\|_{L^2}^2 - \int Q(x)^2|g(x)|^2 - \frac{1}{2}Re\int Q(x)^2g(x)^2dx \geq \lambda_1\|g\|_{\dot{H}^1}^2. \tag{6.30}$$

It is not quite true for $P_{\leq k+9+n(t)}\epsilon$ that (6.29) holds, however, by (3.7), the fact that χ_0 and Q_{x_j} are smooth with rapidly decreasing derivatives, and the bounds on $\lambda(t)$ in (6.19),

$$(P_{\leq k+9+n(t)}\epsilon, f)_{L^2} \lesssim T^{-10}, \tag{6.31}$$

for any f in (3.9). Therefore, for $\eta_0 \ll 1$ sufficiently small, there exists some fixed $\lambda_1 > 0$ such that

$$\frac{1}{2\lambda(t)^2}\|\epsilon\|_{L^2}^2 + \mathcal{P}_2 \geq \frac{\lambda_1}{\lambda(t)^2}\|\epsilon\|_{L^2}^2 + \lambda_1\|\tilde{\tilde{\epsilon}}\|_{\dot{H}^1}^2 - O(2^{2k}T^{-10}). \tag{6.32}$$

Next, by the Sobolev embedding theorem and (6.25),

(6.33)
$$\int|P_{\leq k+9}\tilde{\epsilon}(t,x)|^4dx \lesssim \|\tilde{\tilde{\epsilon}}\|_{L^4}^4 + \|\mathcal{R}\|_{L^4}^4 \lesssim \|\tilde{\tilde{\epsilon}}\|_{\dot{H}^1}^2\|\epsilon\|_{L^2}^2 + \|\mathcal{R}\|_{L^4}^4 \lesssim \eta_0^2\|\tilde{\tilde{\epsilon}}\|_{\dot{H}^1}^2 + 2^{-2k}\eta_0^2\frac{|\xi(t)|^2}{\lambda(t)^2}\|\epsilon\|_{L^2}^4,$$

and

(6.34)
$$\frac{1}{\lambda(t)}\int|Q(\frac{x-x(t)}{\lambda(t)})|\tilde{\epsilon}(t,x)|^3dx \lesssim \|\tilde{\epsilon}\|_{L^2}\|\tilde{\tilde{\epsilon}}\|_{L^4}^2 \lesssim \frac{1}{\lambda(t)}\|\tilde{\epsilon}\|_{L^2}^2\|\tilde{\epsilon}\|_{\dot{H}^1} \lesssim \frac{1}{\lambda(t)}\|\epsilon\|_{L^2}(\eta_0\|\tilde{\tilde{\epsilon}}\|_{\dot{H}^1} + 2^{-k}\eta_0\frac{|\xi(t)|}{\lambda(t)}\|\epsilon\|_{L^2}^2).$$

For $\eta_0 \ll 1$ sufficiently small, we have therefore proved

$$\begin{aligned}
E(P_{\leq k+9}u) &\geq \frac{1}{2}\frac{|\xi(t)|^2}{\lambda(t)^2}\|Q\|_{L^2}^2 + \frac{\lambda_1}{2\lambda(t)^2}\|\epsilon\|_{L^2}^2 + \frac{\lambda_1}{2}\|\tilde{\tilde{\epsilon}}\|_{\dot{H}^1}^2 - \frac{|\xi(t)|^2}{2\lambda(t)^2}\|\epsilon\|_{L^2}^2 - O(2^{2k}T^{-10})\\
&\geq \frac{1}{4}\frac{|\xi(t)|^2}{\lambda(t)^2}\|Q\|_{L^2}^2 + \frac{\lambda_1}{2\lambda(t)^2}\|\epsilon\|_{L^2}^2 + \frac{\lambda_1}{2}\|\tilde{\tilde{\epsilon}}\|_{\dot{H}^1}^2 - O(2^{2k}T^{-10}).
\end{aligned} \tag{6.35}$$

Plugging (6.35) into (6.1) proves Theorem 18. □

6.2. **Almost conservation of energy in dimension $d \geq 3$.**

Theorem 19. *Let $J = [a, b]$ be an interval for which (5.6) and (5.7) hold, as well as $|x(t)| \leq T^{\frac{1}{2000d^2}}$. To simplify notation let $k_0 = k(1 + \frac{1}{10d})$. Then,*

$$\sup_{t \in J} E(P_{\leq k_0+9}u(t)) \lesssim \frac{2^{2k_0}}{T} \int_J \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2} dt + \sup_{t \in J} \frac{|\xi(t)|^2}{\lambda(t)^2} + 2^{2k_0} T^{-10}. \tag{6.36}$$

Proof. Again choose $t_0 \in J$ such that (6.2) holds. As in the two dimensional case, by (5.24), the fact that Q is a real valued function, Q is smooth and all its derivatives are rapidly decreasing, and the Sobolev embedding theorem, taking $\epsilon = \epsilon_1 + i\epsilon_2$,

$$\begin{aligned} E(P_{\leq k_0+9}u) &= \frac{1}{2}\|\nabla P_{\leq k_0+9}u\|_{L^2}^2 - \frac{d}{2(d+2)}\|P_{\leq k_0+9}u\|_{L^{\frac{2(d+2)}{d}}}^{\frac{2(d+2)}{d}} \\ &= \frac{1}{2\lambda(t)^2}\|\nabla Q(x)\|_{L^2}^2 + \frac{1}{2}\frac{|\xi(t)|^2}{\lambda(t)^2}\|Q\|_{L^2}^2 - \frac{d}{2(d+2)\lambda(t)^2}\|Q\|_{L^{\frac{2(d+2)}{d}}}^{\frac{2(d+2)}{d}} \\ &+ \frac{1}{\lambda(t)^2}(\nabla Q, \nabla \epsilon_1)_{L^2} - \frac{\xi(t)}{\lambda(t)^2} \cdot (Q, \nabla \epsilon_2) + \frac{\xi(t)}{\lambda(t)^2} \cdot (\nabla Q, \epsilon_2)_{L^2} + \frac{|\xi(t)|^2}{\lambda(t)^2}(Q, \epsilon_1)_{L^2} - \frac{1}{\lambda(t)^2}(Q^{1+\frac{4}{d}}, \epsilon_1)_{L^2} \\ &+ O(2^{2k_0}\|\epsilon\|_{L^2}^2) + O(2^{2k_0}T^{-10}). \end{aligned} \tag{6.37}$$

As in the two dimensional case, $(\epsilon_2, \nabla Q)_{L^2} = (\nabla \epsilon_2, Q)_{L^2} = 0$, $(\nabla Q, \nabla \epsilon_1)_{L^2} - (Q^{1+\frac{4}{d}}, \epsilon_1)_{L^2} = -(Q, \epsilon_1)_{L^2} = \frac{1}{2}\|\epsilon\|_{L^2}^2$, and $E(Q) = 0$, so

$$E(P_{\leq k_0+9}u) = \frac{1}{2}\frac{|\xi(t)|^2}{\lambda(t)^2}\|Q\|_{L^2}^2 + \frac{1}{2\lambda(t)^2}\|\epsilon\|_{L^2}^2 - \frac{|\xi(t)|^2}{2\lambda(t)^2}\|\epsilon\|_{L^2}^2 + O(2^{2k_0}\|\epsilon\|_{L^2}^2) + O(2^{2k_0}T^{-10}), \tag{6.38}$$

so (6.36) holds at t_0.

Next compute the change of energy. Following the computations in the two dimensional case,

$$\frac{d}{dt}E(P_{\leq k_0+9}u) = (i\Delta P_{\leq k_0+9}u + iP_{\leq k_0+9}F(u), P_{\leq k_0+9}F(u) - F(P_{\leq k_0+9}u))_{L^2}. \tag{6.39}$$

Decompose

$$P_{\leq k_0+9}F(u) - F(P_{\leq k_0+9}u) = F(u) - F(P_{\leq k_0+9}u) - P_{\geq k_0+9}F(u). \tag{6.40}$$

From the proof of Theorem 16, the fact that Q is smooth and all its derivatives are rapidly decreasing, the Sobolev embedding theorem, and Fourier support properties,

$$\begin{aligned} \int_{t_0}^{t'} (i\Delta P_{\leq k_0+9}u + iP_{\leq k_0+9}F(u), P_{\geq k_0+9}F(u))_{L^2} dt &\lesssim 2^{2k_0}\|P_{k_0 \leq \cdot \leq k_0+9}u\|_{L_t^2 L_x^{\frac{2d}{d-2}}}\|P_{\geq k_0}F(u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\ &+ 2^{2k_0}\|P_{k_0 \leq \cdot \leq k_0+9}u\|_{L_t^2 L_x^{\frac{2d}{d+2}}}^2 \lesssim \frac{2^{2k_0}}{T}\int_J \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2} dt + 2^{2k_0}T^{-10}. \end{aligned} \tag{6.41}$$

Next, by Taylor's formula,

$$F(u) - F(P_{\leq k_0+9}u) = (P_{\geq k_0+9}u) \cdot \int_0^1 F'(P_{\leq k_0+9}u + sP_{\geq k_0+9}u) ds. \tag{6.42}$$

By (5.9),

$$\Delta u + F(u) = \Delta \tilde{Q} + \Delta \tilde{\epsilon} + F(\tilde{Q}) + F(u) - F(\tilde{Q}). \tag{6.43}$$

Then by Theorem 16, (5.6), and (5.7),
(6.44)
$$\int_{t_0}^{t'}(iP_{\leq k_0+9}\Delta\tilde{\epsilon},\int_0^1 F'(P_{\leq k_0+9}u+sP_{\geq k_0+9}u)ds\cdot P_{\geq k_0+9}u)_{L^2}dt\lesssim 2^{2k_0}(\frac{1}{T}\int_J\|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}dt)+2^{2k_0}T^{-10}.$$

Also, by Theorem 16, the Sobolev embedding theorem, and interpolation,
(6.45)
$$\int_{t_0}^{t'}(iP_{\leq k_0+9}(F(u)-F(\tilde{Q})),\int_0^1 F'(P_{\leq k_0+9}u+sP_{\geq k_0+9}u)ds\cdot P_{\geq k_0+9}u)_{L^2}dt$$
$$\lesssim\|\tilde{\epsilon}\|_{L_t^2L_x^{\frac{2d}{d-2}}}\|P_{\geq k_0+9}u\|_{L_t^2L_x^{\frac{2d}{d-2}}}\|\tilde{Q}\|_{L_{t,x}^{\infty}}^{4/d}\|u\|_{L_t^{\infty}L_x^2}^{4/d}+\|\Delta P_{\leq k_0+9}\tilde{\epsilon}\|_{L_t^2L_x^{\frac{2d}{d-2}}}\|\tilde{\epsilon}\|_{L_t^{\infty}L_x^2}^{4/d}\|P_{\geq k_0+9}u\|_{L_t^2L_x^{\frac{2d}{d-2}}}\|u\|_{L_t^{\infty}L_x^2}^{4/d}$$
$$+2^{2k_0}\|P_{\geq k_0+9}u\|_{L_t^2L_x^{\frac{2d}{d-2}}}\|P_{\geq k_0+9}\tilde{\epsilon}\|_{L_t^2L_x^{\frac{2d}{d-2}}}\|\tilde{\epsilon}\|_{L_t^{\infty}L_x^2}^{4/d}\|u\|_{L_t^{\infty}L_x^2}^{4/d}\lesssim 2^{2k_0}(\frac{1}{T}\int_J\|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}dt)+2^{2k_0}T^{-10}.$$

Now expand,
(6.46)
$$\Delta\tilde{Q}+F(\tilde{Q})=e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}\lambda(t)^{-d/2-2}\Delta Q(\frac{x-x(t)}{\lambda(t)})-2ie^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}\lambda(t)^{-d/2-2}\xi(t)\cdot\nabla Q(\frac{x-x(t)}{\lambda(t)})$$
$$-e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}\lambda(t)^{-d/2-2}|\xi(t)|^2Q(\frac{x-x(t)}{\lambda(t)})+e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}\lambda(t)^{-d/2-2}Q^{1+4/d}(\frac{x-x(t)}{\lambda(t)}).$$

By (1.17),
(6.47)
$$e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}\lambda(t)^{-d/2-2}\Delta Q(\frac{x-x(t)}{\lambda(t)})+e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}\lambda(t)^{-d/2-2}Q^{1+4/d}(\frac{x-x(t)}{\lambda(t)})=\lambda(t)^{-2}\tilde{Q}.$$

Plugging (6.47) into (6.45), using (5.6) and (5.7),
(6.48)
$$\int_{t_0}^{t'}\lambda(t)^{-2}\int_0^1(i\lambda(t)^{-2}\tilde{Q},F'(P_{\leq k_0+9}u+sP_{\geq k_0+9}u)(P_{\geq k_0+9}u))_{L^2}dsdt$$
$$=\int_{t_0}^{t'}\lambda(t)^{-2}\int_0^1(i(P_{\leq k_0+9}u+sP_{\geq k_0+9}u),F'(P_{\leq k_0+9}u+sP_{\geq k_0+9}u)(P_{\geq k_0+9}u))_{L^2}dsdt$$
$$+\int_{t_0}^{t'}\lambda(t)^{-2}\int_0^1(i(P_{\geq k_0+9}\tilde{Q}-P_{\leq k_0+9}\tilde{\epsilon}-sP_{\geq k_0+9}\tilde{\epsilon}),F'(P_{\leq k_0+9}u+sP_{\geq k_0+9}u)(P_{\geq k_0+9}u))_{L^2}dsdt$$
$$\lesssim(1+\frac{4}{d})\int_{t_0}^{t'}\lambda(t)^{-2}\int_0^1(iF(P_{\leq k_0+9}u+sP_{\geq k_0+9}u),(P_{\geq k_0+9}u))_{L^2}dsdt$$
$$+\eta_1^2\|\tilde{\epsilon}\|_{L_t^2L_x^{\frac{2d}{d-2}}}\|P_{\geq k_0+9}u\|_{L_t^2L_x^{\frac{2d}{d-2}}}\|u\|_{L_t^{\infty}L_x^2}^{4/d}+\eta_1^2\|P_{\geq k_0+9}\tilde{Q}\|_{L_t^2L_x^{\frac{2d}{d-2}}}\|P_{\geq k_0+9}u\|_{L_t^2L_x^{\frac{2d}{d-2}}}\|u\|_{L_t^{\infty}L_x^2}^{4/d}$$
$$\lesssim(1+\frac{4}{d})\int_{t_0}^{t'}\lambda(t)^{-2}(iF(P_{\leq k_0}u),P_{\geq k_0+9}u)_{L^2}dt+\eta_1^2\|P_{\geq k_0}u\|_{L_t^2L_x^{\frac{2d}{d-2}}}^2\|u\|_{L_t^{\infty}L_x^2}^{4/d}$$
$$+\eta_1^2\|\tilde{\epsilon}\|_{L_t^2L_x^{\frac{2d}{d-2}}}\|P_{\geq k_0+9}u\|_{L_t^2L_x^{\frac{2d}{d-2}}}\|u\|_{L_t^{\infty}L_x^2}^{4/d}+\eta_1^2\|P_{\geq k_0+9}\tilde{Q}\|_{L_t^2L_x^{\frac{2d}{d-2}}}\|P_{\geq k_0+9}u\|_{L_t^2L_x^{\frac{2d}{d-2}}}\|u\|_{L_t^{\infty}L_x^2}^{4/d}.$$

Then by Theorem 16,

(6.49)
$$\begin{aligned}(1+\frac{4}{d})\int_{t_0}^{t'}\lambda(t)^{-2}(iF(P_{\leq k_0}u),P_{\geq k_0+9}u)_{L^2}dt+\eta_1^2\|P_{\geq k_0}u\|^2_{L_t^2L_x^{\frac{2d}{d-2}}}\|u\|^{4/d}_{L_t^\infty L_x^2}\\ +\eta_1^2\|\tilde{\epsilon}\|_{L_t^2L_x^{\frac{2d}{d-2}}}\|P_{\geq k_0+9}u\|_{L_t^2L_x^{\frac{2d}{d-2}}}\|u\|^{4/d}_{L_t^\infty L_x^2}+\eta_1^2\|P_{\geq k_0+9}\tilde{Q}\|_{L_t^2L_x^{\frac{2d}{d-2}}}\|P_{\geq k_0+9}u\|_{L_t^2L_x^{\frac{2d}{d-2}}}\|u\|^{4/d}_{L_t^\infty L_x^2}\\ \lesssim \eta_1^2\|P_{\geq k_0}F(u_{\leq k_0})\|_{L_t^2L_x^{\frac{2d}{d+2}}}\|P_{\geq k_0+9}u\|_{L_t^2L_x^{\frac{2d}{d-2}}}+\eta_1^2\|P_{\geq k_0}u\|^2_{L_t^2L_x^{\frac{2d}{d-2}}}\|u\|^{4/d}_{L_t^\infty L_x^2}\\ +\eta_1^2\|\tilde{\epsilon}\|_{L_t^2L_x^{\frac{2d}{d-2}}}\|P_{\geq k_0+9}u\|_{L_t^2L_x^{\frac{2d}{d-2}}}\|u\|^{4/d}_{L_t^\infty L_x^2}+\eta_1^2\|P_{\geq k_0+9}\tilde{Q}\|_{L_t^2L_x^{\frac{2d}{d-2}}}\|P_{\geq k_0+9}u\|_{L_t^2L_x^{\frac{2d}{d-2}}}\|u\|^{4/d}_{L_t^\infty L_x^2}\\ \lesssim 2^{2k_0}(\frac{1}{T}\int_J\|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}dt)+2^{2k_0}T^{-10}.\end{aligned}$$

By the product rule,

(6.50)
$$\begin{aligned}-2ie^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}\lambda(t)^{-d/2-2}\xi(t)\cdot\nabla Q(\frac{x-x(t)}{\lambda(t)})-e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}\lambda(t)^{-d/2-2}|\xi(t)|^2Q(\frac{x-x(t)}{\lambda(t)})\\ =-2i\frac{\xi(t)}{\lambda(t)^2}\cdot\nabla\tilde{Q}+\frac{|\xi(t)|^2}{\lambda(t)^2}\tilde{Q}.\end{aligned}$$

Plugging $\frac{|\xi(t)|^2}{\lambda(t)^2}\tilde{Q}$ into (6.48) and using (5.6) and (5.7), the contribution of $\frac{|\xi(t)|^2}{\lambda(t)^2}\tilde{Q}$ can be handled in the same way as $\lambda(t)^{-2}\tilde{Q}$.

