Busemann process and semi-infinite geodesics in Brownian last-passage percolation

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Abstract. We prove the existence of semi-infinite geodesics for Brownian last-passage percolation (BLPP). Specifically, on a single event of probability one, there exist semi-infinite geodesics started from every space-time point and traveling in every asymptotic direction. Properties of these geodesics include uniqueness for a fixed initial point and direction, non-uniqueness for fixed direction but random initial points, and coalescence of all geodesics traveling in a common, fixed direction. Along the way, we prove that for fixed northeast and southwest directions, there almost surely exist no bi-infinite geodesics in the given directions. The semi-infinite geodesics are constructed from Busemann functions. Our starting point is a result of Alberts, Rassoul-Agha and Simper that established Busemann functions for fixed points and directions. Out of this, we construct the global process of Busemann functions simultaneously for all initial points and directions, and then the family of semi-infinite Busemann geodesics. The uncountable space of the semi-discrete setting requires extra consideration and leads to new phenomena, compared to discrete models.

Résumé. Nous prouvons l'existence de géodésiques semi-infinies pour la percolation brownienne de dernier passage (BLPP). Plus précisément, sur un seul événement de probabilité 1, il existe des géodésiques semi-infinies partant de chaque point d'espace-temps et voyageant dans toutes les directions asymptotiques. Les propriétés de ces géodésiques comprennent l'unicité pour un point et une direction initiaux fixes, la non-unicité pour une direction fixe mais des points initiaux aléatoires, et la coalescence de toutes les géodésiques se déplaçant dans une direction fixe commune. En cours de route, nous prouvons que pour les directions fixes nord-est et sud-ouest, il n'existe presque sûrement pas de géodésiques bi-infinies dans les directions données. Les géodésiques semi-infinies sont construites à partir des fonctions de Busemann. Notre point de départ est le résultat d'Alberts, Rassoul-Agha et Simper qui ont établi des fonctions Busemann pour des points et directions fixes. À partir de là, nous construisons le processus global des fonctions de Busemann simultanément pour tous les points et directions initiaux, puis la famille des géodésiques de Busemann semi-infinies. L'espace indénombrable du cadre semi-discret nécessite une considération supplémentaire et conduit à de nouveaux phénomènes, par rapport aux modèles discrets.

1. Introduction

1.1. *Brownian last-passage percolation*

Brownian last-passage percolation (BLPP) dates back to the 1991 work of Glynn and Whitt [26], where the model appeared as a large-scale limit of multiple queues in series under heavy traffic conditions. Harrison and Williams [33–35] also studied what is known as the Brownian queue and developed a stability result for that model. More specifically, if the arrivals process is given by increments of Brownian motion and the service process is given by increments of an independent Brownian motion with drift, the departures process is also given by increments of Brownian motion. The connection between BLPP and the Brownian queue is expounded on by O'Connell and Yor [45], who also introduced the positive temperature version of this model, known as the Brownian polymer. We discuss the connection to queuing theory in Section 5.3 and Appendix C.

1.2. *BLPP in the Kardar–Parisi–Zhang universality class*

In the early 2000s, Gravner, Tracy, and Widom [27] and Baryshnikov [5] discovered that the Brownian last-passage value $L_{(1,0),(n,1)}(\mathbf{B})$ (to be defined in (2.2) below) has the same distribution as the largest eigenvalue of an $n \times n$ GUE random matrix. Soon after, O'Connell and Yor [46] provided an alternate proof relying on the queuing interpretation of the model. Since then, Brownian last-passage percolation and the Brownian polymer have been widely studied as gateways to properties of the KPZ universality class. As a semi-discrete model with one continuous and one discrete parameter, BLPP serves as an intermediary between discrete models such as LPP on the planar integer lattice and continuum models such as the stochastic heat equation and the KPZ equation. In [14], Corwin and Hammond constructed the Airy line ensemble and proved that certain statistics of Brownian last-passage percolation converge in distribution to the Airy line ensemble. Dauvergne, Nica, and Virág ([18], Corollary 6.4) provided an alternate method to prove this fact, which couples Brownian last-passage percolation with geometric random walks.

Recently, Dauvergne, Ortmann, and Virág [19] constructed the directed landscape, a central object in the KPZ universality class, as the scaling limit of Brownian last-passage percolation. Another central object is the KPZ fixed point, constructed by Matetski, Quastel, and Remenik [40], as the limit of the totally asymmetric simple exclusion process. In [19], a variational duality is described between the directed landscape and KPZ fixed point, which was rigorously proven in [44]. Even more recently, convergence to the KPZ fixed point was shown for a larger class of models, including the height function of the KPZ equation. This was done independently by Quastel and Sarkar [49] and Virág [58].

1.3. *Semi-infinite geodesics in discrete models*

The study of infinite geodesics in planar random growth models has gone through a number of stages over the last 30 years, beginning with the work of Licea and Newman [39,43] on first-passage percolation with i.i.d. edge weights. Under a global curvature assumption on the limit shape, for continuously distributed edge weights, they proved existence of a deterministic full-Lebesgue measure set of directions in which there is a unique semi-infinite geodesic out of every lattice point. They also showed that, for each direction in this set, the semi-infinite geodesics in that direction all coalesce.

A separate strand of work consists of long-term efforts to prove the nonexistence of bi-infinite geodesics. In first-passage percolation, Licea and Newman [39] showed that there are no bi-infinite geodesics in fixed northeast and southwest directions. Howard and Newman [37] later proved similar results for Euclidean last-passage percolation. Around this time, Wehr and Woo [59] proved that, under a first moment assumption on the edge weights, there are no bi-infinite geodesics for first-passage percolation that lie entirely in the upper-half plane.

In 2016, Damron and Hanson [17] strengthened the result of Licea and Newman by proving that, if the weights have continuous distribution and the boundary of the limit shape is differentiable, for each fixed direction, there are no bi-infinite geodesics with one end having that direction. The conjectured nonexistence was finally resolved in exponential last-passage percolation (LPP), known also as the exponential corner growth model (CGM). The proofs came in two independent works: first by Basu, Hoffman, and Sly [7] and shortly thereafter by Balázs, Busani, and the first author [3]. The latter proof is in spirit aligned with the development in the present paper, as it rested on understanding the joint distribution of the Busemann functions from [20].

Another focus of research has been the coalescence structure of geodesics. Ferrari and Pimentel [21] imported the Licea–Newman approach [39,43] to exponential LPP on the lattice. Later Pimentel [47] developed a probabilistic duality between the coalescence time of two semi-infinite geodesics and their last exit times from the initial boundary, again in exponential LPP. A key idea was the equality in distribution of the tree of directed semi-infinite geodesics and the dual tree of southwest-directed geodesics. From this, he obtained a lower bound on tail probabilities of the coalescence time for two geodesics starting from $(-\lfloor k^{2/3}\rfloor, \lfloor k^{2/3}\rfloor)$ and $(\lfloor k^{2/3}\rfloor, -\lfloor k^{2/3}\rfloor)$. Coalescence bounds have seen significant recent improvement in [8] and [54]. A study of the coalescence structure of finite geodesics in BLPP was undertaken by Hammond in four papers [29–32]. The existence of semi-infinite geodesics in the Airy line ensemble, for a countable dense set of directions and initial points, was proven recently by Sarkar and Virág in [51].

Hoffman was the first to use ideas of Busemann functions to study geodesics in first-passage percolation. In [36] he showed the existence of disjoint semi-infinite geodesics. Later, Damron and Hanson [16] constructed generalized Busemann functions from weak subsequential limits of first-passage times. This allowed them to develop results for semi-infinite geodesics under weaker assumptions than the global curvature. Under the assumption that the limit shape is strictly convex and differentiable, they proved that every semi-infinite geodesic has an asymptotic direction and that, in every direction, there exists a semi-infinite geodesic out of every lattice point.

On the side of discrete last-passage percolation with general i.i.d. weights, Georgiou, Rassoul-Agha, and the first author [23,24] used the stationary LPP process to prove existence of Busemann functions under mild moment conditions. The Busemann functions were then used to construct semi-infinite geodesics. They also showed that if the shape function

is strictly concave, every semi-infinite geodesic has an asymptotic direction. Further in this direction, in [53] the first author introduced a new proof of the coalescence of semi-infinite geodesics that utilizes the stationary LPP process.

Our work is related to this last approach. We use Busemann functions to construct and study semi-infinite geodesics in BLPP. Existence of Busemann functions in BLPP, for fixed initial points and directions, was recently established by Alberts, Rassoul-Agha and Simper [2], along with Busemann functions and infinite polymer measures for the semi-discrete Brownian polymer. Their BLPP result is the starting point of our study.

1.4. *New techniques and phenomena in the semi-discrete model*

The present paper develops the global setting of Busemann functions and semi-infinite geodesics in the semi-discrete BLPP, with a view to future study of their finer properties. The novelty lies in going beyond the discrete set-up. In discrete last-passage percolation, one can prove a probability-one statement about semi-infinite geodesics out of a fixed initial point, and then that statement extends to all initial points by a simple union bound. This is not the case in BLPP. To overcome this difficulty, we need new methods of proof. Additionally, the continuum of points gives rise to new results regarding non-uniqueness of semi-infinite geodesics, as seen in Item (3) below.

Specific items proved in this paper include the following, and are recorded as Theorems 3.1 and 3.5.

1. The Busemann functions of BLPP from [2] are extended to a global process on a single event of probability one, for all the uncountably many initial points and directions.
2. Once the Busemann process is in place, we use it to construct semi-infinite geodesics for BLPP. With probability one, there exists a family of semi-infinite geodesics, starting from each initial point and in each asymptotic direction.
3. With probability one, all semi-infinite geodesics, whether constructed by the Busemann functions or not, have an asymptotic direction. For a fixed initial point and direction, there is almost surely a unique semi-infinite geodesic starting from the given point and traveling asymptotically in the given direction. We also show that for a fixed direction, there is a countably infinite set of initial points whose geodesic in that direction is not unique. This non-uniqueness into a fixed direction is a new phenomenon that is not present in exponential last-passage percolation on the lattice.
4. For each fixed direction, we prove that all semi-infinite geodesics, traveling in that common direction, coalesce.
5. For fixed northeast and southwest directions, we prove the almost sure nonexistence of bi-infinite geodesics in those directions.

To construct an infinite up-right path on the lattice $\mathbb{Z}^2$, one chooses at each step whether to move upward or to the right. In the BLPP setting, one chooses a real-valued increment to the right and then takes a unit-size upward step. Section 4.1 explains informally this construction. The locations of the upward steps of a Busemann geodesic are determined by a variational problem for Brownian motion with drift. This formulation is significant in at least two ways. First, this step is where non-uniqueness of geodesics can arise. Understanding it involves properties of Brownian paths. Furthermore, this variational construction can be potentially and profitably adapted to other models. This includes both continuum models of the KPZ class, such as the directed landscape, and lattice LPP. We address the latter in Section 5.2.

The proof of the coalescence in Item (4) above requires technical novelties. The underlying idea from [53] is to construct a dual environment from the original environment and a Busemann function. If two geodesics in a given direction do not coalesce, there exists a bi-infinite geodesic in the dual environment with given northwest and southeast directions. Then, it is proven that there are almost surely no bi-infinite geodesics in fixed directions.

A key point is a non-intersection property between the original semi-infinite geodesics and southwest-travelling semi-infinite geodesics in the dual environment. This is in Theorem 7.12, which is the analogue of Lemma 4.4 in [53]. The dual environment in BLPP is constructed through dual queuing mappings, first presented in [45] and further studied in [2,56]. See Equation (3.9) for the precise definition. While the general approach is not new, our contribution comes first from Theorem D.1 that shows that joint queuing mappings can be inverted by reverse-time queuing mappings. Then, we use this theorem and variational formulas to construct southwest-travelling semi-infinite geodesics in the dual environment. Lastly, the non-intersection of Theorem 7.12 is proved by comparing the maximizers of the variational formulas.

1.5. *Future work*

The present paper lays the foundation for future work on Busemann functions and semi-infinite geodesics, which most immediately will cover the following.

1. To study the full family of semi-infinite geodesics in BLPP, we derive the joint distribution of Busemann functions across all directions. The analogous result was achieved in [20] for the exponential CGM, which led to important advances in the global structure of geodesic trees [38], coalescence [54], convergence [4,13], and bi-infinite geodesics [3]. We likewise study the geometry of geodesics in BLPP and give a full characterization of exceptional directions where geodesics do not coalesce, in terms of the Busemann process.

2. We define the competition interface for each initial point of BLPP. This has a straightforward definition in discrete models, but requires more care in the semi-discrete setting. Specifically, we show that, on each level m of the semi-discrete space $\mathbb{Z} \times \mathbb{R}$, outside a random set of Hausdorff dimension $\frac{1}{2}$, all points (m, t) have a degenerate competition interface. For each of the points in this set of Hausdorff dimension $\frac{1}{2}$, there exists a random direction $\theta^\star$ such that there are two semi-infinite geodesics starting at (m, t) in direction $\theta^\star$ that only share the common initial point.

As can be seen from Item (2), there are many rich questions to explore related to non-uniqueness of semi-infinite geodesics in Brownian last-passage percolation. Other questions related to Hausdorff dimension have recently been explored in the KPZ fixed point and the directed landscape in [6,9,15,22]. Our longer-term goal is to extend the approach of this paper to the continuum models of the KPZ class.

1.6. *Organization of the paper*

Section 2 defines the Brownian last-passage percolation model and related objects. We also state results from other papers, including the existence of Busemann functions from [2]. Section 3 contains the main results on the existence and properties of the global Busemann process and semi-infinite geodesics. Section 4 states more technical theorems related to the specific construction of semi-infinite geodesics from the Busemann functions. In Section 5, we discuss connections to infinite polymer measures in the O'Connell–Yor polymer, semi-infinite geodesics in discrete last-passage percolation, and to queuing theory. The proofs of this paper are contained in Sections 6 and 7. Section 6 constructs the global Busemann process and derives its properties. All results for semi-infinite geodesics are proved in Section 7. The Appendices collect some standard material, background from the literature, and some technical theorems.

2. Definitions and previous results

2.1. *Preliminaries*

The following notation and conventions are used throughout the paper.

(i) For a function $f : \mathbb{R} \to \mathbb{R}$, we write $f(s, t) = f(t) - f(s)$ and $\widetilde{f}(t) = -f(-t)$.
(ii) $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are restricted by subscripts, as in for example $\mathbb{Z}_{>0} = \{1, 2, 3, \ldots\}$.
(iii) Whenever $m \leq n \in \mathbb{Z}$ and $s \leq t \in \mathbb{R}$, we say that $(m, s) \leq (n, t)$.
(iv) Let $X \sim \mathcal{N}(\mu, \sigma^2)$ indicate that the random variable X has normal distribution with mean μ and variance σ^2. For $\alpha > 0$, let $X \sim \mathrm{Exp}(\alpha)$ indicate that X has exponential distribution with rate α, or equivalently, mean α^{-1}.
(v) Equality in distribution between random variables and processes is denoted by $\overset{d}{=}$.
(vi) A two-sided Brownian motion is a continuous random process $\{B(t) : t \in \mathbb{R}\}$ such that $B(0) = 0$ almost surely and such that $\{B(t) : t \geq 0\}$ and $\{B(-t) : t \geq 0\}$ are two independent standard Brownian motions on $[0, \infty)$.
(vii) For $\lambda \in \mathbb{R}$, $\{Y(t) : t \in \mathbb{R}\}$ is a two-sided Brownian motion with drift λ if the process $\{Y(t) - \lambda t : t \in \mathbb{R}\}$ is a two-sided Brownian motion.
(viii) The square $\square$ as a superscript represents a sign $+$ or $-$.

2.2. *Geodesics in Brownian last-passage percolation*

The Brownian last-passage process is defined as follows. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\mathbf{B} = \{B_r\}_{r \in \mathbb{Z}}$ be a field of independent, two-sided Brownian motions. For $(m, s) \leq (n, t)$, define the set

$$\Pi_{(m,s),(n,t)} := \big\{ \mathbf{s}_{m,n} = (s_{m-1}, s_m, \ldots, s_n) \in \mathbb{R}^{n-m+2} : s = s_{m-1} \leq s_m \leq \cdots \leq s_n = t \big\}.$$

Denote the energy of a sequence $\mathbf{s}_{m,n} \in \Pi_{(m,s),(n,t)}$ by

$$(2.1) \qquad\qquad \mathcal{E}(\mathbf{s}_{m,n}) = \sum_{r=m}^{n} B_r(s_{r-1}, s_r).$$

Now, for $\mathbf{x} = (m, s) \leq (n, t) = \mathbf{y}$, define the Brownian last-passage time as

$$(2.2) \qquad\qquad L_{\mathbf{x},\mathbf{y}}(\mathbf{B}) = \sup\big\{ \mathcal{E}(\mathbf{s}_{m,n}) : \mathbf{s}_{m,n} \in \Pi_{\mathbf{x},\mathbf{y}} \big\}.$$

Whenever the specific field of Brownian motions used is either clear from or not important in the context, we write $L_{(m,s),(n,t)}$.
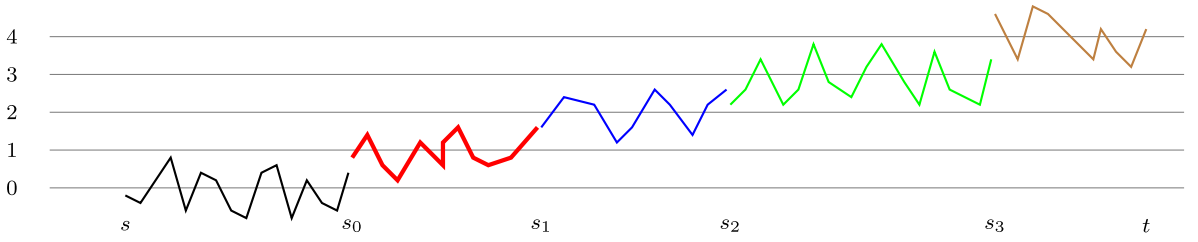
Fig. 1. The Brownian increments $B_r(s_{r-1}, s_r)$ for $r = 0, \ldots, 4$ in (2.1) that make up the energy of the path depicted in Figure 2.
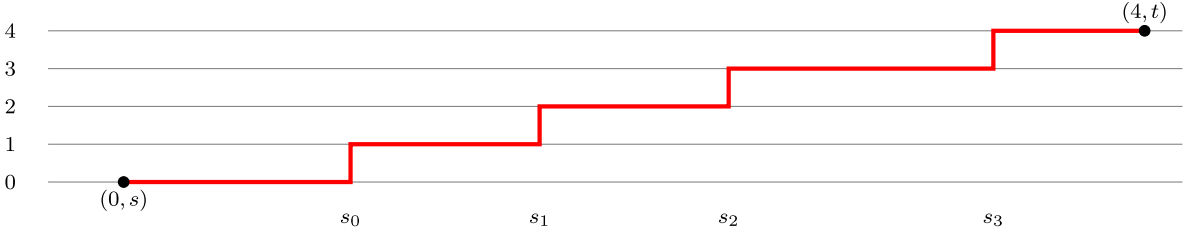

Fig. 2. Example of a planar path from $(0, s)$ to $(4, t)$, represented by the sequence $(s = s_{-1}, s_0, s_1, s_2, s_3, s_4 = t) \in \Pi_{(0,s),(4,t)}$.

The elements of $\Pi_{\mathbf{x},\mathbf{y}}$ are in bijection with paths in $\mathbb{R}^2$ between $\mathbf{x}$ and $\mathbf{y}$, that move to the right in real-valued increments, and move upwards in integer increments. For a given $\mathbf{s}_{m,n} \in \Pi_{(m,s),(n,t)}$, the path consists of the following points:

$$(2.3) \qquad \bigcup_{r=m}^{n} \{(r, u) : u \in [s_{r-1}, s_r]\} \cup \bigcup_{r=m}^{n-1} \{(v, s_r) : v \in [r, r+1]\}.$$

This set consists of horizontal and vertical line segments, such that the vertical segments occur at the points s_r. Because of this bijection, we sometimes say $\Gamma \in \Pi_{\mathbf{x},\mathbf{y}}$ for such an up-right path. For $(m, t) \in \mathbb{Z} \times \mathbb{R}$, we graphically represent the t-coordinate as the horizontal coordinate and the m-coordinate as the vertical coordinate in the plane. Since $\Pi_{\mathbf{x},\mathbf{y}}$ is a compact set and Brownian motion is continuous, on a single event of probability one, for all $(m, t) = \mathbf{x} \le \mathbf{y} = (n, t) \in \mathbb{Z} \times \mathbb{R}$, there exists a sequence $\mathbf{s}_{m,n} \in \Pi_{\mathbf{x},\mathbf{y}}$ such that $\mathcal{E}(\mathbf{s}_{m,n}) = L_{\mathbf{x},\mathbf{y}}$. The associated path is called a *geodesic* between the points.

To an infinite sequence, $s = s_{m-1} \le s_m \le s_{m+1} \le \cdots$ we similarly associate a semi-infinite path. It is possible that $s_r = \infty$ for some $r \ge m$, in which case the last segment of the path is the ray $[s_{r-1}, \infty) \times \{r\}$, where r is the first index with $s_r = \infty$. The infinite path has *direction* $\theta \in [0, \infty]$ or is θ-*directed* if

$$\lim_{n \to \infty} \frac{s_n}{n} \quad \text{exists and equals } \theta.$$

We call an up-right semi-infinite path a *semi-infinite geodesic* if, for any two points $\mathbf{x} \le \mathbf{y} \in \mathbb{Z} \times \mathbb{R}$ that lie along the path, the portion of the path between the two points is a geodesic between the two points. Similarly, a *bi-infinite geodesic* is a bi-infinite path that forms a geodesic between any two of its points. Two up-right, semi-infinite paths *coalesce* if there exists $\mathbf{z} \in \mathbb{Z} \times \mathbb{R}$ such that the two paths agree above and to the right of $\mathbf{z}$, as shown in Figure 3. Alternatively, if the paths are defined by sequences of jump times $s^1 = s^1_{m_1-1} \le s^1_{m_1} \cdots$ and $s^2 = s^2_{m_2-1} \le s^2_{m_2} \le \cdots$, then the two paths coalesce if and only if there exists $N \in \mathbb{Z}$ such that $s^1_r = s^2_r$ for all $r \ge N$.

The following lemma, due to Hammond [31], establishes uniqueness of geodesics for a fixed initial and terminal point.

Lemma 2.1 ([31], Theorem B.1). *Fix endpoints $\mathbf{x} \le \mathbf{y} \in \mathbb{Z} \times \mathbb{R}$. Then, there is almost surely a unique path whose energy achieves $L_{\mathbf{x},\mathbf{y}}(\mathbf{B})$.*

However, it is also true that for each fixed initial point $\mathbf{x} \in \mathbb{Z} \times \mathbb{R}$, with probability one, there exist points $\mathbf{y} \ge \mathbf{x}$, such that the geodesic between $\mathbf{x}$ and $\mathbf{y}$ is not unique. Hence, the following lemma is important for our understanding. It is a deterministic statement which holds for last-passage percolation across any field of continuous functions.
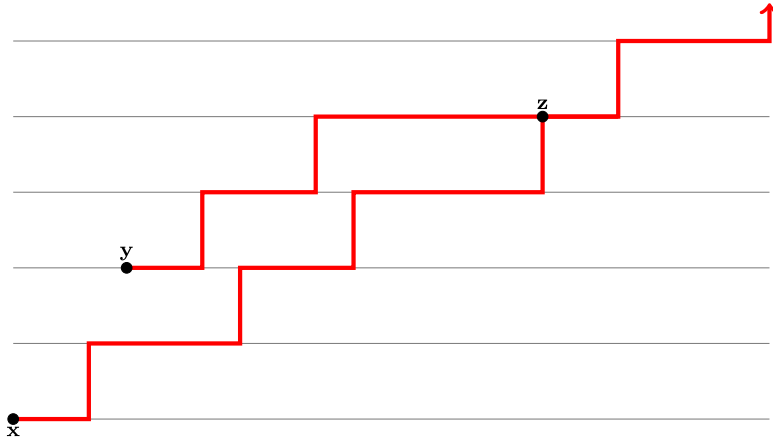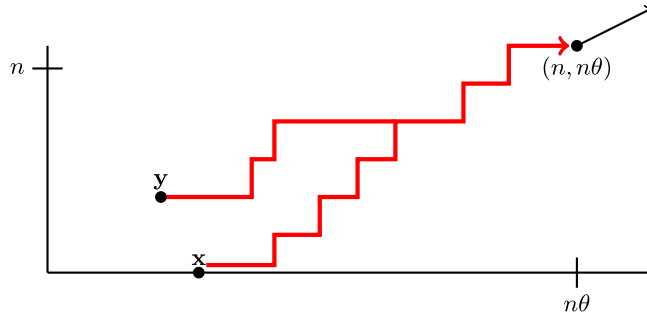
Fig. 3. Coalescence of semi-infinite paths.

Fig. 4. Two geodesics with a common terminal point $(n, n\theta)$.

Lemma 2.2 ([19], Lemma 3.5). *Between any two points $(m, s) \leq (n, t) \in \mathbb{Z} \times \mathbb{R}$, there is a rightmost and a leftmost Brownian last-passage geodesic. That is, there exist $\mathbf{s}_{m,n}^L, \mathbf{s}_{m,n}^R \in \mathbf{\Pi}_{(m,s),(n,t)}$, that are maximal for $\mathcal{E}(\mathbf{s}_{m,n})$, such that, for any other maximal sequence $\mathbf{s}_{m,n}$, $s_r^L \leq s_r \leq s_r^R$ holds for $m \leq r \leq n$.*

2.3. Busemann functions

In the present paper, semi-infinite geodesics are constructed from Busemann functions. Busemann functions are defined to be the asymptotic difference of last-passage times from two different starting points to a common terminal point that is traveling to ∞ in a given direction. See Figure 4. The direction is indexed by a parameter $\theta > 0$. Existence of these Busemann functions was proven in [2], both in the positive temperature and zero temperature cases. We state the zero-temperature result:

Theorem 2.3 ([2], Theorem 4.2). *Fix $\theta > 0$ and $\mathbf{x}, \mathbf{y} \in \mathbb{Z} \times \mathbb{R}$. Then, there exists a random variable $\mathcal{B}^\theta(\mathbf{x}, \mathbf{y})$ and an event $\Omega_{\mathbf{x}, \mathbf{y}}^{(\theta)}$ of probability one, on which*

$$(2.4) \qquad\qquad \mathcal{B}^\theta(\mathbf{x}, \mathbf{y}) = \lim_{n \to \infty} [L_{\mathbf{x}, (n, t_n)} - L_{\mathbf{y}, (n, t_n)}]$$

holds for any sequence $\{t_n\} \subseteq \mathbb{R}$ satisfying $t_n/n \to \theta$. Further, if

$$(2.5) \qquad\qquad v_m^\theta(t) := \mathcal{B}^\theta\big((m-1, t), (m, t)\big), \quad and$$

$$(2.6) \qquad\qquad h_m^\theta(t) := \mathcal{B}^\theta\big((m, 0), (m, t)\big),$$

then $v_m^\theta(t) \sim \text{Exp}(\frac{1}{\sqrt{\theta}})$ and $h_m^\theta(s, t) \sim \mathcal{N}(\frac{t-s}{\sqrt{\theta}}, |t - s|)$ for all $s, t \in \mathbb{R}$ and $m \in \mathbb{Z}$.

3. Main results

3.1. *Semi-infinite geodesics*

The following result summarizes the contributions of this paper related to semi-infinite geodesics in BLPP.

Theorem 3.1. *The following hold*

(i) *With probability one, for every initial point $\mathbf{x} \in \mathbb{Z} \times \mathbb{R}$ and every direction $\theta > 0$, there exists a θ-directed semi-infinite geodesic starting from $\mathbf{x}$.*

(ii) (*Uniqueness for fixed initial points and directions*) *For each fixed $\mathbf{x} \in \mathbb{Z} \times \mathbb{R}$ and $\theta > 0$, there exists an event, $\Omega_{\mathbf{x}}^{(\theta)}$, of probability one, on which there is exactly one θ-directed semi-infinite geodesic starting from $\mathbf{x}$.*

(iii) (*Non-uniqueness for fixed direction and random initial points*) *For each $\theta > 0$, there exists an event $\widetilde{\Omega}^{(\theta)}$, of probability one, on which the set*

$$\{\mathbf{x} \in \mathbb{Z} \times \mathbb{R} : \text{the } \theta\text{-directed semi-infinite geodesic starting from } \mathbf{x} \text{ is not unique}\}$$

is countably infinite. For every (m, t) in this set, at most one of the θ-directed semi-infinite geodesics passes through the point $(m, t + \varepsilon)$ for some $\varepsilon > 0$. All the others pass through $(m + 1, t)$.

(iv) *With probability one, every semi-infinite geodesic is θ-directed for some $\theta \in [0, \infty]$. That is, for any infinite sequence $t = t_{m-1} \le t_m \le t_{m+1} \le \cdots$ defining a semi-infinite geodesic starting from some point $(m, t) \in \mathbb{Z} \times \mathbb{R}$, the limit*

$$\lim_{n \to \infty} \frac{t_n}{n} \quad \text{exists in } [0, \infty]. \tag{3.1}$$

(v) *With probability one, if for any $(m, t) \in \mathbb{Z} \times \mathbb{R}$ and any such sequence, the limit (3.1) equals 0, then $t_r = t$ for all $r \ge m$. Similarly, if this limit is ∞, then $t_m = t_{m+1} = \cdots = \infty$. That is, the only semi-infinite geodesics that are asymptotically vertical or horizontal are trivial (i.e. straight lines).*

(vi) (*Non-existence of bi-infinite geodesics for fixed directions*) *Fix $\theta, \eta > 0$. Then, there exists an event, $\Omega^{(\theta, \eta)}$, of probability one, on which there are no bi-infinite geodesics defined by jump times $\cdots \le \tau_{-1} \le \tau_0 \le \tau_1 \le \cdots$ such that*

$$\lim_{n \to \infty} \frac{\tau_n}{n} = \theta \quad \text{and} \quad \lim_{n \to \infty} \frac{\tau_{-n}}{n} = -\eta. \tag{3.2}$$

(vii) (*Coalescence of geodesics in a fixed direction*) *For each $\theta > 0$, there exists an event $\widehat{\Omega}^{(\theta)}$, of probability one, on which all θ-directed semi-infinite geodesics coalesce.*

Remark 3.2. As discussed in the introduction, the non-uniqueness stated in Part (iii) is a new phenomenon that arises from the semi-discrete nature of the model. We refer the reader to Section 4.3 for further discussion on non-uniqueness.