Finally, use the computations in the proof of Theorem 16 to compute

(6.51)
$$\begin{aligned}2\int_{t_0}^{t'}(P_{\leq k_0+9}\nabla\tilde{Q},\int_0^1F'(P_{\leq k_0+9}u+sP_{\geq k_0+9}u)ds\cdot P_{\geq k_0+9}u)_{L^2}dt\\ \lesssim 2^{2k_0}(\frac{1}{T}\int_J\|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}dt)+2^{2k_0}T^{-10}.\end{aligned}$$

□

The bound on $E(P_{\leq k_0+9}u)$ gives good bounds on the L^2 and $\dot{H}^1$ norms of ϵ in higher dimensions as well.

Theorem 20. *If $|x(t)|\leq T^{\frac{1}{2000d^2}}$ for all $t\in J$, and (5.6) and (5.7) hold, then*

(6.52)
$$\sup_{t\in J}\|P_{\leq k_0+9}(\frac{e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}}{\lambda(t)^{1/2}}\epsilon(t,\frac{x-x(t)}{\lambda(t)}))\|_{\dot{H}^1}^2\lesssim\frac{2^{2k_0}}{T}\int_J\|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}dt+\sup_{t\in J}\frac{|\xi(t)|^2}{\lambda(t)^2}+2^{2k_0}T^{-10},$$

and

$$\sup_{t\in J}\|\epsilon(t)\|_{L^2}^2\lesssim\frac{2^{2k_0}T^{1/25d}}{\eta_1^2T}\int_J\|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}dt+\frac{T^{1/25d}}{\eta_1^2}\sup_{t\in J}\frac{|\xi(t)|^2}{\lambda(t)^2}+2^{2k_0}\frac{T^{1/25d}}{\eta_1^2}T^{-10}. \tag{6.53}$$

Proof. This theorem is a direct consequence of Theorem 17 and an expansion of the energy. Recall from (6.38) that

(6.54)
$$\begin{aligned}E(P_{\leq k_0+9}u)=\frac{1}{2}\frac{|\xi(t)|^2}{\lambda(t)^2}\|Q\|_{L^2}^2+\frac{1}{2\lambda(t)^2}\|\epsilon\|_{L^2}^2-\frac{|\xi(t)|^2}{2\lambda(t)^2}\|\epsilon\|_{L^2}^2\\ +\mathcal{P}_2+O(|P_{\leq k_0+9}\tilde{\epsilon}|^3|P_{\leq k_0+9}\tilde{Q}|)+O(|P_{\leq k_0+9}\tilde{\epsilon}|^{2+4/d})+O(2^{2k_0}T^{-10}).\end{aligned}$$

Again let

(6.55)
$$P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}\lambda(t)^{-1}\epsilon(t,\frac{x-x(t)}{\lambda(t)})) = e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}P_{\leq k_0+9}(\lambda(t)^{-1}\epsilon(t,\frac{x-x(t)}{\lambda(t)}))+\mathcal{R} = \tilde{\tilde{\epsilon}}(t,x)+\mathcal{R},$$

where

$$\|\mathcal{R}\|_{L^2} \lesssim 2^{-k}\frac{|\xi(t)|}{\lambda(t)}\|\epsilon\|_{L^2} \lesssim 2^{-k}\eta_0\|\epsilon\|_{L^2}^2, \qquad \text{and} \qquad \|\nabla\mathcal{R}\|_{L^2} \lesssim \frac{|\xi(t)|}{\lambda(t)}\|\epsilon\|_{L^2} \lesssim \eta_0\|\epsilon\|_{L^2}. \tag{6.56}$$

As in dimension $d = 2$, since Q is real valued, smooth, and has derivatives that are rapidly decreasing,

$$\begin{aligned}\mathcal{P}_2 = \frac{1}{2}\|\nabla\tilde{\epsilon}\|_{L^2}^2 - \frac{d+2}{2d\lambda(t)^{d+2}}\int Q(\frac{x-x(t)}{\lambda(t)})^{4/d}|P_{\leq k_0+9}\epsilon(t,\frac{x-x(t)}{\lambda(t)})|^2dx\\ -\frac{1}{d\lambda(t)^{d+2}}Re\int Q(\frac{x-x(t)}{\lambda(t)})^{4/d}(P_{\leq k_0+9}\epsilon(t,\frac{x-x(t)}{\lambda(t)}))^2dx + O(2^{2k_0}T^{-10})\\ +O(\eta_0^2\|\epsilon\|_{L^2}^2) + O(\eta_0\|\epsilon\|_{L^2}\|\nabla\tilde{\epsilon}\|_{L^2}).\end{aligned} \tag{6.57}$$

Also, as in dimension $d = 2$, by the product rule,

(6.58)
$$\begin{aligned}\frac{1}{2}\|\nabla\tilde{\epsilon}\|_{L^2}^2 = \frac{|\xi(t)|^2}{2\lambda(t)^{d+2}}\|P_{\leq k_0+9}\epsilon(t,\frac{x-x(t)}{\lambda(t)})\|_{L^2}^2\\ +\frac{\xi(t)}{\lambda(t)^{d+1}}\cdot(P_{\leq k_0+9}\epsilon(t,\frac{x-x(t)}{\lambda(t)}), iP_{\leq k_0+9}\nabla\epsilon(t,\frac{x-x(t)}{\lambda(t)}))_{L^2} + \frac{1}{2\lambda(t)^d}\|P_{\leq k_0+9}\epsilon(t,\frac{x-x(t)}{\lambda(t)})\|_{\dot{H}^1}^2.\end{aligned}$$

Rescaling, if $2^{n(t)} = \lambda(t)$,

$$\begin{aligned}\frac{1}{2\lambda(t)^d}\|P_{\leq k_0+9}\epsilon(t,\frac{x-x(t)}{\lambda(t)})\|_{\dot{H}^1}^2 - \frac{d+2}{2d\lambda(t)^{d+2}}\int Q(\frac{x-x(t)}{\lambda(t)})^{4/d}|P_{\leq k_0+9}\epsilon(t,\frac{x-x(t)}{\lambda(t)})|^2dx\\ -\frac{1}{d\lambda(t)^{d+2}}Re\int Q(\frac{x-x(t)}{\lambda(t)})^{4/d}(P_{\leq k_0+9}\epsilon(t,\frac{x-x(t)}{\lambda(t)}))^2dx\\ = \frac{1}{2\lambda(t)^2}\|P_{\leq k_0+9+n(t)}\epsilon(t,x)\|_{\dot{H}^1}^2 - \frac{d+2}{2d\lambda(t)^2}\int Q(x)^{4/d}|P_{\leq k_0+9+n(t)}\epsilon(t,x)|^2dx\\ -\frac{1}{d\lambda(t)^2}Re\int Q(x)^{4/d}(P_{\leq k_0+9+n(t)}\epsilon(t,x)^2dx.\end{aligned} \tag{6.59}$$

Then using the spectral theory of $\mathcal{L}$ in Theorem 10, provided

$$g \perp span\{\chi_0, i\chi_0, Q_{x_j}, iQ_{x_j}\}, \tag{6.60}$$

there exists a fixed constant $\lambda_d > 0$ such that

$$\frac{1}{2}\|\nabla g\|_{L^2}^2 + \frac{1}{2}\|g\|_{L^2}^2 - \frac{d+2}{2d}\int Q(x)^2|g(x)|^2 - \frac{1}{d}Re\int Q(x)^2g(x)^2dx \geq \lambda_d\|g\|_{H^1}^2. \tag{6.61}$$

Again using the fact that χ_0 and Q_{x_j} are smooth with rapidly decreasing derivatives, and the bounds on $\lambda(t)$ in (6.19),

$$(P_{\leq k_0+9+n(t)}\epsilon, f)_{L^2} \lesssim T^{-10}. \tag{6.62}$$

Therefore, for $\eta_0 \ll 1$ sufficiently small, there exists some fixed $\lambda_1 > 0$ such that

$$\frac{1}{2\lambda(t)^2}\|\epsilon\|_{L^2}^2 + \mathcal{P}_2 \geq \frac{\lambda_d}{\lambda(t)^2}\|\epsilon\|_{L^2}^2 + \lambda_d\|\tilde{\epsilon}\|_{\dot{H}^1}^2 - O(2^{2k_0}T^{-10}). \tag{6.63}$$

Next, by the Sobolev embedding theorem and (6.56),

$$\int |\tilde{\epsilon}(t,x)|^{2+\frac{4}{d}} dx \lesssim \|\tilde{\tilde{\epsilon}}\|_{\dot{H}^1}^2 \|\epsilon\|_{L^2}^{\frac{4}{d}} + \frac{|\xi(t)|^2}{\lambda(t)^2} \|\epsilon\|_{L^2}^{2+8/d} \lesssim \eta_0^{\frac{4}{d}} \|\tilde{\tilde{\epsilon}}\|_{\dot{H}^1}^2 + \eta_0^{8/d} \frac{|\xi(t)|^2}{\lambda(t)^2} \|\epsilon\|_{L^2}^2. \tag{6.64}$$

Therefore, by interpolation, for $\eta_0 \ll 1$ sufficiently small,

$$\begin{aligned} E(P_{\leq k_0+9} u) &\geq \frac{1}{2} \frac{|\xi(t)|^2}{\lambda(t)^2} \|Q\|_{L^2}^2 + \frac{\lambda_d}{2\lambda(t)^2} \|\epsilon\|_{L^2}^2 + \frac{\lambda_d}{2} \|\tilde{\tilde{\epsilon}}\|_{\dot{H}^1}^2 - \frac{|\xi(t)|^2}{2\lambda(t)^2} \|\epsilon\|_{L^2}^2 - O(2^{2k_0} T^{-10}) \\ &\geq \frac{1}{4} \frac{|\xi(t)|^2}{\lambda(t)^2} \|Q\|_{L^2}^2 + \frac{\lambda_d}{2\lambda(t)^2} \|\epsilon\|_{L^2}^2 + \frac{\lambda_d}{2} \|\tilde{\tilde{\epsilon}}\|_{\dot{H}^1}^2 - O(2^{2k_0} T^{-10}) \end{aligned} \tag{6.65}$$

Plugging (6.65) into (6.36) proves Theorem 20. □

7. A frequency localized Morawetz estimate

Proceeding to the frequency localized Morawetz estimates, again start with dimension $d = 2$.

7.1. **Two dimensions.**

Theorem 21. *Let $J = [a, b]$ be an interval on which*

$$\frac{|\xi(t)|}{\lambda(t)} \leq \eta_0, \qquad \frac{1}{\eta_1} \leq \lambda(t) \leq \frac{1}{\eta_1} T^{1/100}, \qquad \text{for all} \qquad t \in J, \qquad \int_J \lambda(t)^{-2} dt = T, \qquad \eta_1^{-2} T = 2^{3k}. \tag{7.1}$$

Also suppose that $\epsilon = \epsilon_1 + i\epsilon_2$ and suppose $\xi(a) = x(b) = 0$. Finally suppose there exists a uniform bound on $x(t)$,

$$\sup_{t \in J} |x(t)| \leq R = T^{1/25}. \tag{7.2}$$

Then for T sufficiently large,

$$\int_a^b \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2} dt \leq 3(\epsilon_2(a), Q + x \cdot \nabla Q)_{L^2} - 3(\epsilon_2(b), Q + x \cdot \nabla Q)_{L^2} + \frac{T^{1/15}}{\eta_1^2} \sup_{t \in J} \frac{|\xi(t)|^2}{\lambda(t)^2} + O(T^{-8}). \tag{7.3}$$

Proof. Define a Morawetz potential. Let $\chi(r) \in C^\infty([0, \infty))$ be a smooth, radial function, satisfying $\chi(r) = 1$ for $0 \leq r \leq 1$, and supported on $r \leq 2$. Then let

$$\phi(r) = \int_0^r \chi^2(\frac{s}{2R}) ds = \int_0^r \psi(\frac{s}{2R}) ds, \tag{7.4}$$

and let

$$M(t) = \int \phi(r) Im[\overline{P_{\leq k+9} u} \partial_r P_{\leq k+9} u](t,x) dx = \int \phi(|x|) \frac{x}{|x|} \cdot Im[\overline{P_{\leq k+9} u} \nabla P_{\leq k+9} u](t,x) dx. \tag{7.5}$$

Following (4.37),

$$i\partial_t P_{\leq k+9} u + \Delta P_{\leq k+9} u + F(P_{\leq k+9} u) = F(P_{\leq k+9} u) - P_{\leq k+9} F(u) = -\mathcal{N}. \tag{7.6}$$

Plugging in (7.6) and integrating by parts,

$$\begin{aligned}\frac{d}{dt}M(t) &= \int \phi(r)Re[-\Delta\overline{P_{\leq k+9}u}\partial_r P_{\leq k+9}u + \overline{P_{\leq k+9}u}\Delta\partial_r P_{\leq k+9}u]\\ &+ \int \phi(r)Re[-F(\overline{P_{\leq k+9}u})\partial_r P_{\leq k+9}u + \overline{P_{\leq k+9}u}\partial_r F(P_{\leq k+9}u)]\\ &+ \int \phi(r)Re[\overline{P_{\leq k+9}u}\partial_r\mathcal{N}](t,x)dx - \int \phi(r)Re[\bar{\mathcal{N}}\partial_r P_{\leq k+9}u](t,x)dx\\ &= 2\int \chi^2(\frac{x}{R})|\nabla P_{\leq k+9}u|^2dx - \frac{1}{2R^2}\int \psi''(\frac{x}{R})|P_{\leq k+9}u|^2dx - \int \chi^2(\frac{x}{R})|P_{\leq k+9}u|^4dx\\ &+2\int [\frac{1}{|x|}\phi(x) - \chi^2(\frac{x}{R})](\delta_{jk} - \frac{x_jx_k}{|x|^2})Re(\overline{\partial_j P_{\leq k+9}u}\partial_k P_{\leq k+9}u)dx\\ &+ \int \phi(r)Re[\overline{P_{\leq k+9}u}\partial_r\mathcal{N}](t,x)dx - \int \phi(r)Re[\bar{\mathcal{N}}\partial_r P_{\leq k+9}u](t,x)dx.\end{aligned} \tag{7.7}$$

Next, following (4.46),

$$\begin{aligned}\mathcal{N} = P_{\leq k+9}(2|u_{\leq k+6}|^2u_{\geq k+6} + (u_{\leq k+6})^2\overline{u_{\geq k+6}}) - (2|u_{\leq k+6}|^2u_{k+6\leq\cdot\leq k+9} + (u_{\leq k+6})^2\overline{u_{k+6\leq\cdot\leq k+9}})\\ +P_{\leq k+9}O((u_{\geq k+6})^2u) + O((u_{k+6\leq\cdot\leq k+9})^2u) = \mathcal{N}^{(1)} + \mathcal{N}^{(2)}.\end{aligned} \tag{7.8}$$

As in (4.47), by Theorems 13 and 14,

$$\begin{aligned}\int_a^b\int \phi(r)Re[\overline{P_{\leq k+9}u}\partial_r\mathcal{N}^{(2)}]dxdt - \int_0^T\int \phi(r)Re[\overline{\mathcal{N}^{(2)}}\partial_r P_{\leq k+9}u]dxdt\\ \lesssim 2^kR\|(u_{\geq k+6})(u_{\leq k+3})\|^2_{L^2_{t,x}} + 2^kR\|u_{\geq k+6}\|^2_{L^4_{t,x}}\|u_{\geq k+3}\|^2_{L^4_{t,x}}\\ \lesssim 2^kR(\frac{1}{T}\int_a^b \|\epsilon(t)\|^2_{L^2}\lambda(t)^{-2}dt + 2^{2k}T^{-10}).\end{aligned} \tag{7.9}$$

Splitting $\mathcal{N}^{(1)}$,

$$\mathcal{N}^{(1)} = O((u_{\geq k+6})(u_{\geq k+3})u) + O((u_{\geq k+6})(u_{\leq k+3})^2) = \mathcal{N}^{(1,1)} + \mathcal{N}^{(1,2)}. \tag{7.10}$$

Making calculations identical to the estimate (7.9),

$$\begin{aligned}\int_a^b\int \phi(r)Re[\overline{P_{\leq k+9}u}\partial_r\mathcal{N}^{(1,1)}]dxdt - \int_a^b\int \phi(r)Re[\overline{\mathcal{N}^{(1,1)}}\partial_r P_{\leq k+9}u]dxdt\\ + \int_0^T\int \phi(x)Re[\overline{P_{k+3\leq\cdot\leq k+9}u}\partial_r\mathcal{N}^{(1,1)}]dxdt - \int_a^b\int \phi(x)Re[\overline{\mathcal{N}^{(1,1)}}\partial_r P_{\geq k+3\leq\cdot\leq k+9}u]dxdt\\ \lesssim 2^kR\|(u_{\geq k+6})(u_{\leq k})\|_{L^2_{t,x}}\|(u_{\geq k+3})(u_{\leq k})\|_{L^2_{t,x}} + 2^kR\|u_{\geq k+3}\|^2_{L^4_{t,x}}\|u_{\geq k}\|^2_{L^4_{t,x}}\\ \lesssim 2^kR(\frac{1}{T}\int_a^b \|\epsilon(t)\|^2_{L^2}\lambda(t)^{-2}dt + 2^{2k}T^{-10}).\end{aligned} \tag{7.11}$$