Theorem 3.1(vi) is proven by first solving the "midpoint problem." This problem first appeared in the context of first-passage percolation in a paper of Benjamini, Kalai, and Schramm [10]. In that context the problem asks whether

$$\lim_{n \to \infty} \mathbb{P}\left(\left\lfloor \frac{n}{2} \right\rfloor e_1 \text{ lies on some geodesic between } 0 \text{ and } ne_1 \right) = 0.$$

In 2016, Damron and Hanson [17] proved that this is true, under the assumption that the boundary of the limit shape is differentiable. Later, Ahlberg and Hoffman [1] proved this result without the assumption of differentiability. The following formulation of the midpoint problem more closely matches that for exponential last-passage percolation in the arXiv version of [53], Theorem 4.12.

Lemma 3.3 (Midpoint problem). *Let $\theta, \eta > 0$ and $(m, t) \in \mathbb{Z} \times \mathbb{R}$. Then, the following subset of Ω is contained in an event of probability zero*:

$$\left\{ \text{there exists a sequence } \{t_n\}_{n \in \mathbb{Z}} \text{ satisfying } \lim_{n \to \infty} \frac{t_n}{n} = \theta \text{ and } \lim_{n \to \infty} \frac{t_{-n}}{-n} = \eta \text{ and} \right.$$

$$\left. \text{such that, for each } n \in \mathbb{Z}_{>0}, \text{ some geodesic between } (-n, t_{-n}) \text{ and } (n, t_n) \text{ passes through } (m, t) \right\}.$$

3.2. *Existence and properties of the Busemann process*

To prove Theorem 3.1, we extend the individual Busemann functions of Theorem 2.3 to a global *Busemann process*. The following transformations are used to understand the structure of this process. For functions $Z, B : \mathbb{R} \to \mathbb{R}$ satisfying $Z(0) = B(0) = 0$ and $\limsup_{s \to \infty}(B(s) - Z(s)) = -\infty$, define

$$(3.3) \qquad Q(Z, B)(t) = \sup_{t \leq s < \infty} \big\{ B(t, s) - Z(t, s) \big\},$$

$$(3.4) \qquad D(Z, B)(t) = Z(t) + Q(Z, B)(0) - Q(Z, B)(t),$$

$$(3.5) \qquad R(Z, B)(t) = B(t) + Q(Z, B)(t) - Q(Z, B)(0).$$

Reverse-time analogues of these transformations are defined for continuous functions $Y, C : \mathbb{R} \to \mathbb{R}$ satisfying $Y(0) = C(0) = 0$ and $\limsup_{s \to -\infty}(Y(s) - C(s)) = -\infty$:

$$(3.6) \qquad \overleftarrow{Q}(Y, C)(t) = \sup_{-\infty < s \leq t} \big\{ C(s, t) - Y(s, t) \big\},$$

$$(3.7) \qquad \overleftarrow{D}(Y, C)(t) = Y(t) + \overleftarrow{Q}(Y, C)(t) - \overleftarrow{Q}(Y, C)(0),$$

$$(3.8) \qquad \overleftarrow{R}(Y, C)(t) = C(t) + \overleftarrow{Q}(Y, C)(0) - \overleftarrow{Q}(Y, C)(t).$$

These transformations originate from the Brownian queue, first studied by Glynn and Whitt [26], and further expounded on by Harrison and Williams [33–35] and O'Connell and Yor [45]. See Section 5.3 and Appendix C for more about the queuing interpretation. The following lemma is a straightforward exercise. We state it for completeness, as we will refer to it later in the paper.

Lemma 3.4. *Let* $B, Z : \mathbb{R} \to \mathbb{R}$ *be continuous functions satisfying* $\limsup_{s \to \infty}(B(s) - Z(s)) = -\infty$. *Then,* $Q(Z, B)$, $D(Z, B)$, *and* $R(Z, B)$ *are continuous. Similarly, if* $Y, C : \mathbb{R} \to \mathbb{R}$ *are continuous functions satisfying* $\limsup_{s \to -\infty}(Y(s) - C(s)) = -\infty$, *then* $\overleftarrow{Q}(Y, C)$, $\overleftarrow{D}(Y, C)$, *and* $\overleftarrow{R}(Y, C)$ *are continuous.*

The next theorem summarizes the existence and properties of the Busemann process. This process has discontinuities in the direction parameter θ. Instead of a single cadlag process, it is useful to retain both a left- and a right-continuous version indicated by $\theta-$ and $\theta+$ because this distinction captures spatial limits.

Theorem 3.5. *There exists a process,*

$$\big\{ \mathcal{B}^{\theta \square}(\mathbf{x}, \mathbf{y}) : \theta > 0, \square \in \{+, -\}, \mathbf{x}, \mathbf{y} \in \mathbb{Z} \times \mathbb{R} \big\},$$

and for $\theta > 0$, *there exist events* $\Omega^{(\theta)} \subseteq \Omega_1 \subseteq \Omega$, *each of probability one, such that the following hold. Here,* $v_m^{\theta \square}$, $h_m^{\theta \square}$ *are defined as in* (2.5) *and* (2.6), *placing* $\square$ *in the appropriate superscripts.*

(i) (*Additivity*) *On* Ω_1, *whenever* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in (\mathbb{Z} \times \mathbb{R})$, $\theta > 0$, *and* $\square \in \{+, -\}$,

$$\mathcal{B}^{\theta \square}(\mathbf{x}, \mathbf{y}) + \mathcal{B}^{\theta \square}(\mathbf{y}, \mathbf{z}) = \mathcal{B}^{\theta \square}(\mathbf{x}, \mathbf{z}).$$

(ii) (*Monotonicity*) *On* Ω_1, *whenever* $0 < \gamma < \theta < \infty$, $m \in \mathbb{Z}$, *and* $s < t \in \mathbb{R}$,

$$0 \leq v_m^{\gamma -}(s) \leq v_m^{\gamma +}(s) \leq v_m^{\theta -}(s) \leq v_m^{\theta +}(s), \quad and$$

$$B_m(s, t) \leq h_m^{\theta +}(s, t) \leq h_m^{\theta -}(s, t) \leq h_m^{\gamma +}(s, t) \leq h_m^{\gamma -}(s, t).$$

(iii) (*Convergence*) *On* Ω_1, *for every* $m \in \mathbb{Z}$, $\theta > 0$ *and* $\square \in \{+, -\}$,
  (a) *As* $\gamma \nearrow \theta$, $h_m^{\gamma \square}$ *and* $v_m^{\gamma \square}$ *converge uniformly, on compact subsets of* $\mathbb{R}$, *to* $h_m^{\theta -}$ *and* $v_m^{\theta -}$, *respectively.*
  (b) *As* $\delta \searrow \theta$, $h_m^{\delta \square}$ *and* $v_m^{\delta \square}$ *converge uniformly, on compact subsets of* $\mathbb{R}$, *to* $h_m^{\theta +}$ *and* $v_m^{\theta +}$, *respectively.*
  (c) *As* $\gamma \to \infty$, $h_m^{\gamma \square}$ *converges uniformly, on compact subsets of* $\mathbb{R}$, *to* B_m.
  (d) *As* $\delta \searrow 0$, $v_m^{\delta \square}$ *converges uniformly, on compact subsets of* $\mathbb{R}$, *to* 0.
(iv) (*Continuity*) *On* Ω_1, *for any* $r, m \in \mathbb{Z}$, $\theta > 0$, *and* $\square \in \{+, -\}$, $(s, t) \mapsto \mathcal{B}^{\theta \square}((m, s), (r, t))$ *is a continuous function* $\mathbb{R}^2 \to \mathbb{R}$.

(v) (*Limits*) *On* Ω_1, *for each* $\theta > 0$ *and* $\square \in \{+, -\}$,

$$\lim_{s \to \pm\infty} \left[B_m(s) - h_{m+1}^{\theta\square}(s) \right] = \mp\infty.$$

(vi) (*Queuing relationships between Busemann functions*) *On* Ω_1, *for all* $m \in \mathbb{Z}$, $\theta > 0$, *and signs* $\square \in \{+, -\}$,

$$v_{m+1}^{\theta\square} = Q\left(h_{m+1}^{\theta\square}, B_m\right) \quad \text{and} \quad h_m^{\theta\square} = D\left(h_{m+1}^{\theta\square}, B_m\right).$$

(vii) (*Independence*) *For any* $m \in \mathbb{Z}$,

$$\left\{ h_r^{\theta\square} : \theta > 0, \square \in \{+, -\}, r > m \right\} \text{ is independent of } \{B_r : r \leq m\}$$

(viii) (*Equality for fixed directions*) *Fix* $\theta > 0$. *Then, on the event* $\Omega^{(\theta)}$, *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{Z} \times \mathbb{R}$ *and all sequences* $\{t_n\}$ *with* $t_n/n \to \theta$,

$$\mathcal{B}^{\theta-}(\mathbf{x}, \mathbf{y}) = \lim_{n \to \infty} [L_{\mathbf{x},(n,t_n)} - L_{\mathbf{y},(n,t_n)}] = \mathcal{B}^{\theta+}(\mathbf{x}, \mathbf{y}).$$

(ix) (*Shift invariance*) *For each* $\mathbf{z} \in \mathbb{Z} \times \mathbb{R}$,

$$\left\{ \mathcal{B}^{\theta\square}(\mathbf{x}, \mathbf{y}) : \mathbf{x}, \mathbf{y} \in \mathbb{Z} \times \mathbb{R}, \theta > 0, \square \in \{+, -\} \right\} \overset{d}{=} \left\{ \mathcal{B}^{\theta\square}(\mathbf{x} + \mathbf{z}, \mathbf{y} + \mathbf{z}) : \mathbf{x}, \mathbf{y} \in \mathbb{Z} \times \mathbb{R}, \theta > 0, \square \in \{+, -\} \right\}.$$

Remark 3.6. On account of Part (viii), when working on the event $\Omega^{(\theta)}$ we write $\mathcal{B}^{\theta} = \mathcal{B}^{\theta-} = \mathcal{B}^{\theta+}$. The fact that the limits exist for all initial points $\mathbf{x}, \mathbf{y} \in \mathbb{Z} \times \mathbb{R}$ and fixed $\theta > 0$ on a single event of probability one gives a generalization of Theorem 2.3. As will be seen in Section 6, the key is Lemma C.5, which generalizes a proof in [2].

We introduce a dual field of Brownian motions used later in the proof of coalescence of semi-infinite geodesics. Let $\theta > 0$. Recall the definition (3.5) of the mapping R, and on the event $\Omega^{(\theta)}$, set

$$(3.9) \qquad\qquad\qquad X_m^{\theta} = R\left(h_m^{\theta}, B_{m-1}\right).$$

Denote the field of these random functions by $\mathbf{X}^{\theta} := \{X_m^{\theta}\}_{m \in \mathbb{Z}}$. The following theorem describes the distribution of the Busemann functions for a fixed direction θ.

Theorem 3.7. *Fix* $\theta > 0$.

(i) *The process* $t \mapsto h_m^{\theta}(t)$ *is a two-sided Brownian motion with drift* $\frac{1}{\sqrt{\theta}}$.

(ii) $\mathbf{X}^{\theta}$ *is a field of independent two-sided Brownian motions. For each* $n \in \mathbb{Z}$, $\{X_m^{\theta}\}_{m > n}$ *is independent of* $\{h_m^{\theta}\}_{m \leq n}$.

(iii) *The process* $t \mapsto v_m^{\theta}(t)$ *is a stationary and reversible strong Markov process. For each* $t \in \mathbb{R}$, $v_m^{\theta}(t) \sim \mathrm{Exp}(\frac{1}{\sqrt{\theta}})$.

(iv) (*Burke property*) *Fix* $m < n \in \mathbb{Z}$ *and* $-\infty < t_n \leq t_{n-1} \leq \cdots \leq t_{m+1} < \infty$. *Then, the following random variables and processes are mutually independent*:

$$\left\{ B_m(u, t_{m+1}) : u \leq t_{m+1} \right\}, \qquad v_{m+1}^{\theta}(t_{m+1}), \qquad \left\{ h_m^{\theta}(t_{m+1}, u) : u \geq t_{m+1} \right\},$$

$$\left\{ B_r(u, t_{r+1}) : u \leq t_{r+1} \right\}, \qquad v_{r+1}^{\theta}(t_{r+1}), \qquad \left\{ h_r^{\theta}(t_{r+1}, u) : t_{r+1} \leq u \leq t_r \right\},$$

$$\text{and} \quad \left\{ X_r^{\theta}(t_r, u) : u \geq t_r \right\}, \quad \text{for } m + 1 \leq r \leq n - 1, \qquad \left\{ h_n^{\theta}(u, t_n) : u \leq t_n \right\}, \quad \text{and} \quad \left\{ X_n^{\theta}(t_n, u) : u \geq t_n \right\}.$$

As a special case, $\{v_m^{\theta}(t)\}_{m \in \mathbb{Z}}$ *is an i.i.d. sequence for each* $t \in \mathbb{R}$. *Refer to Figure 5 for clarity.*

Remark 3.8. Using the representation of Equation (3.12) below, many formulas for the process $t \mapsto v_m^{\theta}(t)$ are well known. See [11], page 129 and [50], specifically equations (4) and (5), for more on this process, including the transition density. However, we caution that when comparing formulas, in the setting of [11,50], the process is

$$t \mapsto \sup_{-\infty < s \leq t} \left\{ B(s, t) - \mu(t - s) \right\},$$

where B is a two-sided Brownian motion and $\mu > 0$. In our setting, there is a factor of $\sqrt{2}$ multiplied to B.
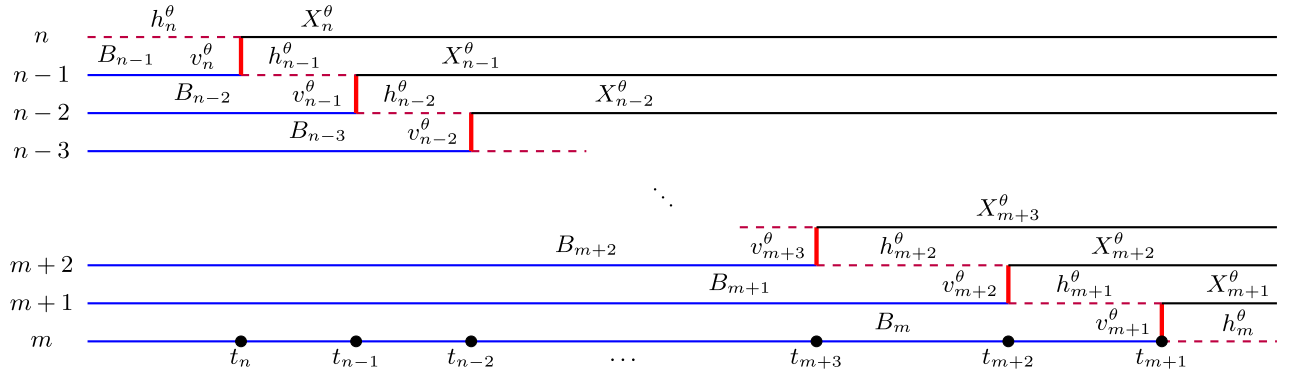
Fig. 5. Independence structure for Busemann functions. Each process h_r^θ is associated to the (purple/dotted) segment on level r, processes B_r and X_r^θ cover the remaining portions of horizontal level r, and the process v_r^θ is associated to the (red/vertical) edge from level $r-1$ to r at time point t_r.

The construction of the Busemann process and the proof of Theorem 3.5 can be found in Section 6. We prove Theorem 3.7 here, assuming Theorem 3.5 and with the help of the results of the appendix.

Proof of Theorem 3.7. *Part (i)*: As will be seen from the construction in Section 6, $t \mapsto h_m^\theta(t)$ has the proper finite-dimensional distributions and is continuous by Theorem 3.5(iv).

Part (ii): Fix integers $p < n < m$. By Theorem 3.5(vii),

$$\left(B_p, \ldots, B_n, \ldots, B_{m-2}, B_{m-1}, h_m^\theta\right) \quad \text{are independent.}$$

Theorem 3.5(vi) and definition (3.9) give $h_{m-1}^\theta = D(h_m^\theta, B_{m-1})$ and $X_m^\theta = R(h_m^\theta, B_{m-1})$, so by Theorem C.2, h_{m-1}^θ and X_m^θ are independent, and X_m^θ is a two-sided Brownian motion. In particular, now

$$\left(B_p, \ldots, B_n, \ldots, B_{m-2}, h_{m-1}^\theta, X_m^\theta\right) \quad \text{are independent.}$$

Continue inductively by applying the transformation (D, R) to successive pairs (B_{j-1}, h_j^θ) for $j = m-1, m-2, \ldots, n+1$ after which

$$\left(B_p, \ldots, B_{n-1}, h_n^\theta, X_{n+1}^\theta, \ldots, X_m^\theta\right) \quad \text{are independent.}$$

To conclude, note that $(h_p^\theta, \ldots, h_n^\theta)$ is a function of $(B_p, \ldots, B_{n-1}, h_n^\theta)$ through iteration of $h_k^\theta = D(h_{k+1}^\theta, B_k)$.

Part (iii): By Theorem 3.5(vi) and (3.9), on the event $\Omega^{(\theta)}$ we have these relations $\forall m \in \mathbb{Z}$:

$$(3.10) \qquad h_{m-1}^\theta = D\left(h_m^\theta, B_{m-1}\right), \qquad X_m^\theta = R\left(h_m^\theta, B_{m-1}\right), \quad \text{and} \quad v_m^\theta = Q\left(h_m^\theta, B_{m-1}\right).$$

The fact that $v_m^\theta(t)$ is exponential with rate $\frac{1}{\sqrt{\theta}}$ then follows from Lemma B.1. Theorem D.1 allows us to reverse these mappings, so $\forall m \in \mathbb{Z}$:

$$(3.11) \qquad h_m^\theta = \overleftarrow{D}\left(h_{m-1}^\theta, X_m^\theta\right), \qquad B_{m-1} = \overleftarrow{R}\left(h_{m-1}^\theta, X_m^\theta\right), \quad \text{and} \quad v_m^\theta = \overleftarrow{Q}\left(h_{m-1}^\theta, X_m^\theta\right).$$

Then, for $t \in \mathbb{R}$,

$$(3.12) \qquad v_m^\theta(t) = \sup_{-\infty < u \le t} \left\{X_m^\theta(u, t) - h_{m-1}^\theta(u, t)\right\}.$$

By Parts (i) and (ii), $t \mapsto X_m^\theta(t) - h_{m-1}^\theta(t)$ is equal in distribution to a two-sided Brownian motion with negative drift, multiplied by a factor of $\sqrt{2}$. Represented this way, $t \mapsto v_m^\theta(t)$ is known as a stationary, reflected Brownian motion with drift. Stationarity follows from the stationarity of increments. The fact that X is a reversible strong Markov process is proven in [33], pg. 81 (see also pg. 49-50 in [33] and Equations (4) and (5) in [50] for a more directly applicable statement).

Part (iv): By Part (ii) and (3.11), for any initial level m, the process $\{h_{r+m}^\theta, v_{r+m+1}^\theta, X_{r+m+1}^\theta, B_{r+m}\}_{r\ge 0}$ has the same distribution as $\{Y_r^{\frac{1}{\sqrt{\theta}}}, q_{r+1}^{\frac{1}{\sqrt{\theta}}}, B_{r+1}, W_r^\lambda\}_{r\ge 0}$ as defined in (C.1). Note that B_{r+m} now plays the role of the W_r^λ, as stated in the definition. Therefore, the independence structure of Theorem C.3 holds. $\qquad\square$

4. Construction and properties of the semi-infinite geodesics

4.1. *Heuristic for construction of semi-infinite geodesics*

The next task is the construction of semi-infinite geodesics from each initial point and in each asymptotic direction. For each given point $(m, t) \in \mathbb{Z} \times \mathbb{R}$ and direction parameter $\theta > 0$, we want to find a semi-infinite geodesic, defined by jump times $t = \tau_{m-1} \leq \tau_m \leq \cdots$ that satisfies

$$\lim_{n \to \infty} \frac{\tau_n}{n} = \theta.$$

We argue heuristically to motivate the useful construction. Start by finding a maximal path for $L_{(m,t),(n,n\theta)}$ for a large value of n. Note that

$$L_{(m,t),(n,n\theta)} = \max_{s \in [t,n\theta]} \left(B_m(t, s) + L_{(m+1,s),(n,n\theta)} \right),$$

and the maximizer $s = \tau_m$ is the location where the geodesic jumps from level m to $m + 1$. For all $t \leq s \leq n\theta$,

$$B_m(t, s) + L_{(m+1,s),(n,n\theta)} \leq B_m(t, \tau_m) + L_{(m+1,\tau_m),(n,n\theta)}.$$

Rearranging yields

$$B_m(\tau_m, s) \leq L_{(m+1,\tau_m),(n,n\theta)} - L_{(m+1,s),(n,n\theta)}.$$

As n changes, so could τ_m, but for the sake of heuristic we hold τ_m constant. Take limits as $n \to \infty$ and rearrange again to get, for some sign $\square \in \{+, -\}$,

$$B_m(s) - h_{m+1}^{\theta\square}(s) \leq B_m(\tau_m) - h_{m+1}^{\theta\square}(\tau_m).$$

4.2. *Busemann geodesics*

The discussion of the previous section motivates this rigorous definition.

Definition 4.1. On the event Ω_1, for all $(m, t) \in \mathbb{Z} \times \mathbb{R}$, $\theta > 0$ and $\square \in \{+, -\}$, let $\mathbf{T}_{(m,t)}^{\theta\square}$ denote the set of sequences

$$t = \tau_{m-1} \leq \tau_m \leq \tau_{m+1} \leq \cdots$$

that satisfy

(4.1) $$B_r(\tau_r) - h_{r+1}^{\theta\square}(\tau_r) = \sup_{s \in [\tau_{r-1}, \infty)} \left\{ B_r(s) - h_{r+1}^{\theta\square}(s) \right\} \quad \text{for each } r \geq m.$$

Theorem 3.5(iv)–(v) imply that such sequences exist. At each level r, there exist leftmost and rightmost maximizers. Let

$$t = \tau_{(m,t),m-1}^{\theta\square,L} \leq \tau_{(m,t),m}^{\theta\square,L} \leq \tau_{(m,t),m+1}^{\theta\square,L} \leq \cdots \quad \text{and} \quad t = \tau_{(m,t),m-1}^{\theta\square,R} \leq \tau_{(m,t),m}^{\theta\square,R} \leq \tau_{(m,t),m+1}^{\theta\square,R} \leq \cdots$$

denote the leftmost and rightmost sequences in $\mathbf{T}_{(m,t)}^{\theta\square}$. Furthermore, define

$$\mathbf{T}_{(m,t)}^{\theta} := \mathbf{T}_{(m,t)}^{\theta+} \cup \mathbf{T}_{(m,t)}^{\theta-}.$$
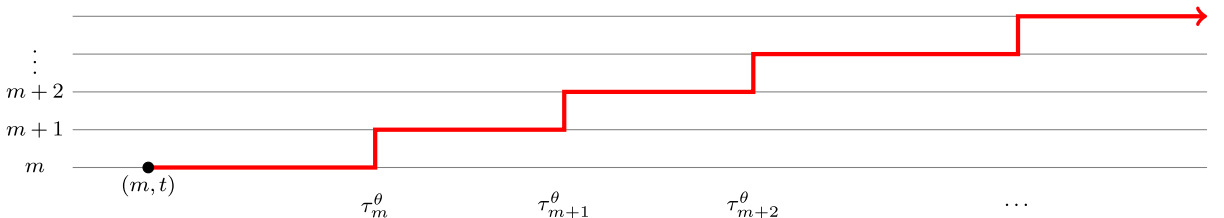

Fig. 6. Example of an element of $\mathbf{T}_{(m,t)}^{\theta}$.

Remark 4.2. Since every non-decreasing sequence in $\mathbf{T}^{\theta}_{(m,t)}$ defines a semi-infinite up-right path, $\mathbf{T}^{\theta}_{(m,t)}$ will be used to denote the set of up-right paths constructed in this way. Theorem 4.3(i) below shows that all sequences in $\mathbf{T}^{\theta}_{(m,t)}$ are semi-infinite geodesics starting from (m,t). By Theorem 3.5(viii), on the event $\Omega^{(\theta)} \subseteq \Omega_1$, $\mathbf{T}^{\theta-}_{\mathbf{x}} = \mathbf{T}^{\theta+}_{\mathbf{x}} = \mathbf{T}^{\theta}_{\mathbf{x}}$ for all $\mathbf{x} \in \mathbb{Z} \times \mathbb{R}$. This does not imply that $\mathbf{T}^{\theta}_{\mathbf{x}}$ contains only one element, so the leftmost and rightmost distinction is still necessary in general for a fixed direction θ. See Theorem 4.7 below. However, by Theorem 3.1(ii), for fixed $\theta > 0$ and fixed $(m,t) \in \mathbb{Z} \times \mathbb{R}$, $\mathbf{T}^{\theta}_{(m,t)}$ almost surely contains a single element. In this case, Theorem B.2 gives the distribution of the first jump time τ_m.

Two elements $\{\tau_r\}_{r \geq m-1}$ and $\{\tau'_r\}_{r \geq m-1}$ of $\mathbf{T}^{\theta\square}_{\mathbf{x}}$ are distinct if $\tau_r \neq \tau'_r$ for at least one index r. Uniqueness of the $\theta\square$ Busemann geodesic from $\mathbf{x}$ means that $\mathbf{T}^{\theta\square}_{\mathbf{x}}$ contains exactly one sequence.

The following theorems collect the properties of the Busemann semi-infinite geodesics, to be proved in Section 7.

Theorem 4.3. *There exists an event, Ω_2, of full probability, on which the following hold.*

(i) (*Existence*) *For all $\mathbf{x} \in \mathbb{Z} \times \mathbb{R}$, $\theta > 0$, and $\square \in \{+, -\}$, every element of $\mathbf{T}^{\theta\square}_{\mathbf{x}}$ defines a semi-infinite geodesic starting from $\mathbf{x}$. More specifically, for any two points $\mathbf{y} \leq \mathbf{z}$ along a path in $\mathbf{T}^{\theta}_{\mathbf{x}}$, the energy of this path between $\mathbf{y}$ and $\mathbf{z}$ is $\mathcal{B}^{\theta\square}(\mathbf{y}, \mathbf{z})$, and this energy is maximal over all paths between $\mathbf{y}$ and $\mathbf{z}$.*

(ii) (*Leftmost and rightmost finite geodesics along paths*) *If, for some $\theta > 0$, $\square \in \{+, -\}$, and $\mathbf{x} \in \mathbb{Z} \times \mathbb{R}$, the points $\mathbf{y} \leq \mathbf{z} \in \mathbb{Z} \times \mathbb{R}$ both lie on the leftmost semi-infinite geodesic in $\mathbf{T}^{\theta\square}_{\mathbf{x}}$, then the portion of this geodesic between $\mathbf{y}$ and $\mathbf{z}$ coincides with the leftmost finite geodesic between these two points. Similarly, the rightmost semi-infinite geodesic is the rightmost geodesic between any two of its points.*

(iii) (*Monotonicity*) *The following inequalities hold.*

(a) *For all $0 < \gamma < \theta$, all $(m,t) \in \mathbb{Z} \times \mathbb{R}$, and $r \geq m$,*

$$t \leq \tau^{\gamma-,L}_{(m,t),r} \leq \tau^{\gamma+,L}_{(m,t),r} \leq \tau^{\theta-,L}_{(m,t),r} \leq \tau^{\theta+,L}_{(m,t),r} \quad and \quad t \leq \tau^{\gamma-,R}_{(m,t),r} \leq \tau^{\gamma+,R}_{(m,t),r} \leq \tau^{\theta-,R}_{(m,t),r} \leq \tau^{\theta+,R}_{(m,t),r}.$$

(b) *For all $\theta > 0$, $m \leq r \in \mathbb{Z}$, $s < t \in \mathbb{R}$, and $\square \in \{+, -\}$,*

$$\tau^{\theta\square,L}_{(m,s),r} \leq \tau^{\theta\square,L}_{(m,t),r} \quad and \quad \tau^{\theta\square,R}_{(m,s),r} \leq \tau^{\theta\square,R}_{(m,t),r}.$$

(c) *For $\theta > 0$, on the θ-dependent full-probability event $\widetilde{\Omega}^{(\theta)}$ of Theorem 3.1(iii), for all pairs of initial points (m,s) and (m,t) in $\mathbb{Z} \times \mathbb{R}$ that satisfy $s < t$, we have*

$$\tau^{\theta,R}_{(m,s),r} \leq \tau^{\theta,L}_{(m,t),r} \quad for\ all\ r \geq m.$$

(iv) (*Convergence*) *The following limits hold.*

(a) *For all $(m,t) \in \mathbb{Z} \times \mathbb{R}$, $r \geq m$, $\theta > 0$, and $\square \in \{+, -\}$,*

$$\lim_{\gamma \nearrow \theta} \tau^{\gamma\square,L}_{(m,t),r} = \tau^{\theta-,L}_{(m,t),r} \quad and \quad \lim_{\delta \searrow \theta} \tau^{\delta\square,R}_{(m,t),r} = \tau^{\theta+,R}_{(m,t),r}.$$

(b) *For all $(m,t) \in \mathbb{Z} \times \mathbb{R}$, $r \geq m$, $\square \in \{+, -\}$, and $S \in \{L, R\}$,*

$$\lim_{\theta \searrow 0} \tau^{\theta\square,S}_{(m,t),r} = t \quad and \quad \lim_{\theta \to \infty} \tau^{\theta\square,S}_{(m,t),r} = \infty.$$

(c) *For all $(m,t) \in \mathbb{Z} \times \mathbb{R}$, $r \geq m$, $\theta > 0$, and $\square \in \{+, -\}$,*

$$\lim_{s \nearrow t} \tau^{\theta\square,L}_{(m,s),r} = \tau^{\theta\square,L}_{(m,t),r} \quad and \quad \lim_{u \searrow t} \tau^{\theta\square,R}_{(m,u),r} = \tau^{\theta\square,R}_{(m,t),r},$$

(v) (*Directness*) *For all $\mathbf{x} \in \mathbb{Z} \times \mathbb{R}$, $\theta > 0$, $\square \in \{+, -\}$, and all $\{\tau_r\}_{r \geq m} \in \mathbf{T}^{\theta\square}_{\mathbf{x}}$,*

$$\lim_{n \to \infty} \frac{\tau_n}{n} = \theta.$$

Remark 4.4 (A look ahead). In future work, we will use the joint distribution of Busemann functions to build on these results and strengthen parts of Theorem 4.3. Specifically, Part (iv)(a) can be made stronger in the following way: There

exists an event of full probability on which, for all $\theta > 0$, $m \le r \in \mathbb{Z}$, $s < t \in \mathbb{R}$, $S \in \{L, R\}$ and $\square \in \{+, -\}$, there exists $\varepsilon > 0$ such that

$$\tau^{\gamma\square,S}_{(m,s),r} = \tau^{\theta-,S}_{(m,s),r} \quad \text{for all } \theta - \varepsilon < \gamma < \theta, \quad \text{and} \quad \tau^{\delta\square,S}_{(m,s),r} = \tau^{\theta+,S}_{(m,s),r} \quad \text{for all } \theta < \delta < \theta + \varepsilon.$$

This is used to strengthen Part (iii)(c) to show that, on this event, for each $m \le r \in \mathbb{Z}$, $s < t \in \mathbb{R}$, $\theta > 0$, and $\square \in \{+, -\}$,

$$\tau^{\theta\square,R}_{(m,s),r} \le \tau^{\theta\square,L}_{(m,t),r}.$$

Part (iii)(a) cannot be strengthened to compare $\tau^{\gamma\square,R}_{(m,t),r}$ and $\tau^{\theta\square,L}_{(m,t),r}$ for general $\gamma < \theta$. Specifically, there exists (m, t) and $\gamma < \theta$ such that

$$\tau^{\gamma-,R}_{(m,t),r} = \tau^{\gamma+,R}_{(m,t),r} > \tau^{\theta-,L}_{(m,t),r} = \tau^{\theta+,L}_{(m,t),r}.$$

The following theorem shows that the Busemann geodesics give control over all semi-infinite geodesics.