Finally, using Bernstein's inequality, the fact that ϕ is smooth, rapidly decreasing, $R = T^{1/25}$, and $\|u\|_{L^4_{t,x}([a,b]\times\mathbb{R}^2)} \lesssim T^{1/4}$,

$$\begin{aligned}\int_a^b \int \phi(r) Re[\overline{u_{\leq k+3}} \partial_r \mathcal{N}^{(1,2)}] dxdt - \int_a^b \int \phi(r) Re[\overline{\mathcal{N}^{(1,2)}} \partial_r u_{\leq k+3}] dxdt \\ \lesssim \|P_{\geq k+3}\phi(x)\|_{L^\infty} \|(u_{\geq k+6}u)(u_{\leq k+3})\|_{L^2_{t,x}} \|u_{\leq k+3}\|^2_{L^4_{t,x}} \lesssim \frac{1}{T^9}.\end{aligned} \tag{7.12}$$

Therefore, the error arising from frequency truncation is bounded by

$$2^k R(\frac{1}{T}\int_a^b \|\epsilon(t)\|^2_{L^2} \lambda(t)^{-2} dt + 2^{2k} T^{-10}). \tag{7.13}$$

Using the fact that $(\frac{1}{r}\phi(r) - \chi^2(r))(\delta_{jk} - \frac{x_j x_k}{|x|^2})$ is a positive definite matrix, by the fundamental theorem of calculus,

$$\begin{aligned}2\int_a^b \int \chi^2(\frac{x}{R}) |\nabla P_{\leq k+9} u|^2 dxdt - \frac{1}{2R^2}\int_a^b \int \psi''(\frac{x}{R}) |P_{\leq k+9} u|^2 dxdt - \int_a^b \int \chi^2(\frac{x}{R}) |P_{\leq k+9} u|^4 dxdt \\ \lesssim M(b) - M(a) + 2^k R(\frac{1}{T}\int_J \|\epsilon(t)\|^2_{L^2} \lambda(t)^{-2} dt)) + O(T^{-8}).\end{aligned} \tag{7.14}$$

Next compute $M(b) - M(a)$ under the assumption that $\xi(a) = x(b) = 0$. Since Q is smooth, real valued, rapidly decreasing, all its derivatives are rapidly decreasing, by (7.1), (7.2), (7.4), and $\xi(b) = x(a) = 0$,

$$\int \phi(|x|) \frac{x}{|x|} \cdot Im[\overline{P_{\leq k+9}(e^{-i\gamma(t)} e^{-ix\cdot \frac{\xi(t)}{\lambda(t)}} \frac{1}{\lambda(t)} Q(\frac{x - x(t)}{\lambda(t)}))} \nabla (P_{\leq k+9}(e^{-i\gamma(t)} e^{-ix\cdot \frac{\xi(t)}{\lambda(t)}} \frac{1}{\lambda(t)} Q(\frac{x-x(t)}{\lambda(t)})))] dx|_a^b \tag{7.15}$$

$$\begin{aligned}= \frac{1}{\lambda(t)^2}\int \phi(|x|) \frac{x}{|x|} \cdot \frac{\xi(t)}{\lambda(t)} |Q(\frac{x-x(t)}{\lambda(t)})|^2 dx + O(2^k R T^{-10}) \\ = \frac{x(t)}{\lambda(t)^2} \cdot \frac{\xi(t)}{\lambda(t)} \int Q(\frac{x}{\lambda(t)})^2 dx|_a^b + O(2^k R T^{-10}) = O(2^k R T^{-10}).\end{aligned} \tag{7.16}$$

Next, by Theorem 18, (7.1), (7.2), (7.4), $\xi(b) = x(a) = 0$, $(\epsilon_1, \nabla Q)_{L^2} = (\epsilon_2, \nabla Q)_{L^2}$, the fact that Q is smooth and rapidly decreasing, and that $\phi(|x|)\frac{x}{|x|}$ is smooth,

$$\int \phi(|x|) \frac{x}{|x|} \cdot Im[\overline{P_{\leq k+9}(e^{-i\gamma(t)} e^{-ix\cdot \frac{\xi(t)}{\lambda(t)}} \frac{1}{\lambda(t)} \epsilon(\frac{x - x(t)}{\lambda(t)}))} \nabla (P_{\leq k+9}(e^{-i\gamma(t)} e^{-ix\cdot \frac{\xi(t)}{\lambda(t)}} \frac{1}{\lambda(t)} Q(\frac{x-x(t)}{\lambda(t)})))] dx|_a^b \tag{7.17}$$

$$\begin{aligned}= \xi(t) \cdot (\epsilon_1, xQ)_{L^2} + \frac{x(t)\cdot\xi(t)}{\lambda(t)}(\epsilon_1, Q)_{L^2} - (\epsilon_2, x\cdot\nabla Q)_{L^2} - x(t)(\epsilon_2, \nabla Q)_{L^2} + O(T^{-10})|_a^b, \\ = \xi(t)\cdot(\epsilon_1, xQ)_{L^2} + (\epsilon_2, x\cdot\nabla Q) + O(T^{-10})|_a^b = O(\frac{|\xi(t)|}{\lambda(t)}\lambda(t)\|\epsilon\|_{L^2}) - (\epsilon_2, x\cdot\nabla Q)_{L^2}|_a^b + O(T^{-10}) \\ = \frac{T^{1/50}}{\eta_1^2} O(\frac{|\xi(t)|^2}{\lambda(t)^2}) + \|\epsilon\|^2_{L^2} - (\epsilon_2, x\cdot\nabla Q)_{L^2}|_a^b + O(T^{-10}) \\ = -(\epsilon_2, x\cdot\nabla Q)_{L^2}|_a^b + O(\frac{2^{2k}T^{1/50}}{\eta_1^2 T}\int_J \|\epsilon(t)\|^2_{L^2}\lambda(t)^{-2} dt + \frac{T^{1/50}}{\eta_1^2}\sup_{t\in J}\frac{|\xi(t)|^2}{\lambda(t)^2} + 2^{2k}\frac{T^{1/50}}{\eta_1^2} T^{-10}).\end{aligned}$$

Also, integrating by parts, since Q and all its derivatives are rapidly decreasing, as well as $\phi(|x|)\frac{x}{|x|}$ is smooth,

(7.18)
$$\begin{aligned}\int \phi(|x|)\frac{x}{|x|}\cdot Im[\overline{P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}\frac{1}{\lambda(t)}Q(\frac{x-x(t)}{\lambda(t)}))}\nabla(P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}\frac{1}{\lambda(t)}\epsilon(\frac{x-x(t)}{\lambda(t)})))]dx|_a^b\\ = (7.17) - \int \nabla\cdot(\phi(|x|)\frac{x}{|x|})\cdot Im[\overline{\frac{1}{\lambda(t)}Q(\frac{x-x(t)}{\lambda(t)}))}(\frac{1}{\lambda(t)}\epsilon(\frac{x-x(t)}{\lambda(t)})))]dx|_a^b + O(T^{-10})\\ = (7.17) - 2(\epsilon_2, Q)_{L^2}|_a^b + O(T^{-10}).\end{aligned}$$

Finally, by Theorem 18, for any $t\in J$,

(7.19)
$$\begin{aligned}\int \phi(|x|)\frac{x}{|x|}\cdot Im[\overline{P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}\frac{1}{\lambda(t)}\epsilon(\frac{x-x(t)}{\lambda(t)}))}\nabla(P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}\frac{1}{\lambda(t)}\epsilon(\frac{x-x(t)}{\lambda(t)})))]dx\\ \lesssim R\|\epsilon\|_{L^2}\|P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}\frac{1}{\lambda(t)}\epsilon(\frac{x-x(t)}{\lambda(t)}))\|_{\dot H^1} + R\frac{|\xi(t)|}{\lambda(t)}\|\epsilon\|_{L^2}^2\\ \lesssim R(\frac{2^{2k}T^{1/50}}{\eta_1^2T}\int_J\|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}dt + \frac{T^{1/50}}{\eta_1^2}\sup_{t\in J}\frac{|\xi(t)|^2}{\lambda(t)^2} + 2^{2k}\frac{T^{1/50}}{\eta_1^2}T^{-10}).\end{aligned}$$

Therefore,

(7.20)
$$\begin{aligned}M(b)-M(a) = 2(\epsilon_2(a), Q+x\cdot\nabla Q)_{L^2} - 2(\epsilon_2(b), Q+x\cdot\nabla Q)_{L^2}\\ +O(\frac{2^{2k}T^{1/15}}{\eta_1^2T}\int_J\|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}dt + \frac{T^{1/15}}{\eta_1^2}\sup_{t\in J}\frac{|\xi(t)|^2}{\lambda(t)^2} + 2^{2k}\frac{T^{1/15}}{\eta_1^2}T^{-10})).\end{aligned}$$

Therefore, to complete the proof of Theorem 21, it only remains to obtain a lower bound for

(7.21)
$$2\int_a^b\int\chi^2(\frac{x}{R})|\nabla P_{\leq k+9}u|^2dxdt - \frac{1}{2R^2}\int_a^b\int\psi''(\frac{x}{R})|P_{\leq k+9}u|^2dxdt - \int_a^b\int\chi^2(\frac{x}{R})|P_{\leq k+9}u|^4dxdt.$$

Splitting

(7.22)
$$|P_{\leq k+9}u|^2 \leq 2\lambda(t)^{-2}|P_{\leq k+9}(e^{-i\gamma(t)}e^{ix\cdot\frac{\xi(t)}{\lambda(t)}}Q(\frac{x-x(t)}{\lambda(t)})|^2 + 2\lambda(t)^{-2}|P_{\leq k+9}(e^{-i\gamma(t)}e^{ix\cdot\frac{\xi(t)}{\lambda(t)}}\epsilon(t,\frac{x-x(t)}{\lambda(t)})|^2.$$

By (7.1), (7.2), and (7.4), the support of $\psi''(x)$, and the fact that Q is smooth and all its derivatives are rapidly decreasing,

$$\frac{1}{R^2}\lambda(t)^{-2}\int\psi''(\frac{x}{R})|P_{\leq k+9}(e^{-i\gamma(t)}e^{ix\cdot\frac{\xi(t)}{\lambda(t)}}Q(\frac{x-x(t)}{\lambda(t)})|^2dx \lesssim \lambda(t)^{-2}\frac{1}{T^{10}}. \tag{7.23}$$

On the other hand, by (7.1), (7.2), and (7.4), for T sufficiently large,

$$\frac{1}{R^2}\lambda(t)^{-2}\int\psi''(\frac{x}{R})|P_{\leq k+9}(e^{-i\gamma(t)}e^{ix\cdot\frac{\xi(t)}{\lambda(t)}}\epsilon(t,\frac{x-x(t)}{\lambda(t)})|^2dx \lesssim \frac{1}{R}\lambda(t)^{-2}\|\epsilon\|_{L^2}^2. \tag{7.24}$$

Since Q is smooth and all its derivatives are rapidly decreasing, by (7.1), (7.2), and (7.4),

(7.25)
$$\begin{aligned}\frac{1}{2}\int(1-\chi^2(\frac{x}{2R}))|\nabla P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)}Q(\frac{x-x(t)}{\lambda(t)})|^2dx\\ -\frac{1}{4}\int(1-\chi^2(\frac{x}{2R}))|P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)}Q(\frac{x-x(t)}{\lambda(t)})|^4dx = O(2^{2k}T^{-10}).\end{aligned}$$

Also,

(7.26)
$$\int (1-\chi^2(\frac{x}{2R}))Re(\overline{\nabla P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)}Q(\frac{x-x(t)}{\lambda(t)})} \cdot \nabla P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)}\epsilon(t,\frac{x-x(t)}{\lambda(t)}))dx$$
$$- \int (1-\chi^2(\frac{x}{2R}))|P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)}Q(\frac{x-x(t)}{\lambda(t)})|^2$$
$$\times Re(\overline{P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)}Q(\frac{x-x(t)}{\lambda(t)})} \cdot P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)}\epsilon(t,\frac{x-x(t)}{\lambda(t)}))dx = O(2^{2k}T^{-10}).$$

Therefore, from (6.23),

(7.27)
$$\frac{1}{2}\int \chi^2(\frac{x}{2R})|\nabla P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)}Q(\frac{x-x(t)}{\lambda(t)})|^2dx$$
$$-\frac{1}{4}\int \chi^2(\frac{x}{2R})|P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)}Q(\frac{x-x(t)}{\lambda(t)})|^4dx$$
$$+\int \chi^2(\frac{x}{2R})Re(\overline{\nabla P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)}Q(\frac{x-x(t)}{\lambda(t)})} \cdot \nabla P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)}\epsilon(t,\frac{x-x(t)}{\lambda(t)}))dx$$
$$-\int \chi^2(\frac{x}{2R})|P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)}Q(\frac{x-x(t)}{\lambda(t)})|^2$$
$$\times Re(\overline{P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)}Q(\frac{x-x(t)}{\lambda(t)})} \cdot P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)}\epsilon(t,\frac{x-x(t)}{\lambda(t)}))dx$$
$$= \frac{1}{2}\frac{|\xi(t)|^2}{\lambda(t)^2}\|Q\|_{L^2}^2 + \frac{1}{2\lambda(t)^2}\|\epsilon\|_{L^2}^2 - \frac{|\xi(t)|^2}{2\lambda(t)^2}\|\epsilon\|_{L^2}^2 + O(2^{2k}T^{-10}).$$

Turning to the terms with two ϵ's, by the product rule,

(7.28)
$$\frac{1}{2}\int \chi^2(\frac{x}{2R})|\nabla P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)}\epsilon(t,\frac{x-x(t)}{\lambda(t)})|^2dx$$
$$= \frac{1}{2}\|\chi(\frac{x}{2R})P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)}\epsilon(t,\frac{x-x(t)}{\lambda(t)}))\|_{\dot{H}^1}^2 + O(\frac{1}{R\lambda(t)^2}\|\epsilon\|_{L^2}^2).$$

Then by (7.1), (7.2), (7.4), (6.36), and the fact that Q_{x_j} and χ_0 are rapidly decreasing,

$$(\chi(\frac{x\lambda(t)+x(t)}{2R})\epsilon, f)_{L^2} \lesssim T^{-10}. \tag{7.29}$$

Therefore, following the analysis in (6.24)–(6.32),

(7.30)
$$\frac{1}{2}\int \chi^2(\frac{x}{2R})|\nabla P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)}\epsilon(t,\frac{x-x(t)}{\lambda(t)})|^2dx$$
$$-\frac{1}{\lambda(t)^4}\int |P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}Q(\frac{x-x(t)}{\lambda(t)}))|^2|P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\epsilon(t,\frac{x-x(t)}{\lambda(t)}))|^2dx$$
$$-\frac{1}{2\lambda(t)^4}Re\int \overline{(P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}Q(\frac{x-x(t)}{\lambda(t)})))^2}(P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\epsilon(t,\frac{x-x(t)}{\lambda(t)})))^2dx$$
$$\geq \frac{\lambda_1}{\lambda(t)^2}\|\epsilon\|_{L^2}^2 + \lambda_1\int \chi(\frac{x}{2R})^2|\nabla P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)}\epsilon(t,\frac{x-x(t)}{\lambda(t)})|^2dx - O(2^{2k}T^{-10}).$$

Following (1.13),

$$\begin{aligned}\int \chi^2(\frac{x}{2R})|g(x_1,x_2)|^4dx_1dx_2 &\leq \int \chi(\frac{x}{2R})|g(x_1,x_2)|^2\cdot(\sup_{x_1\in\mathbb{R}}\chi(\frac{x}{2R})|g(x_1,x_2)|^2)dx_1dx_2\\ &\leq \int \chi(\frac{x}{2R})|g(x_1,x_2)|^2\cdot(\int|\partial_{x_1}(\chi(\frac{x}{2R})|g(s,x_2)|^2)|ds)dx_1dx_2\\ &= \int(\int\chi(\frac{x}{2R})|g(x_1,x_2)|^2dx_1)\cdot(\int|\partial_{x_1}(\chi(\frac{x}{2R})|g(x_1,x_2)|^2)|dx_1)dx_2\\ &\leq \sup_{x_2\in\mathbb{R}}(\int\chi(\frac{x}{2R})|g(x_1,x_2)|^2dx_1)\cdot\int(\int|\partial_{x_1}(\chi(\frac{x}{2R})|g(x_1,x_2)|^2)|dx_1)dx_2\\ &\leq \int\int|\partial_{x_2}(\chi(\frac{x}{2R})|g(x_1,x_2)|^2)|dx_1dx_2\cdot\int\int|\partial_{x_1}(\chi(\frac{x}{2R})|g(x_1,x_2)|^2)|dx_1dx_2\\ &\lesssim \|\chi(\frac{x}{2R})\nabla g\|_{L^2}^2\|g\|_{L^2}^2+\frac{1}{R^2}\|g\|_{L^2}^4.\end{aligned}\tag{7.31}$$

Therefore,

$$\begin{aligned}&\int\chi(\frac{x}{2R})^2|P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\lambda(t)^{-1}\epsilon(t,\frac{x-x(t)}{\lambda(t)}))|^4dx\\ &\lesssim \eta_0^2\|\chi(\frac{x}{2R})\nabla P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\lambda(t)^{-1}\epsilon(t,\frac{x-x(t)}{\lambda(t)})\|_{L^2}^2+\frac{\eta_0^2}{R^2}\|\epsilon\|_{L^2}^2.\end{aligned}\tag{7.32}$$

Finally, by interpolation,

$$\begin{aligned}&\int\chi(\frac{x}{2R})^2|P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\lambda(t)^{-1}\epsilon(t,\frac{x-x(t)}{\lambda(t)}))|^3|P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}\frac{1}{\lambda(t)}Q(\frac{x-x(t)}{\lambda(t)}))|dx\\ &\lesssim \eta_0\|\chi(\frac{x}{2R})\nabla P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\lambda(t)^{-1}\epsilon(t,\frac{x-x(t)}{\lambda(t)})\|_{L^2}\frac{\|\epsilon\|_{L^2}}{\lambda(t)}+\frac{\eta_0}{R\lambda(t)}\|\epsilon\|_{L^2}^2.\end{aligned}\tag{7.33}$$

Therefore, we have finally proved that there exists some $\lambda_1>0$ such that for η_0 sufficiently small,

$$(7.21)\gtrsim \lambda_1\int_a^b\|\epsilon\|_{L^2}^2\lambda(t)^{-2}dt-O(T^{-8}).\tag{7.34}$$

Combining (7.34), (7.14), (7.23), (7.24), and (7.20) proves Theorem 21. □

7.2. **Dimensions $d\geq 3$.** The computations in dimensions $d\geq 3$ are quite similar.

Theorem 22. *Let $J=[a,b]$ be an interval on which (5.6) and (5.7) hold, and $|x(t)|\leq T^{\frac{1}{2000d^2}}$. Also suppose that $\epsilon=\epsilon_1+i\epsilon_2$, and that $\xi(a)=0$ and $x(b)=0$. Then for T sufficiently large,*

$$\int_a^b\|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}dt\leq 3(\epsilon_2(a),Q+x\cdot\nabla Q)_{L^2}-3(\epsilon_2(b),Q+x\cdot\nabla Q)_{L^2}+\frac{T^{1/25d}}{\eta_1^2}\sup_{t\in J}\frac{|\xi(t)|^2}{\lambda(t)^2}+O(T^{-8}).\tag{7.35}$$

Proof. Once again let $R=T^{1/25d}$, $\chi(r)\in C^\infty([0,\infty))$ be a smooth, radial function, satisfying $\chi(r)=1$ for $0\leq r\leq 1$, and supported on $r\leq 2$, and let

$$\phi(r)=\int_0^r\chi^2(\frac{s}{2R})ds=\int_0^r\psi(\frac{s}{2R})ds,\tag{7.36}$$

and

(7.37)
$$M(t) = \int \phi(r) Im[\overline{P_{\leq k_0+9}u}\partial_r P_{\leq k_0+9}u](t,x)dx = \int \phi(|x|)\frac{x}{|x|}\cdot Im[\overline{P_{\leq k_0+9}u}\nabla P_{\leq k_0+9}u](t,x)dx.$$

Once again,

$$i\partial_t P_{\leq k_0+9}u + \Delta P_{\leq k_0+9}u + F(P_{\leq k_0+9}u) = F(P_{\leq k_0+9}u) - P_{\leq k_0+9}F(u) = -\mathcal{N}. \tag{7.38}$$

Plugging in (7.38) and integrating by parts,

(7.39)
$$\begin{aligned}
\frac{d}{dt}M(t) &= \int \phi(r) Re[-\Delta\overline{P_{\leq k_0+9}u}\partial_r P_{\leq k_0+9}u + \overline{P_{\leq k_0+9}u}\Delta\partial_r P_{\leq k_0+9}u] \\
&\quad + \int \phi(r) Re[-F(\overline{P_{\leq k_0+9}u})\partial_r P_{\leq k_0+9}u + \overline{P_{\leq k_0+9}u}\partial_r F(P_{\leq k_0+9}u)] \\
&\quad + \int \phi(r) Re[\overline{P_{\leq k_0+9}u}\partial_r \mathcal{N}](t,x) - \int \phi(r) Re[\bar{\mathcal{N}}\partial_r P_{\leq k_0+9}u](t,x) \\
&= 2\int \chi^2(\frac{x}{2R})|\nabla P_{\leq k_0+9}u|^2 dx - \frac{1}{2R^2}\int \psi''(\frac{x}{2R})|P_{\leq k_0+9}u|^2 dx - \frac{2d}{d+2}\int \chi^2(\frac{x}{2R})|P_{\leq k_0+9}u|^4 dx \\
&\quad + 2\int [\frac{1}{|x|}\phi(x) - \chi^2(\frac{x}{2R})](\delta_{jk} - \frac{x_j x_k}{|x|^2}) Re(\overline{\partial_j P_{\leq k_0+9}u}\partial_k P_{\leq k_0+9}u)dx \\
&\quad + \int \phi(r) Re[\overline{P_{\leq k_0+9}u}\partial_r \mathcal{N}](t,x) - \int \phi(r) Re[\bar{\mathcal{N}}\partial_r P_{\leq k_0+9}u](t,x).
\end{aligned}$$

Decompose

$$\mathcal{N} = P_{\leq k_0+9}F(u) - F(P_{\leq k_0+9}u) = F(u) - F(P_{\leq k_0+9}u) - P_{\geq k_0+9}F(u). \tag{7.40}$$

Then by Theorem 16, Bernstein's inequality, Fourier support properties of $P_{\geq k_0+9}$ and $P_{\leq k_0}$, and the fact that ϕ is smooth and all its derivatives are rapidly decreasing,

(7.41)
$$\begin{aligned}
\int_J \int \phi(r) Re[\overline{P_{\geq k_0+9}F(u)}\partial_r P_{\leq k_0+9}u](t,x)dxdt &\lesssim 2^{k_0} R \|P_{\geq k_0}F(u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \|P_{\geq k_0}u\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \\
&\quad + \int_J \int \phi(r) Re[\overline{P_{\geq k_0+9}F(u)}\partial_r P_{\leq k_0}u](t,x)dxdt,
\end{aligned}$$

and

(7.42)
$$\begin{aligned}
\int_J \int \phi(r) Re[\overline{P_{\geq k_0+9}F(u)}\partial_r P_{\leq k_0}u](t,x)dxdt &\lesssim 2^{k_0} R \|P_{\geq k_0}F(u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} T^{-5} \\
&\lesssim 2^{k_0} R(\frac{1}{T}\int_J \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2}dt) + T^{-9}.
\end{aligned}$$

Next, by Taylor's formula,

(7.43)
$$\begin{aligned}
&\int \phi(r) Im[\overline{F(u) - F(P_{\leq k_0+9}u)}\partial_r(P_{\leq k_0+9}u)]dxdt \\
&= \int \int_0^1 \phi(r) Im[\overline{F'(P_{\leq k_0+9}u + sP_{\geq k_0+9}u)(P_{\geq k_0+9}u)}\partial_r(P_{\leq k_0+9}u)]dsdxdt.
\end{aligned}$$

By the chain rule and the computations in (7.41)–(7.42),

(7.44)
$$\int \phi(r) Im[\overline{F'(P_{\leq k_0+9}u)(P_{\geq k_0+9}u)}\partial_r(P_{\leq k_0+9}u)]dxdt = \int \phi(r) Im[\overline{P_{\geq k_0+9}u}]\partial_r F(P_{\leq k_0+9}u)dxdt$$
$$\lesssim 2^{k_0} R(\frac{1}{T}\int_J \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2}dt) + T^{-9}.$$

Next, by Theorem 16, if $\frac{1}{p} = \frac{1}{2}\frac{1}{1+4/d}$, and $\frac{1}{q} = \frac{d+2}{2d}\frac{1}{1+4/d}$,

(7.45)
$$\int\int_0^1 \phi(r) Im[\overline{[F'(P_{\leq k_0+9}u + sP_{\geq k_0+9}u) - F'(P_{\leq k_0+9}u)](P_{\geq k_0+9}u)}\partial_r(P_{\leq k_0+9}\tilde{\epsilon})]dsdxdt$$
$$\lesssim R\|P_{\geq k_0+9}u\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \|P_{\geq k_0+9}u\|_{L_t^p L_x^q}^{4/d} \|\nabla P_{\leq k_0+9}\tilde{\epsilon}\|_{L_t^p L_x^q} \lesssim 2^{k_0} R(\frac{1}{T}\int_J \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2}dt) + T^{-9}.$$

Then using the fact that Q is smooth with all derivatives rapidly decreasing,

(7.46)
$$\int\int_0^1 \phi(r) Im[\overline{[F'(P_{\leq k_0+9}u + sP_{\geq k_0+9}u) - F'(P_{\leq k_0+9}\tilde{Q} + P_{\leq k_0+9}\tilde{\epsilon})](P_{\geq k_0+9}u)}\partial_r(P_{\leq k_0+9}\tilde{Q})]dsdxdt$$
$$\lesssim \int\int_0^1 \phi(r) Im[\overline{[F'(\tilde{Q} + P_{\leq k_0+9}\tilde{\epsilon} + sP_{\geq k_0+9}\tilde{\epsilon}) - F'(\tilde{Q})](P_{\geq k_0+9}u)}\partial_r(\tilde{Q})]dsdxdt$$
$$+ \int\int_0^1 \phi(r) Im[\overline{[F'(\tilde{Q} + P_{\leq k_0+9}\tilde{\epsilon}) - F'(\tilde{Q})](P_{\geq k_0+9}u)}\partial_r(\tilde{Q})]dsdxdt + T^{-9}.$$

Using the computations in Theorem 16,

(7.47)
$$\int\int_0^1 \phi(r) Im[\overline{[F'(\tilde{Q} + P_{\leq k_0+9}\tilde{\epsilon} + sP_{\geq k_0+9}\tilde{\epsilon}) - F'(\tilde{Q})](P_{\geq k_0+9}u)}\partial_r(\tilde{Q})]dsdxdt$$
$$\lesssim 2^{k_0} R(\frac{1}{T}\int_J \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2}dt) + T^{-9},$$

and

(7.48)
$$\int\int_0^1 \phi(r) Im[\overline{[F'(\tilde{Q} + P_{\leq k_0+9}\tilde{\epsilon}) - F'(\tilde{Q})](P_{\geq k_0+9}u)}\partial_r(\tilde{Q})]dsdxdt$$
$$\lesssim 2^{k_0} R(\frac{1}{T}\int_J \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2}dt) + T^{-9},$$

Next, integrating by parts,

(7.49)
$$\int\int \phi(x)\frac{x}{|x|}\cdot Im[(\overline{P_{\leq k_0+9}u})\nabla\mathcal{N}]dxdt = -\int\int \phi(x)\frac{x}{|x|}\cdot Im[(\nabla\overline{P_{\leq k_0+9}u})\mathcal{N}]dxdt$$
$$-\int\int \nabla\cdot(\phi(x)\frac{x}{|x|})Im[(\overline{P_{\leq k_0+9}u})\mathcal{N}]dxdt.$$

The first term in (7.49) is handled by the computations in (7.41)–(7.47). Again using the smoothness of $\phi(x)\frac{x}{|x|}$, Theorem 16, and Fourier support arguments,

(7.50)
$$\int\int \nabla\cdot(\phi(x)\frac{x}{|x|})Im[(\overline{P_{\leq k_0+9}u})P_{\geq k_0+9}F(u)]dxdt \lesssim \frac{1}{T}\int_J \|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}dt + T^{-9}.$$

Also, since $\nabla \cdot (\phi(x)\frac{x}{|x|}) \lesssim \inf\{1, \frac{R}{|x|}\}$, using the analysis in (7.43)–(7.47),

(7.51)
$$\int \int \nabla \cdot (\phi(x)\frac{x}{|x|}) Im[(\overline{P_{\leq k_0+9}u})(F(u) - F(P_{\leq k_0+9}u))]dxdt$$
$$= \int \int \int_0^1 \nabla \cdot (\phi(x)\frac{x}{|x|}) Im[(\overline{P_{\leq k_0+9}u})F'(P_{\leq k_0+9}u + sP_{\geq k_0+9}u)(P_{\geq k_0+9}u)]dsdxdt$$
$$\lesssim (1+\frac{4}{d}) \int \int \int_0^1 \nabla \cdot (\phi(x)\frac{x}{|x|}) Im[\overline{F(P_{\leq k_0+9}u + sP_{\geq k_0+9}u)}(P_{\geq k_0+9}u)]dsdxdt + \|P_{\geq k_0+9}u\|^2_{L_t^2 L_x^{\frac{2d}{d-2}}}$$
$$\lesssim (1+\frac{4}{d}) \int \int \nabla \cdot (\phi(x)\frac{x}{|x|}) Im[\overline{F(P_{\leq k_0}u)}(P_{\geq k_0+9}u)]dxdt + \|P_{\geq k_0+9}u\|^2_{L_t^2 L_x^{\frac{2d}{d-2}}}.$$

By Theorem 16,

(7.52)
$$\|P_{\geq k_0+9}u\|^2_{L_t^2 L_x^{\frac{2d}{d-2}}} + \|P_{\geq k_0}F(P_{\leq k_0}u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \|P_{\geq k_0+9}u\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \lesssim \frac{1}{T}\int_J \|\epsilon(t)\|^2_{L^2}\lambda(t)^{-2}dt + T^{-9}.$$

Meanwhile, since $\nabla \cdot (\frac{x}{|x|}\phi(x))$ is smooth,

(7.53)
$$(1+\frac{4}{d}) \int \int \nabla \cdot (\phi(x)\frac{x}{|x|}) Im[\overline{P_{\leq k_0}F(P_{\leq k_0}u)}(P_{\geq k_0+9}u)]dxdt \lesssim T^{-5}\|P_{\geq k_0+9}u\|_{L_t^2 L_x^{\frac{2d}{d-2}}}$$
$$\lesssim \frac{1}{T}\int_J \|\epsilon(t)\|^2_{L^2}\lambda(t)^{-2}dt + T^{-9}.$$

Therefore, the error arising from frequency truncation is bounded by

$$2^{k_0}R(\frac{1}{T}\int_J \|\epsilon(t)\|^2_{L^2}\lambda(t)^{-2}dt) + T^{-9}. \tag{7.54}$$

Using the fact that $(\frac{1}{r}\phi(r) - \chi^2(r))(\delta_{jk} - \frac{x_j x_k}{|x|^2})$ is a positive definite matrix, we have proved

(7.55)
$$2\int_a^b \int \chi^2(\frac{x}{R})|\nabla P_{\leq k_0+9}u|^2 dxdt - \frac{1}{2R^2}\int_a^b \int \psi''(\frac{x}{R})|P_{\leq k_0+9}u|^2 dxdt$$
$$-\frac{2d}{d+2}\int_a^b \int \chi^2(\frac{x}{R})|P_{\leq k_0+9}u|^{\frac{2(d+2)}{d}}dxdt \lesssim M(b) - M(a) + \frac{1}{\eta_1^2 T^9} + 2^{k_0}R(\frac{1}{T}\int_0^T \|\epsilon(t)\|^2_{L^2}\lambda(t)^{-2}dt)).$$

The estimate of $M(b) - M(a)$ under the assumption that $\xi(a) = x(b) = 0$ is the same as in dimension $d = 2$. Indeed, since Q is smooth, real valued, rapidly decreasing, all its derivatives are rapidly decreasing, by (5.6), (5.7), and (7.36), $\xi(b) = x(a) = 0$, as in dimension $d = 2$,

(7.56)
$$\int \phi(|x|)\frac{x}{|x|} \cdot \overline{Im[P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}\frac{1}{\lambda(t)^{d/2}}Q(\frac{x-x(t)}{\lambda(t)}))}$$
$$\times \nabla(P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}\frac{1}{\lambda(t)^{d/2}}Q(\frac{x-x(t)}{\lambda(t)})))]dx|_a^b = O(2^{k_0}RT^{-10}).$$

Also,

$$\int \phi(|x|)\frac{x}{|x|}\cdot Im[\overline{P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}\frac{1}{\lambda(t)^{d/2}}\epsilon(\frac{x-x(t)}{\lambda(t)}))} \times\nabla(P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}\frac{1}{\lambda(t)^{d/2}}Q(\frac{x-x(t)}{\lambda(t)})))]dx|_a^b$$
$$= -(\epsilon_2, x\cdot\nabla Q)_{L^2}|_a^b + O(\frac{2^{2k_0}T^{1/25d}}{\eta_1^2 T}\int_J \|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}dt + \frac{T^{1/25d}}{\eta_1^2}\sup_{t\in J}\frac{|\xi(t)|^2}{\lambda(t)^2} + 2^{2k_0}\frac{T^{1/25d}}{\eta_1^2}T^{-10}). \tag{7.57}$$

Also, integrating by parts, since Q and all its derivatives are rapidly decreasing, as well as that $\phi(|x|)\frac{x}{|x|}$ is smooth,

$$\int \phi(|x|)\frac{x}{|x|}\cdot Im[\overline{P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}\frac{1}{\lambda(t)^{d/2}}Q(\frac{x-x(t)}{\lambda(t)}))} \times\nabla(P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}\frac{1}{\lambda(t)^{d/2}}\epsilon(\frac{x-x(t)}{\lambda(t)})))]dx$$
$$= (7.57) - \int \nabla\cdot(\phi(|x|)\frac{x}{|x|})\cdot Im[\frac{1}{\lambda(t)^{d/2}}\overline{Q(\frac{x-x(t)}{\lambda(t)}))}(\frac{1}{\lambda(t)^{d/2}}\epsilon(\frac{x-x(t)}{\lambda(t)})))]dx + O(T^{-10})$$
$$= (7.57) - d(\epsilon_2, Q)_{L^2} + O(T^{-10}). \tag{7.58}$$