Theorem 4.5. *The following hold on the full probability event Ω_2.*

(i) (*Control of finite geodesics*) *Let $\theta > 0$, $(m, t) \in \mathbb{Z} \times \mathbb{R}$, and let $\{t_n\}$ be any sequence that has direction θ. For all n sufficiently large so that $n \ge m$ and $t_n \ge t$, let $t = t_{n,m-1} \le t_{n,m} \le \cdots \le t_{n,n} = t_n$ be any sequence that defines a (finite) geodesic between (m, t) and (n, t_n). Then, for each $r \ge m$,*

$$\tau^{\theta-,L}_{(m,t),r} \le \liminf_{n \to \infty} t_{n,r} \le \limsup_{n \to \infty} t_{n,r} \le \tau^{\theta+,R}_{(m,t),r}.$$

(ii) (*Control of semi-infinite geodesics*) *If, for some $\theta > 0$ and $(m, t) \in \mathbb{Z} \times \mathbb{R}$, any other geodesic (constructed from the Busemann functions or not) is defined by the sequence $t = t_{m-1} \le t_m \le \cdots$, starts at (m, t), and has direction θ, then for all $r \ge m$,*

$$\tau^{\theta-,L}_{(m,t),r} \le t_r \le \tau^{\theta+,R}_{(m,t),r}.$$

(iii) (*Convergence of finite geodesics*) *Assume that $\mathbf{T}^\theta_{(m,t)}$ contains a single element $\{\tau_r\}_{r \ge m-1}$. If $\{t_n\}$ is a θ-directed sequence and, for each n, the sequence $t = t_{n,m-1} \le t_{n,m} \le \cdots \le t_{n,n}$ defines a finite geodesic between (m, t) and (n, t_n), then*

$$\lim_{n \to \infty} t_{n,r} = \tau_r \quad \text{for all } r \ge m.$$

Remark 4.6. In Part (iii), the assumption of uniqueness holds, for example, on the event $\Omega^{(\theta)}_{(m,t)}$ of Theorem 3.1(ii). However, this assumption does not extend to all $(m, t) \in \mathbb{Z} \times \mathbb{R}$ and $\theta > 0$ simultaneously with probability one, as discussed in the following section.

4.3. *Non-uniqueness of semi-infinite geodesics*

There are two types of non-uniqueness of semi-infinite geodesics from an initial point $\mathbf{x}$ into an asymptotic direction θ.

(i) The first type, described in Theorem 3.1(iii) and in the next Theorem 4.7, is caused by the continuum time variable and does *not* appear in the lattice corner growth model. It is captured by the L/R distinction. For each fixed direction $\theta > 0$ and level $m \in \mathbb{Z}$, this happens with probability one at infinitely many locations. To illustrate, let $s^\star \ge 0$ be the maximizer below:

$$B_m(s^\star) - h^\theta_{m+1}(s^\star) = \sup_{0 \le s < \infty} \{B_m(s) - h^\theta_{m+1}(s)\}.$$

By Theorem 3.5(vii) and Lemma 7.1, with probability one, the maximizer $s^\star$ is unique. By Theorem B.2, $s^\star > 0$ with probability one. By Theorem 3.5(v),

$$t^\star := \sup\{t < 0 : B_m(t) - h^\theta_{m+1}(t) = B_m(s^\star) - h^\theta_{m+1}(s^\star)\} \quad \text{exists in } \mathbb{R}_{<0}.$$

Then, both $t^\star$ and $s^\star$ are maximizers of $B_m(s) - h^\theta_{m+1}(s)$ on $[t^\star, \infty)$. This gives at least two distinct sequences in the set $\mathbf{T}^\theta_{(m,t^\star)}$, with $\tau^{\theta,L}_{(m,t^\star),m} = t^\star$ and $\tau^{\theta,R}_{(m,t^\star),m} = s^\star$.

This presents a new type of non-uniqueness that is not present in discrete last-passage percolation with exponential weights. However, when θ is fixed, θ-directed geodesics can disagree only for a finite amount of time, because Theorem 3.1(vii) forces them to eventually come back together.

(ii) The second type of non-uniqueness of semi-infinite geodesics is captured by the $\theta\pm$ distinction. Hence, it happens with probability zero at a fixed θ and thereby requires investigation of the full Busemann process and the full collection of all semi-infinite geodesics. In contrast to the first type of non-uniqueness, this bears some similarity to the behavior present in discrete last-passage percolation shown in [38].

In future work, we show that there exists a random countable set of directions θ such that, out of every initial point, there are two θ-directed geodesics. These geodesics may initially stay together for a while, but eventually they separate for good and never come back together. Furthermore, there is a distinguished subset of initial points at which geodesics with the same direction split immediately. This set will be shown to have almost surely Hausdorff dimension $\frac{1}{2}$.

The following theorem clarifies the non-uniqueness described by (i) above. Fix $\theta > 0$. On a full probability event where the $\theta\pm$ distinction is not present, define the following sets:

$$\mathrm{NU}_0^\theta = \left\{(m,t) \in \mathbb{Z} \times \mathbb{R} : \tau_{(m,t),r}^{\theta,L} < \tau_{(m,t),r}^{\theta,R} \text{ for some } r \geq m\right\}, \quad \text{and}$$

$$\mathrm{NU}_1^\theta = \left\{(m,t) \in \mathrm{NU}_0^\theta : \tau_{(m,t),m}^{\theta,L} < \tau_{(m,t),m}^{\theta,R}\right\}.$$

Since $\theta > 0$ is fixed, by Theorem 4.5(ii), NU_0^θ is almost surely the set of points $\mathbf{x} \in \mathbb{Z} \times \mathbb{R}$ such that the θ-directed semi-infinite geodesic from $\mathbf{x}$ is not unique. Its subset NU_1^θ is the set of initial points from which two θ-directed geodesics separate on the first level.

Theorem 4.7. *There exists a full probability event $\widetilde{\Omega}^{(\theta)}$ on which the following hold.*

(i) *The sets NU_0^θ and NU_1^θ are countably infinite and can be written as*

$$\mathrm{NU}_0^\theta = \left\{(m,t) \in \mathbb{Z} \times \mathbb{R} : t = \tau_{(m,t),r}^{\theta,L} < \tau_{(m,t),r}^{\theta,R} \text{ for some } r \geq m\right\}, \quad \text{and}$$

$$\mathrm{NU}_1^\theta = \left\{(m,t) \in \mathrm{NU}_0^\theta : t = \tau_{(m,t),m}^{\theta,L} < \tau_{(m,t),m}^{\theta,R}\right\}.$$

For each $(m,t) \in \mathbb{Z} \times \mathbb{R}$ and $\theta > 0$, at most one geodesic in $\mathbf{T}_{(m,t)}^\theta$ passes horizontally through $(m, t + \varepsilon)$ for some $\varepsilon > 0$.

(ii) *The set NU_1^θ is neither discrete nor dense in $\mathbb{Z} \times \mathbb{R}$. More specifically, for each point $(m,t) \in \mathrm{NU}_1^\theta$ and every $\varepsilon > 0$, there exists $s \in (t - \varepsilon, t)$ such that $(m,s) \in \mathrm{NU}_1^\theta$. For each $(m,t) \in \mathrm{NU}_1^\theta$, there exists $\delta > 0$ such that, for all $s \in (t, t + \delta)$, $(m,s) \notin \mathrm{NU}_0^\theta$.*

Remark 4.8. Part (i) states that, on $\widetilde{\Omega}^{(\theta)}$, if there exist multiple θ-directed geodesics out of (m,t), then these geodesics separate one by one from the upward vertical ray at (m,t). The set NU_1^θ is the subset of NU_0^θ such that two geodesics separate immediately at the initial point. See Figures 7 and 8.

4.4. *Dual geodesics and coalescence*

To prove the coalescence of Theorem 3.1(vii), we use the dual field $\mathbf{X}^\theta$ of independent Brownian motions from (3.9) and their southwest-directed semi-infinite geodesics. We use this to construct the BLPP analogue of Pimentel's dual tree [47] and then adapt the argument of [53].

Since Brownian motion is symmetric, in distribution, about the origin, there exist θ-directed dual southwest semi-infinite geodesics for the environment $\mathbf{X}^\theta$. These are constructed in a very similar manner as the northeast geodesics in Definition 4.1. Specifically, for $(m,t) \in \mathbb{Z} \times \mathbb{R}$, let $\mathbf{T}_{(m,t)}^{\theta,\star}$ be the set of sequences $t = \tau_m^\star \geq \tau_{m-1}^\star \geq \cdots$ satisfying

$$(4.2) \qquad h_{r-1}^\theta(\tau_{r-1}^\star) - X_r^\theta(\tau_{r-1}^\star) = \sup_{-\infty < s \leq \tau_r^\star} \left\{h_{r-1}^\theta(s) - X_r^\theta(s)\right\} \quad \text{for each } r \leq m.$$

Define the leftmost and rightmost sequences similarly, by $\tau_{(m,t),r}^{\theta,L\star}$ and $\tau_{(m,t),r}^{\theta,R\star}$. These sequences define southwest semi-infinite paths, similar as for the northeast paths. We graphically represent southwest paths on the plane, where the continuous coordinate is not changed, but the discrete coordinate is shifted down by $\frac{1}{2}$. That is, for $m \in \mathbb{Z}$, denote $m^\star = m - \frac{1}{2}$,
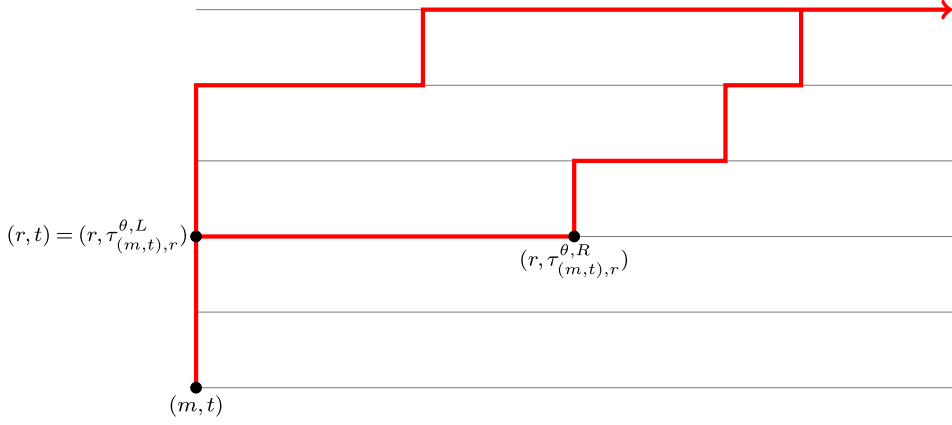
Fig. 7. In this figure, $(m, t) \in \mathrm{NU}_0^\theta \setminus \mathrm{NU}_1^\theta$. The two geodesics split on the vertical line containing the initial point, but they must come back together to coalesce.
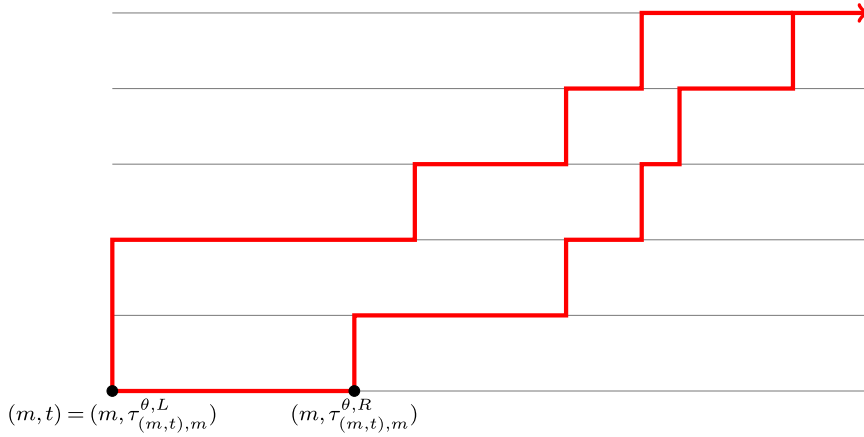

Fig. 8. In this figure, $(m, t) \in \mathrm{NU}_1^\theta$. The two geodesics split immediately from the initial point, but later coalesce.

and for $\mathbf{x} = (m, t) \in \mathbb{Z} \times \mathbb{R}$, denote $\mathbf{x}^\star = (m - \frac{1}{2}, t)$. Then, for $\{\tau_r^\star\}_{r \le m} \in \mathbf{T}_{(m,t)}^{\theta,\star}$, the southwest path consists of horizontal and vertical line segments, where $\tau_r^\star$ denotes the position of the vertical segment connecting levels $(r + 1)^\star$ and $r^\star$. Specifically, the path consists of the points

$$\bigcup_{r=-\infty}^{m} \{(r^\star, u) : u \in [\tau_{r-1}^\star, \tau_r^\star]\} \cup \bigcup_{r=-\infty}^{m-1} \{(v, \tau_r^\star) : v \in [r^\star, (r+1)^\star]\}.$$

Figure 9 shows the regular axes and the dual axes together, with a southwest dual geodesic traveling on this dual plane. Each element of $\mathbf{T}_{(m,t)}^{\theta\star}$ is a southwest semi-infinite geodesic for the dual environment $\mathbf{X}^\theta$. This fact is recorded in Theorem 7.11.

Since $\mathbf{X}^\theta$ is an environment of i.i.d. Brownian motions, the following theorem allows us to conclude Part (vii) of Theorem 3.1 from Part (vi). Full details of this connection are found in the proofs. Refer to Figure 10 for clarity.

Theorem 4.9. *Fix $\theta > 0$. With probability one, if for any $\mathbf{x}, \mathbf{y} \in \mathbb{Z} \times \mathbb{R}$, the rightmost semi-infinite geodesics in $\mathbf{T}_{\mathbf{x}}^\theta$ and $\mathbf{T}_{\mathbf{y}}^\theta$ are disjoint (i.e. the paths share no points), then there exists a bi-infinite, upright path defined by jump times $\cdots \le \tau_{-1}^\star \le \tau_0^\star \le \tau_1^\star \le \cdots$ (where $\tau_i^\star$ denotes the jump time from level i to level $i + 1$) that satisfies the following:*

(i) *For any point $\mathbf{x}^\star$ along the path, the portion of that path to the south and west of $\mathbf{x}^\star$ is the leftmost semi-infinite geodesic in the set $\mathbf{T}_{\mathbf{x}}^{\theta,\star}$. Specifically, when shifted back up by $\frac{1}{2}$, the path is a bi-infinite geodesic for the environment $\mathbf{X}^\theta$.*
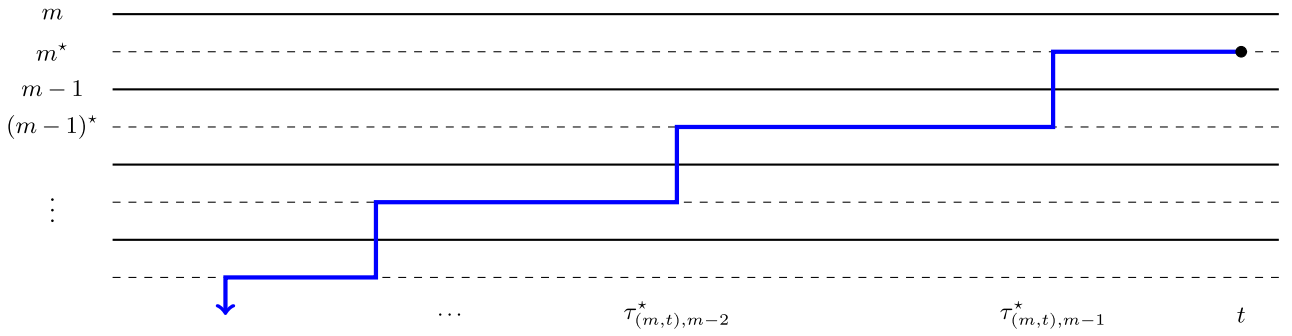
Fig. 9. The original discrete levels (solid) and the dual levels (dashed). Dual levels are labeled with a $\star$.
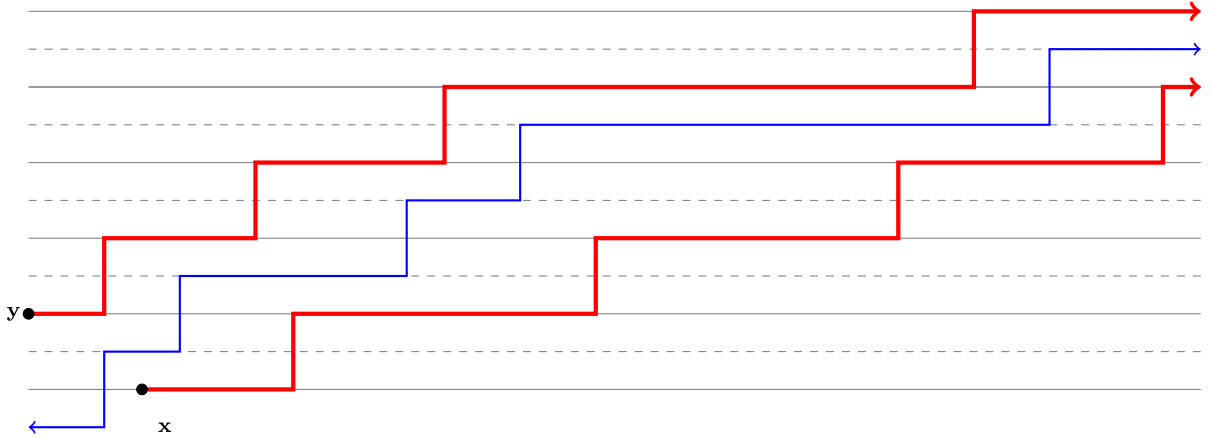

Fig. 10. The outcome of Theorem 4.9: two disjoint semi-infinite northeast paths (red/thick) and the dual bi-infinite path (blue/thin).

(ii) *The sequence $\{\tau_n^\star\}_{n \in \mathbb{Z}}$ satisfies*

$$\lim_{n \to \infty} \frac{\tau_n^\star}{n} = \theta = \lim_{n \to \infty} \frac{\tau_{-n}^\star}{-n}.$$

The analogous result holds if we assume that two leftmost θ-directed semi-infinite geodesics are disjoint. In this case, the portion of the path to the south and west of each of its points is the rightmost dual semi-infinite geodesic.

5. Connections to other models

5.1. *Connection to infinite-length polymer measures for the Brownian polymer*

Recalling the definitions at the beginning of Section 2.2, for $(m, s) \le (n, t) \in \mathbb{Z} \times \mathbb{R}$, the point-to-point partition function of the Brownian polymer with unit temperature is

$$Z_{(m,s),(n,t)}(\mathbf{B}) = \int e^{\mathcal{E}(\mathbf{s}_{m,n})} \mathbf{1}\{\mathbf{s}_{m,n-1} \in \Pi_{(m,s),(n,t)}\} d\mathbf{s}_{m,n-1}.$$

The associated quenched polymer measure on $\Pi_{(m,s),(n,t)}$ is

$$Q^{\mathbf{B}}(\tau_m \in ds_m, \dots, \tau_{n-1} \in ds_{n-1}) = \frac{1}{Z_{(m,s),(n,t)}(\mathbf{B})} \exp\left(\sum_{k=m}^{n} B_k(s_{k-1}, s_k)\right) \mathbf{1}\{\mathbf{s}_{m,n-1} \in \Pi_{(m,s),(n,t)}\} d\mathbf{s}_{m,n-1}.$$

Brownian last-passage percolation is the zero-temperature analogue of the Brownian or O'Connell–Yor polymer. This is made precise by the limit

$$\lim_{\beta \to \infty} \frac{1}{\beta} \log Z_{(m,s),(n,t)}(\beta \mathbf{B}) = L_{(m,s),(n,t)}(\mathbf{B}),$$

which, by convergence of L^p norms, holds in a deterministic sense as long as **B** is a field of continuous functions $\mathbb{R} \to \mathbb{R}$. The parameter β is used to denote inverse temperature.

Alberts, Rassoul-Agha, and Simper [2] showed the existence of Busemann functions and infinite-length limits of the quenched measures when the right endpoint (n, t_n) satisfies $t_n/n \to \theta$ for some fixed $\theta > 0$. The Busemann functions are defined as

$$\widehat{\mathcal{B}}^{\theta}(\mathbf{x}, \mathbf{y}) := \lim_{n \to \infty} \log \frac{Z_{\mathbf{x},(n,t_n)}(\mathbf{B})}{Z_{\mathbf{y},(n,t_n)}(\mathbf{B})}.$$

Similarly as in (2.6), define $\widehat{h}_m^{\theta}(t) = \widehat{\mathcal{B}}^{\theta}((m, 0), (m, t))$. Using the "Proof of Theorem 2.5, assuming Theorem 3.1" on page 1937 of [2] and an analogous construction of the Busemann functions for all initial points as in Section 6 of the present paper, it can be shown that, the process $t \mapsto \widehat{h}_m^{\theta}(t)$ is a two-sided Brownian motion with drift, independent of $\{B_r\}_{r \le m}$. By Theorems 3.5(vii) and 3.7(i), the same is true for the zero temperature Busemann process $t \mapsto h_m^{\theta}(t)$. An infinite, up-right path under the quenched infinite-length polymer measure is a continuous-time Markov chain, starting from a point (m, t) and defined by jump times $t = \tau_{m-1} \le \tau_m \le \cdots$. By Equation (2.3) in [2], the quenched conditional distribution of τ_r given τ_{r-1} is

$$Q_{(m,s),(n,t_n)}^{\mathbf{B}}(\tau_r \in ds_r | \tau_{r-1} = s_{r-1}) = e^{B_r(s_{r-1}, s_r)} \frac{Z_{(r+1,s_r),(n,t_n)}(\mathbf{B})}{Z_{(r,s_{r-1}),(n,t_n)}(\mathbf{B})} \mathbf{1}\{s_{r-1} \le s_r\} ds_r.$$

The proof of existence of infinite length measures requires a rigorous tightness argument, but to motivate the connection to BLPP, we formally take limits as $n \to \infty$ to yield the conditional measure

$$\exp\big(B_r(s_{r-1}, s_r) + \widehat{\mathcal{B}}^{\theta}((r+1, s_r), (r, s_{r-1}))\big)\mathbf{1}\{s_{r-1} \le s_r\} ds_r$$
$$= \exp\big(B_r(s_{r-1}, s_r) - \widehat{h}_{r+1}^{\theta}(s_r) + \widehat{\mathcal{B}}^{\theta}((r+1, 0), (r, s_{r-1})\big)\mathbf{1}\{s_{r-1} \le s_r\} ds_r.$$

Now, note that the term $\widehat{\mathcal{B}}^{\theta}((r+1, 0), (r, s_{r-1}))$ does not depend on s_r. We can then think of the connection between BLPP and the O'Connell–Yor polymer in the following sense: In the positive temperature case, given the environment **B**, the transition density from τ_{r-1} to τ_r is given by

$$C \exp\big(B_r(s) - \widehat{h}_{r+1}^{\theta}(s)\big)\mathbf{1}\{\tau_{r-1} \le s\} ds,$$

where C is a normalizing constant depending on r and τ_{r-1}. On the other hand, by Definition 4.1, in the zero-temperature case, given the environment **B** and the previous jump τ_{r-1}, the jump τ_r is chosen in a deterministic fashion by maximizing $B_r(s) - h_{r+1}^{\theta}(s)$ over $s \in [\tau_{r-1}, \infty)$.

5.2. *Connection to semi-infinite geodesics in discrete last-passage percolation*

The corner growth model, or discrete last-passage percolation, is defined as follows. Let $\{Y_{\mathbf{x}}\}_{\mathbf{x} \in \mathbb{Z}^2}$ be a collection of nonnegative i.i.d. random variables, each associated to a vertex on the integer lattice. For $\mathbf{x} \le \mathbf{y} \in \mathbb{Z} \times \mathbb{Z}$, define the last-passage time as

$$G_{\mathbf{x},\mathbf{y}} = \sup_{\mathbf{x}_{\bullet} \in \Pi_{\mathbf{x},\mathbf{y}}} \sum_{k=0}^{|\mathbf{y}-\mathbf{x}|_1} Y_{\mathbf{x}_k},$$

where $\Pi_{x,y}$ is the set of up-right paths $\{\mathbf{x}_k\}_{k=0}^n$ that satisfy $\mathbf{x}_0 = \mathbf{x}$, $\mathbf{x}_n = \mathbf{y}$, and $\mathbf{x}_k - \mathbf{x}_{k-1} \in \{\mathbf{e}_1, \mathbf{e}_2\}$. Under the assumption that Y_0 has finite second moment, Theorem 3.1 and Corollary 3.1 of [26] introduced BLPP as a universal scaling limit of the corner growth model, where one variable is scaled, and the other is held constant. That is, if Y_0 is normalized to have unit mean and variance,

$$\big\{L_{(m,s),(n,t)} : (m, s) \le (n, t) \in \mathbb{Z} \times \mathbb{R}\big\}$$

is the functional limit, as $k \to \infty$, of the properly interpolated version of the process

$$\left\{\frac{1}{\sqrt{k}} G_{(m,\lfloor sk \rfloor),(n,\lfloor tk \rfloor)} - (t - s)k : (m, s) \le (n, t) \in \mathbb{Z} \times \mathbb{R}\right\}.$$

The most tractable case of discrete last-passage percolation is the case where Y_0 has the exponential distribution with rate 1. In this case, Busemann functions exist and are indexed by a direction vector **u**. They are defined by

$$U^{\mathbf{u}}(\mathbf{x}, \mathbf{y}) := \lim_{n \to \infty} G_{(\mathbf{x}, \mathbf{z}_n)} - G_{(\mathbf{y}, \mathbf{z}_n)},$$

where $\mathbf{z}_n$ satisfies $\mathbf{z}_n/n \to \mathbf{u}$ for a fixed direction $\mathbf{u}$. For a given $\omega \in \Omega$, $\mathbf{x} \in \mathbb{Z}^2$, and direction $\mathbf{u}$, a semi-infinite geodesic $\boldsymbol{\gamma}$ is defined by the sequence $\{\boldsymbol{\gamma}_k^{\mathbf{u},\mathbf{x}}\}_{k \in \mathbb{Z}_{\geq 0}}$. At each step, a choice is made to move upward or to the right. First, set $\boldsymbol{\gamma}_0^{\mathbf{u},\mathbf{x}} = \mathbf{x}$, and for $k \geq 0$,

$$(5.1) \qquad \boldsymbol{\gamma}_{k+1}^{\mathbf{u},\mathbf{x}} = \begin{cases} \boldsymbol{\gamma}_k^{\mathbf{u},\mathbf{x}} + \mathbf{e}_1, & \text{if } U^{\mathbf{u}}\big(\boldsymbol{\gamma}_k^{\mathbf{u},\mathbf{x}}, \boldsymbol{\gamma}_k^{\mathbf{u},\mathbf{x}} + \mathbf{e}_1\big) \leq U^{\mathbf{u}}\big(\boldsymbol{\gamma}_k^{\mathbf{u},\mathbf{x}}, \boldsymbol{\gamma}_k^{\mathbf{u},\mathbf{x}} + \mathbf{e}_2\big), \\ \boldsymbol{\gamma}_k^{\mathbf{u},\mathbf{x}} + \mathbf{e}_2, & \text{if } U^{\mathbf{u}}\big(\boldsymbol{\gamma}_k^{\mathbf{u},\mathbf{x}}, \boldsymbol{\gamma}_k^{\mathbf{u},\mathbf{x}} + \mathbf{e}_2\big) < U^{\mathbf{u}}\big(\boldsymbol{\gamma}_k^{\mathbf{u},\mathbf{x}}, \boldsymbol{\gamma}_k^{\mathbf{u},\mathbf{x}} + \mathbf{e}_1\big). \end{cases}$$

In the case of exponential weights, this sequence is a semi-infinite geodesic with direction $\mathbf{u}$. This construction is inherently discrete, so it does not extend directly to the case of BLPP. However, this construction is equivalent to taking a sequence of maximizers of the appropriate function, analogous to Definition 4.1. The following is a discrete analogue of Theorem 3.5(vi) that holds in the case of exponential weights.

$$(5.2) \qquad U^{\mathbf{u}}\big((m,r),(m,r+1)\big) = \max_{\substack{m \leq k < \infty \\ k \in \mathbb{Z}}} \left\{ \sum_{i=m}^{k} Y_{(i,r)} + U^{\mathbf{u}}\big((k,r+1),(m,r+1)\big) \right\}.$$

Now, suppose that, starting at the point (m,r), the semi-infinite geodesic constructed in (5.1) makes an $\mathbf{e}_2$ step from (k,r) to $(k,r+1)$ for some $k \geq m$. Then, we show that k is maximal for (5.2). Indeed, by (5.1), if the path $\boldsymbol{\gamma}$ makes an $\mathbf{e}_2$ step from (k,r) to $(k,r+1)$, then

$$U^{\mathbf{u}}\big((i,r),(i+1,r)\big) \leq U^{\mathbf{u}}\big((i,r),(i,r+1)\big) \quad \text{for } m \leq i \leq k-1, \quad \text{and}$$

$$U^{\mathbf{u}}\big((k,r),(k+1,r)\big) > U^{\mathbf{u}}\big((k,r),(k,r+1)\big).$$

Using the identity $Y_{\mathbf{x}} = U^{\mathbf{u}}(\mathbf{x}, \mathbf{x} + \mathbf{e_1}) \wedge U^{\mathbf{u}}(\mathbf{x}, \mathbf{x} + \mathbf{e_2})$ and additivity of the Busemann functions,

$$\sum_{i=m}^{k} Y_{(i,r)} + U^{\mathbf{u}}\big((k,r+1),(m,r+1)\big)$$

$$= \sum_{i=m}^{k-1} U^{\mathbf{u}}\big((i,r),(i+1,r)\big) + U^{\mathbf{u}}\big((k,r),(k,r+1)\big) + U^{\mathbf{u}}\big((k,r+1),(m,r+1)\big)$$

$$= U^{\mathbf{u}}\big((m,r),(m,r+1)\big),$$

so k is indeed maximal in for the right hand-side of (5.2). The inductive step follows in the same manner. Hence, we see that the construction of semi-infinite geodesics in Definition 4.1 is a continuous analogue of the procedure for the discrete case when viewed from the perspective of maximizers.

5.3. *Connection to queuing theory*

Fix $\theta > 0$, and consider the almost surely unique θ-directed semi-infinite geodesic (Theorem 3.1(ii)) starting from $(0,0)$ and defined by the sequence of jump times $\{\tau_r\}_{r \geq -1}$ with $\tau_{-1} = 0$. For each $r \geq 0$, associate a queuing station as follows. For $s < t$, let $\frac{1}{\sqrt{\theta}}(t-s) - X_{r+1}^{\theta}(s,t)$ denote the service available in the interval $(s,t]$ and let $\frac{1}{\sqrt{\theta}}(t-s) - h_r^{\theta}(s,t)$ denote the arrivals to the queue in the interval $(s,t]$. Here, the parameter $\frac{1}{\sqrt{\theta}}$ dictates the rate of service. See Appendix C for more details about the queuing setup. By (3.11), $h_{r+1}^{\theta} = \vec{D}(h_r^{\theta}, X_{r+1})$, so as in the proof of Theorem C.2, $\frac{1}{\sqrt{\theta}}(t-s) - h_{r+1}^{\theta}(s,t)$ gives the departures from the rth station in the interval $(s,t]$, and the departures process from the rth station becomes the arrivals process for the $r+1$st station. In other words, once a customer is served at the rth station, they move into the queue at the $r+1$st station. By Equations (3.10) and (3.11), for all $r \in \mathbb{Z}$ and $t \in \mathbb{R}$,

$$v_{r+1}^{\theta}(t) = \sup_{t \leq u < \infty} \left\{ B_r(t,u) - h_{r+1}^{\theta}(t,u) \right\} = \sup_{-\infty < u \leq t} \left\{ X_{r+1}^{\theta}(t,u) - h_r^{\theta}(t,u) \right\} = \overset{\leftarrow}{Q}\big(h_r^{\theta}, X_{r+1}^{\theta}\big)(t).$$

In queuing terms, $v_{r+1}^{\theta}(t)$ gives the length of the rth queue at time t. As τ_r is the maximizer of $B_r(u) - h_{r+1}^{\theta}(u)$ over $u \in [\tau_{r-1}, \infty)$, τ_r is the first time $u \geq \tau_{r-1}$ such that $v_{r+1}^{\theta}(u) = 0$ (see Lemma 7.4). In queuing terms, τ_r is the first time greater than or equal to τ_{r-1} at which the queue is empty. Thus, the semi-infinite geodesic represents the movement of a customer through the infinite series of queuing stations: the customer starts at station 0 at time 0 and is served at the rth station at a time no later than τ_r.

6. Construction and proofs for the Busemann process

The remainder of the paper is devoted to the proofs of the theorems. This section constructs the global Busemann process, with the proof of Theorem 3.5 as the ultimate result. Section 7 proves all results about the semi-infinite geodesics, culminating in the proof of Theorem 3.1.

6.1. *Construction of Busemann functions for a fixed direction*

Definition 6.1 (Definition of Busemann functions for fixed θ and a countable dense set of points). For a fixed $\theta > 0$, and $\mathbf{x}, \mathbf{y} \in \mathbb{Z} \times \mathbb{Q}$, let

$$\Omega_0^{(\theta)} = \bigcap_{\mathbf{x}, \mathbf{y} \in \mathbb{Z} \times \mathbb{Q}} \Omega_{\mathbf{x}, \mathbf{y}}^{(\theta)},$$

where the $\Omega_{\mathbf{x}, \mathbf{y}}^{(\theta)}$ are the events of Theorem 2.3. Then, $\mathbb{P}(\Omega_0^{(\theta)}) = 1$. On this event, for $\mathbf{x}, \mathbf{y} \in \mathbb{Z} \times \mathbb{Q}$, define $\mathcal{B}^\theta(\mathbf{x}, \mathbf{y})$ by (2.4). By definition in terms of limits, it is clear that the Busemann functions are additive, that is, for $\omega \in \Omega_0^{(\theta)}$ and $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{Z} \times \mathbb{Q}$,

$$(6.1) \qquad \mathcal{B}^\theta(\mathbf{x}, \mathbf{y}) + \mathcal{B}^\theta(\mathbf{y}, \mathbf{z}) = \mathcal{B}^\theta(\mathbf{x}, \mathbf{y}).$$

Hence, the entire collection of Busemann functions is constructed from the collection of horizontal and vertical Busemann functions $\{h_m^\theta, v_m^\theta\}_{m \in \mathbb{Z}}$ (Equations (2.6) and (2.5)).

Set $Y(t) = -B_0(t) + \frac{1}{\sqrt{\theta}} t$. By Lemma C.5, as elements of the space of functions $\mathbb{Q} \to \mathbb{R}$, equipped with the standard product σ-algebra,

$$\{h_0^\theta(t) : t \in \mathbb{Q}\} \overset{d}{=} \{Y(t) : t \in \mathbb{Q}\} \quad \text{and} \quad \{v_1^\theta(t) : t \in \mathbb{Q}\} \overset{d}{=} \{\overleftarrow{Q}(Y, B_1)(t) : t \in \mathbb{Q}\}.$$

For $\omega \in \Omega_0^{(\theta)}$, $m \in \mathbb{Z}$, and $t \in \mathbb{Q}$,

$$h_m^\theta(t) = \lim_{n \to \infty} L_{(m,0),(n,n\theta)}(\mathbf{B}) - L_{(m,t),(n,n\theta)}(\mathbf{B}) = \lim_{n \to \infty} L_{(m,0),(n+m,(n+m)\theta)}(\mathbf{B}) - L_{(m,t),(n+m,(n+m)\theta)}(\mathbf{B}).$$

Since the environment $\mathbf{B} = \{B_m\}_{m \in \mathbb{Z}}$ is a field of i.i.d. two-sided Brownian motions, the right-hand side implies that $h_m^\theta(t)$ has the same distribution for all m. By this same reasoning, for each $m \in \mathbb{Z}$,

$$\{h_m^\theta(t), v_{m+1}^\theta(t) : t \in \mathbb{Q}\} \overset{d}{=} \{h_0^\theta(t), v_1^\theta(t) : t \in \mathbb{Q}\}.$$

Since Brownian motion satisfies $\lim_{s \to \infty} \frac{B(s)}{s} = 0$ almost surely, for all $m \in \mathbb{Z}$, the following limits also hold almost surely:

$$\lim_{\mathbb{Q} \ni s \to \pm\infty} [B_m(s) - h_{m+1}^\theta(s)] = \mp\infty.$$

Set

$$\Omega^{(\theta)} = \bigcap_{m,r \in \mathbb{Z}} \{(s,t) \mapsto \mathcal{B}^\theta((m,s),(r,t)) \text{ is uniformly continuous on all bounded subsets of } \mathbb{Q} \times \mathbb{Q}\}$$

$$(6.2)$$

$$\cap \bigcap_{m \in \mathbb{Z}} \left\{ \lim_{\mathbb{Q} \ni s \to \pm\infty} [B_m(s) - h_{m+1}^\theta(s)] = \mp\infty \right\}.$$

By almost sure continuity of the functions Y and $\overleftarrow{Q}(Y, B_1)$ (Lemma 3.4), $\mathbb{P}(\Omega^{(\theta)}) = 1$.

Definition 6.2 (Definition of Busemann functions for fixed θ, arbitrary points). On the event $\Omega^{(\theta)}$, for arbitrary $\mathbf{x}, \mathbf{y} \in \mathbb{Z} \times \mathbb{R}$, define $\mathcal{B}^\theta(\mathbf{x}, \mathbf{y})$ such that, for each $m, r \in \mathbb{Z}$,

$$(s,t) \mapsto \mathcal{B}^\theta((m,s),(r,t))$$

is the unique continuous extension of this function from $\mathbb{Q} \times \mathbb{Q}$ to $\mathbb{R}^2$.

The following lemma states properties for a fixed θ, as a precursor to the more general Theorems 3.5 and Theorem 3.7.

Lemma 6.3. *Let* $\theta > 0$. *Then, the following hold.*

(i) *On the event* $\Omega^{(\theta)}$, *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{Z} \times \mathbb{R}$ *and all sequences* $\{t_n\}$ *satisfying* $t_n/n \to \theta$,

$$\mathcal{B}^\theta(\mathbf{x}, \mathbf{y}) = \lim_{n \to \infty} [L_{\mathbf{x},(n,t_n)} - L_{\mathbf{y},(n,t_n)}].$$

(ii) *On the event* $\Omega^{(\theta)}$, *whenever* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{Z} \times \mathbb{R}$,

$$\mathcal{B}^\theta(\mathbf{x}, \mathbf{y}) + \mathcal{B}^\theta(\mathbf{y}, \mathbf{z}) = \mathcal{B}^\theta(\mathbf{x}, \mathbf{z}).$$

(iii) *For* $0 < \gamma < \theta < \infty$, *on the event* $\Omega^{(\theta)} \cap \Omega^{(\gamma)}$, *for all* $m \in \mathbb{Z}$ *and* $s < t \in \mathbb{R}$,

$$0 \le v_m^\gamma(s) \le v_m^\theta(s), \quad and \quad B_m(s,t) \le h_m^\theta(s,t) \le h_m^\gamma(s,t).$$

(iv) *On the event* $\Omega^{(\theta)}$, *for each* $m \in \mathbb{Z}$,

$$\lim_{s \to \pm\infty} B_m(s) - h_{m+1}^\theta(s) = \mp\infty.$$

(v) *For every* $m \in \mathbb{Z}$, *the process* $t \mapsto h_m^\theta(t)$ *is a two-sided Brownian motion with drift* $\frac{1}{\sqrt{\theta}}$. *For each* $m \in \mathbb{Z}$ *and* $t \in \mathbb{R}$, $v_m^\theta(t) \sim \mathrm{Exp}(\frac{1}{\sqrt{\theta}})$.

(vi) *For any* $m \in \mathbb{Z}$, $\{h_r^\theta\}_{r>m}$ *is independent of* $\{B_r\}_{r \le m}$.

(vii) *There exists a full-probability event* $\Omega_1^{(\theta)} \subseteq \Omega^{(\theta)}$ *on which the following hold for all* $m \in \mathbb{Z}$ *and* $t \in \mathbb{R}$:

$$v_{m+1}^\theta(t) = Q(h_{m+1}^\theta, B_m)(t) \quad and \quad h_m^\theta(t) = D(h_{m+1}^\theta, B_m)(t).$$

Proof. *Part (i)*: On $\Omega^{(\theta)}$, let $(m,s), (r,t) \in \mathbb{Z} \times \mathbb{R}$, and let $\{t_n\}$ be any sequence with $t_n/n \to \theta$. Note that, whenever $t_1 < t_2$, $m \in \mathbb{Z}$ and $\mathbf{y} \ge (m, t_2)$,

$$L_{(m,t_1),\mathbf{y}} \ge B_m(t_1, t_2) + L_{(m,t_2),\mathbf{y}}.$$

Then, let $q_1, q_2 \in \mathbb{Q}$ be such that $q_1 < s$ and $q_2 > t$. Then,

$$L_{(m,s),(n,t_n)} - L_{(r,t),(n,t_n)} \le L_{(m,q_1),(n,t_n)} - B_m(q_1, s) - L_{(r,q_2),(n,t_n)} - B_r(t, q_2),$$

and therefore,

$$\limsup_{n \to \infty}[L_{(m,s),(n,t_n)} - L_{(r,t),(n,t_n)}] \le \mathcal{B}^\theta\big((m, q_1), (r, q_2)\big) - B_m(q_1, s) - B_r(t, q_2).$$

Taking $q_1 \nearrow s$ and $q_2 \searrow t$ and using the continuity of Brownian motion and the Busemann functions,

$$\limsup_{n \to \infty}[L_{(m,s),(n,t_n)} - L_{(r,t),(n,t_n)}] \le \mathcal{B}^\theta\big((m, s), (r, t)\big).$$

A similar procedure gives the appropriate lower bound.

Part (ii): This follows from the additivity of the process on the countable dense set (Equation (6.1)) and the construction of Definition 6.2 as the unique continuous extension.

Part (iii): By Lemma A.4, for $s, t \in \mathbb{R}$ and all n such that $n \ge m$ and $n\gamma \ge s \vee t$,

$$0 \le L_{(m,s),(n,n\gamma)} - L_{(m+1,s),(n,n\gamma)} \le L_{(m,s),(n,n\theta)} - L_{(m+1,s),(n,n\theta)}$$

$$and \quad B_m(s,t) \le L_{(m,s),(n,n\theta)} - L_{(m,t),(n,n\theta)} \le L_{(m,s),(n,n\gamma)} - L_{(m,t),(n,n\gamma)}.$$

The proof is complete by Part (i), taking limits as $n \to \infty$.

Part (iv): This follows from the definition of $\Omega^{(\theta)}$ and continuity.

Part (v): Observe that $t \mapsto h_m^\theta(t)$ is a continuous process with the correct finite-dimensional distributions by Lemma C.5, taking limits when necessary. Part (i) and Theorem 2.3 guarantee that $v_m^\theta(t) \sim \mathrm{Exp}(\frac{1}{\sqrt{\theta}})$ for all $t \in \mathbb{R}$.

Part (vi): By Part (i),

$$\{h_r^\theta(t) : r > m, t \in \mathbb{R}\} = \Big\{\lim_{n \to \infty}[L_{(r,0),(n,n\theta)} - L_{(r,t),(n,n\theta)}] : r > m, t \in \mathbb{R}\Big\}.$$

The right-hand side is a function of $\{B_r\}_{r>m}$, which is independent of $\{B_r\}_{r \le m}$.

Part (vii): We start with the first statement. We need to show that

(6.3) $$v_{m+1}^{\theta}(t) = \sup_{t \le s < \infty} \left\{ B_m(t,s) - h_{m+1}^{\theta}(t,s) \right\}.$$

By Part (i),

(6.4) $$\sup_{t \le s < \infty} \left\{ B_m(t,s) - h_{m+1}^{\theta}(t,s) \right\} = \sup_{t \le s < \infty} \left\{ B_m(t,s) + \lim_{n \to \infty} [L_{(m+1,s),(n,n\theta)} - L_{(m+1,t),(n,n\theta)}] \right\}.$$

On the other hand,

(6.5) $$v_{m+1}^{\theta}(t) = \lim_{n \to \infty} [L_{(m,t),(n,n\theta)} - L_{(m+1,t),(n,n\theta)}].$$

For each $s \ge t$ and all n sufficiently large so that $n\theta \ge s$ and $n \ge m+1$,

$$B_m(t,s) + L_{(m+1,s),(n,n\theta)} - L_{(m+1,t),(n,n\theta)} \le L_{(m,t),(n,n\theta)} - L_{(m+1,t),(n,n\theta)}.$$

Taking limits as $n \to \infty$ and comparing (6.4) and (6.5) establishes

$$v_{m+1}^{\theta}(t) \ge \sup_{t \le s < \infty} \left\{ B_m(t,s) - h_{m+1}^{\theta}(t,s) \right\}.$$

Next, by Part (v), $v_{m+1}^{\theta}(t) \sim \mathrm{Exp}(\frac{1}{\sqrt{\theta}})$. By Parts (v) and (vi) and Lemma B.1,

$$\sup_{t \le s < \infty} \left\{ B_m(t,s) - h_{m+1}^{\theta}(t,s) \right\} \overset{d}{=} \sup_{0 \le s < \infty} \left\{ \sqrt{2} B(s) - \frac{1}{\sqrt{\theta}} s \right\} \overset{d}{=} v_{m+1}^{\theta}(t),$$

where B is a standard Brownian motion. Thus, since $v_{m+1}^{\theta}(t) \ge \sup_{t \le s < \infty} \{ B_m(t,s) - h_{m+1}^{\theta}(t,s) \}$, equality holds with probability one for each fixed $t \in \mathbb{R}$. Let $\Omega_1^{(\theta)} \subseteq \Omega^{(\theta)}$, be the event of full probability on which (6.3) holds for all $t \in \mathbb{Q}$. Continuity of both sides of (6.3) (Definition 6.2 and Lemma 3.4) extend the result to all $t \in \mathbb{R}$ on $\Omega_1^{(\theta)}$. The equality for h_m^{θ} then follows by the definitions and additivity of the Busemann functions, as shown below.

$$\begin{aligned} D\left(h_{m+1}^{\theta}, B_m\right)(t) &= h_{m+1}^{\theta}(t) + Q\left(h_{m+1}^{\theta}, B_m\right)(0) - Q\left(h_{m+1}^{\theta}, B_m\right)(t) \\ &= h_{m+1}^{\theta}(t) + v_{m+1}^{\theta}(0) - v_{m+1}^{\theta}(t) \\ &= \mathcal{B}^{\theta}\left((m+1,0),(m+1,t)\right) + \mathcal{B}^{\theta}\left((m,0),(m+1,0)\right) - \mathcal{B}^{\theta}\left((m,t),(m+1,t)\right) \\ &= \mathcal{B}^{\theta}\left((m,0),(m,t)\right) = h_m^{\theta}(t). \qquad \square \end{aligned}$$

6.2. *Global construction of the Busemann process*

We now have the proper framework to define the global Busemann process of Theorem 3.5. Fix a countable dense subset D of $(0,\infty)$. Let

$$\Omega_0 = \bigcap_{\theta \in D} \Omega_1^{(\theta)},$$

and on the event Ω_0, define $\mathcal{B}^{\theta}$ for all $\theta \in D$, as in Definition 6.2. Then, the conclusions of Lemma 6.3 hold for all $\theta \in D$.

Lemma 6.4. *On Ω_0, for each $\theta \in D$, $m \in \mathbb{Z}$, and $t \in \mathbb{R}$, let $\tau_{(m,t)}^{\theta}$ denote the rightmost maximizer of $B_m(s) - h_{m+1}^{\theta}(s)$ over $s \in [t,\infty)$. Such a maximizer exists by continuity and Lemma 6.3(iv). Let Ω_1 be the subset of Ω_0 on which, for each $\theta \in D$, m, $N \in \mathbb{Z}$, and $t \in \mathbb{R}$, the following limits hold:*

(i) $\lim_{D \ni \gamma \to \theta} h_m^{\gamma}(N) = h_m^{\theta}(N)$
(ii) $\lim_{D \ni \gamma \to \infty} h_m^{\gamma}(N) = B_m(N)$.
(iii) $\lim_{D \ni \delta \searrow 0} \tau_{(m,t)}^{\delta} = t$.
(iv) $\lim_{D \ni \gamma \to \infty} \tau_{(m,t)}^{\gamma} = \infty$.

Then, $\mathbb{P}(\Omega_1) = 1$.

Proof. The almost sure uniqueness of maximizers follows from Lemma 6.3(vi) and the $n = m$ case of Lemma 7.1. By Lemma 6.3(iii), $h_m^\gamma(N) = h_m^\gamma(0, N)$ is monotone as $D \ni \gamma \nearrow \theta$, and for $N > 0$,

$$(6.6) \qquad \lim_{D \ni \gamma \nearrow \theta} h_m^\gamma(N) \geq h_m^\theta(N).$$

For negative N, the inequality flips. By Theorem 2.3, $h_m^\gamma(N) \sim \mathcal{N}(\frac{N}{\sqrt{\gamma}}, |N|)$. As $\gamma \nearrow \theta$, this converges in distribution to $\mathcal{N}(\frac{N}{\sqrt{\theta}}, |N|)$. Then, by (6.6), $\lim_{D \ni \gamma \nearrow \theta} h_m^\gamma(N) = h_m^\theta(N)$ with probability one. An analogous argument proves the almost sure convergence for limits from the right and the convergence to $B_m(N)$.

Next, we show that, on Ω_0, Parts (iii) and (iv) hold if and only if they hold for all $t \in \mathbb{Q}$. Assume the statement holds for all $t \in \mathbb{Q}$. By Lemmas 6.3(iii) and A.1, $\tau_{(m,t)}^\delta$ is monotone as $\delta \searrow 0$, and so the limit exists. By definition of $\tau_{(m,t)}^\theta$, we have the inequality $\tau_{(m,s)}^\theta \leq \tau_{(m,t)}^\theta$ whenever $s < t$. Then, for any $t \in \mathbb{R}$ and any $q_1, q_2 \in \mathbb{Q}$ with $q_1 < t < q_2$,

$$q_1 = \lim_{D \ni \delta \searrow 0} \tau_{(m,q_1)}^\delta \leq \lim_{D \ni \delta \searrow 0} \tau_{(m,t)}^\delta \leq \lim_{D \ni \delta \searrow 0} \tau_{(m,q_2)}^\delta = q_2.$$

Taking limits as $q_1 \nearrow t$ and $q_2 \searrow t$ shows the statement for all $t \in \mathbb{R}$. The same argument can be applied to the limits as $\gamma \to \infty$.

Lastly, we show that the limit in (iii) holds with probability one for fixed $t \in \mathbb{R}$. By Lemma 6.3, Parts (v) and (vi),

$$\left\{ B_m(s) - h_{m+1}^\delta(s) : s \in \mathbb{R} \right\} \overset{d}{=} \left\{ \sqrt{2} B(s) - \frac{s}{\sqrt{\delta}} : s \in \mathbb{R} \right\},$$

where B is a standard, two-sided Brownian motion. Then, by Theorem B.2, for $(m, t) \in \mathbb{Z} \times \mathbb{R}$ and $s \geq 0$,

$$\mathbb{P}(\tau_{(m,t)}^\delta > s + t) = \left(2 + \frac{s}{\delta}\right) \Phi\left(-\sqrt{\frac{s}{2\delta}}\right) - \sqrt{\frac{s}{\pi\delta}} e^{-\frac{s}{4\delta}}.$$

Taking limits as $\delta \searrow 0$, it follows that $\tau_{(m,t)}^\theta$ converges weakly to the constant t. Since the limit exists almost surely by monotonicity, the desired conclusion follows. A similar argument applies to show $\lim_{\gamma \to \infty} \tau_{(m,t)}^\gamma = \infty$ with probability one. $\square$

On Ω_1, we extend the definition of $\mathcal{B}^\theta$ to all $\theta > 0$. We proceed similarly as shown for the exponential corner growth model in [52]. For BLPP, there is an additional step that must be taken to guarantee the convergence and continuity of Theorem 3.5, Parts (iii) and (iv).

Lemma 6.5. *On Ω_1, for each $\theta \in D$, $m \in \mathbb{Z}$, and $t \in \mathbb{R}$,*

$$\lim_{D \ni \gamma \to \theta} h_m^\gamma(t) = h_m^\theta(t) \quad and \quad \lim_{D \ni \gamma \to \theta} v_m^\gamma(t) = v_m^\theta(t).$$

For each $m \in \mathbb{Z}$, the convergence is uniform in t on compact subsets of $\mathbb{R}$. Additionally, for each $m \in \mathbb{Z}$,

$$\lim_{D \ni \gamma \to \infty} h_m^\gamma(t) = B_m(t), \quad and \quad \lim_{D \ni \delta \searrow 0} v_m^\theta(t) = 0,$$

uniformly in t on compact subsets of $\mathbb{R}$.

Proof. We first prove the statements for the h_m. We show that

$$\lim_{D \ni \gamma \nearrow \theta} h_m^\gamma(t)$$

exists and equals $h_m^\theta(t)$, uniformly in t on compact subsets of $\mathbb{R}$. The limits from the right (as well as the case of B_m in place of h_m^θ) follow by analogous arguments.

By rearranging the inequality of Lemma 6.3(iii), for $0 < \gamma < \theta < \infty$ and $a < b$ and any $t \in [a, b]$,

$$(6.7) \qquad h_m^\gamma(a) - h_m^\theta(a) \leq h_m^\gamma(t) - h_m^\theta(t) \leq h_m^\gamma(b) - h_m^\theta(b),$$

and the inequality still holds if we replace h_m^θ with B_m. Thus, on the event Ω_1, h_m^γ converges to h_m^θ, uniformly in t on compact subsets of $\mathbb{R}$. Now, we prove the convergence statements for v_m. By Lemma 6.3(vii), on Ω_0, for all $t \in \mathbb{R}$, and all $\theta \in D$,

$$(6.8) \qquad v_m^\theta(t) = Q\big(h_m^\theta, B_{m-1}\big)(t) = \sup_{t \le s < \infty} \big\{ B_{m-1}(t, s) - h_m^\theta(t, s) \big\}.$$

Let $\omega \in \Omega_1$, and $t \in \mathbb{R}$. By the monotonicity of Lemma 6.3(iii) and Lemma A.1, since $\tau_{(m,t)}^{\theta+1}$ is a maximizer of $B_{m-1}(s) - h_m^{\theta+1}(s)$ over $s \in [t, \infty)$, for all $\gamma \in D$ with $\gamma < \theta + 1$,

$$v_m^\gamma(t) = \sup_{t \le s < \infty} \big\{ B_{m-1}(t, s) - h_m^\gamma(t, s) \big\} = \sup_{t \le s \le \tau_{(m,t)}^{\theta+1}} \big\{ B_{m-1}(t, s) - h_m^\gamma(t, s) \big\}.$$

Since h_m^γ converges uniformly on compact sets to h_m^θ, $v_m^\gamma(t)$ converges pointwise to $v_m^\theta(t)$. By Lemma 6.3(iii), the convergence from both right and left is monotone, so by the continuity of Definition 6.2 and Dini's Theorem, the convergence is uniform.