Finally, by Theorem 20,

$$\int \phi(|x|)\frac{x}{|x|}\cdot Im[\overline{P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}\frac{1}{\lambda(t)^{d/2}}\epsilon(\frac{x-x(t)}{\lambda(t)}))} \times\nabla(P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}\frac{1}{\lambda(t)^{d/2}}\epsilon(\frac{x-x(t)}{\lambda(t)})))]dx$$
$$\lesssim R\|\epsilon\|_{L^2}\|P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}\frac{1}{\lambda(t)^{d/2}}\epsilon(\frac{x-x(t)}{\lambda(t)}))\|_{\dot{H}^1} + R\frac{|\xi(t)|}{\lambda(t)}\|\epsilon\|_{L^2}^2$$
$$\lesssim R(\frac{2^{2k_0}T^{1/25d}}{\eta_1^2 T}\int_J \|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}dt + \frac{T^{1/25d}}{\eta_1^2}\sup_{t\in J}\frac{|\xi(t)|^2}{\lambda(t)^2} + 2^{2k_0}\frac{T^{1/25d}}{\eta_1^2}T^{-10}). \tag{7.59}$$

Therefore, letting Λ denote the operator

$$\Lambda f = x\cdot\nabla f + \frac{d}{2}f, \tag{7.60}$$

$$M(b) - M(a) = 2(\epsilon_2(a), \Lambda Q)_{L^2} - 2(\epsilon_2(b), \Lambda Q)_{L^2}$$
$$+O(\frac{2^{2k_0}T^{2/15d}}{\eta_1^2 T}\int_J \|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}dt + \frac{T^{2/15d}}{\eta_1^2}\sup_{t\in J}\frac{|\xi(t)|^2}{\lambda(t)^2} + 2^{2k_0}\frac{T^{2/15d}}{\eta_1^2}T^{-10})). \tag{7.61}$$

Therefore, to complete the proof of Theorem 22, it only remains to obtain a lower bound for

$$2\int_a^b\int \chi^2(\frac{x}{2R})|\nabla P_{\leq k_0+9}u|^2dxdt - \frac{1}{2R^2}\int_a^b\int \psi''(\frac{x}{2R})|P_{\leq k_0+9}u|^2dxdt$$
$$-\frac{2d}{d+2}\int_a^b\int \chi^2(\frac{x}{2R})|P_{\leq k_0+9}u|^{\frac{2(d+2)}{d}}dxdt. \tag{7.62}$$

Again splitting

(7.63)
$$|P_{\leq k_0+9}u|^2 \leq 2\lambda(t)^{-d}|P_{\leq k_0+9}(e^{-i\gamma(t)}e^{ix\cdot\frac{\xi(t)}{\lambda(t)}}Q(\frac{x-x(t)}{\lambda(t)})|^2+2\lambda(t)^{-d}|P_{\leq k_0+9}(e^{-i\gamma(t)}e^{ix\cdot\frac{\xi(t)}{\lambda(t)}}\epsilon(t,\frac{x-x(t)}{\lambda(t)})|^2.$$

By (5.6), (5.7), and (7.36), the support of $\psi''(x)$, and the fact that Q is smooth and all its derivatives are rapidly decreasing,

$$\frac{1}{R^2}\lambda(t)^{-d}\int \psi''(\frac{x}{R})|P_{\leq k_0+9}(e^{-i\gamma(t)}e^{ix\cdot\frac{\xi(t)}{\lambda(t)}}Q(\frac{x-x(t)}{\lambda(t)})|^2dx \lesssim \lambda(t)^{-2}\frac{1}{T^{10}}. \tag{7.64}$$

On the other hand, by (5.6), (5.7), and (7.36), for T sufficiently large,

$$\frac{1}{R^2}\lambda(t)^{-d}\int \psi''(\frac{x}{R})|P_{\leq k+9}(e^{-i\gamma(t)}e^{ix\cdot\frac{\xi(t)}{\lambda(t)}}\epsilon(t,\frac{x-x(t)}{\lambda(t)})|^2dx \lesssim \frac{1}{R}\lambda(t)^{-2}\|\epsilon\|_{L^2}^2. \tag{7.65}$$

As in two dimensions, since Q is smooth and all its derivatives are rapidly decreasing, by (5.6), (5.7), and (7.36),

$$\begin{aligned}\frac{1}{2}\int(1-\chi^2(\frac{x}{2R}))|\nabla P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)^{d/2}}Q(\frac{x-x(t)}{\lambda(t)})|^2dx\\ -\frac{d}{2(d+2)}\int(1-\chi^2(\frac{x}{2R}))|P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)}Q(\frac{x-x(t)}{\lambda(t)})|^{\frac{2(d+2)}{d}}dx = O(T^{-10}).\end{aligned} \tag{7.66}$$

Also,

$$\begin{aligned}\int(1-\chi^2(\frac{x}{2R}))Re(\overline{\nabla P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)^{d/2}}Q(\frac{x-x(t)}{\lambda(t)})}\\ \cdot\nabla P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)^{d/2}}\epsilon(t,\frac{x-x(t)}{\lambda(t)}))dx\\ -\int(1-\chi^2(\frac{x}{2R}))|P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)^{d/2}}Q(\frac{x-x(t)}{\lambda(t)})|^{\frac{4}{d}}\\ \times Re(\overline{P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)^{d/2}}Q(\frac{x-x(t)}{\lambda(t)})}\\ \cdot P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)^{d/2}}\epsilon(t,\frac{x-x(t)}{\lambda(t)}))dx = O(2^{2k_0}T^{-10}).\end{aligned} \tag{7.67}$$

Therefore, from (6.54),

(7.68)
$$\begin{aligned}
&\frac{1}{2}\int \chi^2(\frac{x}{2R})|\nabla P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)^{d/2}}Q(\frac{x-x(t)}{\lambda(t)}))|^2dx\\
&-\frac{d}{2(d+2)}\int \chi^2(\frac{x}{2R})|P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)^{d/2}}Q(\frac{x-x(t)}{\lambda(t)}))|^{\frac{2(d+2)}{d}}dx\\
&+\int \chi^2(\frac{x}{2R})Re(\overline{\nabla P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)}Q(\frac{x-x(t)}{\lambda(t)}))}\cdot\nabla P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)}\epsilon(t,\frac{x-x(t)}{\lambda(t)}))dx\\
&-\int \chi^2(\frac{x}{2R})|P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)^{d/2}}Q(\frac{x-x(t)}{\lambda(t)})|^2\\
&\times Re(\overline{P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)^{d/2}}Q(\frac{x-x(t)}{\lambda(t)}))}\cdot P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)}\epsilon(t,\frac{x-x(t)}{\lambda(t)}))dx\\
&=\frac{1}{2}\frac{|\xi(t)|^2}{\lambda(t)^2}\|Q\|_{L^2}^2+\frac{1}{2\lambda(t)^2}\|\epsilon\|_{L^2}^2-\frac{|\xi(t)|^2}{2\lambda(t)^2}\|\epsilon\|_{L^2}^2+O(2^{2k_0}T^{-10}).
\end{aligned}$$

Turning to the terms with two ϵ's, by the product rule,

(7.69)
$$\begin{aligned}
&\frac{1}{2}\int \chi^2(\frac{x}{2R})|\nabla P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)^{d/2}}\epsilon(t,\frac{x-x(t)}{\lambda(t)})|^2dx\\
&=\frac{1}{2}\|\chi(\frac{x}{2R})P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)^{d/2}}\epsilon(t,\frac{x-x(t)}{\lambda(t)}))\|_{\dot H^1}^2+O(\frac{1}{R\lambda(t)^2}\|\epsilon\|_{L^2}^2).
\end{aligned}$$

Then by (5.6), (5.7), (7.36), (6.36), and the fact that Q_{x_j} and χ_0 are rapidly decreasing,

$$(\chi(\frac{x\lambda(t)+x(t)}{2R})\epsilon,f)_{L^2}\lesssim T^{-10}. \tag{7.70}$$

Therefore, following the analysis in (6.55)–(6.63),

(7.71)
$$\begin{aligned}
&\frac{1}{2}\int \chi^2(\frac{x}{2R})|\nabla P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)^{d/2}}\epsilon(t,\frac{x-x(t)}{\lambda(t)})|^2dx\\
&-\frac{d+2}{2d\lambda(t)^{d+2}}\int |P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}Q(\frac{x-x(t)}{\lambda(t)}))|^2|P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\epsilon(t,\frac{x-x(t)}{\lambda(t)}))|^2dx\\
&-\frac{1}{d\lambda(t)^{d+2}}Re\int \overline{(P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}Q(\frac{x-x(t)}{\lambda(t)})))^2}(P_{\leq k+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\epsilon(t,\frac{x-x(t)}{\lambda(t)})))^2dx\\
&\geq\frac{\lambda_d}{\lambda(t)^2}\|\epsilon\|_{L^2}^2+\lambda_d\int\chi(\frac{x}{2R})^2|\nabla P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\frac{1}{\lambda(t)^{d/2}}\epsilon(t,\frac{x-x(t)}{\lambda(t)})|^2dx-O(2^{2k_0}T^{-10}).
\end{aligned}$$

By the Sobolev embedding theorem, for $g\in H^1$,

$$\int\chi^2(\frac{x}{2R})|g(x)|^{\frac{2(d+2)}{d}}dx\lesssim\|\chi(\frac{x}{2R})g\|_{\dot H^1}^2\|g\|_{L^2}^{4/d}. \tag{7.72}$$

Therefore, by the product rule,

(7.73)
$$\begin{aligned}
&\int\chi(\frac{x}{2R})^2|P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\lambda(t)^{-d/2}\epsilon(t,\frac{x-x(t)}{\lambda(t)}))|^{\frac{2(d+2)}{d}}dx\\
&\lesssim\eta_0^{4/d}\|\chi(\frac{x}{2R})\nabla P_{\leq k_0+9}(e^{-i\gamma(t)}e^{-ix\cdot\xi(t)}\lambda(t)^{-d/2}\epsilon(t,\frac{x-x(t)}{\lambda(t)}))\|_{L^2}^2+\frac{\eta_0^{4/d}}{R^2}\|\epsilon\|_{L^2}^2.
\end{aligned}$$

Therefore, we have proved that for any $d \geq 3$, there exists some $\lambda_d > 0$ such that for η_0 sufficiently small,

$$(7.55) \gtrsim \lambda_d \int_a^b \|\epsilon\|_{L^2}^2 \lambda(t)^{-2} dt - O(T^{-8}). \tag{7.74}$$

Combining (7.74), (7.55), (7.64), (7.65), and (7.55) proves Theorem 22. □

8. An L_s^p bound on $\|\epsilon(s)\|_{L^2}$ when $p > 1$

As in one dimension, Theorems 21 and 22 imply that $\|\epsilon(s)\|_{L^2}$ lies in L_s^p for any $p > 1$.

Theorem 23. *Let u be a solution to (1.1) that satisfies $\|u\|_{L^2} = \|Q\|_{L^2}$, and suppose*

$$\sup_{s \in [0,\infty)} \|\epsilon(s)\|_{L^2} \leq \eta_*, \tag{8.1}$$

and $\|\epsilon(0)\|_{L^2} = \eta_*$. Then

$$\int_0^\infty \|\epsilon(s)\|_{L^2}^2 ds \lesssim \eta_*, \tag{8.2}$$

with implicit constant independent of η_* when $\eta_* \ll 1$ is sufficiently small.

Furthermore, for any $j \in \mathbb{Z}_{\geq 0}$, let

$$s_j = \inf\{s \in [0,\infty) : \|\epsilon(s)\|_{L^2} = 2^{-j}\eta_*\}. \tag{8.3}$$

By definition, $s_0 = 0$, and the continuity of $\|\epsilon(s)\|_{L^2}$ combined with Theorem 9 implies that such an s_j exists for any $j > 0$. Then,

$$\int_{s_j}^\infty \|\epsilon(s)\|_{L^2}^2 ds \lesssim 2^{-j}\eta_*, \tag{8.4}$$

for each $j \geq 0$, with implicit constant independent of η_*.

Proof. Set $T_* = \frac{1}{\eta_*}$ and suppose that T_* is sufficiently large such that Theorems 21 and 22 hold. Then by (3.25) and (8.1), for any $s' \geq 0$,

$$|\sup_{s \in [s', s'+T_*]} \ln(\lambda(s)) - \inf_{s \in [s', s'+T_*]} \ln(\lambda(s))| \lesssim 1, \tag{8.5}$$

with implicit constant independent of $s' \geq 0$. Let J be the largest dyadic integer that satisfies

$$J = 2^{j_*} \leq -\ln(\eta_*)^{1/4}. \tag{8.6}$$

By (8.5) and the triangle inequality,

$$|\sup_{s \in [s', s'+JT_*]} \ln(\lambda(s)) - \inf_{s \in [s', s'+JT_*]} \ln(\lambda(s))| \lesssim J, \tag{8.7}$$

and therefore,

$$\frac{\sup_{s \in [s', s'+3JT_*]} \lambda(s)}{\inf_{s \in [s', s'+3JT^*]} \lambda(s)} \lesssim T_*^{\frac{1}{5000d^2}}. \tag{8.8}$$

Rescale so that

$$\frac{1}{\eta_1} \leq \lambda(s) \leq \frac{1}{\eta_1} T_*^{\frac{1}{5000d^2}} \qquad \text{for any} \qquad s \in [s', s' + 3JT_*]. \tag{8.9}$$

Now take a subset $[a,b] \subset [s', s'+3JT_*]$, make a Galilean transformation so that $\xi(a) = 0$, and a translation in space so that $x(b) = 0$. By (3.24) and (8.9),

$$\sup_{s\in[s',s'+3JT_*]} |\frac{\xi(s)}{\lambda(s)}| \lesssim \eta_1 J\eta_* \ll \eta_0. \tag{8.10}$$

Also by (3.26),

$$|x(s)| \lesssim \frac{1}{\eta_1^2} T_*^{\frac{1}{5000d^2}} \eta_1 J + \frac{1}{\eta_1} T_*^{\frac{1}{5000d^2}} J \ll T_*^{\frac{1}{2000d^2}}. \tag{8.11}$$

Therefore, Theorems 21 and 22 may be utilized on $[s', s'+JT_*]$, proving that for any $s' \geq 0$,

$$\int_{s'}^{s'+JT_*} \|\epsilon(s)\|_{L^2}^2 ds \lesssim \|\epsilon(s')\|_{L^2} + \|\epsilon(s'+JT_*)\|_{L^2} + \eta_1^2 J^2 \eta_*^2 + O(\frac{1}{J^9 T_*^9}). \tag{8.12}$$

Note that the left hand side of (8.12) is scale invariant.

Moreover, for any $s' > JT_*$,

$$\int_{s'}^{s'+JT_*} \|\epsilon(s)\|_{L^2}^2 ds \lesssim \inf_{s\in[s'-JT_*,s']} \|\epsilon(s)\|_{L^2} + \inf_{s\in[s'+JT_*,s'+2JT_*]} \|\epsilon(s)\|_{L^2} + \eta_1^2 J^2 \eta_*^2 + O(\frac{1}{J^9 T_*^9}). \tag{8.13}$$

In particular, for a fixed $s' \geq 0$,

$$\sup_{a>0} \int_{s'+aJT_*}^{s'+(a+1)JT_*} \|\epsilon(s)\|_{L^2}^2 \lesssim \frac{1}{J^{1/2} T_*^{1/2}} (\sup_{a\geq 0} \int_{s'+aJT_*}^{s'+(a+1)JT_*} \|\epsilon(s)\|_{L^2}^2 ds)^{1/2} + \eta_1^2 J^2 \eta_*^2 + O(\frac{1}{J^9 T_*^9}). \tag{8.14}$$

Meanwhile, when $a = 0$,

$$\int_{s'}^{s'+JT_*} \|\epsilon(s)\|_{L^2}^2 \lesssim \|\epsilon(s')\|_{L^2} + \frac{1}{J^{1/2} T_*^{1/2}} (\sup_{a\geq 0} \int_{s'+aJT_*}^{s'+(a+1)JT_*} \|\epsilon(s)\|_{L^2}^2 ds)^{1/2} + \eta_1^2 J^2 \eta_*^2 + O(\frac{1}{J^9 T_*^9}). \tag{8.15}$$

Therefore, taking $s' = s_{j_*}$,

$$\sup_{a\geq 0} \int_{s_{j_*}+aJT_*}^{s_{j_*}+(a+1)JT_*} \|\epsilon(s)\|_{L^2}^2 ds \lesssim 2^{-j_*}\eta_* + \eta_1^2 J^2 \eta_*^2 + O(2^{-9j_*}\eta_*^9). \tag{8.16}$$

Then by the triangle inequality,

$$\sup_{s'\geq s_{j_*}} \int_{s'}^{s'+JT_*} \|\epsilon(s)\|_{L^2}^2 ds \lesssim 2^{-j_*}\eta_*, \tag{8.17}$$

and by Hölder's inequality,

$$\sup_{s'\geq s_{j_*}} \int_{s'}^{s'+JT_*} \|\epsilon(s)\|_{L^2} ds \lesssim 1. \tag{8.18}$$

Repeating this argument, Theorem 23 can be proved by induction. Indeed, fix a constant $C < \infty$ and suppose that there exists a positive integer n_0 such that for all integers $0 \leq n \leq n_0$,

$$\sup_{s'\geq s_{nj_*}} \int_{s'}^{s'+J^nT_*} \|\epsilon(s)\|_{L^2} ds \leq C, \qquad \sup_{s'\geq s_{nj_*}} \int_{s'}^{s'+J^nT_*} \|\epsilon(s)\|_{L^2}^2 ds \leq CJ^{-n}\eta_*. \tag{8.19}$$

Then by (8.5), for $s' \geq s_{nj_*}$,

$$\frac{\sup_{s\in[s',s'+3J^{n+1}T_*]}\lambda(s)}{\inf_{s\in[s',s'+3J^{n+1}T_*]}\lambda(s)} \lesssim T_*^{\frac{1}{5000d^2}}. \tag{8.20}$$

Remark 7. *The C in* (8.19) *will ultimately be given by the implicit constants in Theorems* 21 *and* 22*, so for T_* sufficiently large,* (8.20) *will hold.*

Rescale so that (8.9) holds. Also, for $[a,b] \subset [s', s'+3J^{n+1}T_*]$, setting $\xi(a) = 0$ and $x(b) = 0$, by (3.24) and (8.19),

$$\sup_{s\in[s',s'+3J^{n+1}T_*]} \frac{|\xi(s)|}{\lambda(s)} \lesssim CJ^{-n}\eta_1\eta_*, \tag{8.21}$$

and by (3.26) and (8.19),

$$\begin{aligned} |x(s)| &\lesssim \sup\lambda(s)\int_{s'}^{s'+3J^{n+1}T_*} \|\epsilon(s)\|_{L^2}ds + \sup\lambda(s)^2\cdot\sup\frac{|\xi(s)|}{\lambda(s)}\int_{s'}^{s'+3J^{n+1}T_*} 1ds \\ &\lesssim \frac{1}{\eta_1}T_*^{\frac{1}{5000d^2}}CJ + \frac{1}{\eta_1^2}T_*^{1/25d}CJ^{-n}\eta_1\eta_*J^{n+1}T_* \lesssim \frac{1}{\eta_1}T_*^{\frac{1}{5000d^2}}CJ + \frac{1}{\eta_1}T_*^{\frac{1}{2500d^2}}CJ \ll T_*^{\frac{1}{2000d^2}}. \end{aligned} \tag{8.22}$$

Then by Theorems 21 and 22,

$$\sup_{s'\geq s_{(n+1)j_*}}\int_{s'}^{s'+J^{n+1}T_*} \|\epsilon(s)\|_{L^2}^2 ds \leq CJ^{-(n+1)}T_*^{-1}, \tag{8.23}$$

and by Hölder's inequality,

$$\sup_{s'\geq s_{(n+1)j_*}}\int_{s'}^{s'+J^{n+1}T_*} \|\epsilon(s)\|_{L^2} ds \leq C. \tag{8.24}$$

Therefore, (8.19) holds for any integer $n > 0$.