Lastly, for limits as $\delta \searrow 0$, we again apply Lemmas 6.3(iii) and A.1 so that, for $\delta \in D$ with $\delta \le 1$,

$$v_m^\delta(t) = \sup_{t \le s < \infty} \big\{ B_{m-1}(t, s) - h_m^\delta(t, s) \big\} = \sup_{t \le s \le \tau_{(m,t)}^\delta} \big\{ B_{m-1}(t, s) - h_m^\delta(t, s) \big\}$$

$$\le \sup_{t \le s \le \tau_{(m,t)}^\delta} \big\{ B_{m-1}(t, s) - h_m^1(t, s) \big\},$$

and the right-hand side converges to 0 as $\delta \searrow 0$ by the continuity of $B_{m-1} - h_m^1$ and the convergence of $\tau_{(m,t)}^\delta$ to t given in the definition of Ω_1. Dini's Theorem again strengthens the pointwise convergence to uniform convergence. $\qquad \square$

By the additivity of Lemma 6.3(ii), for arbitrary $(m, s), (r, t) \in \mathbb{Z} \times \mathbb{R}$ with $m \le r$,

$$\mathcal{B}^\theta\big((m, s), (r, t)\big) = h_m^\theta(s, t) + \sum_{k=m}^{r-1} v_{k+1}(t).$$

For $m > r$, $\mathcal{B}^\theta((m, s), (r, t)) = -\mathcal{B}^\theta((r, t), (m, s))$, so $\mathcal{B}^\theta((m, s), (r, t))$ is still a sum of horizontal and vertical increments. Then, by Lemma 6.5, on the event Ω_1, for all $\theta \in D$ and $\mathbf{x}, \mathbf{y} \in \mathbb{Z} \times \mathbb{R}$,

$$(6.9) \qquad \lim_{D \ni \gamma \to \theta} \mathcal{B}^\gamma(\mathbf{x}, \mathbf{y}) = \mathcal{B}^\theta(\mathbf{x}, \mathbf{y}).$$

Definition 6.6. On the event Ω_1, for an arbitrary $\theta > 0$ and $\mathbf{x}, \mathbf{y} \in \mathbb{Z} \times \mathbb{R}$, define the following.

$$\mathcal{B}^{\theta-}(\mathbf{x}, \mathbf{y}) = \lim_{D \ni \gamma \nearrow \theta} \mathcal{B}^\gamma(\mathbf{x}, \mathbf{y}), \quad \text{and} \quad \mathcal{B}^{\theta+}(\mathbf{x}, \mathbf{y}) = \lim_{D \ni \delta \searrow \theta} \mathcal{B}^\delta(\mathbf{x}, \mathbf{y}).$$

Remark 6.7. By the additivity of Lemma 6.3(ii) and the monotonicity of Lemma 6.3(iii), these limits exist for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z} \times \mathbb{R}$ and $\theta > 0$. By (6.9), on Ω_1, for all $\theta \in D$ and $\mathbf{x}, \mathbf{y} \in \mathbb{Z} \times \mathbb{R}$, $\mathcal{B}^{\theta+}(\mathbf{x}, \mathbf{y}) = \mathcal{B}^{\theta-}(\mathbf{x}, \mathbf{y}) = \mathcal{B}^\theta(\mathbf{x}, \mathbf{y})$.

Proof of Theorem 3.5. *Parts (i)–(ii)*: These follow by taking limits in Parts (ii) and (iii) of Lemma 6.3.

Parts (iii)–(iv): By Part (ii), it suffices to take limits along the countable dense set D. Then, Lemma 6.5 establishes Parts (c) and (d). The monotonicity of Part (ii) can be rearranged, just as in Equation (6.7), which strengthens the pointwise convergence of the $h_m^{\gamma \square}$ to uniform convergence on compact sets. Thus, since h_m^γ is continuous for $\gamma \in D$, $h_m^{\theta \square}$ is also continuous for $\square \in \{+, -\}$. Now, for the convergence of the $v_m^{\theta \square}$, recall that on the event Ω_1, for all $t \in \mathbb{R}$ and $\gamma \in D$,

$$v_m^\gamma(t) = Q\big(h_m^\gamma, B_{m-1}\big)(t) = \sup_{t \le s < \infty} \big\{ B_{m-1}(t, s) - h_m^\gamma(t, s) \big\}.$$

By Part (ii) and Lemma A.1, this supremum may be restricted to a common compact set for all $\gamma < \theta$. Then, since h_m^γ converges uniformly to $h_m^{\theta-}$ as $D \ni \gamma \nearrow \theta$, $v_m^\gamma(t)$ converges pointwise to $Q(h_m^{\theta-}, B_{m-1})(t)$ as $\gamma \nearrow \theta$. But, by definition, $v_m^{\theta-}(t) = \lim_{D \ni \gamma \nearrow \theta} v_m^\gamma(t)$, so $v_m^{\theta-}(t) = Q(h_m^{\theta-}, B_{m-1})(t)$, which is continuous by continuity of $h_m^{\theta-}$ and B_{m-1}. Since the convergence is monotone, Dini's Theorem implies the convergence is uniform on compact sets. The proof for $\delta \searrow \theta$

is analogous. Part (iv) follows by continuity of the Busemann functions from Definition 6.2, the uniform convergence of Part (iii), and the additivity in Part (i).

Part (v): This follows from Lemma 6.3(iv) and the monotonicity of Part (ii).

Part (vi): The equality for $v_{m+1}^{\theta\square}$ was shown in the proof of Part (iii). The equality for h_m^θ follows by the additivity of Part (i) and the same argument as in the proof of Lemma 6.3(vii).

Part (vii): This follows because $\{h_r^{\theta\square} : \theta > 0, \square \in \{+, -\}, r > m\}$ is a function of $\{B_r\}_{r>m}$.

Part (viii): Fix $\theta > 0$. We first show that, for each $\theta > 0$ and $\mathbf{x}, \mathbf{y} \in \mathbb{Z} \times \mathbb{R}$, there exists an event $\Omega_{\mathbf{x},\mathbf{y}}^{(\theta)}$ on which

$$(6.10) \qquad \mathcal{B}^{\theta-}(\mathbf{x}, \mathbf{y}) = \lim_{n\to\infty} [L_{\mathbf{x},(n,t_n)} - L_{\mathbf{y},(n,t_n)}] = \mathcal{B}^{\theta+}(\mathbf{x}, \mathbf{y}),$$

for all sequences $\{t_n\}$ with $\lim_{n\to\infty} \frac{t_n}{n} = \theta$. By Theorem 2.3, for each $\mathbf{x}$, $\mathbf{y}$, and $\theta > 0$, there exists an event of full probability on which the limit in (6.10) exists and is independent of the choice of sequence. Then, by additivity, it is sufficient to show that, for each fixed $\theta > 0$, $m \in \mathbb{Z}$, and $t \in \mathbb{R}$, with probability one,

$$h_m^{\theta-}(t) = \lim_{n\to\infty} [L_{(m,0),(n,n\theta)} - L_{(m,t),(n,n\theta)}] = h_m^{\theta+}(t), \quad \text{and}$$

$$v_m^{\theta-}(t) = \lim_{n\to\infty} [L_{(m-1,t),(n,n\theta)} - L_{(m,t),(n,n\theta)}] = v_m^{\theta+}(t).$$

We show that

$$(6.11) \qquad h_m^{\theta-}(t) = \lim_{n\to\infty} [L_{(m,0),(n,n\theta)} - L_{(m,t),(n,n\theta)}] \quad \text{a.s.},$$

and the other statements follow by analogous arguments. By Theorem 2.3, the right-hand side of (6.11) has distribution $\mathcal{N}(\frac{t}{\sqrt{\theta}}, |t|)$. By definition, $h_m^{\theta-}(t)$ is the limit, as $\gamma \nearrow \theta$, of $h_m^\gamma(t)$, and $h_m^\gamma(t) \sim \mathcal{N}(\frac{t}{\sqrt{\gamma}}, |t|)$, which converges weakly to $\mathcal{N}(\frac{t}{\sqrt{\theta}}, |t|)$. Using Theorem 6.3(i) and Lemma A.4, for each fixed $t > 0$ and all $D \ni \gamma < \theta$,

$$h_m^\gamma(t) = \lim_{n\to\infty} [L_{(m,0),(n,n\gamma)} - L_{(m,t),(n,n\gamma)}] \geq \lim_{n\to\infty} [L_{(m,0),(n,n\theta)} - L_{(m,t),(n,n\theta)}] \quad \text{a.s.}$$

The inequality flips for $t < 0$. Hence, both sides in (6.11) have the same distribution, while one dominates the other, so they are equal with probability one.

Next, set

$$\Omega^{(\theta)} = \bigcap_{\mathbf{x},\mathbf{y}\in\mathbb{Z}\times\mathbb{Q}} \Omega_{\mathbf{x},\mathbf{y}}^{(\theta)}.$$

Then, using the continuity of Part (iv), the desired conclusion follows by the same reasoning as in the proof of Lemma 6.3(i).

Part (ix): This follows from the construction of the Busemann functions as limits of BLPP times and the shift invariance of BLPP.   □

7. Proofs of the results for semi-infinite geodesics

Throughout this section, Ω_1 is the event defined in Lemma 6.4 and referenced in Theorem 3.5.

7.1. *Key lemmas*

For two fields of Brownian motions, $\mathbf{B}$ and $\overline{\mathbf{B}}$, $\lambda > 0$, and an initial point $(m, s) \in \mathbb{Z} \times \mathbb{R}$, define the point-to-line last passage time:

$$(7.1) \qquad \overline{L}_{(m,s),n}^\lambda(\mathbf{B}, \overline{\mathbf{B}}) = \sup\left\{\sum_{r=m}^n B_r(s_{k-1}, s_k) - \overline{B}_{n+1}(s_n) - \lambda s_n : \mathbf{s}_{m,n} \in \Pi_{(m,s),n}\right\},$$

where $\Pi_{(m,s),n}$ denotes the set of sequences $s = s_{m-1} \leq s_m \leq \cdots \leq s_n$. The following is due to Hammond [31].

Lemma 7.1 ([31], Lemma B.2). *Let $\mathbf{B}$ be a field of independent, two-sided Brownian motions and $\overline{\mathbf{B}}$ be any other field of Brownian motions such that, for each n, $\overline{B}_{n+1}$ is independent of $\{B_m\}_{m\leq n}$. Fix an initial point (m, s) and let $n \geq m$. Then, with probability one, the quantity in (7.1) is finite, and there is a unique sequence $\mathbf{s}_{m,n} \in \Pi_{(m,s),n}$ that is maximal for (7.1).*

Remark 7.2. Lemma 7.1 is stated slightly differently in [31]. In that paper, the function $s \mapsto -\overline{B}_{n+1}(s) - \lambda s$ is replaced by an arbitrary deterministic and measurable function $h : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ satisfying $h(t) > -\infty$ for some $t > s$ and $\limsup_{t \to \infty} h(t)/t < 0$. The assumption that $\{\overline{B}_{n+1}\}_{n+1}$ is independent of $\{B_m\}_{m \le n}$ allows us to condition on $\overline{B}_{n+1}$ and obtain the desired result.

Lemma 7.3. *Let* $\omega \in \Omega_1$, $(m, t) \in \mathbb{Z} \times \mathbb{R}$, $\theta > 0$, *and* $\square \in \{+, -\}$. *Then, the following hold.*

(i) *Let* $\{\tau_r\}_{r=m-1}^{\infty}$ *be any sequence in* $\mathbf{T}_{(m,t)}^{\theta\square}$. *Then, for each* $n \ge m$, *the jump times* $t = \tau_{m-1} \le \tau_m \le \cdots \le \tau_n$ *are a maximizing sequence for*

$$(7.2) \qquad \overline{L}_{(m,t),n}^{\lambda}(\mathbf{B}, \overline{\mathbf{B}}) := \sup\left\{ \sum_{r=m}^{n} B_r(s_{r-1}, s_r) - h_{n+1}^{\theta\square}(s_n) : \mathbf{s}_{m,n} \in \Pi_{(m,t),n} \right\}.$$

Here, $\overline{L}_{(m,t),n}^{\lambda}(\mathbf{B}, \overline{\mathbf{B}})$ *is as in* (7.1), *where* $\lambda = \frac{1}{\sqrt{\theta}}$, $\mathbf{B} = \{B_r\}_{r \in \mathbb{Z}}$, *and* $\overline{\mathbf{B}} = \{h_r^{\theta\square}(t) - \frac{t}{\sqrt{\theta}} : t \in \mathbb{R}\}_{r \in \mathbb{Z}}$.

(ii) *Conversely, for each* $n \ge m$, *whenever* $t = t_{m-1} \le t_m \le \cdots \le t_n$ *is a maximizing sequence for* (7.2), *there exists* $\{\tau_r\}_{r=m-1}^{\infty} \in \mathbf{T}_{(m,t)}^{\theta\square}$ *such that* $t_r = \tau_r$ *for* $m \le r \le n$.

(iii) *For each* $n \ge m$, *the sequences* $t = \tau_{(m,t),m-1}^{\theta\square,L} \le \cdots \le \tau_{(m,t),n}^{\theta\square,L}$ *and* $t = \tau_{(m,t),m-1}^{\theta\square,R} \le \cdots \le \tau_{(m,t),n}^{\theta\square,R}$ *are, respectively, the leftmost and rightmost maximizing sequences for* (7.2).

Proof. *Part (i)*: Recall by (4.1), that $\tau_m \ge t$ is a maximizer of

$$B_m(t, s) - h_{m+1}^{\theta\square}(s) \quad \text{over } s \in [t, \infty).$$

Hence, the statement holds for $n = m$. Now, assume that the statement holds for some $n \ge m$. Then, $t = \tau_{m-1} \le \cdots \le \tau_n$ satisfies

$$(7.3) \qquad \sum_{r=m}^{n} B_r(\tau_{r-1}, \tau_r) - h_{n+1}^{\theta\square}(\tau_n) = \sup\left\{ \sum_{r=m}^{n} B_r(s_{r-1}, s_r) - h_{n+1}^{\theta\square}(s_n) : \mathbf{s}_{m,n} \in \Pi_{(m,t),n} \right\}.$$

Using Theorem 3.5(vi) and rearranging the terms in the definition (3.4) of the operator D,

$$(7.4) \qquad h_{n+1}^{\theta\square}(s_n) = B_{n+1}(s_n) + \sup_{0 \le s < \infty} \left\{ B_{n+1}(s) - h_{n+2}^{\theta\square}(s) \right\} - \sup_{s_n \le s_{n+1} < \infty} \left\{ B_{n+1}(s_{n+1}) - h_{n+2}^{\theta\square}(s_{n+1}) \right\},$$

for $s_n \in \mathbb{R}$. Specifically, since τ_{n+1} is a maximizer of $B_{n+1}(s) - h_{n+2}^{\theta\square}(s)$ over $s \in [\tau_n, \infty)$,

$$(7.5) \qquad h_{n+1}^{\theta\square}(\tau_n) = -B_{n+1}(\tau_n, \tau_{n+1}) + h_{n+2}^{\theta\square}(\tau_{n+1}) + \sup_{0 \le s < \infty} \left\{ B_{n+1}(s) - h_{n+2}^{\theta\square}(s) \right\}$$

Substituting (7.4) and (7.5) into (7.3) and discarding the term $\sup_{0 \le s < \infty}\{B_{n+1}(s) - h_{n+2}^{\theta\square}(s)\}$ on both sides,

$$\sum_{r=m}^{n+1} B_r(\tau_{r-1}, \tau_r) - h_{n+2}^{\theta\square}(\tau_{n+1})$$

$$= \sup\left\{ \sum_{r=m}^{n} B_r(s_{r-1}, s_r) - B_{n+1}(s_n) + \sup_{s_n \le s_{n+1} < \infty} \left\{ B_{n+1}(s_{n+1}) - h_{n+2}^{\theta\square}(s_{n+1}) \right\} : \mathbf{s}_{m,n} \in \Pi_{(m,t),n} \right\}$$

$$= \sup\left\{ \sum_{r=m}^{n+1} B_r(s_{r-1}, s_r) - h_{n+2}^{\theta\square}(s_{n+1}) : \mathbf{s}_{m,n+1} \in \Pi_{(m,t),n+1} \right\}.$$

Part (ii): We prove this part by induction. First, note that in the case $n = m$, maximizers of $B_n(t, s_m) - h_{m+1}^{\theta\square}(s_m)$ over $s_m \in [t, \infty)$ are precisely those that are the first jump times of the Busemann semi-infinite geodesics. Now, assume that the statement holds for n. We show that if $t = t_{m-1} \le \cdots \le t_{n+1}$ is a maximizing sequence for (7.3) (with n replaced by $n + 1$), then t_{n+1} is a maximizer of $B_{n+1}(u) - h_{n+2}^{\theta\square}(u)$ over $u \in [t_n, \infty)$, and $t = t_{m-1} \le \cdots \le t_n$ is a maximizing sequence for (7.3).

With this procedure mapped out, observe that, as in the proof of Part (i),

$$\sup\left\{\sum_{r=m}^{n+1} B_r(s_{r-1}, s_r) - h_{n+2}^{\theta\square}(s_{n+1}) : \mathbf{s}_{m,n+1} \in \Pi_{(m,t),n+1}\right\}$$

$$= \sup\left\{\sum_{r=m}^{n} B_r(s_{r-1}, s_r) - B_{n+1}(s_n) + \sup_{s_n \leq s_{n+1} < \infty} \left\{B_{n+1}(s_{n+1}) - h_{n+2}^{\theta\square}(s_{n+1})\right\} : \mathbf{s}_{m,n} \in \Pi_{(m,t),n}\right\}$$

Hence, any maximizing sequence $t = t_{m-1} \leq \cdots \leq t_{n+1}$ must satisfy

$$B_{n+1}(t_{n+1}) - h_{n+2}^{\theta\square}(t_{n+1}) = \sup_{t_n \leq u < \infty} \left\{B_{n+1}(u) - h_{n+2}^{\theta\square}(u)\right\}.$$

Furthermore, $t = t_{m-1} \leq \cdots \leq t_n$ is a maximizing sequence for

$$\sum_{r=m}^{n} B_r(s_{r-1}, s_r) - B_{n+1}(s_n) + \sup_{s_n \leq u < \infty} \left\{B_{n+1}(u) - h_{n+2}^{\theta\square}(u)\right\}$$

over all sequences $\mathbf{s}_{m,n} \in \Pi_{(m,t),n}$ Subtracting off a constant, $t = t_{m-1} \leq \cdots \leq t_n$ is also a maximizing sequence for

$$\sum_{r=m}^{n} B_r(s_{r-1}, s_r) - \left(B_{n+1}(s_n) + \sup_{0 \leq u < \infty} \left\{B_{n+1}(u) - h_{n+2}^{\theta\square}(u)\right\} - \sup_{s_n \leq u < \infty} \left\{B_{n+1}(u) - h_{n+2}^{\theta\square}(u)\right\}\right)$$

$$= \sum_{r=m}^{n} B_r(s_{r-1}, s_r) - h_{n+1}^{\theta\square}(s_n),$$

where the last line comes by (7.4). This completes the inductive step.

Part (iii): This follows by Parts (i) and (ii) since $t = \tau_{(m,t),m-1}^{\theta\square,L} \leq \tau_{(m,t),m}^{\theta\square,L} \leq \cdots$ and $t = \tau_{(m,t),m-1}^{\theta\square,R} \leq \tau_{(m,t),m}^{\theta\square,R} \leq \cdots$ are respectively the leftmost and rightmost sequences in $\mathbf{T}_{(m,t)}^{\theta\square}$. $\square$

Lemma 7.4. *Let $\omega \in \Omega_1$, $(m, t) \in \mathbb{Z} \times \mathbb{R}$, $\theta > 0$, and $\square \in \{+, -\}$, and $\{\tau_r\}_{r \geq m-1} \in \mathbf{T}_{(m,t)}^{\theta\square}$. Then, for all $r \geq m$,*

$$v_{r+1}^{\theta\square}(\tau_r) = 0, \quad and \quad h_r^{\theta\square}(u, v) = B_r(u, v) \quad for \ all \ u, v \in [\tau_{r-1}, \tau_r].$$

Furthermore, the following identities hold for $r \geq m$.

(7.6) $$\tau_{(m,t),r}^{\theta\square,L} = \inf\left\{u \geq \tau_{(m,t),r-1}^{\theta\square,L} : v_{r+1}^{\theta\square}(u) = 0\right\} \quad and$$

(7.7) $$\tau_{(m,t),r}^{\theta\square,R} = \sup\left\{u \geq \tau_{(m,t),r-1}^{\theta\square,R} : h_r^{\theta\square}\left(\tau_{(m,t),r-1}^{\theta\square,R}, u\right) = B_r\left(\tau_{(m,t),r-1}^{\theta\square,R}, u\right)\right\}$$

More specifically, if $u \geq \tau_{(m,t),r-1}^{\theta\square,R}$, then $h_r^{\theta\square}(\tau_{(m,t),r-1}^{\theta\square,R}, u) = B_r(\tau_{(m,t),r-1}^{\theta\square,R}, u)$ if and only if $u \leq \tau_{(m,t),r}^{\theta\square,R}$.

Remark 7.5. The first part of Lemma 7.4 says that, along any semi-infinite path in $\mathbf{T}_{(m,t)}^{\theta\square}$, the Busemann process agrees with the energy of the semi-infinite geodesic. That is, at every vertical jump from (r, τ_r) to $(r + 1, \tau_r)$, the vertical Busemann function $v_{r+1}^{\theta\square}$ equals zero. This is as it should be because, according to (2.1), a vertical step of a path does not collect any energy from the environment. Along each horizontal step from (r, τ_{r-1}) to (r, τ_r), the increment of the horizontal Busemann function $h_r^{\theta\square}$ agrees with the increment of the Brownian motion B_r. Equations (7.6) and (7.7) are more subtle. Let s_{r-1} be the time when a $\theta\square$ geodesic jumps from level $r - 1$ to level r. Equation (7.6) says that the leftmost $\theta\square$ geodesic jumps from level r to $r + 1$ at the *first* time $u \geq s_{r-1}$ such that $v_{r+1}^{\theta\square}(u) = 0$. Equation (7.7) says that the rightmost geodesic jumps from level r to $r + 1$ at the *last* time $u \geq s_{r-1}$ such that $B_r(s_{r-1}, u) = h_r^{\theta\square}(s_{r-1}, u)$. At all subsequent times $v > u$ the equality is lost, and we have $B_r(s_{r-1}, v) < h_r^{\theta\square}(s_{r-1}, v)$.

Proof of Lemma 7.4. By Theorem 3.5(vi) and the definition of τ_r as a maximizer,

$$0 \leq v_{r+1}^{\theta\square}(\tau_r) = \sup_{\tau_r \leq s < \infty} \left\{B_r(\tau_r, s) - h_{r+1}^{\theta\square}(\tau_r, s)\right\}$$

$$= \sup_{\tau_r \le s < \infty} \left\{ B_r(s) - h_{r+1}^{\theta\square}(s) \right\} - \left[B_r(\tau_r) - h_{r+1}^{\theta\square}(\tau_r) \right]$$

$$\le \sup_{\tau_{r-1} \le s < \infty} \left\{ B_r(s) - h_{r+1}^{\theta\square}(s) \right\} - \left[B_r(\tau_r) - h_{r+1}^{\theta\square}(\tau_r) \right] = 0.$$

To establish (7.6), assume, by way of contradiction, that for some $\tau_{(m,t),r-1}^{\theta\square,L} \le u < \tau_{(m,t),r}^{\theta\square,L}$,

$$0 = v_{r+1}^{\theta\square}(u) = \sup_{u \le s < \infty} \left\{ B_r(u, s) - h_{r+1}^{\theta\square}(u, s) \right\}.$$

Then,

$$B_m(u) - h_{m+1}^{\theta\square}(u) \ge B_m(s) - h_{m+1}^{\theta\square}(s)$$

for all $s \ge u$, and specifically for $s = \tau_{(m,t),r}^{\theta\square,L}$. This contradicts the definition of $\tau_{(m,t),r}^{\theta\square,L}$ as the leftmost maximizer of $B_r(s) - h_{r+1}^{\theta\square}(s)$ over $s \in [\tau_{(m,t),r-1}^{\theta\square,L}, \infty)$.

By Theorem 3.5(vi),

(7.8) $$h_r^{\theta\square}(u, v) = B_r(u, v) + \sup_{u \le s < \infty} \left\{ B_r(s) - h_{r+1}^{\theta\square}(s) \right\} - \sup_{v \le s < \infty} \left\{ B_r(s) - h_{r+1}^{\theta\square}(s) \right\},$$

so, since τ_r maximizes $B_r(s) - h_{r+1}^{\theta\square}(s)$ on $[\tau_{r-1}, \infty)$, the two supremum terms above are both equal to $B_r(\tau_r) - h_{r+1}^{\theta\square}(\tau_r)$ whenever $u, v \in [\tau_{r-1}, \tau_r]$. Therefore, $h_r^{\theta\square}(u, v) = B_r(u, v)$. Now we establish (7.7). If, for some $r \ge m$, (7.7) fails, then by (7.8), for some $u > \tau_{(m,t),r}^{\theta\square,R}$,

$$\sup_{\tau_{(m,t),r-1}^{\theta\square,R} \le s < \infty} \left\{ B_r(s) - h_{r+1}^{\theta\square}(s) \right\} = \sup_{u \le s < \infty} \left\{ B_r(s) - h_{r+1}^{\theta\square}(s) \right\}.$$

Thus, all maximizers of $B_r(s) - h_{r+1}^{\theta\square}(s)$ over $s \in [u, \infty)$ are also maximizers over the larger set $[\tau_{(m,t),r-1}^{\theta\square,R}, \infty)$. But this is a contradiction because $\tau_{(m,t),r}^{\theta\square,R}$ is the rightmost maximizer over $s \in [\tau_{(m,t),r-1}^{\theta\square,R}, \infty)$. $\square$

Recall, by Remark 4.2, that on the event $\Omega^{(\theta)} \subseteq \Omega_1$ of Theorem 3.5, $\mathbf{T}_{\mathbf{x}}^{\theta+} = \mathbf{T}_{\mathbf{x}}^{\theta-} = \mathbf{T}_{\mathbf{x}}^{\theta}$ for all $\mathbf{x} \in \mathbb{Z} \times \mathbb{R}$.

Lemma 7.6. *Fix $\theta > 0$ and $\mathbf{x} = (m, t) \in \mathbb{Z} \times \mathbb{R}$. Then there exists an event $\widetilde{\Omega}_{\mathbf{x}}^{(\theta)} \subseteq \Omega^{(\theta)}$, of probability one, on which the set $\mathbf{T}_{\mathbf{x}}^{\theta}$ contains exactly one sequence $\{\tau_r\}_{r \ge m-1}$. This sequence satisfies*

$$\lim_{n \to \infty} \frac{\tau_n}{n} = \theta.$$

To prove this lemma, we need some machinery from [2]. For two fields of Brownian motions $\mathbf{B}$ and $\overline{\mathbf{B}}$ and a subset $A \subset \mathbb{R}$, define

(7.9) $$\overline{L}_{(m,s),n}^{\lambda}(\mathbf{B}, \overline{\mathbf{B}}; s_n \in A) = \sup \left\{ \sum_{r=m}^{n} B_r(s_{r-1}, s_r) - \overline{B}_{n+1}(s_n) - \lambda s_n : \mathbf{s}_{m,n} \in \Pi_{(m,s),n}, s_n \in A \right\}.$$

The only difference between this definition and (7.1) is the restriction on the s_n.

Lemma 7.7 ([2], page 1949). *Let $\mathbf{B}$ be a field of independent, two-sided Brownian motions and $\overline{\mathbf{B}}$ an arbitrary field of Brownian motions. Fix $s \in \mathbb{R}$, $0 \le S \le T \le \infty$, and $m \in \mathbb{Z}$. Then, with probability one,*

$$\lim_{n \to \infty} n^{-1} \overline{L}_{(m,s),n}^{\lambda}(\mathbf{B}, \overline{\mathbf{B}}; s + nS \le s_n \le s + nT) = \sup_{S \le t \le T} \{2\sqrt{t} - \lambda t\}.$$

Remark 7.8. The appearance of the term $2\sqrt{t}$ in Lemma 7.7 comes from the shape theorem for BLPP. Namely, the following limit holds with probability one.

$$\lim_{n \to \infty} n^{-1} L_{(0,0),(n,nt)} = 2\sqrt{t}.$$

This almost sure convergence was first proved in [28] (see also [41] for an alternative proof). If $\overline{\mathbf{B}}_{n+1}$ is independent of $\mathbf{B}_m, \ldots, \mathbf{B}_n$, then the quantity $\overline{L}^\lambda_{(m,s),n}(\mathbf{B}, \overline{\mathbf{B}})$ is distributed as the sum of $n - m + 1$ independent exponential random variables with rate λ and has the interpretation as the sum of vertical increments in the increment-stationary BLPP model. (see Appendix C of the present paper and Section 4 of [45]). Then, in the case $S = 0$ and $T = \infty$, Lemma 7.7 degenerates to an application of the law of large numbers, namely

$$\lim_{n \to \infty} n^{-1} \overline{L}^\lambda_{(m,s),n}(\mathbf{B}, \overline{\mathbf{B}}) = \lambda^{-1}.$$

Lemma 7.9 ([2], Lemma 4.7). *Let* $\mathbf{B}$ *and* $\overline{\mathbf{B}}$ *satisfy the same conditions of Lemma 7.7. Fix* $\lambda > 0$, $s \in \mathbb{R}$, *and* $m \in \mathbb{Z}$. *If* $0 < \theta < \lambda^{-2}$, *then there exist a nonrandom* $\varepsilon = \varepsilon(\lambda, \theta) > 0$ *such that, with probability one, for all sufficiently large* n,

$$\overline{L}^\lambda_{(m,s),n}(\mathbf{B}, \overline{\mathbf{B}}; s_n \leq s + n\theta) + n\varepsilon < \overline{L}^\lambda_{(m,s),n}(\mathbf{B}, \overline{\mathbf{B}}).$$

Similarly, if $\theta > \lambda^{-2}$, *there exists a nonrandom* $\varepsilon = \varepsilon(\lambda, \theta) > 0$ *such that, with probability one, for all sufficiently large* n,

$$\overline{L}^\lambda_{(m,s),n}(\mathbf{B}, \overline{\mathbf{B}}; s_n \geq s + n\theta) + n\varepsilon < \overline{L}^\lambda_{(m,s),n}(\mathbf{B}, \overline{\mathbf{B}})$$

Lemma 7.9 is slightly stronger than the result stated in [2], so we include a proof.

Proof. We prove the first statement, and the second follows analogously. The unique maximum of $2\sqrt{t} - \lambda t$ for $t \in [0, \infty)$ is achieved at $t = \lambda^{-2}$. Then, by Lemma 7.7 and the assumption $\theta < \lambda^{-2}$,

$$\lim_{n \to \infty} n^{-1} \overline{L}^\lambda_{(m,s),n}(\mathbf{B}, \overline{\mathbf{B}}; s_n \leq s + n\theta) = \sup_{0 \leq t \leq \theta} \{2\sqrt{t} - \lambda t\} < \sup_{0 \leq t < \infty} \{2\sqrt{t} - \lambda t\} = \lim_{n \to \infty} n^{-1} \overline{L}^\lambda_{(m,s),n}(\mathbf{B}, \overline{\mathbf{B}}).$$

Hence, for sufficiently large n,

$$\overline{L}^\lambda_{(m,s),n}(\mathbf{B}, \overline{\mathbf{B}}; s_n \leq s + n\theta) + n\varepsilon < \overline{L}^\lambda_{(m,s),n}(\mathbf{B}, \overline{\mathbf{B}}),$$

where

$$\varepsilon = \frac{1}{2} \left(\sup_{0 \leq t < \infty} \{2\sqrt{t} - \lambda t\} - \sup_{0 \leq t \leq \theta} \{2\sqrt{t} - \lambda t\} \right). \qquad \square$$

Proof of Lemma 7.6. By Lemma 7.3(i), for a sequence $\{\tau_r\}_{r \geq m-1} \in \mathbf{T}^\theta_{(m,t)}$, the jump times $t = \tau_{m-1} \leq \cdots \leq \tau_n$ are a maximizing sequence for the point-to-line last passage time (7.2). By Theorem 3.5(vii), h^θ_{n+1} and $\{B_r\}_{r \leq n}$ are independent for each n. This allows us to apply the almost sure uniqueness of point-to-line last-passage maximizers in Lemma 7.1. Let $\widetilde{\Omega}^{(\theta)}_{\mathbf{x}}$ be the event on which these maximizers are unique and on which $\tau_n/n \to \theta$ for the (now) almost surely unique sequence $\{\tau_r\}_{r \geq m-1} \in \mathbf{T}^\theta_{(m,t)}$. It remains to show that

$$\mathbb{P}\left(\lim_{n \to \infty} \frac{\tau_n}{n} = \theta \right) = 1.$$

For $\gamma < \theta = \lambda^{-2}$, Lemma 7.9 guarantees that, with probability one, for all sufficiently large n,

$$\overline{L}^\lambda_{(m,t),n}(\mathbf{B}, \overline{\mathbf{B}}; s_n \leq s + n\gamma) < \overline{L}^\lambda_{(m,t),n}(\mathbf{B}, \overline{\mathbf{B}}).$$

Therefore, by Lemma 7.3(i), $\tau_n > s + n\gamma$ for all sufficiently large n. Thus, for $\gamma < \theta$,

$$\mathbb{P}\left(\liminf_{n \to \infty} \frac{\tau_n}{n} \geq \gamma \right) = 1, \qquad \text{so by taking } \gamma \nearrow \theta, \quad \mathbb{P}\left(\liminf_{n \to \infty} \frac{\tau_n}{n} \geq \theta \right) = 1.$$

A symmetric argument using the second statement of Lemma 7.9 shows that

$$\mathbb{P}\left(\limsup_{n \to \infty} \frac{\tau_n}{n} \leq \theta \right) = 1. \qquad \square$$

7.2. *Proofs of Theorems* 4.3–4.7:

We now begin to prove the theorems of Section 4.2. First, we define the event Ω_2 used in the theorems. Let $\widetilde{\Omega}_{\mathbf{x}}^{(\gamma)}$ be the events of Lemma 7.6. For the countable dense set $D \subseteq (0, \infty)$ of Section 6, set

$$\Omega_2 := \bigcap_{\gamma \in D, \mathbf{x} \in \mathbb{Z} \times \mathbb{Q}} \widetilde{\Omega}_{\mathbf{x}}^{(\gamma)} \tag{7.10}$$

Then, $\mathbb{P}(\Omega_2) = 1$ and $\Omega_2 \subseteq \bigcap_{\theta \in \mathbb{Q}_{>0}} \Omega^{(\theta)} \subseteq \Omega_1$.

Proof of Theorem 4.3. *Part (i)*: On the event Ω_2, let $(m, t) \in \mathbb{Z} \times \mathbb{R}$, $\theta > 0$ and $\square \in \{+, -\}$, and let $t = \tau_{m-1} \leq \tau_m \leq \cdots$ be a sequence in $\mathbf{T}_{(m,t)}^{\theta\square}$. Let Γ be the associated path. By Lemma 7.4,

$$v_{r+1}^{\theta\square}(\tau_r) = 0, \quad \text{and} \quad h_r^{\theta\square}(s, t) = B_r(s, t) \quad \text{for } \tau_{r-1} \leq s \leq t \leq \tau_r. \tag{7.11}$$

We take $\mathbf{x} = \mathbf{y} = (m, t)$ and $\mathbf{z} = (n, \tau_n)$. The case for general $\mathbf{y} \leq \mathbf{z}$ along the path Γ follows by the same argument. By (7.11) and additivity of Busemann functions (Theorem 3.5(i)), the energy of path Γ between $\mathbf{x}$ and $\mathbf{y}$ is given by

$$\sum_{r=m}^{n} \left(B_r(\tau_{r-1}, \tau_r)\right) = \sum_{r=m}^{n-1} \left(h_r^{\theta\square}(\tau_{r-1}, \tau_r) + v_{r+1}^{\theta\square}(\tau_r)\right) + h_n^{\theta\square}(\tau_{n-1}, \tau_n) = \mathcal{B}^{\theta\square}\left((m, t), (n, \tau_n)\right).$$

Let jump times $t = s_{m-1} \leq s_m \leq s_{m+1} \leq \cdots \leq s_n = \tau_n$ define any other path between (m, t) and (n, τ_n). Then, by Theorem 3.5, Parts (i) and (ii), the energy of this path is

$$\sum_{r=m}^{n} B_r(s_{r-1}, s_r) \leq \sum_{r=m}^{n-1} \left(h_r^{\theta\square}(s_{r-1}, s_r) + v_{r+1}^{\theta\square}(s_r)\right) + h_n^{\theta\square}(s_{n-1}, \tau_n) = \mathcal{B}^{\theta\square}\left((m, t), (n, \tau_n)\right). \tag{7.12}$$

Part (ii): By Theorem 3.5(ii), equality holds in Equation (7.12) only if $B_r(s_{r-1}, s_r) = h_r^{\theta\square}(s_{r-1}, s_r)$ for $m \leq r \leq n$ and $v_{r+1}^{\theta\square}(s_r) = 0$ for $m \leq r \leq n - 1$. Then, the statement follows by the Equations (7.6) and (7.7) of Lemma 7.4.

Part (iii)(a): The key is Theorem 3.5(ii). We show that, for $r \geq m$,

$$\tau_r^- := \tau_{(m,t),r}^{\theta-,L} \leq \tau_{(m,t),r}^{\theta+,L} =: \tau_r^+,$$

and all other inequalities of the statement follow by the same procedure. By definition, $\tau_{m-1}^- = \tau_{m-1}^+ = t$. Inductively, assume that $\tau_r^- \leq \tau_r^+$ for some $r \geq m$. We use the notation $L \arg \sup$ to denote leftmost maximizer. Then,

$$\tau_{r+1}^- = L \arg \sup_{\tau_r^- \leq s < \infty} \left\{B_r(s) - h_{r+1}^{\theta-;L}(s)\right\} \leq L \arg \sup_{\tau_r^+ \leq s < \infty} \left\{B_r(s) - h_{r+1}^{\theta-;L}(s)\right\}$$

$$\leq L \arg \sup_{\tau_r^+ \leq s < \infty} \left\{B_r(s) - h_{r+1}^{\theta+,L}(s)\right\} = \tau_{r+1}^+.$$

The first inequality above holds because $\tau_r^- \leq \tau_r^+$. The second inequality is an application of Lemma A.1, using the fact that $h_m^{\theta-}(s, t) \leq h_m^{\theta+}(s, t)$ for all $s \leq t$ (Theorem 3.5(ii)).

Part (iii)(b): Let $s < t$. We show that

$$\tau_{s,r} := \tau_{(m,s),r}^{\theta\square,L} \leq \tau_{(m,t),r}^{\theta\square,L} =: \tau_{t,r},$$

and the statement with 'L' replaced by 'R' has an analogous proof. Again, the base case of $r = m - 1$ follows by definition. Assume the inequality holds for some $r \geq m - 1$. Then,

$$\tau_{s,r+1} = L \arg \sup_{\tau_{s,r} \leq u < \infty} \left\{B_m(u) - h_{m+1}^{\theta\square}(u)\right\} \leq L \arg \sup_{\tau_{t,r} \leq u < \infty} \left\{B_m(u) - h_{m+1}^{\theta\square}(u)\right\} = \tau_{t,r+1}.$$

Part (iii)(c): The proof of this item is postponed until the end of Section 7.2.
Part (iv)(a): The monotonicity of Part (iii)(a) ensures that the limits exist and that

$$\tau_r := \lim_{\gamma \nearrow \theta} \tau_{(m,t),r}^{\gamma\square,L} \leq \tau_{(m,t),r}^{\theta-,L}. \tag{7.13}$$

We prove equality in the above expression, and the other statements follow analogously. By Lemma 7.3(i), for any $n \geq m$, $t = \tau^{\gamma\square,L}_{(m,t),m-1} \leq \cdots \leq \tau^{\gamma\square,L}_{(m,t),n}$ is a maximizing sequence for

$$\overline{L}^{\frac{1}{\sqrt{\gamma}}}_{(m,t),n}(\mathbf{B}, \overline{\mathbf{B}}) := \sup\left\{ \sum_{r=m}^{n} B_r(s_{r-1}, s_r) - h^{\gamma\square}_{n+1}(s_n) : \mathbf{s}_{m,n} \in \Pi_{(m,t),n} \right\}.$$

Using the monotonicity of Part (iii)(a) again, for all $\gamma \leq \theta$, the supremum may be restricted to the compact subset of $\Pi_{(m,t),n}$ such that $s_n \leq \tau^{(\theta+1)-,L}_{(m,t),n}$. By Theorem 3.5(iii)(a), as $\gamma \nearrow \theta$, $h^{\gamma\square}_{n+1}(s)$ converges to $h^{\theta-}_{n+1}(s)$, uniformly over $s \in [t, \tau^{(\theta+1)-,L}_{(m,t),n}]$. Then, by Lemma A.2, $t = \tau_{m-1} \leq \cdots \leq \tau_n$ is a maximizing sequence for

$$\overline{L}^{\frac{1}{\sqrt{\theta}}}_{(m,t),n}(\mathbf{B}, \overline{\mathbf{B}}) := \sup\left\{ \sum_{r=m}^{n} B_r(s_{r-1}, s_r) - h^{\theta-}_{n+1}(s_n) : \mathbf{s}_{m,n} \in \Pi_{(m,t),n} \right\}.$$

Part (iii) of Lemma 7.3 then implies the inequality (7.13) must be an equality. The statement for limits as $\delta \searrow \theta$ follows by the same reasoning.

Part (iv)(b): The proof of this item is postponed until the very end of Section 7. This item is not used in any subsequent proofs.

Part (iv)(c): This follows the same proof as that of Part (iv)(a), replacing the use of Part (iii)(a) with Part (iii)(b) and replacing the use of Lemma A.2 with Lemma A.3.

Part (v): Let $\omega \in \Omega_2$, $\theta > 0$, $(m, t) \in \mathbb{Z} \times \mathbb{Q}$, $(\tau_r)_{r \geq m-1} \in \mathbf{T}^{\theta\square}_{(m,t)}$, $\varepsilon > 0$, and let $\gamma, \delta \in D$ be such that $\theta - \varepsilon < \gamma < \theta < \delta < \theta + \varepsilon$. Part (iii)(a) and Lemma 7.6 imply that

$$\limsup_{n\to\infty} \frac{\tau_n}{n} \leq \limsup_{n\to\infty} \frac{\tau^{\theta+,R}_{(m,t),n}}{n} \leq \limsup_{n\to\infty} \frac{\tau^{\delta+,R}_{(m,t),n}}{n} = \delta < \theta + \varepsilon.$$

By a similar argument, on Ω_2,

$$\liminf_{n\to\infty} \frac{\tau_n}{n} \geq \theta - \varepsilon.$$

An analogous application of Part (iii)(b) extends Theorem 4.3(v) to all $\mathbf{x} \in \mathbb{Z} \times \mathbb{R}$, on the event Ω_2. $\qquad\square$

Proof of Theorem 4.5. *Part (i)*: Let $\omega \in \Omega_2$, $\theta > 0$, $(m, t) \in \mathbb{Z} \times \mathbb{R}$, and $\{t_{n,r}\}_{n \geq m, m-1 \leq r \leq n}$ satisfy the given assumptions. Let $0 < \gamma < \theta < \delta$ be arbitrary. Then, by assumption that $t_n/n \to \theta$ and Theorem 4.3(v), there exists $N \in \mathbb{Z}$ such that for all $n \geq N$,

$$(7.14) \qquad\qquad\qquad \tau^{\gamma-,L}_{(m,t),n} < t_n < \tau^{\delta+,R}_{(m,t),n}.$$

By Theorem 4.3(ii), the sequence $t = \tau^{\gamma-,L}_{(m,t),m-1} \leq \tau^{\gamma-,L}_{(m,t),m} \leq \cdots \leq \tau^{\gamma-,L}_{(m,t),n}$ defines the leftmost geodesic between (m, t) and $(n, \tau^{\gamma-,L}_{(m,t),n})$, and $t = \tau^{\delta+,R}_{(m,t),m-1} \leq \tau^{\delta+,R}_{(m,t),m} \leq \cdots \tau^{\delta+,R}_{(m,t),n}$ defines the rightmost geodesic between (m, t) and $(n, \tau^{\delta+,R}_{(m,t),n})$. Let $t = t^L_{m-1} \leq t^L_m \leq \cdots t^L_n = t_n$ and $t = t^R_{m-1} \leq t^R_m \leq \cdots t^R_n = t_n$ define the leftmost (resp. rightmost) geodesic between the points (m, t) and (n, t_n). Then, by (7.14) and Lemma 2.2, for all $n \geq N$ and $m \leq r \leq n$,

$$\tau^{\gamma-,L}_{(m,t),r} \leq t^L_r \leq t_{n,r} \leq t^R_r \leq \tau^{\delta+,R}_{(m,t),r}.$$

Taking limits as $n \to \infty$ produces

$$\tau^{\gamma-,L}_{(m,t),r} \leq \liminf_{n\to\infty} t_{n,r} \leq \limsup_{n\to\infty} t_{n,r} \leq \tau^{\delta+,R}_{(m,t),r}.$$

Taking limits as $\gamma \nearrow \theta$ and $\delta \searrow \theta$ and using Theorem 4.3(iv)(a) completes the proof of Part (i).

Part (ii): This is an immediate consequence of Part (i), setting $t_{n,r} = t_r$ for any θ-directed semi-infinite geodesic defined by the sequence $\{t_r\}_{r \geq m-1}$.

Part (iii): This is also an immediate consequence of Part (i), because if $\{\tau_r\}^{\infty}_{r=m-1}$ is the unique sequence in $\mathbf{T}^{\theta}_{(m,t)}$, for each $r \geq m$, $\tau_r = \tau^{\theta-,L}_{(m,t),r} = \tau^{\theta+,R}_{(m,t),r}$. $\qquad\square$

We now define the events $\widetilde{\Omega}^{(\theta)}$ of Theorem 4.7. First, let $\widetilde{\Omega}_{\mathbf{x}}^{(\theta)}$ be the events of Lemma 7.6, and for each $\mathbf{x} \in \mathbb{Z} \times \mathbb{R}$, define the full probability events

$$(7.15) \qquad \Omega_{\mathbf{x}}^{(\theta)} = \widetilde{\Omega}_{\mathbf{x}}^{(\theta)} \cap \Omega_2.$$

By Theorem 3.5(vii) and Theorem 3.7(i), for each $r \in \mathbb{Z}$, $s \mapsto B_r(s) - h_{r+1}^{\theta}(s)$ is a scaled, two-sided Brownian motion with strictly negative drift. By Theorem B.4, for $\theta > 0$ and $r \in \mathbb{Z}$, there exists an event $\mathrm{CM}_r^{(\theta)}$, of probability one, on which the set

$$\left\{ t \in \mathbb{R} : B_r(s) - h_{r+1}^{\theta}(s) \text{ over } s \in [t, \infty) \text{ has a non-unique maximum at } s = t \right\}$$

is countably infinite. Then, for $\theta > 0$, set

$$(7.16) \qquad \widetilde{\Omega}^{(\theta)} := \bigcap_{\mathbf{x} \in \mathbb{Z} \times \mathbb{Q}} \Omega_{\mathbf{x}}^{(\theta)} \cap \bigcap_{r \in \mathbb{Z}} \mathrm{CM}_r^{(\theta)} .$$

Because $\Omega_{\mathbf{x}}^{(\theta)} = \widetilde{\Omega}_{\mathbf{x}}^{(\theta)} \cap \Omega_2$ by definition and $\Omega_{\mathbf{x}}^{(\theta)} \subseteq \Omega^{(\theta)}$ by Lemma 7.6, $\widetilde{\Omega}^{(\theta)} \subseteq \Omega^{(\theta)} \cap \Omega_2$ where $\Omega^{(\theta)}$ are the events of Theorem 3.5. Furthermore, $\mathbb{P}(\widetilde{\Omega}^{(\theta)}) = 1$.

Proof of Theorem 4.7. In this proof, since θ is fixed, we drop the $\pm$ distinction for θ in the superscript.

Part (i): By Theorem 4.5(ii), on the event $\widetilde{\Omega}^{(\theta)}$, there exist multiple θ-directed geodesics starting from (m, t) if and only if $\tau_{(m,t),r}^{\theta,L} < \tau_{(m,t),r}^{\theta,R}$ for some $r \geq m$. If $\omega \in \widetilde{\Omega}^{(\theta)} \subseteq \Omega_2$ and (m, t) is a point whose θ-directed geodesic is not unique, then at most one such semi-infinite geodesic can pass through $(m, t + \varepsilon)$ for some $\varepsilon > 0$. Otherwise, two different geodesics would pass through (m, q) for some $q \in \mathbb{Q}$, giving two θ-directed geodesics starting from (m, q). This cannot hold on the event $\Omega_{\mathbf{x}}^{(\theta)} \supseteq \widetilde{\Omega}^{(\theta)}$. Inductively, to get two different θ-directed geodesics starting from (m, t), there must be some level $r \geq m$ such that all θ-directed geodesics pass through (r, t) and then one geodesic passes through $(r + 1, t)$, and the other passes through $(r, t + \varepsilon)$ for some $\varepsilon > 0$. Therefore, by Definition 4.1, on the event $\widetilde{\Omega}^{(\theta)}$, there exists a point $(m, t) \in \mathbb{Z} \times \mathbb{R}$ whose semi-infinite geodesic in direction θ is not unique if and only if there exists $r \geq m$ such that, for $m \leq k \leq r - 1$, $B_k(s) - h_{k+1}^{\theta}(s)$ over $s \in [t, \infty)$ has a unique maximum at $s = t$, and $B_r(s) - h_{r+1}^{\theta}(s)$ over $s \in [t, \infty)$ has a non-unique maximum at $s = t$. Therefore, $t = \tau_{(m,t),r}^{\theta,L} < \tau_{(m,t),r}^{\theta,R}$ for some $r \geq m$. The countability of the sets then follows from Theorem B.4(iv) because, for each k, $B_k - h_{k+1}^{\theta}$ is a (scaled) Brownian motion with negative drift (Theorem 3.5(vii) and Theorem 3.7(i)).

Part (ii): By Theorem 4.7(i), $(m, t) \in \mathrm{NU}_1^{\theta}$ if and only if $B_m(s) - h_{m+1}^{\theta}(s)$ has two maximizers over $s \in [t, \infty)$—one at $s = t$ and one at some $s > t$. The result then follows from Theorem B.4(iii). □

Proof of Theorem 4.3(iii)(c). Let $\omega \in \widetilde{\Omega}^{(\theta)}$, $m \in \mathbb{Z}$, and $s < t \in \mathbb{R}$. We use a modified induction, in the following manner.

(A) First, note by definition that $s = \tau_{(m,s),m-1}^{\theta,R} < \tau_{(m,t),m-1}^{\theta,L} = t$.

(B) For each $r \geq m$, we assume that $\tau_{(m,s),r-1}^{\theta,R} < \tau_{(m,t),r-1}^{\theta,L}$.

(C) Under assumption (B), we show that if $\tau_{(m,s),r}^{\theta,R} \geq \tau_{(m,t),r}^{\theta,L}$, then $\tau_{(m,s),k}^{\theta,R} = \tau_{(m,t),k}^{\theta,L}$ for all $k \geq r$.

With this procedure mapped out, assume that, for some $r \geq m$,

$$\tau_{(m,s),r-1}^{\theta,R} < \tau_{(m,t),r-1}^{\theta,L} \quad \text{and} \quad \tau_{(m,s),r}^{\theta,R} \geq \tau_{(m,t),r}^{\theta,L}.$$

By definition, $\tau_{(m,s),r}^{\theta,R}$ is a maximizer of $B_r(u) - h_{r+1}^{\theta}(u)$ over $u \in [\tau_{(m,s),r-1}^{\theta,R}, \infty)$. But since

$$\tau_{(m,s),r}^{\theta,R} \geq \tau_{(m,t),r}^{\theta,L} \geq \tau_{(m,t),r-1}^{\theta,L},$$

$\tau_{(m,s),r}^{\theta,R}$ is also a maximizer over $u \in [\tau_{(m,t),r-1}^{\theta,L}, \infty)$. By definition, $\tau_{(m,t),r}^{\theta,L}$ is another maximizer over this set, so

$$B_r(\tau_{(m,s),r}^{\theta,R}) - h_{r+1}^{\theta}(\tau_{(m,s),r}^{\theta,R}) = B_r(\tau_{(m,t),r}^{\theta,L}) - h_{r+1}^{\theta}(\tau_{(m,t),r}^{\theta,L}),$$

and both $\tau_{(m,s),r}^{\theta,R}$ and $\tau_{(m,t),r}^{\theta,L}$ are maximizers of $B_r(u) - h_{r+1}^{\theta}(u)$ over $u \in [q, \infty)$ for any rational $q \in [\tau_{(m,s),r-1}^{\theta,R}, \tau_{(m,t),r-1}^{\theta,L}]$. By assumption (B), such a rational q exists. Then, the sequence $\{\tau_{(m,s),k}^{\theta,R}\}_{k \geq r}$ and the sequence $\{\tau_{(m,t),k}^{\theta,L}\}_{k \geq r}$ both define jump times for a semi-infinite geodesic starting from (r, q). Since $\omega \in \widetilde{\Omega}^{(\theta)}$, there is a unique θ-directed semi-infinite geodesic starting from this point, and conclusion (C) holds. □

7.3. *The dual environment*

Throughout Sections 7.3 and 7.4, $\theta > 0$ is fixed, and we work on the full probability event $\Omega^{(\theta)}$ of Theorem 3.5. By Remark 4.2, on this event $\mathbf{T}_{\mathbf{x}}^{\theta} = \mathbf{T}_{\mathbf{x}}^{\theta+} = \mathbf{T}_{\mathbf{x}}^{\theta-}$ for all $\mathbf{x} \in \mathbb{Z} \times \mathbb{R}$. Recall the dual environment $\mathbf{X}^{\theta}$ of independent Brownian motions from Theorem 3.7.

Lemma 7.10. *For each $\theta > 0$, there exists an event $\widetilde{\Omega}^{(\theta,\star)}$ of probability one, on which*

$$\lim_{n \to \infty} \left[L_{(-n,-t_n),\mathbf{x}}(\mathbf{X}^{\theta}) - L_{(-n,-t_n),\mathbf{y}}(\mathbf{X}^{\theta}) \right] = \mathcal{B}^{\theta}(\mathbf{y}, \mathbf{x})$$

for any $\mathbf{x}, \mathbf{y} \in \mathbb{Z} \times \mathbb{R}$ and for any sequence $\{t_n\}$ satisfying $\lim_{n \to \infty} \frac{t_n}{n} = \theta$. To clarify, $L_{\mathbf{x},\mathbf{y}}(\mathbf{X}^{\theta})$ is the last-passage process on the environment $\mathbf{X}^{\theta}$ while $\mathcal{B}^{\theta}$ is the original Busemann function of the environment $\mathbf{B}$ as in Theorem 3.5.

Proof. Recall from Theorem 3.5(vi) that

$$h_{m-1}^{\theta} = D\big(h_m^{\theta}, B_{m-1}\big), \qquad X_m^{\theta} = R\big(h_m^{\theta}, B_{m-1}\big), \quad \text{and} \quad v_m^{\theta} = Q\big(h_m^{\theta}, B_{m-1}\big).$$

Further, recall (3.11), which states

(7.17) $$h_m^{\theta} = \overleftarrow{D}\big(h_{m-1}^{\theta}, X_m^{\theta}\big), \qquad B_{m-1} = \overleftarrow{R}\big(h_{m-1}^{\theta}, X_m^{\theta}\big), \quad \text{and} \quad v_m^{\theta} = \overleftarrow{Q}\big(h_{m-1}^{\theta}, X_m^{\theta}\big).$$

Recall that $\widetilde{f}(t) = -f(-t)$. Apply Lemma D.2 to deduce that

(7.18) $$\widetilde{h}_{m-1}^{\theta} = \overleftarrow{D}\big(\widetilde{h}_m^{\theta}, \widetilde{B}_{m-1}\big), \qquad \widetilde{X}_m^{\theta} = \overleftarrow{R}\big(\widetilde{h}_m^{\theta}, \widetilde{B}_{m-1}\big), \quad \text{and} \quad -\widetilde{v}_m^{\theta} = \overleftarrow{Q}\big(\widetilde{h}_m^{\theta}, \widetilde{B}_{m-1}\big).$$

From independence and matching marginals (Part Theorem 3.5(vii) and Theorem 3.7, Parts (i) and (ii)) follows

(7.19) $$\big(\widetilde{h}_k^{\theta}, \widetilde{B}_{k-1}, \widetilde{B}_{k-2}, \widetilde{B}_{k-3}, \dots\big) \stackrel{d}{=} \big(h_{m-1}^{\theta}, X_m^{\theta}, X_{m+1}^{\theta}, X_{m+2}^{\theta}, \dots\big) \quad \forall k, m \in \mathbb{Z}.$$

Thus, iterating mappings (7.18) backward in the index m and mappings (7.17) forward in the index m gives this equality in distribution:

$$\big\{\widetilde{h}_{-(m-1)}^{\theta}, -\widetilde{v}_{-(m-1)}^{\theta}, \widetilde{X}_{-(m-1)}^{\theta}, \widetilde{B}_{-m}\big\}_{m \in \mathbb{Z}} \stackrel{d}{=} \big\{h_{m-1}^{\theta}, v_m^{\theta}, B_{m-1}, X_m^{\theta}\big\}_{m \in \mathbb{Z}}$$

Apply Theorem 3.5(viii) to the environment $\widetilde{\mathbf{X}}_-^{\theta} = \{\widetilde{X}_{-m}^{\theta}\}_{m \in \mathbb{Z}}$ in the process on the left above, to deduce that there exists a full probability event $\widetilde{\Omega}^{(\theta,\star)}$ on which, for any $t \in \mathbb{R}$ and $m \in \mathbb{Z}$,

$$\lim_{n \to \infty} L_{(-m,-t),(n,t_n)}\big(\widetilde{\mathbf{X}}_-^{\theta}\big) - L_{(-m,0),(n,t_n)}\big(\widetilde{\mathbf{X}}_-^{\theta}\big) = -\widetilde{h}_{-(-m)}^{\theta}(-t) = h_m^{\theta}(t) \quad \text{and}$$

$$\lim_{n \to \infty} L_{(-m,-t),(n,t_n)}\big(\widetilde{\mathbf{X}}_-^{\theta}\big) - L_{(-m+1,-t),(n,t_n)}\big(\widetilde{\mathbf{X}}_-^{\theta}\big) = -\widetilde{v}_{-(-m)}^{\theta}(-t) = v_m^{\theta}(t).$$

Adding together horizontal and vertical steps for the general case, the proof is complete by noting that, for any $m, k \in \mathbb{Z}$ and $s, t \in \mathbb{R}$,

$$\lim_{n \to \infty} L_{(-m,-s),(n,t_n)}\big(\widetilde{\mathbf{X}}_-^{\theta}\big) - L_{(-k,-t),(n,t_n)}\big(\widetilde{\mathbf{X}}_-^{\theta}\big) = \lim_{n \to \infty} L_{(-n,-t_n),(m,s)}\big(\mathbf{X}^{\theta}\big) - L_{(-n,-t_n),(k,t)}\big(\mathbf{X}^{\theta}\big)$$

because

$$L_{(-m,-s),(n,t_n)}\big(\widetilde{\mathbf{X}}_-^{\theta}\big) = \sup\left\{ \sum_{r=-m}^{n} \widetilde{X}_{-r}^{\theta}(s_{r-1}, s_r) : -s = s_{-m-1} \le \dots \le s_n = t_n \right\}$$

$$= \sup\left\{ \sum_{r=-m}^{n} X_{-r}^{\theta}(-s_r, -s_{r-1}) : -s = s_{-m-1} \le \dots \le s_n = t_n \right\}$$

$$= \sup\left\{ \sum_{r=-m}^{n} X_{-r}^{\theta}(-s_r, -s_{r-1}) : -t_n = -s_n \le -s_{n-1} \le \dots \le -s_{m-1} = s \right\}$$

$$= \sup\left\{ \sum_{r=-n}^{m} X_r^{\theta}(\widetilde{s}_{r-1}, \widetilde{s}_r) : -t_n = \widetilde{s}_{-n-1} \le \widetilde{s}_{-n} \le \dots \le \widetilde{s}_m = s \right\} = L_{(-n,-t_n),(m,s)}\big(\mathbf{X}^{\theta}\big).$$

To get the second-to-last line above, simply set $\widetilde{s}_k = -s_{-k-1}$. $\qquad \square$

7.4. *Dual geodesics*

Recall the definition of the sets $\mathbf{T}_{\mathbf{x}}^{\theta\star}$ from Section 4.4. Analogous results as for the original northwest semi-infinite geodesics hold, as demonstrated by the following theorem.