Now take any $j \in \mathbb{Z}$ and suppose $nj_* < j \leq (n+1)j_*$. Then (8.20)–(8.22) hold on $[s_j + aJ^{n+1}T_*, s_j + (a+1)J^{n+1}T_*]$ for any $a \geq 0$, so by Theorems 21 and 22,

$$\sup_{a\geq 0}\int_{s_j+aJ^{n+1}T_*}^{s_j+(a+1)J^{n+1}T_*} \|\epsilon(s)\|_{L^2}^2 ds \lesssim 2^{-j}\eta_*, \tag{8.25}$$

and therefore by Hölder's inequality, for any $s' \geq s_j$,

$$\sup_{s'\geq s_j}\int_{s'}^{s'+2^jT_*} \|\epsilon(s)\|_{L^2} ds \lesssim 1, \tag{8.26}$$

with bound independent of j. Inequalities (8.25) and (8.26) imply that (8.20)–(8.22) hold on $[s', s' + 3\cdot 2^j JT_*]$ for any $s' \geq s_j$, so

$$\int_{s_j}^{s_j+2^jJT_*} \|\epsilon(s)\|_{L^2}^2 \lesssim 2^{-j}\eta_*, \tag{8.27}$$

and therefore, by the mean value theorem,

$$\inf_{s\in[s_j,s_j+2^jJT_*]} \|\epsilon(s)\|_{L^2} \lesssim 2^{-j}\eta_* J^{-1/2}, \tag{8.28}$$

which implies

$$s_{j+1} \in [s_j, s_j + 2^j JT_*]. \tag{8.29}$$

Therefore, by (8.27) and Hölder's inequality,

$$\int_{s_j}^{s_{j+1}} \|\epsilon(s)\|_{L^2}^2 ds \lesssim 2^{-j}\eta_*, \qquad \text{and} \qquad \int_{s_j}^{s_{j+1}} \|\epsilon(s)\|_{L^2} ds \lesssim 1, \tag{8.30}$$

with constant independent of j. Summing in j gives (8.2) and (8.4). □

Now then, (3.27) and (8.2) imply

$$\lim_{s\to\infty} \|\epsilon(s)\|_{L^2} = 0. \tag{8.31}$$

Next, by definition of s_j, (8.4) implies

$$\int_{s_j}^{s_{j+1}} \|\epsilon(s)\|_{L^2} ds \lesssim 1, \tag{8.32}$$

and for any $1 < p < \infty$,

$$(\int_{s_j}^{s_{j+1}} \|\epsilon(s)\|_{L^2}^p ds) \lesssim \eta_*^{p-1} 2^{-j(p-1)}, \tag{8.33}$$

which implies that $\|\epsilon(s)\|_{L^2}$ belongs to L_s^p for any $p > 1$, but not L_s^1.

Comparing (8.33) to the pseudoconformal transformation of the soliton, (1.18), for $0 < t < 1$,

$$\lambda(t) \sim t, \qquad \text{and} \qquad \|\epsilon(t)\|_{L^2} \sim t, \tag{8.34}$$

so

$$\int_0^1 \|\epsilon(t)\|_{L^2} \lambda(t)^{-2} dt = \infty, \tag{8.35}$$

but for any $p > 1$,

$$\int_0^1 \|\epsilon(t)\|_{L^2}^p \lambda(t)^{-2} dt < \infty. \tag{8.36}$$

For the soliton, $\epsilon(s) \equiv 0$ for any $s \in \mathbb{R}$, so obviously, $\|\epsilon(s)\|_{L^2} \in L_s^p$ for $1 \le p \le \infty$.

9. Monotonicity of λ

As in the one dimensional case, it is possible to use the virial identity from [MR05], to show that $\lambda(s)$ is an approximately monotone decreasing function.

Theorem 24. *For any $s \ge 0$, let*

$$\tilde{\lambda}(s) = \inf_{\tau\in[0,s]} \lambda(\tau). \tag{9.1}$$

Then for any $s \ge 0$,

$$1 \le \frac{\lambda(s)}{\tilde{\lambda}(s)} \le 3. \tag{9.2}$$

Proof. Suppose there exist $0 \le s_- \le s_+ < \infty$ satisfying

$$\frac{\lambda(s_+)}{\lambda(s_-)} = e. \tag{9.3}$$

Then we can show that u is a soliton solution to (1.1), which is a contradiction, since $\lambda(s)$ is constant in that case.

The proof that (9.3) implies that u is a soliton uses a virial identity from [MR05], combined with the L_s^p bounds on $\|\epsilon(s)\|_{L^2}$ obtained in Theorem 23. Using (3.21), compute

(9.4)
$$\frac{d}{ds}(\epsilon, |x|^2 Q) + \frac{\lambda_s}{\lambda}\|xQ\|_{L^2}^2 + 4(\frac{d}{2}Q + x\cdot\nabla Q, \epsilon_2)_{L^2}$$
$$= O(|\gamma_s + 1 - \frac{x_s}{\lambda}\cdot\xi(s) - |\xi(s)|^2|\|\epsilon\|_{L^2}) + O(|\xi_s - \frac{\lambda_s}{\lambda}\xi(s)|\|\epsilon\|_{L^2}) + O(|\frac{\lambda_s}{\lambda}|\|\epsilon\|_{L^2}) + O(|\frac{x_s}{\lambda} + 2\xi|\|\epsilon\|_{L^2})$$
$$+O(\|\epsilon\|_{L^2}^2) + O(\|\epsilon\|_{L^2}\|\epsilon\|_{L^{2(1+\frac{4}{d})}}^{1+\frac{4}{d}}), \qquad \text{if} \qquad 2 \le d \le 4, \qquad +O(\|\epsilon\|_{L^2}^{1+4/d}), \qquad \text{if} \qquad d > 5.$$

Indeed, by direct computation,

(9.5) $(\nabla Q, x^2 Q)_{L^2} = (iQ, x^2 Q)_{L^2} = (i\nabla Q, x^2 Q) = 0, \qquad \text{and} \qquad -\mathcal{L}_-(y^2 Q) = -2dQ - 4x\cdot\nabla Q.$

Also,

(9.6)
$$(\frac{d}{2}Q + x\cdot\nabla Q, |x|^2 Q)_{L^2} = \frac{d}{2}\|xQ\|_{L^2}^2 + \frac{1}{2}(|x|^2 x, \nabla Q^2)_{L^2} = \frac{d}{2}\|xQ\|_{L^2}^2 + \frac{1}{8}(\nabla|x|^4, \nabla Q)_{L^2}$$
$$= \frac{d}{2}\|xQ\|_{L^2}^2 - \frac{1}{8}(\Delta|x|^4, Q^2)_{L^2} = \frac{d}{2}\|xQ\|_{L^2}^2 - \frac{d+2}{2}\|xQ\|_{L^2}^2 = -\|xQ\|_{L^2}^2.$$

Then by Theorem 23, the fundamental theorem of calculus, and (3.23)–(3.26),

$$\|xQ\|_{L^2}^2 + 4\int_{s_-}^{s_+}(\epsilon_2, \frac{Q}{2} + xQ_x)_{L^2} = O(\eta_*). \tag{9.7}$$

Therefore, there exists $s' \in [s_-, s_+]$ such that

$$(\epsilon_2, \frac{d}{2}Q + x\cdot\nabla Q)_{L^2} < 0. \tag{9.8}$$

Rescale so that $\lambda(s') = \frac{1}{\eta_1}$.

Since $s' \ge 0$, there exists some $j \ge 0$ such that $s_j \le s' + T_* < s_{j+1}$. Using the proof of Theorem 23, in particular (8.20)–(8.22), setting $\xi(s') = 0$ and $x(s_{j+1+J}) = 0$,

$$\int_{s'}^{s_{j+1+J}} |\frac{\lambda_s}{\lambda}| ds \lesssim J. \tag{9.9}$$

Then by Theorems 21 and 22, (9.8) implies

$$\int_{s'}^{s_{j+1+J}} \|\epsilon(s)\|_{L^2}^2 ds \lesssim 2^{-(j+1+J)}\eta_*, \tag{9.10}$$

and therefore by definition of s_{j+1+J},

$$\int_{s'}^{s_{j+1+J}} \|\epsilon(s)\|_{L^2} ds \lesssim 1. \tag{9.11}$$

Arguing by induction, suppose that for some $1 \le k \le k_0$,

$$\int_{s'}^{s_{j+k}} \|\epsilon(s)\|_{L^2}^2 ds \lesssim 2^{-j-k}\eta_*, \tag{9.12}$$

and

$$\int_{s'}^{s_{j+k}} \|\epsilon(s)\|_{L^2} ds \lesssim 1, \tag{9.13}$$

with implicit constant independent of k. By Theorem 23,

$$\int_{s'}^{s_{j+k+J}} \|\epsilon(s)\|_{L^2}^2 ds \lesssim 2^{-j-k}\eta_*, \tag{9.14}$$

and

$$\int_{s'}^{s_{j+k+J}} \|\epsilon(s)\|_{L^2} ds \lesssim J. \tag{9.15}$$

Then by (8.20)–(8.22) and setting $\xi(s') = x(s_{j+k+J}) = 0$,

$$\sup_{s\in[s',s_{j+k+J}]} |x(s)| \lesssim T_*^{\frac{1}{2000d^2}}, \qquad \sup_{s\in[s',s_{j+k+J}]} |\xi(s)| \le \eta_0, \qquad \frac{\sup_{s\in[s',s_{j+k+J}]}\lambda(s)}{\inf_{s\in[s',s_{j+k+J}]}\lambda(s)} \le T_*^{\frac{1}{5000d^2}}, \tag{9.16}$$

so by Theorems 21 and 22 and (9.8),

$$\int_{s'}^{s_{j+k}} \|\epsilon(s)\|_{L^2}^2 ds \lesssim 2^{-j-k}\eta_*, \tag{9.17}$$

and

$$\int_{s'}^{s_{j+k}} \|\epsilon(s)\|_{L^2} ds \lesssim 1, \tag{9.18}$$

for $1 \le k \le k_0 + J$. Therefore, (9.17) and (9.18) hold for any k, with implicit constant independent of k.

Taking $k \to \infty$,

$$\int_{s'}^{\infty} \|\epsilon(s)\|_{L^2}^2 ds = 0, \tag{9.19}$$

which implies that $\epsilon(s) = 0$ for all $s \ge s'$. Therefore,

$$u_0(x) = e^{i\gamma} e^{ix\cdot\xi} \lambda^{d/2} Q(\lambda x + x_0), \tag{9.20}$$

for some $\gamma \in \mathbb{R}$, $\lambda > 0$, $\xi \in \mathbb{R}^d$, $x_0 \in \mathbb{R}^d$, which proves that u is a soliton solution. □

10. Proof of Theorem 6

The key difference between the soliton solution (1.21) and the pseudoconformal transformation of the soliton (1.22) is that the soliton is global, while the pseudoconformal transformation of the soliton blows up in finite time. To prove Theorem 6, it suffices to show that a solution to (1.1) satisfying the conditions of Theorem 6 that blows up in infinite time must be a soliton. A pseudoconformal transformation of a finite time blowup solution will then show that it must be a pseudoconformal transformation of a soliton.

Theorem 25. *In dimensions $2 \le d \le 15$, if u is a solution to (1.1) that satisfies the conditions of Theorem 6, blows up forward in time, and*

$$\sup(I) = \infty, \tag{10.1}$$

then u is equal to a soliton solution.

Remark 8. *This theorem is the only place where the proof of Theorem 6 does not work in dimensions $d \ge 16$.*

Proof. For any integer $k \geq 0$, let

$$I(k) = \{s \geq 0 : 2^{-k+2} \leq \tilde{\lambda}(s) \leq 2^{-k+3}\}. \tag{10.2}$$

Then by (9.2),

$$2^{-k} \leq \lambda(s) \leq 2^{-k+3}, \tag{10.3}$$

for all $s \in I(k)$. By (3.20), the fact that $\sup(I) = \infty$ implies that

$$\sum 2^{-2k} |I(k)| = \infty. \tag{10.4}$$

If $\lambda(s) \to 0$ as $s \to \infty$, then there exists a sequence $k_n \nearrow \infty$ such that

$$|I(k_n)| 2^{-2k_n} \geq \frac{1}{k_n^2}. \tag{10.5}$$

Now let $I(k_n) = [a_n, b_n]$. Then by (8.3) and (8.30), for any $j \geq 0$,

$$|s_{j+1} - s_j| \lesssim 2^j \eta_*^{-1}, \qquad \|\epsilon(s_{j+1})\|_{L^2} = 2^{-(j+1)} \eta_*, \tag{10.6}$$

and therefore there exists $s' \in [0, b_n]$ such that

$$\|\epsilon(s_n')\|_{L^2} \lesssim k_n^2 2^{-2k_n}. \tag{10.7}$$

When $d = 2$, the proof of Theorem 25 is much simpler, so we will start with that. Make a Galilean transformation so that $\xi(s') = 0$. By (3.24) and Theorem 21, for any $0 \leq s \leq s_n'$,

$$\frac{|\xi(s)|}{\lambda(s)} \lesssim \int_0^{s_n'} \frac{1}{\lambda(s)} \|\epsilon(s)\|_{L^2}^2 ds \lesssim \eta_1 \eta_*. \tag{10.8}$$

Using (1.4), rescale so that

$$\frac{1}{\eta_1} \leq \lambda(s) \leq \frac{1}{\eta_1} 2^{k_n}, \qquad \text{for any} \qquad 0 \leq s \leq s_n'. \tag{10.9}$$

Setting $t_n' = s^{-1}(s_n')$, using (10.8)–(10.9), Corollary 1 implies that for $r_n = \frac{2k_n}{3}$,

$$\|P_{\geq r_n} u\|_{U_\Delta^2([0,t_n'] \times \mathbb{R}^2)} \lesssim \eta_*. \tag{10.10}$$

Furthermore, arguing by induction on frequency, and using (4.79) and the preceding computations,

$$\|P_{\geq r_n + \frac{k_n}{4}} u\|_{U_\Delta^2([0,t_n'] \times \mathbb{R}^2)} \lesssim k_n^2 2^{-2k_n}. \tag{10.11}$$

Then using the computations in (6.4),

$$E(P_{\leq r_n + \frac{k_n}{4}} u(t_n')) \lesssim (k_n^2 2^{-2k_n} 2^{r_n + \frac{k_n}{4}})^2 \sim (k_n^2 2^{-\frac{13k_n}{12}})^2. \tag{10.12}$$

Next, following the computations in the proof of Theorem 17, and using (10.11),

$$\sup_{t \in [0,t_n']} E(P_{\leq r_n + \frac{k_n}{4}} u(t)) \lesssim (k_n^2 2^{-\frac{13k_n}{12}})^2. \tag{10.13}$$

Therefore, by (6.35) and (10.9),

$$\|\epsilon(0)\|_{L^2}^2 \lesssim (k_n^2 2^{-\frac{k_n}{12}} \eta_1^{-1})^2. \tag{10.14}$$

Since $k_n \to \infty$ as $n \to \infty$, (10.14) implies that $\epsilon(0) = 0$, or that u is a soliton solution to (1.1).

If $\lambda(s) \geq \delta > 0$ for some $\delta > 0$, then rescale so that

$$\frac{1}{\eta_1} \leq \lambda(s) \leq \frac{1}{\eta_1} 2^{k_0}, \tag{10.15}$$

for some $k_0 \in \mathbb{Z}_{\geq 0}$. Then for $k_n = n$, $|s_n| \lesssim 2^n \eta_*^{-1}$, by (10.8), (10.23), Corollary 1, and (10.10)–(10.13),

$$\sup_{t\in[0,t_n']} E(P_{\leq r_n + \frac{k_n}{4}} u(t)) \lesssim 2^{-\frac{13k_n}{6}}, \qquad \text{and} \qquad \sup_{t\in[0,t_n']} \|\epsilon(t)\|_{L^2} \lesssim \eta_1^{-1} 2^{-\frac{13k_n}{12}} 2^{k_0}. \tag{10.16}$$

In this case as well, since $k_n \to \infty$, $\epsilon(0) = 0$.

In dimensions $d \geq 3$, the proof is complicated by two factors. The first is that the long time Strichartz estimates in Theorem 16 depend on a bound on $|x(t)|$, which was not needed in two dimensions. The second is that in dimensions $d \geq 3$, $F(x) = |x|^{\frac{4}{d}} x$ is not a smooth function of x.

First suppose $\lambda(s) \searrow 0$ as $s \to \infty$. Again rescale so that

$$\frac{1}{\eta_1} \leq \lambda(s) \leq \frac{1}{\eta_1} 2^{k_n}, \qquad \text{for any} \qquad 0 \leq s \leq s_n'. \tag{10.17}$$

Suppose $s_n' \in I(\tilde{k}_n) = [a_n, b_n]$, for $\tilde{k}_n \leq k_n$. By (10.8),

$$\frac{|\xi(s)|}{\lambda(s)} \lesssim \eta_1 \eta_*, \qquad \text{for all} \qquad 0 \leq s \leq s_n'. \tag{10.18}$$

Also, for any $s_j \in I(\tilde{k}_n)$, $s_j \in [a_n, b_n]$, setting $x(s_j) = \xi(s_j) = 0$, by (10.8),

$$\sup_{s\in[s_j, s_{j+1}]} |x(s)| \lesssim \frac{1}{\eta_1} + \frac{1}{\eta_1^2} \eta_1 \eta_* 2^{-j} \frac{2^j}{\eta_*} \lesssim \frac{1}{\eta_1}. \tag{10.19}$$

Since $|s_n' - a_n| \lesssim \frac{1}{k_n^2} 2^{2k_n}$, setting $T \sim \frac{1}{k_n^2} 2^{2k_n}$ and $\eta_1^{-2} T = 2^{\alpha_d k}$, as in (5.7), Theorem 16 implies that for any $i \geq \frac{2k_n}{10d}$, letting $a_n' = s^{-1}(a_n)$,

$$\begin{aligned} &\|P_{\geq i} u\|_{L_t^2 L_x^{\frac{2d}{d-2}}([a_n', t_n']\times\mathbb{R}^d)} + \|P_{\geq i} u\|_{U_\Delta^p([a_n', t_n']\times\mathbb{R}^d)} \\ &\lesssim k_n 2^{\frac{\alpha_d}{2}((1+\frac{1}{10d})k - i)} \Big(\frac{1}{T}\int_{a_n'}^{t_n'} \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2} dt\Big)^{1/2} + k_n^2 2^{-2k_n} + T^{-10} \\ &= \eta_1^{-1+\frac{1}{10d}} k_n^{1-\frac{1}{5d}} 2^{\frac{k_n}{5d}} 2^{-\frac{\alpha_d}{2} i} \Big(\int_{a_n'}^{t_n'} \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2} dt\Big)^{1/2} + k_n^2 2^{-2k_n} + T^{-10}. \end{aligned} \tag{10.20}$$

In fact, revisiting the proof of Theorem 16, by (10.7), the right hand side of (5.12) can be replaced by $k_n^2 2^{-2k_n} + T^{-10}$, and (5.21) can be replaced by

$$\|P_{\geq i} u\|_{U_\Delta^p(J\times\mathbb{R}^d)} + \|P_{\geq i} u\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J\times\mathbb{R}^d)} \lesssim \Big\| \int_{t_0}^t e^{i(t-\tau)\Delta} P_{\geq i} F(u_{\leq i}) d\tau \Big\|_{U_\Delta^p \cap L_t^2 L_x^{\frac{2d}{d-2}}} + k_n^2 2^{-2k_n} + T^{-10}. \tag{10.21}$$

Next, the contributions of (5.22), (5.30)–(5.31), and (5.39) can be replaced by

$$\eta_1^{-1+\frac{1}{10d}} k_n^{1-\frac{1}{5d}} 2^{\frac{k_n}{5d}} 2^{-\frac{\alpha_d}{2} i} \Big(\int_{a_n'}^{t_n'} \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2} dt\Big)^{1/2} \|\epsilon\|^{2/d}_{L_t^\infty L_x^2([a_n', t_n']\times\mathbb{R}^d)} + k_n^2 2^{-2k_n} \|\epsilon\|^{2/d}_{L_t^\infty L_x^2([a_n', t_n']\times\mathbb{R}^d)} + T^{-10}. \tag{10.22}$$

Furthermore, by the support properties of $\psi^2(x)$, where ψ is as defined in the proof of Theorem 16, the contribution of (5.27) and (5.29) may be controlled by the right hand side of (10.22) plus

$$2^{-i} \int (v, (\nabla(u_{\leq i} - \tilde{Q}) \tilde{Q}^{4/d}))_{L^2} dt + 2^{-i} (v, (\nabla \tilde{Q} \frac{(u_{\leq i} - \tilde{Q})}{\tilde{Q}^{1-4/d}}))_{L^2} dt, \tag{10.23}$$

where $\|v\|_{V_\Delta^2} = 1$ and $\hat{v}$ is supported on $|\xi| \geq 2^i$. Theorem 15 implies that $Q^{4/d}$ is differentiable, and the gradient is smooth and rapidly decreasing. Theorem 15 and (1.17) also imply that since Q is radially symmetric,

(10.24)
$$\nabla \frac{\nabla Q}{Q^{1-4/d}} = \nabla_k \frac{Q_r}{Q^{1-4/d}} \frac{x_j}{|x|} = (\frac{\delta_{jk}}{|x|} + \frac{x_j x_k}{|x|^3}) \frac{Q_r}{Q^{1-4/d}} + \frac{Q_{rr}}{Q^{1-4/d}} \frac{x_j x_k}{|x|^2} - (1 - \frac{4}{d}) \frac{Q_r^2}{Q^{2-4/d}} \frac{x_j x_k}{|x|^2}$$
$$\lesssim \frac{Q_r}{|x| Q^{1-4/d}} + Q^{4/d} + \frac{Q_r^2}{Q^{2-4/d}} \in L_x^{d/2}.$$

Therefore, by Bernstein's inequality, arguing as in (5.27) and (5.29),

$$(10.23) \lesssim \eta_1^{1/p} 2^{-\frac{i}{p} - 2i} T^{\frac{1}{2000 d^2}} \| \langle \nabla \rangle^2 (P_{\leq i} u - \tilde{Q}) \|_{L_t^2 L_x^{\frac{2d}{d-2}}}. \tag{10.25}$$

Therefore, plugging in (10.20), we have proved

(10.26)
$$\|P_{\geq i} u\|_{U_\Delta^p([a_n', t_n'] \times \mathbb{R}^d)} + \|P_{\geq i} u\|_{L_t^2 L_x^{\frac{2d}{d-2}}([a_n', t_n'] \times \mathbb{R}^d)} \lesssim \eta_1^{-1 + \frac{1}{10d}} k_n^{1 - \frac{1}{5d}} 2^{\frac{k_n}{5d}} 2^{-(\alpha_d - 1) i} (\int_{a_n'}^{t_n'} \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2} dt)^{1/2}$$
$$+ \eta_1^{-1 + \frac{1}{10d}} k_n^{1 - \frac{1}{5d}} 2^{\frac{k_n}{5d}} 2^{-\frac{\alpha_d}{2} i} (\int_{a_n'}^{t_n'} \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2} dt)^{1/2} \|\epsilon\|_{L_t^\infty L_x^2([a_n', t_n] \times \mathbb{R}^d)}^{2/d} + k_n^2 2^{-2k_n} + T^{-10},$$

and revisiting (5.16) and (5.17),

$$\|P_{\geq i} F(u)\|_{U_\Delta^{p'} \cap L_t^2 L_x^{\frac{2d}{d+2}}([a_n', t_n'] \times \mathbb{R}^d)} \lesssim \eta_1^{-1 + \frac{1}{10d}} k_n^{1 - \frac{1}{5d}} 2^{\frac{k_n}{5d}} 2^{-(\alpha_d - 1) i} (\int_{a_n'}^{t_n'} \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2} dt)^{1/2}$$
$$+ \eta_1^{-1 + \frac{1}{10d}} k_n^{1 - \frac{1}{5d}} 2^{\frac{k_n}{5d}} 2^{-\frac{\alpha_d}{2} i} (\int_{a_n'}^{t_n'} \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2} dt)^{1/2} \|\epsilon\|_{L_t^\infty L_x^2([a_n', t_n] \times \mathbb{R}^d)}^{2/d}$$
$$+ k_n^2 2^{-2k_n} \|\epsilon\|_{L_t^\infty L_x^2}^{4/d} + k_n^2 2^{-2k_n} 2^{-\frac{k_n}{10d}} + T^{-10}. \tag{10.27}$$

Combining (10.26) and (10.27) with the proof of Theorem 19, for $t \in [a_n', t_n']$, and Theorem 23,

(10.28)
$$E(P_{\leq k_n (1 - \frac{1}{10d})} u(t)) \lesssim 2^{-4k_n} 2^{2k_n (1 - \frac{1}{10d})} k_n^4 + 2^{(4 - 2\alpha_d) k_n (1 - \frac{1}{10d})} 2^{\frac{2k_n}{5d}} k_n^2 \eta_1^{-2} \|\epsilon\|_{L_t^\infty L_x^2([a_n', t_n'] \times \mathbb{R}^d)}$$
$$+ \eta_1^{-2} k_n^2 2^{\frac{2k_n}{5d}} 2^{-(\alpha_d - 2) k_n (1 - \frac{1}{10d})} \|\epsilon\|_{L_t^\infty L_x^2([a_n', t_n'] \times \mathbb{R}^d)}^{1 + 4/d} + T^{-10}.$$

Now then, when $d = 3, 4$, by (6.65), $\|\epsilon(t)\|_{L^2}^2 \lesssim \lambda(t)^2 E(u(t)) + T^{-10}$, so

$$E(P_{\leq k_n (1 - \frac{1}{10d})} u(t)) \lesssim 2^{-4k_n} 2^{2k_n (1 - \frac{1}{10d})} k_n^4 + 2^{(8 - 4\alpha_d) k_n (1 - \frac{1}{10d})} 2^{\frac{4k_n}{5d}} k_n^4 \eta_1^{-8} + T^{-10}, \tag{10.29}$$

so for k_n sufficiently large,

$$E(P_{\leq k_n (1 - \frac{1}{10d})} u(t)) \lesssim 2^{-4k_n} 2^{2k_n (1 - \frac{1}{10d})} k_n^4 + T^{-10}, \tag{10.30}$$

for any $t \in [a_n', t_n']$. Furthermore, for any $j \leq \tilde{k}_n$, suppose $I(j) = [a_j, b_j]$. Then, $\lambda(s) \sim \frac{1}{\eta_1} 2^{(\tilde{k}_n - j)}$. Rescaling so that $\lambda(s) \sim \frac{1}{\eta_1}$ on this interval, repeating the calculations obtaining (10.26) and

(10.27), and then rescaling back, if $a_j' = s^{-1}(a_j)$ and $b_j' = s^{-1}(b_j)$,

(10.31)
$$\|P_{\geq i}u\|_{U^p_\Delta([a_j',b_j']\times\mathbb{R}^d)} + \|P_{\geq i}u\|_{L^2_t L_x^{\frac{2d}{d-2}}([a_j',b_j']\times\mathbb{R}^d)} \lesssim \eta_1^{-1+\frac{1}{10d}} k_n^{1-\frac{1}{5d}} 2^{\frac{k_n}{5d}} 2^{-(\alpha_d-1)(i+\tilde{k}_n-j)} \|\epsilon\|^{1/2}_{L_t^\infty L_x^2([a_j',b_j']\times\mathbb{R}^d)}$$
$$+\eta_1^{-1+\frac{1}{10d}} k_n^{1-\frac{1}{5d}} 2^{\frac{k_n}{5d}} 2^{-\frac{\alpha_d}{2}(i+\tilde{k}_n-j)} \|\epsilon\|^{1/2+2/d}_{L_t^\infty L_x^2([a_j',b_j']\times\mathbb{R}^d)} + \sup\{2^{-i}, 2^{-k_n(1-\frac{1}{10d})}\} E(P_{\leq k_n(1-\frac{1}{10d})}u(b_j'))^{1/2} + T^{-10},$$

and letting Y denote the dual space to $U^p_\Delta \cap L^2_t L_x^{\frac{2d}{d-2}}$,

(10.32)
$$\begin{aligned}\|P_{\geq i}F(u)\|_{Y(J\times\mathbb{R}^d)} &\lesssim \eta_1^{-1+\frac{1}{10d}} k_n^{1-\frac{1}{5d}} 2^{\frac{k_n}{5d}} 2^{-(\alpha_d-1)(i+\tilde{k}_n-j)} \|\epsilon\|^{1/2}_{L_t^\infty L_x^2([a_j',b_j']\times\mathbb{R}^d)} \\ &+\eta_1^{-1+\frac{1}{10d}} k_n^{1-\frac{1}{5d}} 2^{\frac{k_n}{5d}} 2^{-\frac{\alpha_d}{2}(i+\tilde{k}_n-j)} \|\epsilon\|^{1/2+2/d}_{L_t^\infty L_x^2([a_j',b_j']\times\mathbb{R}^d)} \\ &+\sup\{2^{-i}, 2^{-k_n(1-\frac{1}{10d})}\} E(P_{\leq k_n(1-\frac{1}{10d})}u(b_j'))^{1/2} (\|\epsilon\|^{4/d}_{L_t^\infty L_x^2} + 2^{-\frac{k_n}{10d}}) + T^{-10}.\end{aligned}$$

Using (6.65), for $0 \leq j \leq \tilde{k}_n$,

$$\eta_1 2^{j-k_n} \|\epsilon(t)\|_{L^2} \lesssim E(u(t))^{1/2}, \tag{10.33}$$

so arguing by induction on j, $0 \leq j \leq \tilde{k}_n$, and following (10.28)–(10.30),

$$\sup_{t\in[0,t_n']} E(u(t)) \lesssim 2^{-4k_n} 2^{2k_n(1-\frac{1}{10d})} k_n^4 + T^{-10}. \tag{10.34}$$

Again by (6.65) and (10.17), (10.34) implies

$$\|\epsilon(0)\|_{L^2} \lesssim \frac{1}{\eta_1^2} 2^{-2k_n} 2^{2k_n(1-\frac{1}{10d})} k_n^4 + \frac{1}{\eta_1^2} 2^{2k_n} T^{-10}. \tag{10.35}$$

Taking $k_n \to \infty$ implies $\epsilon(0) = 0$, so u is a soliton. The case when $\lambda(s)$ has a positive lower bound is easier, as in the two dimensional case.

In dimensions $5 \leq d \leq 8$,

(10.36)
$$\begin{aligned}E(P_{\leq k_n(1-\frac{1}{10d})}u(t)) &\lesssim 2^{-4k_n} 2^{2k_n(1-\frac{1}{10d})} k_n^4 + 2^{(4-2\alpha_d)k_n(1-\frac{1}{10d})} 2^{\frac{2k_n}{5d}} k_n^2 \eta_1^{-2} \|\epsilon\|_{L_t^\infty L_x^2([a_n',t_n']\times\mathbb{R}^d)} \\ &+\eta_1^{-2} k_n^2 2^{\frac{2k_n}{5d}} 2^{-(\alpha_d-2)k_n(1-\frac{1}{10d})} \|\epsilon\|^{1+4/d}_{L_t^\infty L_x^2([a_n',t_n']\times\mathbb{R}^d)} + T^{-10},\end{aligned}$$

for $t \in [a_n', t_n']$ implies

$$\begin{aligned}E(P_{\leq k_n(1-\frac{1}{10d})}u(t)) &\lesssim 2^{-4k_n} 2^{2k_n(1-\frac{1}{10d})} k_n^4 + 2^{(8-4\alpha_d)k_n(1-\frac{1}{10d})} 2^{\frac{4k_n}{5d}} k_n^4 \eta_1^{-4} \\ &+(\eta_1^{-2} k_n^2 2^{\frac{2k_n}{5d}} 2^{-(\alpha_d-2)k_n(1-\frac{1}{10d})})^{\frac{2d}{d-4}} + T^{-10}.\end{aligned} \tag{10.37}$$

Doing some algebra, since $\alpha_d = 3 - \frac{1}{5d}$,

(10.38)
$$\frac{2k_n}{5d} \cdot \frac{2d}{d-4} - (1 - \frac{3}{10d} + \frac{1}{50d^2}) \frac{d}{d-4} = 2 + \frac{8}{d-4} - \frac{3}{10(d-4)} - \frac{1}{50d(d-4)} - \frac{4}{5d(d-4)} \geq 2 + \frac{4}{d-4}.$$

Therefore, for $t \in [a_n', t_n']$,

$$E(P_{\leq k_n(1-\frac{1}{10d})}u(t)) \lesssim 2^{-2k_n} 2^{-\frac{4k_n}{d-4}}. \tag{10.39}$$

Then as in (10.31) and (10.32),

$$\sup_{t\in[0,t_n']} E(P_{\leq k_n(1-\frac{1}{10d})}u(t)) \lesssim 2^{-2k_n} 2^{-\frac{4k_n}{d-4}}. \tag{10.40}$$

Then by by (6.65) and (10.9), taking $k_n \to \infty$, $\epsilon(0) = 0$.