Theorem 7.11. *Fix $\theta > 0$. Then, for every $\mathbf{x} \in \mathbb{Z} \times \mathbb{R}$, every semi-infinite path in $\mathbf{T}_{\mathbf{x}}^{\theta,\star}$ is a semi-infinite geodesic for Brownian last-passage percolation with environment $\mathbf{X}^\theta$. Specifically, the following hold.*

(i) *On the full probability event $\Omega^{(\theta)}$, let $\mathbf{x} \in \mathbb{Z} \times \mathbb{R}$, and let $\Gamma^\star$ be any semi-infinite path in $\mathbf{T}_{\mathbf{x}}^{\theta,\star}$. Then, for any $\mathbf{y} \leq \mathbf{z} \in \mathbb{Z} \times \mathbb{R}$ with $\mathbf{y}^\star$, $\mathbf{z}^\star$ lying along the semi-infinite path, the energy of the portion of that path between $\mathbf{y}^\star$ and $\mathbf{z}^\star$, in the environment $\mathbf{X}^\theta$, is*

$$L_{\mathbf{y},\mathbf{z}}(\mathbf{X}^\theta) = \mathcal{B}^\theta(\mathbf{y}, \mathbf{z}),$$

and this energy is maximal among all paths between $\mathbf{y}$ and $\mathbf{z}$ in the environment $\mathbf{X}^\theta$. To be clear, $\mathcal{B}^\theta$ is the original Busemann function for the environment $\mathbf{B}$.

(ii) *On $\Omega^{(\theta)}$, for all $(m, t) \in \mathbb{Z} \times \mathbb{R}$ and $r \leq m$,*

$$\tau_{(m,t),r-1}^{\theta,R\star} = \sup\{u \leq \tau_{(m,t),r}^{\theta,R\star} : v_r^\theta(u) = 0\} \quad and$$

$$\tau_{(m,t),r-1}^{\theta,L\star} = \inf\{u \leq \tau_{(m,t),r}^{\theta,L\star} : h_r^\theta(u, \tau_{(m,t),r}^{\theta,L\star}) = X_r^\theta(u, \tau_{(m,t),r}^{\theta,L\star})\}$$

More specifically, if $u \leq \tau_{(m,t),r}^{\theta,L\star}$, then $h_r^\theta(u, \tau_{(m,t),r}^{\theta,L\star}) = X_r^\theta(u, \tau_{(m,t),r}^{\theta,L\star})$ if and only if $u \geq \tau_{(m,t),r-1}^{\theta,L\star}$.

(iii) *On $\Omega^{(\theta)}$, if, for some $\mathbf{z} \in \mathbb{Z} \times \mathbb{R}$, $\mathbf{x}^\star \geq \mathbf{y}^\star$ lie along the leftmost semi-infinite geodesic in $\mathbf{T}_{\mathbf{z}}^{\theta,\star}$, then the portion of the path between $\mathbf{x}^\star$ and $\mathbf{y}^\star$, shifted back up by $\frac{1}{2}$ to lie on integer levels, is the leftmost geodesic between $\mathbf{x}$ and $\mathbf{y}$ in the environment $\mathbf{X}^\theta$. The analogous statement holds for the rightmost geodesics.*

(iv) *The following distributional equality holds.*

$$\{(\tau_{(m,t),r}^{\theta,R}, \tau_{(m,t),r}^{\theta,L}) : (m, t) \in \mathbb{Z}, r \geq m\}$$

$$\stackrel{d}{=} \{(-\tau_{(-m,-t),-(r+1)}^{\theta,L\star}, -\tau_{(-m,-t),-(r+1)}^{\theta,R\star}) : (m, t) \in \mathbb{Z}, r \geq m\}$$

(v) *There exists an event of full probability, $\Omega^{(\theta,\star)} \subseteq \Omega^{(\theta)}$, on which, for every $(m, t) \in \mathbb{Z} \times \mathbb{R}$ and every sequence $\{\tau_r^\star\}_{r \leq m} \in \mathbf{T}_{(m,t)}^{\theta,\star}$,*

$$\lim_{n \to \infty} \frac{\tau_{-n}^\star}{-n} = \theta.$$

Proof. *Parts (i)–(iii)*: On $\Omega^{(\theta)}$, let $(m, t) \in \mathbb{Z} \times \mathbb{R}$, and $\{\tau_r^\star\}_{r \leq m} \in \mathbf{T}_{(m,t)}^{\theta,\star}$. By (7.17) and the definitions (3.6)–(3.8), for all $u, v \in \mathbb{R}$ and $r \in \mathbb{Z}$,

$$(7.20) \qquad v_r^\theta(u) = \sup_{-\infty < s \leq u} \{X_r^\theta(s, u) - h_{r-1}^\theta(s, u)\},$$

$$h_r^\theta(u, v) = h_{r-1}^\theta(u, v) + \sup_{-\infty < s \leq v} \{X_r^\theta(s, v) - h_{r-1}^\theta(s, v)\} - \sup_{-\infty < s \leq u} \{X_r^\theta(s, u) - h_{r-1}^\theta(s, u)\}$$

$$(7.21)$$

$$= X_r^\theta(u, v) + \sup_{-\infty < s \leq v} \{h_{r-1}^\theta(s) - X_r^\theta(s)\} - \sup_{-\infty < s \leq u} \{h_{r-1}^\theta(s) - X_r^\theta(s)\}, \quad and$$

$$B_{r-1}(u, v) = X_r^\theta(u, v) + \sup_{-\infty < s \leq u} \{X_r^\theta(s, u) - h_{r-1}^\theta(s, u)\} - \sup_{-\infty < s \leq v} \{X_r^\theta(s, v) - h_{r-1}^\theta(s, v)\}$$

$$(7.22)$$

$$= h_{r-1}^\theta(u, v) + \sup_{-\infty < s \leq u} \{h_{r-1}^\theta(s) - X_r^\theta(s)\} - \sup_{-\infty < s \leq v} \{h_{r-1}^\theta(s) - X_r^\theta(s)\}.$$

As $\tau_{r-1}^\star$ is a maximizer of $h_{r-1}(s) - X_r^\theta(s)$ over $s \in (\infty, \tau_r^\star]$, from (7.20), it follows that $v_r^\theta(\tau_{r-1}^\star) = 0$ for each $r \leq m$. By (7.21), $h_r^\theta(u, v) = X_r^\theta(u, v)$ for $u, v \in [\tau_{r-1}^\star, \tau_r^\star]$. In general, $v_r^\theta(u) \geq 0$ and $h_r^\theta(u, v) \geq X_r^\theta(u, v)$ for $u \leq v$. Then, Parts (i)–(iii) follow just as for the analogous statements in the proofs of Theorem 4.3, Parts (i)–(ii) and Lemma 7.4.

Parts (iv)–(v): By (7.19), $\{\widetilde{h}_{r-1}^\theta, \widetilde{X}_r^\theta\}_{r \in \mathbb{Z}}$ has the same distribution as $\{h_{-(r-1)}^\theta, B_{-r}\}_{r \in \mathbb{Z}}$. Furthermore, the leftmost (rightmost) maximizers of $h_{r-1}^\theta(s) - X_r^\theta(s)$ for $s \in (-\infty, t]$ are the negative of the rightmost (resp. leftmost) maximizers of $\widetilde{X}_r^\theta(s) - \widetilde{h}_{r-1}^\theta(s)$ for $s \in [-t, \infty)$, establishing Part (iv). Part (v) then follows from Theorem 4.3(v). $\qquad \square$
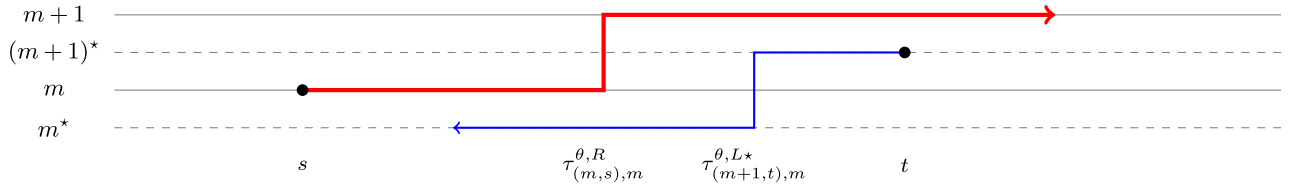
Fig. 11. A rightmost northeast geodesic (red/thick) lies strictly above and to the left of a leftmost dual southwest geodesic (blue/thin).
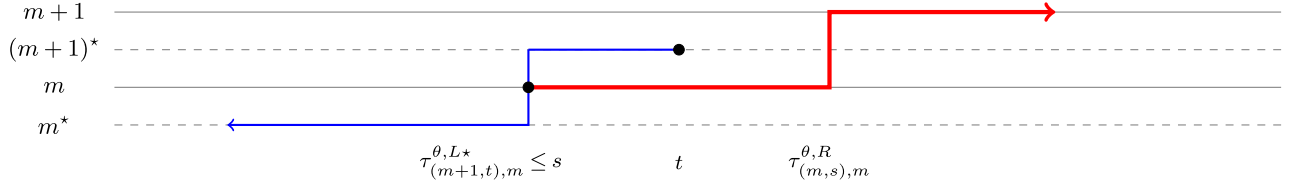

Fig. 12. A rightmost northeast geodesic (red/thick) lies weakly to the right and below a leftmost dual southwest geodesic (blue/thin).
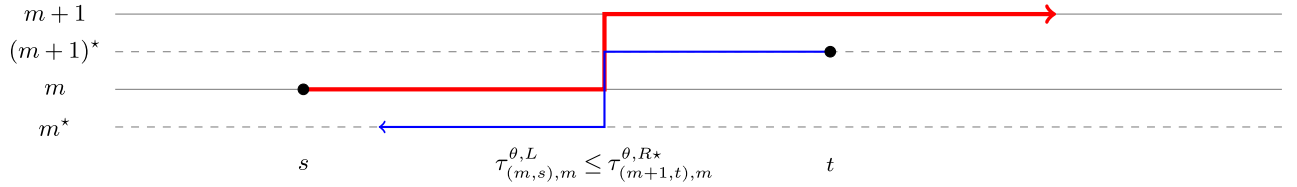

Fig. 13. A leftmost northeast geodesic (red/thick) lies weakly above and to the left of a rightmost dual southwest geodesic (blue/thin).
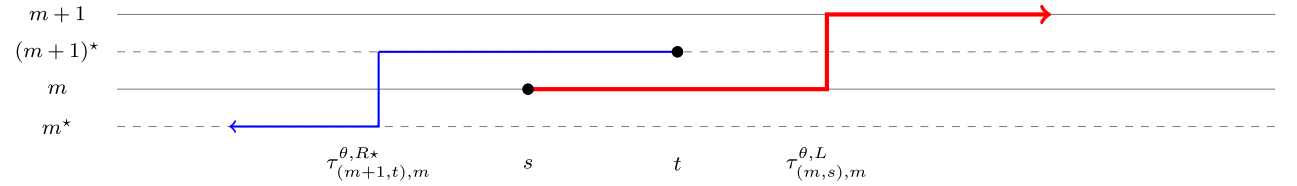

Fig. 14. A leftmost northeast geodesic (red/thick) lies strictly to the right and below a rightmost dual southwest geodesic (blue/thin).

Theorem 7.12. *On the event* $\Omega^{(\theta)}$, *the following hold for all* $s \leq t \in \mathbb{R}$ *and* $m \in \mathbb{Z}$.

(i) *If* $\tau^{\theta,R}_{(m,s),m} < t$, *then also* $\tau^{\theta,R}_{(m,s),m} < \tau^{\theta,L\star}_{(m+1,t),m}$. *See Figure* 11 *for clarity.*

(ii) *If* $t \leq \tau^{\theta,R}_{(m,s),m}$, *then* $\tau^{\theta,L^\star}_{(m+1,t),m} \leq s$. *See Figure* 12 *for clarity.*

(iii) *If* $\tau^{\theta,L}_{(m,s),m} \leq t$, *then also* $\tau^{\theta,L}_{(m,s),m} \leq \tau^{\theta,R\star}_{(m+1,t),m}$. *See Figure* 13 *for clarity.*

(iv) *If* $t < \tau^{\theta,L}_{(m,s),m}$, *then* $\tau^{\theta,R\star}_{(m,+1,t),m} < s$. *See Figure* 14 *for clarity.*

Remark 7.13. All the statements of Theorem 7.12 only mention the first jump time for northeast geodesics and the first time of descent for southwest geodesics. However, for any $\mathbf{x} \in \mathbb{Z} \times \mathbb{R}$ and any point $\mathbf{y}$ along the rightmost (resp. leftmost) semi-infinite geodesic in $\mathbf{T}^\theta_{\mathbf{x}}$, the rightmost (resp. leftmost) semi-infinite geodesic in $\mathbf{T}^\theta_{\mathbf{y}}$ agrees with the remainder of the original semi-infinite geodesic started from $\mathbf{x}$. Thus, the results of Theorem 7.12 can be extended by induction. For example, Part (i) implies that if a leftmost dual southwest geodesic starts strictly to the right and below a rightmost northeast geodesic, it remains to the right and below the northeast geodesic. See Figure 11.

Remark 7.14. Theorem 7.12 is the analogue of Lemma 4.4 in [53], which states a result for exponential last-passage percolation. In words, Theorem 7.12 says that northeast Busemann geodesics do not cross dual southwest geodesics. In this sense, the theorem gives an analogue to Pimentel's dual tree [47]. In the setting of exponential LPP, in Section 5 of [53], it is shown that semi-infinite geodesics in the original environment are competition interfaces for geodesics in the dual environment with boundary conditions given by the Busemann process. This gives some intuition on why the

northeast geodesics do not cross the southwest dual geodesics. Due to the general non-uniqueness of geodesics, the construction of competition interfaces is somewhat delicate for BLPP and will be studied in future work.

Proof. *Part (i)*: By Lemma 7.4, if $u \geq s$, then

$$(7.23) \qquad B_m(s, u) = h_m^\theta(s, u) \quad \text{if and only if} \quad u \leq \tau_{(m,s),m}^{\theta,R}.$$

By Theorem 7.11(ii), if $u \leq t$, then

$$(7.24) \qquad X_{m+1}^\theta(u, t) = h_{m+1}^\theta(u, t) \quad \text{if and only if} \quad u \geq \tau_{(m+1,t),m}^{\theta,L\star}.$$

By (7.21) and (7.22),

$$
\begin{aligned}
X_{m+1}^\theta(u, v) = h_{m+1}^\theta(u, v) \quad &\Longleftrightarrow \quad B_m(u, v) = h_m^\theta(u, v) \\
&\Longleftrightarrow \quad \sup_{-\infty \leq w \leq u} \left\{ h_m^\theta(w) - X_{m+1}^\theta(w) \right\} = \sup_{-\infty \leq w \leq v} \left\{ h_m^\theta(w) - X_{m+1}^\theta(w) \right\}.
\end{aligned}
$$

This along with (7.24) implies that if $u \leq t$,

$$(7.25) \qquad B_m^\theta(u, t) = h_m^\theta(u, t) \quad \text{if and only if} \quad u \geq \tau_{(m+1,t),m}^{\theta,L\star}.$$

By assumption, $\tau_{(m,s),m}^{\theta,R} < t$. Equation (7.23) and the monotonicity of Theorem 3.5(ii) imply that

$$(7.26) \qquad B_m(s, t) < h_m^\theta(s, t).$$

Assume, by way of contradiction, that $\tau_{(m+1,t),m}^{\theta,L\star} \leq \tau_{(m,s),m}^{\theta,R}$. Then, there exists $u \leq \tau_{(m,s),m}^{\theta,R}$ with $u \geq s \vee \tau_{(m+1,t),m}^{\theta,L\star}$. Then, for such u, by (7.23) and (7.25),

$$B_r(s, u) = h_r^\theta(s, u) \quad \text{and} \quad B_r^\theta(u, t) = h_r^\theta(u, t).$$

Adding these two equations gives us a contradiction to (7.26).

Part (ii): Let $t \leq \tau_{(m,s),m}^{\theta,R}$, and assume by way of contradiction, that $\tau_{(m+1,t),m}^{\theta,L\star} > s$. This also implies that $s < t$ since $t \geq \tau_{(m+1,t),m}^{\theta,L\star}$. By Equation (7.25), $B_m(s, t) < h_m^\theta(s, t)$. However, since $s < t \leq \tau_{(m,s),m}^{\theta,R}$, Equation (7.23), implies that $B_m(s, t) = h_m^\theta(s, t)$, giving the desired contradiction.

Part (iii): Assume that $\tau_{(m,s),m}^{\theta,L} \leq t$. By Theorem 7.11(ii),

$$\tau_{(m+1,t),m}^{\theta,R\star} = \sup \left\{ u \leq t : v_{m+1}^\theta(u) = 0 \right\}.$$

By Lemma 7.4, $v_{m+1}^\theta(\tau_{(m,s),m}^{\theta,L}) = 0$, so the desired conclusion follows.

Part (iv): Assume that $t < \tau_{(m,s),m}^{\theta,L}$. By Lemma 7.4,

$$\tau_{(m,s),m}^{\theta,L} = \inf \left\{ u \geq s : v_{m+1}^\theta(u) = 0 \right\}.$$

By assumption, $\tau_{(m+1,t),m}^{\theta,R\star} \leq t < \tau_{(m,s),m}^{\theta,L}$. By Theorem 7.11(ii), $v_{m+1}^\theta(\tau_{(m+1,t),m}^{\theta,R\star}) = 0$, so $\tau_{(m+1,t),m}^{\theta,R\star} < s$. $\qquad \square$

We use these non-intersection properties to prove Theorem 4.9.

Proof of Theorem 4.9. We follow a similar compactness argument as in the proof of Lemma 4.6 in [53], appropriately modified for the semi-discrete setting of BLPP. The full probability event of this theorem is $\Omega^{(\theta)} \cap \Omega^{(\theta,\star)} \cap \Omega_2$, where $\Omega^{(\theta,\star)}$ is the event of Theorem 7.11(v). On this event, let $\mathbf{x} = (m, t)$, $\mathbf{y} = (k, t)$. Assume that Γ_1 and Γ_2 are disjoint, where Γ_1 denotes the path defined by the jump times $t = \tau_{(k,t),k-1}^{\theta,R} \leq \tau_{(k,t),k}^{\theta,R} \leq \cdots$, and Γ_2 denotes the path defined by the jump times $s = \tau_{(m,s),m-1}^{\theta,R} \leq \tau_{(m,s),m}^{\theta,R} \leq \cdots$. By Theorem 4.3(v), these paths satisfy

$$(7.27) \qquad \lim_{n \to \infty} \frac{\tau_{(m,s),n}^{\theta,R}}{n} = \lim_{n \to \infty} \frac{\tau_{(k,t),n}^{\theta,R}}{n} = \theta,$$
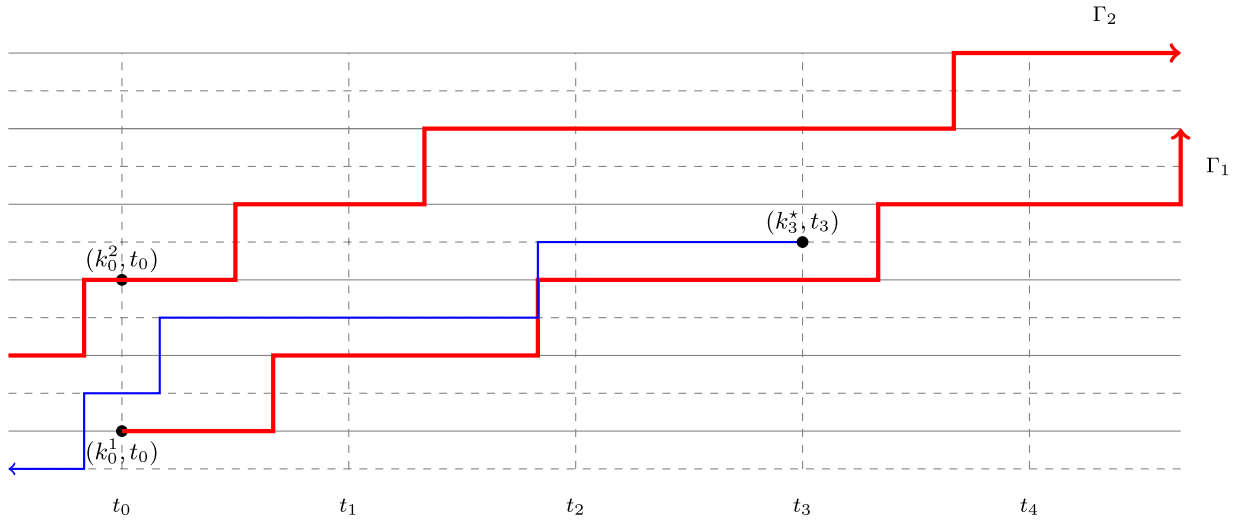
Fig. 15. Constructing a backwards semi-infinite path from each discrete time point. The upper red/thick path is Γ_2 and the lower red/thick path is Γ_1.

so both the space and time coordinates of these up-right paths go to ∞. Assume, without loss of generality, that Γ_2 lies above and to the left of Γ_1. Then, by picking t_0 large enough, there exists k_0^1 and k_0^2 such that $(k_0^1, t_0) \in \Gamma_1$ and $(k_0^2, t_0) \in \Gamma_2$. For each t_0, there are only finitely many choices of k_0^1 and k_0^2, so assume that k_0^1 is the largest such choice and k_0^2 is the smallest such choice. Then, $k_0^1 < k_0^2$ by the assumption that Γ_2 lies above and to the left of Γ_1. See Figure 15 for clarity. For $i = 1, 2, \ldots$, define $t_i = t_0 + i$. For each time i, similarly define k_i^1 and k_i^2 so that $k_i^1 < k_i^2$ and (k_i^j, t_i) lies on Γ_j for $j = 1, 2$. Recall the notation $m^\star = m - \frac{1}{2}$. Then, for $i = 0, 1, 2, \ldots$, there exists $k_i \in \mathbb{Z}$ with $k_i^1 < k_i^\star < k_i^2$. This gives us an infinite sequence $(k_i^\star, t_i)_{i \geq 0}$ such that, for each i, the point $(k_i^\star, t_i)$, lies between the paths Γ_1 and Γ_2 (see Figure 15). Starting from each of these points $(k_i^\star, t_i)$, let $\Gamma_i^\star$ be the leftmost dual geodesic path in $\mathbf{T}_{(k_i, t_i)}^{\theta, \star}$. Each $\Gamma_i^\star$ is an infinite, down-left path satisfying the limit condition of Theorem 7.11(v), so for each $j < i$, each of the paths $\Gamma_i^\star$ intersects the vertical line $t = t_j$. Let $k_{i,i}^\star = k_i^\star$, and for $j < i$, let $k_{i,j}$ be the maximal integer such that $(k_{i,j}^\star, t_j)$ lies on the path $\Gamma_i^\star$. Since $(k_i^\star, t_i)$ lies between Γ_1 and Γ_2, and Γ_1, Γ_2 are rightmost semi-infinite geodesics constructed from the Busemann functions, Theorem 7.12(i)–(ii) implies that for $i \geq 0$, $\Gamma_i^\star$ lies strictly below and to the right of Γ_2, and weakly above and to the left of Γ_1 (see also Remark 7.13).

Since $k_j^1 < k_{i,j}^\star < k_j^2$ for $0 \leq j \leq i$, there are only finitely many values of $k_{i,j}$ for each value of j. Then, there exists a subsequence k_{i_r} such that for some $N_0 \in \mathbb{Z}$ with $k_0^1 < N_0^\star < k_0^2$ and all ℓ, $k_{i_\ell, 0} = N_0$. Take a further subsequence $k_{i_{r_\ell}}$ such that for some $k_1^1 < N_1^\star < k_1^2$ and every element of this subsequence, $k_{i_{r_\ell}, 1} = N_1$. Continuing in this way, there exists a sequence (N_i, t_i) of elements of $\mathbb{Z} \times \mathbb{R}$, that $(N_i^\star, t_i)$ lies between the paths Γ_1 and Γ_2, and such that the leftmost southwest dual semi-infinite geodesic starting from $(N_i^\star, t_i)$ passes through $(N_j^\star, t_j)$ for $0 \leq j \leq i$. Since we chose the leftmost paths in $\mathbf{T}_{(N_i, t_i)}^{\theta, \star}$ at each step, the paths are consistent at all space-time points. Hence, this construction gives a bi-infinite path such that, for any point along the path, the part of the path to the southwest of that point is a semi-infinite southwest dual geodesic. To get the last part of the theorem about asymptotic direction of the paths, we already showed the direction to the southwest. The direction to the northeast follows by (7.27) because the path lies between Γ_1 and Γ_2.

The proof for the disjoint left geodesics is analogous, replacing the use of leftmost southwest dual geodesics with rightmost southwest geodesics, and now, the bi-infinite path lies weakly below and to the right of Γ_2 and strictly above and to the left of Γ_1. □

7.5. Proof of Lemma 3.3, Theorem 3.1, and Parts (iii)(c) and (iv)(b) of Theorem 4.3

Proof of Lemma 3.3. We follow the procedure of the proof of Theorem 4.12 in the arXiv version of [52]. We take $(m, t) = \mathbf{0}$, and the general case follows analogously. We simplify further: for any sequence $\{t_k\}_{k \in \mathbb{Z}}$ satisfying

$$\lim_{n \to \infty} \frac{t_n}{n} = \theta \quad \text{and} \quad \lim_{n \to \infty} \frac{t_{-n}}{-n} = \eta,$$

set $\widetilde{t}_k = -t_{-(k+1)}$. Then, $\{\widetilde{t}_k\}_{k\in\mathbb{Z}}$ is a sequence satisfying

$$\lim_{n\to\infty} \frac{\widetilde{t}_n}{n} = \eta \quad \text{and} \quad \lim_{n\to\infty} \frac{t_{-n}}{-n} = \theta.$$

Furthermore, a geodesic path between $(-n, t_{-n})$ and (n, t_n) for the original field of Brownian motions $\mathbf{B} = \{B_r\}_{r\in\mathbb{Z}}$, when reflected through the origin, becomes a geodesic path between $(-n, \widetilde{t}_{-n})$ and $(n, \widetilde{t}_n)$ for the field of Brownian motions $\{\widetilde{B}_{-r}\}_{r\in\mathbb{Z}}$. Then, without loss of generality, assume that

(7.28) $$\mathbb{E}\big[\mathcal{B}^\eta\big((1, -1), \mathbf{0}\big)\big] \geq \mathbb{E}\big[\mathcal{B}^\theta\big((1, -1), \mathbf{0}\big)\big].$$

If a geodesic between $(-n, t_{-n})$ and (n, t_n) passes through $\mathbf{0}$, then the geodesic cannot pass through any point above and to the left of $\mathbf{0}$. Then, for any $\mathbf{x} \in \mathbb{Z}_{>0} \times \mathbb{R}_{<0}$,

$$L_{(-n, t_{-n}), \mathbf{0}} + L_{\mathbf{0}, (n, t_n)} \geq L_{(-n, t_{-n}), \mathbf{x}} + L_{\mathbf{x}, (n, t_n)}$$

Specifically, for all $k \in \mathbb{Z}_{>0}$,

$$L_{(-n, t_{-n}), \mathbf{0}} - L_{(-n, t_{-n}), (k, -k)} \geq L_{(k, -k), (n, t_n)} - L_{\mathbf{0}, (n, t_n)}.$$

For fixed $\mathbf{x}, \mathbf{y} \in \mathbb{Z} \times \mathbb{R}$, let $\widetilde{\mathcal{B}}^\eta(\mathbf{x}, \mathbf{y})$ denote the almost sure limit

$$\lim_{n\to\infty} L_{(-n, t_{-n}), \mathbf{x}} - L_{(-n, -t_{-n}), \mathbf{y}}.$$

By Theorem 3.7(ii) and Lemma 7.10, the limit exists almost surely, and $\widetilde{\mathcal{B}}^\theta(\mathbf{x}, \mathbf{y}) \overset{d}{=} \mathcal{B}^\theta(\mathbf{y}, \mathbf{x})$. Let $\Omega_{\mathbb{Z}}$ be the full probability event on which, for every $k \in \mathbb{Z}_{>0}$ and for every sequence $\{t_n\}_{n\in\mathbb{Z}}$ satisfying

$$\lim_{n\to\infty} \frac{t_n}{n} = \theta \quad \text{and} \quad \lim_{n\to\infty} \frac{t_{-n}}{-n} = \eta,$$

we have

$$\lim_{n\to\infty} L_{(-n, t_{-n}, \mathbf{0})}(\mathbf{B}) - L_{(-n, t_{-n}, (k, -k))}(\mathbf{B}) = \widetilde{\mathcal{B}}^\eta\big(\mathbf{0}, (k, -k)\big), \quad \text{and}$$

$$\lim_{n\to\infty} L_{(k, -k), (n, t_n)}(\mathbf{B}) - L_{\mathbf{0}, (n, t_n)}(\mathbf{B}) = \mathcal{B}^\theta\big((k, -k), \mathbf{0}\big).$$