When $d \geq 9$, recall that $\alpha_d = 2 + \frac{8}{d} - \frac{1}{5d}$. However, examining the proof of Theorem 16, it is only the contribution of (5.39) that needs $\alpha_d = 2 + \frac{8}{d} - \frac{1}{5d}$, the other terms have the regularity $\frac{3}{2} - \frac{1}{10d}$. Therefore, for $t \in [a_n', t_n']$, replace (10.36) by

(10.41)
$$E(P_{\leq k_n(1-\frac{1}{10d})}u(t)) \lesssim 2^{-4k_n}2^{2k_n(1-\frac{1}{10d})}k_n^4 + 2^{(4-2\alpha_d)k_n(1-\frac{1}{10d})}2^{\frac{2k_n}{5d}}k_n^2\eta_1^{-2}\|\epsilon\|_{L_t^\infty L_x^2([a_n',t_n']\times\mathbb{R}^d)}$$
$$+\eta_1^{-2}k_n^2 2^{\frac{2k_n}{5d}}2^{-(1-\frac{1}{5d})k_n(1-\frac{1}{10d})}\|\epsilon\|^{1+4/d}_{L_t^\infty L_x^2([a_n',t_n']\times\mathbb{R}^d)} + \eta_1^{-2}k_n^2 2^{\frac{2k_n}{5d}}2^{-(\frac{8}{d}-\frac{1}{5d})k_n(1-\frac{1}{10d})}\|\epsilon\|^{1+8/d}_{L_t^\infty L_x^2} + T^{-10}.$$

Then

$$\eta_1^{-2}k_n^2 2^{\frac{2k_n}{5d}}2^{-(\frac{8}{d}-\frac{1}{5d})k_n(1-\frac{1}{10d})}\|\epsilon\|^{1+8/d}_{L_t^\infty L_x^2} \lesssim \|\epsilon\|^2_{L_t^\infty L_x^2} + (\eta_1^{-2}k_n^2 2^{\frac{2k_n}{5d}}2^{-(\frac{8}{d}-\frac{1}{5d})k_n(1-\frac{1}{10d})})^{\frac{2d}{d-8}}. \tag{10.42}$$

Doing some algebra, for $8 \leq d \leq 15$,

$$(\frac{8}{d} - \frac{3}{5d} - \frac{7}{10d^2}) \cdot \frac{2d}{d-8} \geq 2 + \frac{3}{70}. \tag{10.43}$$

Therefore, for $8 \leq d \leq 15$, for k_n sufficiently large,

$$(\eta_1^{-2}k_n^2 2^{\frac{2k_n}{5d}}2^{-(\frac{8}{d}-\frac{1}{5d})k_n(1-\frac{1}{10d})})^{\frac{2d}{d-8}} \lesssim 2^{-2k_n}2^{-\frac{3}{70}k_n}. \tag{10.44}$$

Once again, taking $k_n \to \infty$ proves $\epsilon(0) = 0$.

Dimensions $d \geq 16$ remain unresolved. □

Now turn to a finite time blowup solution. As in dimension one, $\sup(I) < \infty$ implies that u is a pseudoconformal transformation of the soliton. Suppose without loss of generality that $\sup(I) = 0$, and

$$\sup_{-1<t<0} \|\epsilon(t)\|_{L^2} \leq \eta_*. \tag{10.45}$$

Then decomposing u,

$$u(t,x) = \frac{e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}}{\lambda(t)^{d/2}}Q(\frac{x-x(t)}{\lambda(t)}) + \frac{e^{-i\gamma(t)}e^{-ix\cdot\frac{\xi(t)}{\lambda(t)}}}{\lambda(t)^{d/2}}\epsilon(t, \frac{x-x(t)}{\lambda(t)}). \tag{10.46}$$

Then apply the pseudoconformal transformation to $u(t,x)$. For $-\infty < t < -1$, let

$$\begin{aligned} v(t,x) = \frac{1}{t^{d/2}}\overline{u(\frac{1}{t}, \frac{x}{t})}e^{i|x|^2/4t} &= \frac{1}{t^{d/2}}\frac{e^{i\gamma(1/t)}e^{i\frac{x}{t}\cdot\frac{\xi(\frac{1}{t})}{\lambda(\frac{1}{t})}}}{\lambda(1/t)^{d/2}}Q(\frac{x-tx(\frac{1}{t})}{t\lambda(1/t)})e^{i|x|^2/4t} \\ &+ \frac{1}{t^{d/2}}\frac{e^{i\gamma(1/t)}e^{i\frac{x}{t}\cdot\frac{\xi(\frac{1}{t})}{\lambda(\frac{1}{t})}}}{\lambda(1/t)^{d/2}}\overline{\epsilon(\frac{1}{t}, \frac{x-tx(\frac{1}{t})}{t\lambda(1/t)})}e^{i|x|^2/4t}. \end{aligned} \tag{10.47}$$

Since the L^2 norm is preserved by the pseudoconformal transformation,

$$\begin{aligned} &\lim_{t\searrow-\infty} \|\frac{1}{t^{d/2}}\frac{e^{i\gamma(1/t)}e^{i\frac{x}{t}\cdot\frac{\xi(\frac{1}{t})}{\lambda(\frac{1}{t})}}}{\lambda(1/t)^{d/2}}\overline{\epsilon(\frac{1}{t}, \frac{x-tx(\frac{1}{t})}{t\lambda(1/t)})}e^{i|x|^2/4t}\|_{L^2} = 0, \qquad \text{and} \\ &\sup_{-\infty<t<-1} \|\frac{1}{t^{d/2}}\frac{e^{i\gamma(1/t)}e^{i\frac{x}{t}\cdot\frac{\xi(\frac{1}{t})}{\lambda(\frac{1}{t})}}}{\lambda(1/t)^{d/2}}\overline{\epsilon(\frac{1}{t}, \frac{x-tx(\frac{1}{t})}{t\lambda(1/t)})}e^{ix^2/4t}\|_{L^2} \leq \eta_*. \end{aligned} \tag{10.48}$$

Decompose

$$\frac{|x|^2}{4t} = \frac{|x - tx(\frac{1}{t})|^2}{4t} + \frac{x \cdot x(\frac{1}{t})}{2} - \frac{t}{4}|x(\frac{1}{t})|^2. \tag{10.49}$$

Since

$$\frac{1}{t^{d/2}} \frac{e^{i\gamma(1/t)} e^{i\frac{x}{t}\cdot\frac{\xi(\frac{1}{t})}{\lambda(\frac{1}{t})}} e^{ix\cdot\frac{x(\frac{1}{t})}{2}} e^{i\frac{t}{4}|x(\frac{1}{t})|^2}}{\lambda(1/t)^{d/2}} Q(\frac{x - tx(\frac{1}{t})}{t\lambda(1/t)}) \tag{10.50}$$

is in the form of $\frac{e^{i\tilde{\gamma}(t)} e^{ix\cdot\frac{\tilde{\xi}(t)}{\lambda(t)}}}{\tilde{\lambda}(t)^{d/2}} Q(\frac{x-\tilde{x}(t)}{\tilde{\lambda}(t)})$, it only remains to estimate

$$\| \frac{1}{t^{d/2}} \frac{e^{i\gamma(1/t)} e^{i\frac{x}{t}\cdot\frac{\xi(t)}{\lambda(t)}}}{\lambda(1/t)^{d/2}} Q(\frac{x - tx(\frac{1}{t})}{t\lambda(1/t)})(e^{i|x-tx(\frac{1}{t})|^2/4t} - 1) \|_{L^2}. \tag{10.51}$$

As in (10.3), for any $k \geq 0$, $\lambda(s) \sim 2^{-k}$ for all $s \in I(k)$. Furthermore, by (3.25), $\|\epsilon(t)\|_{L^2} \to 0$ as $t \nearrow 0$ implies that there exists a sequence $c_k \nearrow \infty$ such that

$$|I(k)| \geq c_k, \qquad \text{for all} \qquad k \geq 0. \tag{10.52}$$

Then by (3.20), there exists $r(t) \searrow 0$ as $t \nearrow 0$ such that

$$\lambda(t) \leq t^{1/2} r(t), \qquad \text{so} \qquad \lambda(1/t) \leq t^{-1/2} r(1/t). \tag{10.53}$$

Therefore, since Q is rapidly decreasing,

$$\lim_{t \searrow -\infty} \| \frac{1}{t^{d/2}\lambda(1/t)^{d/2}} Q(\frac{x - tx(\frac{1}{t})}{t\lambda(1/t)}) \frac{|x - tx(\frac{1}{t})|^2}{4t} \|_{L^2} = 0, \tag{10.54}$$

as well as

$$\lim_{t \searrow -\infty} \| \frac{1}{t^{d/2}\lambda(1/t)^{d/2}} Q(\frac{x - tx(\frac{1}{t})}{t\lambda(1/t)})(e^{i|x-tx(\frac{1}{t})|^2/4t} - 1) \|_{L^2} = 0, \tag{10.55}$$

Therefore, v is a solution that blows up backward in time at $\inf(I) = -\infty$ and v satisfies the conditions of Theorem 10 on $(-\infty, t_0]$ for some $t_0 \in \mathbb{R}$. Therefore, by time reversal symmetry and Theorem 25, v must be a soliton. Therefore, u is the pseudoconformal transformation of a soliton.

11. A Liouville result

Recall the Liouville theorem for the generalized KdV equation from [MM00].

Theorem 26. *Let $u_0 \in H^1(\mathbb{R})$ and let $\alpha = \|u_0 - Q\|_{H^1}$. Suppose that the solution to the focusing, mass-critical generalized KdV problem,*

$$u_t + \partial_x(u_{xx} + u^5) = 0, \qquad u(0, x) = u_0, \tag{11.1}$$

is global in time, and for all $t \in \mathbb{R}$, and assume that for some $c_1, c_2 > 0$,

$$c_1 \leq \|u(t)\|_{H^1} \leq c_2. \tag{11.2}$$

Also suppose that there exists $x(t)$ such that

$$v(t, x) = u(t, x + x(t)), \tag{11.3}$$

satisfies

$$\forall \epsilon_0, \qquad \exists R_0(\epsilon_0) > 0, \qquad \forall t \in \mathbb{R}, \qquad \int_{|x|>R_0} v(t, x)^2 dx \leq \epsilon_0. \tag{11.4}$$

There exists $\alpha_0 > 0$ such that if $\alpha < \alpha_0$, there exists λ_0, x_0 such that

$$u(t,x) = \lambda_0^{1/2} Q(\lambda_0(x - x_0) - \lambda_0^3 t). \tag{11.5}$$

Using Theorems 2 and 3, we can prove such a result for the nonlinear Schrödinger equation, without requiring the initial data to lie close to the soliton.

Theorem 27. *Let $u_0 \in H^1(\mathbb{R}^d)$ and suppose $\|u_0\|_{L^2} \leq \|Q\|_{L^2} + \alpha$, for some $0 < \alpha < \alpha_0 \ll 1$. Suppose that a solution $u(t)$ to (1.1) is defined for all $t \in \mathbb{R}$ and for some $c_1, c_2 > 0$,*

$$c_1 \leq \|u(t)\|_{H^1} \leq c_2, \qquad \text{for all} \qquad t \in \mathbb{R}. \tag{11.6}$$

Also suppose that for all $t \in \mathbb{R}$ there exists $x(t) \in \mathbb{R}^d$ such that

$$v(t,x) = u(t, x + x(t)), \tag{11.7}$$

satisfies

$$\forall \epsilon_0 > 0, \qquad \exists R_0 > 0, \qquad \forall t \in \mathbb{R}, \qquad \int_{|x| > R_0} |v(t,x)|^2 dx \leq \epsilon_0. \tag{11.8}$$

Then there exists $\alpha_0 > 0$ sufficiently small such that if $\alpha < \alpha_0$, u should be in the form (1.19).

Proof. By Theorem 3 and scattering for $\|u_0\|_{L^2} < \|Q\|_{L^2}$, it suffices to check

$$\|Q\|_{L^2} < \|u\|_{L^2} + \|Q\|_{L^2} + \alpha. \tag{11.9}$$

By [Fan21], [Dod20], and [Dod21b], there exists a sequence $t_n \to +\infty$ and a sequence $\gamma_{*,n} \in \mathbb{R}$, $\xi_{*,n} \in \mathbb{R}^d$, $\lambda_{*,n} \in (0, \infty)$, $x_{*,n} \in \mathbb{R}^d$, such that

$$e^{i\gamma_{*,n}} e^{ix \cdot \xi_{*,n}} \lambda_{*,n}^{d/2} u(t_n, \lambda_{*,n} x + x_{*,n}) \rightharpoonup Q, \qquad \text{weakly in} \qquad L^2. \tag{11.10}$$

By (11.6) and (11.8), this can be upgraded to convergence in L^2, which implies $\|u\|_{L^2} = \|Q\|_{L^2}$, which proves the theorem. □

In fact, it is possible to say more. Suppose u_0 does not lie in H^1, but only in L^2, but we have uniform bounds on the length of the intervals for which local well-posedness of (1.1) holds. The Liouville theorem still holds.

Theorem 28. *Let $u_0 \in L^2(\mathbb{R}^d)$ and suppose $\|u_0\|_{L^2} \leq \|Q\|_{L^2} + \alpha$, for some $0 < \alpha < \alpha_0 \ll 1$. Suppose that a solution $u(t)$ to (1.1) is defined for all $t \in \mathbb{R}$ and for some $c_1, c_2 > 0$,*

$$\begin{aligned} \sup_{t_0 \in \mathbb{R}} \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}([t_0, t_0+1] \times \mathbb{R}^d)} &\leq c_2, \\ \inf_{t_0 \in \mathbb{R}} \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}([t_0, t_0+1] \times \mathbb{R}^d)} &\geq c_1. \end{aligned} \tag{11.11}$$

Also suppose that for all $t \in \mathbb{R}$ there exists $x(t) \in \mathbb{R}^d$ such that

$$v(t,x) = u(t, x + x(t)), \tag{11.12}$$

satisfies

$$\forall \epsilon_0 > 0, \qquad \exists R_0 > 0, \qquad \forall t \in \mathbb{R}, \qquad \int_{|x| > R_0} |v(t,x)|^2 dx \leq \epsilon_0. \tag{11.13}$$

Then there exists $\alpha_0 > 0$ sufficiently small such that if $\alpha < \alpha_0$, u should be in the form (1.19).

Remark 9. *Note that (11.6) and (11.8) imply (11.11).*

Proof. Once again, it suffices to only consider initial data that satisfy (11.9). Once again, (11.10) holds for a sequence $t_n \nearrow \infty$.

We claim that for any N,

$$\|P_{\leq N}(e^{i\gamma_{*,n}}e^{ix\cdot\xi_{*,n}}\lambda_{*,n}^{d/2}u(t_n,\lambda_{*,n}x+x_{*,n})-Q(x))\|_{L^2}\to 0. \tag{11.14}$$

Otherwise, by (11.13),

$$P_{\leq N}(e^{i\gamma_{*,n}}e^{ix\cdot\xi_{*,n}}\lambda_{*,n}^{d/2}u(t_n,\lambda_{*,n}x+x_{*,n})-Q(x))\rightharpoonup f\neq 0,\qquad \text{weakly in}\qquad L^2, \tag{11.15}$$

which would contradict (11.10). Therefore, there exists a sequence $N_n \nearrow \infty$ such that

$$\|P_{\leq N_n}(e^{i\gamma_{*,n}}e^{ix\cdot\xi_{*,n}}\lambda_{*,n}^{d/2}u(t_n,\lambda_{*,n}x+x_{*,n})-Q(x))\|_{L^2}\to 0. \tag{11.16}$$

Next, since (11.11) implies that u blows up in both time directions, for α_0 sufficiently small, Theorem 3 combined with standard perturbative arguments implies that u is close to a member of the soliton family (1.21) for all $t\in\mathbb{R}$. Furthermore, (11.11) implies that $\lambda(t)\sim 1$ for all $t\in\mathbb{R}$. Therefore, let $u_n(t)$ be the solution to (1.1) with initial data of the form

$$u_n(0)=e^{i\gamma_{*,n}}e^{ix\cdot\xi_{*,n}}\lambda_{*,n}^{d/2}u(t_n,\lambda_{*,n}x+x_{*,n}). \tag{11.17}$$

Then by (11.16) and standard perturbative arguments, there exists a sequence $T_n \nearrow \infty$ such that

$$u_n(t)=e^{it}Q+v_n(t)+r_n(t),\qquad \text{where}\qquad \|v_n\|_{L_{t,x}^{\frac{2(d+2)}{d}}([0,T_n]\times\mathbb{R}^d)}\lesssim 1,\qquad \|r_n\|_{L_{t,x}^{\frac{2(d+2)}{d}}([0,T_n]\times\mathbb{R}^d)}\searrow 0. \tag{11.18}$$

However, by Hölder's inequality and (11.13), for n sufficiently large,

$$\|v_n(t)\|_{L^{\frac{2(d+2)}{d}}(\mathbb{R}^d)}\gtrsim(\|u_0\|_{L^2}-\|Q\|_{L^2}), \tag{11.19}$$

for any $t\in[0,T_n]$, with lower bound independent of n. This gives a contradiction for n sufficiently large, when $\|u_0\|_{L^2}>\|Q\|_{L^2}$. When $\|u_0\|_{L^2}=\|Q\|_{L^2}$, apply Theorem 3. □

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