Then, we have the following inclusion of sets:

$$\left\{ \text{There exists a sequence } \{t_n\}_{n\in\mathbb{Z}} \text{ satisfying } \lim_{n\to\infty} \frac{t_n}{n} = \theta \text{ and } \lim_{n\to\infty} \frac{t_{-n}}{-n} = \eta \right.$$

$$\left. \text{such that, for all } n \in \mathbb{Z}_{>0}, \text{ some geodesic between } (-n, t_{-n}) \text{ and } (n, t_n) \text{ passes through } \mathbf{0} \right\}$$

$$\subseteq \bigcap_{m=1}^{\infty} \left\{ \text{There exists a sequence } \{t_n\}_{n\in\mathbb{Z}} \text{ satisfying } \lim_{n\to\infty} \frac{t_n}{n} = \theta \text{ and } \lim_{n\to\infty} \frac{t_{-n}}{-n} = \eta \text{ such that for all} \right.$$

$$\left. n \in \mathbb{Z}_{>0} \text{ and } k = 1, \ldots, m, L_{(-n, t_{-n}), \mathbf{0}}(\mathbf{B}) - L_{(-n, t_{-n}), (k, -k)}(\mathbf{B}) \geq L_{(k, -k), (n, t_n)}(\mathbf{B}) - L_{\mathbf{0}, (n, t_n)}(\mathbf{B}) \right\}$$

$$\subseteq \bigcap_{m=1}^{\infty} \left\{ \widetilde{\mathcal{B}}^\eta\big(\mathbf{0}, (k, -k)\big) \geq \mathcal{B}^\theta\big((k, -k), \mathbf{0}\big), \quad k = 1, \ldots, m \right\} \cup \Omega_{\mathbb{Z}}^C.$$

We show that this last event has probability 0. Define the process $\{S_k\}_{k\geq 0}$ by $S_0 = 0$ and for $k \geq 1$,

$$S_k = \mathcal{B}^\eta\big(\mathbf{0}, (k, -k)\big) - \mathcal{B}^\theta\big((k, -k), \mathbf{0}\big) = \sum_{i=1}^{k}\big(\widetilde{\mathcal{B}}^\eta\big((i-1, -i+1), (i, -i)\big) - \mathcal{B}^\theta\big((i, -i), (i-1, -i+1)\big)\big).$$

The sequences

$$\left\{\mathcal{B}^\theta\big((i,-i),(i-1,-i+1)\big)\right\}_{i\in\mathbb{Z}_{>0}} \quad\text{and}\quad \left\{\widetilde{\mathcal{B}}^\eta\big((i-1,-i+1),(i,-i)\big)\right\}_{i\in\mathbb{Z}_{>0}}$$

are independent because they are constructed from disjoint increments of the field of independent Brownian motions **B**. Furthermore,

$$\left\{\mathcal{B}^\theta\big((i,-i),(i-1,-i+1)\big)\right\}_{i\in\mathbb{Z}_{>0}} = \left\{v_i^\theta(-i)+h_{i-1}^\theta(-i,-i+1)\right\}_{i\in\mathbb{Z}_{>0}}$$

is an i.i.d. collection of random variables by Theorem 3.7(iv). By reflection applied to the Busemann functions $\widetilde{B}^\eta$, the process $\{S_k\}_{k\geq0}$ has independent increments. By Theorem 3.7, the increments of this random walk are independent sums of normal and exponential random variables and therefore have finite first and second moment. Let μ be the mean and σ^2 the variance of S_1. Then, $\mu\leq0$ by the assumption (7.28). Thus, for $m\geq1$,

$$\mathbb{P}\big(\widetilde{\mathcal{B}}^\eta\big(\mathbf{0},(k,-k)\big)\geq\mathcal{B}^\theta\big((k,-k),\mathbf{0}\big), k=1,\dots,m\big)=\mathbb{P}(S_k\geq0, k=1,\dots,m)$$

$$=\mathbb{P}\left(\frac{S_{\lfloor um\rfloor}-um\mu}{\sqrt{m\sigma^2}}\geq-\frac{um\mu}{\sqrt{m\sigma^2}}\text{ for }u\in[0,1]\right)\leq\mathbb{P}\left(\inf_{u\in[0,1]}\frac{S_{\lfloor um\rfloor}-um\mu}{\sqrt{m\sigma^2}}\geq0\right)\overset{m\to\infty}{\longrightarrow}0.$$

The convergence in the last step holds because

$$\inf_{u\in[0,1]}\frac{S_{\lfloor um\rfloor}-um\mu}{\sqrt{m\sigma^2}}\overset{m\to\infty}{\Longrightarrow}\inf_{u\in[0,1]}B(u)$$

where the convergence holds in distribution by Donsker's Theorem, and B is a standard Brownian motion.   □

We are now ready to prove all parts of the main theorem of this paper.

Proof of Theorem 3.1. Unless specified otherwise, the full probability event of the parts of this theorem is the event Ω_2 defined in Equation (7.10).

Part (i): This is a direct corollary of Theorem 4.3(i).

Part (ii): Let $\Omega_{\mathbf{x}}^{(\theta)}$ be the event defined in Equation (7.15). By Lemma 7.6 on $\Omega_{\mathbf{x}}^{(\theta)}$, there is a unique element of $\mathbf{T}_{\mathbf{x}}^\theta$. Setting $\mathbf{x}=(m,t)$, $\tau_{(m,t),r}^{\theta-,L}=\tau_{(m,t),r}^{\theta+,R}$ for all $r\geq m$, and the desired conclusion follows from Theorem 4.5(ii).

Part (iii): By Theorems 3.5(viii) and 4.5(ii), on the full probability event $\Omega^{(\theta)}\cap\Omega_2$, there exists multiple θ-directed semi-infinite geodesics from (m,t) if and only if $\tau_{(m,t),r}^{\theta,L}<\tau_{(m,t),r}^{\theta,R}$ for some $r\geq m$, or in other words, if and only if $\mathbf{T}_{(m,t)}^\theta$ contains more than one element. Since the event $\widetilde{\Omega}^{(\theta)}$ constructed in Equation (7.16) is contained in $\Omega^{(\theta)}$, Theorem 4.7(i) implies that this set of points whose θ-directed geodesic is not unique is countable. Then, since $\widetilde{\Omega}^{(\theta)}\subseteq\bigcap_{\mathbf{x}\in\mathbb{Z}\times\mathbb{Q}}\Omega_{\mathbf{x}}^{(\theta)}$, if two θ-directed geodesics from (m,t) pass through $(m,t+\varepsilon)$ for some $\varepsilon>0$, then they both pass through (m,q) for some $(m,q)\in\mathbb{Z}\times\mathbb{Q}$. Further, the portions of the geodesics after (m,q) are both θ-directed semi-infinite geodesics from (m,q) and are therefore the same geodesic by Part (ii).

Part (iv): Assume to the contrary, that, for some $\omega\in\Omega_2$, there exists a point $(m,t)\in\mathbb{Z}\times\mathbb{R}$ and a sequence $t=t_{m-1}\leq t_m\leq t_{m+1}\leq\cdots$ defining a semi-infinite geodesic Γ, starting from (m,t) and satisfying

$$0\leq\underline{\theta}:=\liminf_{n\to\infty}\frac{t_n}{n}<\limsup_{n\to\infty}\frac{t_n}{n}=:\overline{\theta}\leq\infty.$$

Choose some $\theta\in(\underline{\theta},\overline{\theta})$. By Theorem 4.3(v),

$$\lim_{n\to\infty}\frac{\tau_{(m,t),n}^{\theta-,L}}{n}=\theta$$

Therefore, there are infinitely many values of n with $t_n<\tau_{(m,t),n}^{\theta-,L}$ and another infinitely many values of n with $t_n>\tau_{(m,t),n}^{\theta-,L}$. Let n_1 be the minimal index such that $t_{n_1}<\tau_{(m,t),n_1}^{\theta-,L}$, and let n_2 be the minimal index larger than n_1 such that $t_{n_2}>\tau_{(m,t),n_2}^{\theta-,L}$. See Figure 16 for clarity. By planarity, the points (n_1,t_1) and $(n_2,\tau_{(m,t),n_2-1}^{\theta-,L})$ lie on both geodesics, and between the two points, the path Γ lies strictly to the left of the leftmost geodesic in $\mathbf{T}_{(m,t)}^{\theta-}$. This contradicts Theorem 4.3(ii) which states the leftmost geodesic in $\mathbf{T}_{(m,t)}^{\theta-}$ is the leftmost geodesic between any two of its points.
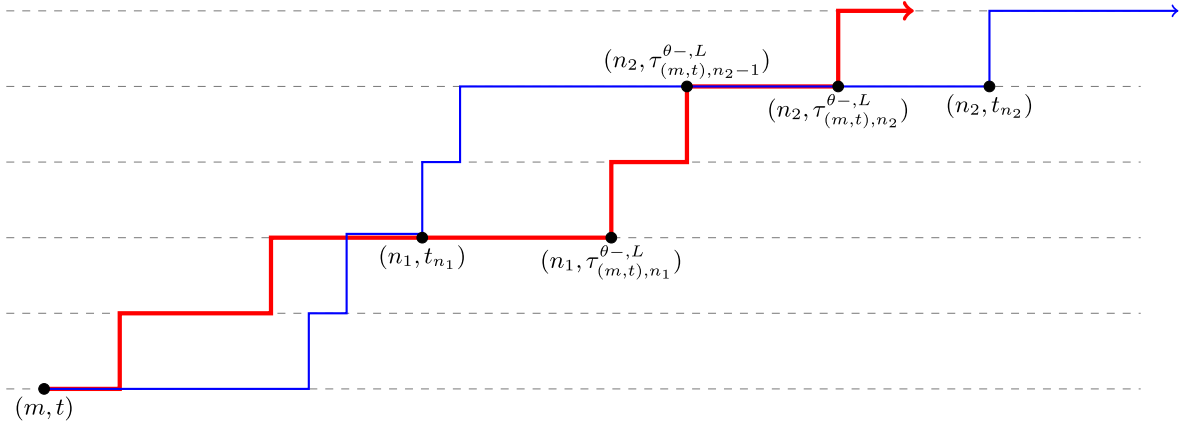
Fig. 16. Crossing of geodesic paths. The blue/thin path is the semi-infinite geodesic defined by the jump times $t \leq t_m \leq t_{m+1} \leq \cdots$, and the red/thick path is the semi-infinite geodesic defined by the times $t \leq \tau_{(m,t),m}^{\theta-,L} \leq \tau_{(m,t),m+1}^{\theta-,L} \leq \cdots$.

Part (v): Next, assume that for some $\omega \in \Omega_2$, there exists $(m,t) \in \mathbb{Z} \times \mathbb{R}$, and a sequence $t = t_{m-1} \leq t_m \leq \cdots$ that defines a semi-infinite geodesic Γ, starting from (m,t) and satisfying

$$\lim_{n \to \infty} \frac{t_n}{n} = 0.$$

To show that $t_r = t$ for all $r \geq m$, it is sufficient to show that $t_m = t$. For then, this semi-infinite geodesic travels vertically to $(m+1,t)$, the remaining part of the geodesic is a semi-infinite geodesic starting at $(m+1,t)$, and the result follows by induction. By Theorem 4.3(v), for every $\theta > 0$ and any $(m,t) \in \mathbb{Z} \times \mathbb{R}$,

$$\lim_{n \to \infty} \frac{\tau_{(m,t),n}^{\theta+,R}}{n} = \theta.$$

Hence, for all $\theta > 0$ and all sufficiently large n, $t_n < \tau_{(m,t),n}^{\theta+,R}$. However, by an analogous argument as in Part (iv), again using Theorem 2.3(ii), $t_r \leq \tau_{(m,t),r}^{\theta+,R}$ for all $r \geq m$. Specifically, the inequality holds for $r = m$. By monotonicity of Theorem 4.3(iii)(a), $\lim_{\theta \searrow 0} \tau_{(m,t),m}^{\theta+,R}$ exists. By definition of the event Ω_1 from Lemma 6.4, this limit equals t on the event $\Omega_1 \supseteq \Omega_2$. The case where $\lim_{n \to \infty} \frac{t_n}{n} = \infty$ is handled similarly.

Part (vi): Let $\Omega^{(\theta,\eta)}$ be a full probability event on which, for every $(m,q) \in \mathbb{Z} \times \mathbb{Q}$, there exists no sequences satisfying the conditions (3.2) and such that, for every $n \in \mathbb{Z}$, there exists a geodesic between $(-n, \tau_{-n})$ and (n, τ_n) that passes through (m,q). Such an event exists by Lemma 3.3. Then, on this event, if a bi-infinite geodesic that satisfies (3.2) exists, it cannot pass through (m,q) for any $q \in \mathbb{Q}$. But then this bi-infinite geodesic cannot ever move horizontally, and so there must exist $t \in \mathbb{R}$ such that the bi-infinite geodesic consists only of points (r,t). Now (3.2) fails.

Part (vii): Define

$$\widehat{\Omega}^{(\theta)} = \widetilde{\Omega}^{(\theta)} \cap \Omega^{(\theta,\star)} \cap \Omega^{(\theta,\theta)}(\mathbf{X}^\theta),$$

where $\widetilde{\Omega}^{(\theta)}$ is the event defined in (7.16), $\Omega^{(\theta,\star)}$ is the event of Theorem 7.11(v), and $\Omega^{(\theta,\theta)}(\mathbf{X}^\theta)$ is the event of Part (vi), applied to the random environment $\mathbf{X}^\theta$ of i.i.d. Brownian motions (Lemma 3.7(ii)).

By the construction of the $\mathbf{T}_{\mathbf{x}}^\theta$ in terms of the variational formula (Definition 4.1), if for some $\mathbf{x}, \mathbf{y} \in \mathbb{Z} \times \mathbb{R}$, the rightmost semi-infinite geodesics in $\mathbf{T}_{\mathbf{x}}^\theta$ and $\mathbf{T}_{\mathbf{y}}^\theta$ ever intersect, they agree above and to the right of the point of intersection. The same is true of leftmost geodesics. Because $\widetilde{\Omega}^{(\theta)} \subseteq \Omega^{(\theta)} \cap \Omega_2$ (see discussion after (7.16)),

$$\widehat{\Omega}^{(\theta)} \subseteq \Omega^{(\theta)} \cap \Omega^{(\theta,\star)} \cap \Omega_2.$$

If, by way of contradiction, for some $\omega \in \widehat{\Omega}^{(\theta)}$, there exists $\mathbf{x}$ and $\mathbf{y} \in \mathbb{Z} \times \mathbb{R}$ such that the rightmost geodesics in $\mathbf{T}_{\mathbf{x}}^\theta$ and $\mathbf{T}_{\mathbf{y}}^\theta$ are disjoint, then by Theorem 4.9, there exists a bi-infinite geodesic for the environment $\mathbf{X}^\theta$ that is defined by jump times $\{\tau_r\}_{r \in \mathbb{Z}}$ and satisfies

$$\lim_{n \to \infty} \frac{\tau_n}{n} = \theta = \lim_{n \to \infty} \frac{\tau_{-n}}{-n}.$$

This is a contradiction since $\omega \in \Omega^{(\theta,\theta)}(\mathbf{X}^\theta)$. Therefore, for all $\omega \in \widehat{\Omega}^{(\theta)}$, whenever (m, s), $(k, t) \in \mathbb{Z} \times \mathbb{R}$, the rightmost geodesics in $\mathbf{T}^\theta_{(m,s)}$ and $\mathbf{T}^\theta_{(r,t)}$ coalesce. The same is true by replacing "right" with "left." In other words, for all sufficiently large r,

(7.29) $$\tau^{\theta,R}_{(m,s),r} = \tau^{\theta,R}_{(k,t),r} \quad \text{and} \quad \tau^{\theta,L}_{(m,s),r} = \tau^{\theta,L}_{(k,t),r}.$$

Now, let $s = s_{m-1} \leq s_m \leq \cdots$ and $t = t_{k-1} \leq t_k \leq \cdots$ be any sequences defining θ-directed semi-infinite geodesics starting from (m, s) and (k, t), respectively. We show that these semi-infinite geodesics coalesce. Without loss of generality, assume that $m \leq k$ and $s_{k-1} < t_{k-1} = t$. The other cases are handled similarly. Then, by Theorems 4.5(ii) and 4.3(iii)(c), for all $r \geq k$,

$$\tau^{\theta,L}_{(k,s_{k-1}),r} \leq s_r \leq \tau^{\theta,R}_{(k,s_{k-1}),r} \leq \tau^{\theta,L}_{(k,t),r} \leq t_r \leq \tau^{\theta,R}_{(k,t),r}.$$

Then, by (7.29), these inequalities are all equalities for all sufficiently large r, and the geodesics coalesce. $\qquad\square$

We conclude this section by completing the proof of Theorem 4.3(iv)(b) alluded to earlier.

Proof of Theorem 4.3(iv)(b):. By the monotonicity of Theorem 4.3(iii)(a), it suffices to show that

$$\lim_{\theta \searrow 0} \tau^{\theta+,R}_{(m,t),r} = t \quad \text{and} \quad \lim_{\theta \to \infty} \tau^{\theta-,L}_{(m,t),r} = \infty \quad \text{for } r \geq m.$$

The proof of Theorem 3.1(v), established the statement for $r = m$. For the limits as $\theta \to \infty$, this implies the statement holds for all $r \geq m$ since $\tau^{\theta-,L}_{(m,t),m} \leq \tau^{\theta-,L}_{(m,t),r}$. By the monotonicity of Theorem 4.3(iii)(a), the limit

$$\tau^0_{(m,t),r} := \lim_{\theta \searrow 0} \tau^{\theta+,R}_{(m,t),r}$$

exists and satisfies $\tau^0_{(m,t),r} \leq \tau^{\theta+,R}_{(m,t),r}$ for all $\theta > 0$. Furthermore, since $\{\tau^{\theta+,R}_{(m,t),r}\}_{r \geq m-1}$ is a nondecreasing sequence for each $\theta > 0$, the sequence $\{\tau^0_{(m,t),r}\}_{r \geq m-1}$ is also nondecreasing. By Theorem 4.3(v),

$$0 \leq \limsup_{n \to \infty} \frac{\tau^0_{(m,t),n}}{n} \leq \lim_{n \to \infty} \frac{\tau^{\theta+,R}_{(m,t),n}}{n} = \theta \quad \text{for all } \theta > 0 \quad \implies \quad \lim_{n \to \infty} \frac{\tau^0_{(m,t),n}}{n} = 0.$$

Therefore, by Theorem 3.1(v), if the sequence of jump times $t = \tau^{0,R}_{(m,t),m-1} \leq \tau^{0,R}_{(m,t),m} \leq \cdots$ defines a semi-infinite geodesic starting from (m, t), the desired conclusion follows. It is sufficient to show that, for any $n \geq m$, the sequence $t = \tau^0_{(m,t),m-1} \leq \tau^0_{(m,t),m} \leq \cdots \leq \tau^0_{(m,t),n}$ defines jump times for a finite geodesic between (m, t) and $(n, \tau^0_{(m,t),n})$. By Theorem 4.3(i), For each $\theta > 0$, the sequence $t = \tau^{\theta+,R}_{(m,t),m-1} \leq \cdots \leq \tau^{\theta+,R}_{(m,t),n}$ is a maximizing sequence for

$$L_{(m,t),(n,\tau^{\theta+,R}_{(m,t),n})} = \sup\left\{ \sum_{r=m}^n B_r(s_r, s_{r-1}) : t = s_{m-1} \leq s_m \leq \cdots \leq s_n = \tau^{\theta+,R}_{(m,t),n} \right\},$$

so since $\lim_{\theta \searrow 0} \tau^{\theta+,R}_{(m,t),n} = \tau^0_{(m,t),n}$, Lemma A.3 completes the proof. $\qquad\square$

Appendix A: Deterministic facts about continuous functions

Lemma A.1. *Let $\eta^1, \eta^2 : \mathbb{R} \to \mathbb{R}$ be continuous functions satisfying*

$$\limsup_{t \to \infty} \eta^i(t) = -\infty \quad \text{for } i = 1, 2.$$

Further, assume that $\eta^1(s, u) \leq \eta^2(s, u)$ for all $s < u$. For each $t \in \mathbb{R}$ and $i = 1, 2$, let $s^{i,L}_t$ and $s^{i,R}_t$ be the leftmost and rightmost maximizers of η^i on the set $[t, \infty)$. Then,

$$s^{1,L}_t \leq s^{2,L}_t \quad \text{and} \quad s^{1,R}_t \leq s^{2,R}_t.$$

Proof. Since $s_t^{2,L}$ is a maximizer for η^2 on $[t, \infty)$, $\eta^2(s_t^{2,L}) \geq \eta^2(s)$, or equivalently, $\eta^2(s_t^{2,L}, s) \leq 0$ for all $s \geq t$, Then, for all $s \geq s_t^{2,L} \geq t$, the hypothesis of the lemma gives

$$\eta^1(s_t^{2,L}, s) \leq \eta^2(s_t^{2,L}, s) \leq 0.$$

Thus, the leftmost maximum of η^1 on the larger set $[t, \infty)$ can be no larger than $s_t^{2,L}$.

For rightmost maximizers, the lemma follows by similar reasoning: for $s > s_t^{2,R}$,

$$\eta^1(s_t^{2,R}, s) \leq \eta^2(s_t^{2,R}, s) < 0,$$

so $\eta^1(s_t^{2,R}) > \eta^1(s)$ for all $s > s_t^{2,R}$, and no maximizer of $\eta^1(s)$ over $s \in [t, \infty)$ can be larger than $s_t^{2,R}$. $\quad\square$

Lemma A.2. *Let $S \subseteq \mathbb{R}^n$, and let $f_n : S \to \mathbb{R}$ be a sequence of continuous functions, converging uniformly to the function $f : S \to \mathbb{R}$. Assume that there exists a sequence $\{c_n\}$, of maximizers of f_n, converging to some $c \in S$. Then, c is a maximizer of f.*

Proof. $f_n(c_n) \geq f_n(x)$ for all $x \in S$, so it suffices to show that $f_n(c_n) \to f(c)$. This follows from the uniform convergence of f_n to f, the continuity of f, and

$$|f_n(c_n) - f(c)| \leq |f_n(c_n) - f(c_n)| + |f(c_n) - f(c)|. \quad\square$$

Lemma A.3. *Let S_n for $n \geq 0$ be subsets of some set $\widetilde{S} \subseteq \mathbb{R}^n$, on which the function $f : \widetilde{S} \to \mathbb{R}$ is continuous. Assume that each point $x \in S_0$ is the limit of a sequence $\{x_n\}$, where $x_n \in S_n$ for each n. Assume that $\{c_n\}$ is a sequence of maximizers of f on S_n. Assume further that c_n converges to some $c \in S_0$. Then, c is a maximizer of f on S_0.*

Proof. For each $x_0 \in S_0$, write $x_0 = \lim_{n \to \infty} x_n$, where $x_n \in S_n$ for each n. Then, $f(c_n) \geq f(x_n)$ for all $n \geq 1$, and the result follows by taking limits. $\quad\square$

Lemma A.4. *Let $\mathbf{X} = \{X_m\}_{m \in \mathbb{Z}}$, where each $X_m : \mathbb{R} \to \mathbb{R}$ is a continuous function. Let $0 < s < t < T < u$ and $m \leq n$. Then,*

$$X_m(s, t) \leq L_{(m,s),(n,u)}(\mathbf{X}) - L_{(m,t),(n,u)}(\mathbf{X}) \leq L_{(m,s),(n,T)}(\mathbf{X}) - L_{(m,t),(n,T)}(\mathbf{X})$$

Similarly, let $0 < s < t < u < \infty$ and $m < n$. Then,

$$0 \leq L_{(m,s),(n,t)}(\mathbf{X}) - L_{(m+1,s),(n,t)}(\mathbf{X}) \leq L_{(m,s),(n,u)}(\mathbf{X}) - L_{(m+1,s),(n,u)}(\mathbf{X}).$$

Remark A.5. This is a deterministic statement. The only necessary ingredient is the continuity of the X_m so that each of the last-passage times is finite and has a sequence of maximizing times.

Proof. This proof follows a standard paths-crossing argument. For example, the proofs of Lemma 4.6 in [2] and Proposition 3.8 in [19] follow the same procedure. We prove the first statement, and the second is proven similarly. By definition of last-passage time,

$$X_m(s, t) + L_{(m,t),(n,u)} \leq L_{(m,s),(n,u)}.$$

Since $0 < s < t < T < u$, any geodesic between (m, s) and (n, u) must cross any geodesic between (m, t) and (n, T). Let $\mathbf{z} \in \mathbb{Z} \times \mathbb{R}$ be a point of intersection. Then,

$$L_{(m,s),(n,u)} = L_{(m,s),\mathbf{z}} + L_{\mathbf{z},(n,u)} \quad \text{and} \quad L_{(m,t),(n,T)} = L_{(m,t),\mathbf{z}} + L_{\mathbf{z},(n,T)},$$

so

$$L_{(m,s),(n,u)} - L_{(m,t),(n,u)} \leq L_{(m,s),\mathbf{z}} + L_{\mathbf{z},(n,u)} - (L_{(m,t),\mathbf{z}} + L_{\mathbf{z},(n,u)}) = L_{(m,s),\mathbf{z}} - L_{(m,t),\mathbf{z}}$$

$$= L_{(m,s),\mathbf{z}} + L_{\mathbf{z},(n,T)} - (L_{(m,t),\mathbf{z}} + L_{\mathbf{z},(n,T)}) \leq L_{(m,s),(n,T)} - L_{(m,t),(n,T)}. \quad\square$$

Appendix B: Probabilistic results

The following is a classical result that is often used in this paper.

Lemma B.1 ([11], Equation 1.1.4 (1) on pg 251). *For a standard Brownian motion B and $\lambda > 0$,*

$$\sup_{0 \leq s < \infty} \left\{ \sqrt{2} B(s) - \lambda s \right\} \sim \text{Exp}(\lambda).$$

The proof of the following theorem follows from a straightforward, but tedious convolution. The details are included in the arXiv version of this paper [55].

Theorem B.2. *Let B be a standard Brownian motion and let $\lambda > 0$. Let T be the unique maximizer of $\sqrt{2} B(t) - \lambda t$ for $t \in [0, \infty)$. Then, for $t \geq 0$,*

$$\mathbb{P}(T > t) = \left(2 + \lambda^2 t \right) \Phi(-\lambda \sqrt{t/2}) - \lambda \sqrt{t/\pi} e^{-\frac{\lambda^2 t}{4}}.$$

Remark B.3. Theorem B.2 should be credited to Norros and Salminen, who computed the Laplace transform of this random variable in Proposition 3.9 of [50]. One can also obtain it by integrating the probability density of the time of maximum of Brownian motion with drift on the interval $[0, t]$ found in [12], Equation (1.3), and then taking $t \to \infty$. See also the discussion after Equation (1.3) in [12] for more historical details on this formula.

Theorem B.4. *Let X be a two-sided Brownian motion with strictly negative drift. Let*

$$M = \left\{ t \in \mathbb{R} : X(t) = \sup_{t \leq s < \infty} X(s) \right\}.$$

Furthermore, let

$$M^U = \left\{ t \in M : X(t) > X(s) \text{ for all } s > t \right\}$$

be the set of $t \in M$ that are unique maximizers of $X(s)$ over $s \in [t, \infty)$. Define $M^N = M \setminus M^U$ to be the set of $t \in M$ that are non-unique maximizers of $X(s)$ over $s \in [t, \infty)$. Then, there exists an event of probability one, on which the following hold.

 (i) *M is a closed set.*
 (ii) *For all $\hat{t} \in M^U$ and $\varepsilon > 0$, there exists $t \in M^N$ satisfying $\hat{t} < t < \hat{t} + \varepsilon$. For all $t \in M^N$ and $\varepsilon > 0$, there exists $\hat{t} \in M^U$ satisfying $t - \varepsilon < \hat{t} < t$.*
 (iii) *For all $t \in M^N$ and $\varepsilon > 0$, there exists $t^\star \in M^N$ with $t - \varepsilon < t^\star < t$. For each $t \in M^N$, there exists $\delta > 0$ such that $M \cap (t, t + \delta) = \varnothing$.*
 (iv) *M^N is a countably infinite set.*

Proof. By the $n = m$ case of Lemma 7.1, for each fixed $t \in \mathbb{R}$, $X(s)$ is almost surely uniquely maximized for $s \in [t, \infty)$. By Theorem B.2, this maximizer is almost surely strictly greater than t. Let $\Omega_{\mathbb{Q}}$ be the full probability event on which,

$$\lim_{s \to -\infty} X(s) = \infty \quad \text{and} \quad \lim_{s \to \infty} X(s) = -\infty,$$

and such that, for every $q \in \mathbb{Q}$, there is a unique maximizer of $X(s)$ over $s \in [q, \infty)$ that is strictly larger than q.

Part (i): This follows from Lemma A.3.

Part (ii): Let $\omega \in \Omega_{\mathbb{Q}}$, $\hat{t} \in M^U$ and $\varepsilon > 0$, and let $q \in \mathbb{Q}$ satisfy $\hat{t} < q < \hat{t} + \varepsilon$. Then, there is a unique maximizer, $s > q$ of $X(u)$ over $u \in [q, \infty)$. By assumption, $\hat{t}$ uniquely maximizes $X(u)$ over the larger set $[\hat{t}, \infty)$, so $X(q) < X(s) < X(\hat{t})$. By the intermediate value theorem, there exists a point u with $\hat{t} < u < q$ and $X(u) = X(s)$. Then, set

$$t = \max \left\{ u \in (\hat{t}, s) : X(u) = X(s) \right\}.$$

Then, $t < q < s$, and

$$X(t) = X(s) = \sup_{t \leq u < \infty} X(u)$$

because $X(u) < X(s)$ for all $t < u < q$. Therefore, $\hat{t} < t < q < \hat{t} + \varepsilon$, and $t \in M^N$.

Now, let $t \in M^N$ and $\varepsilon > 0$. Let $q \in \mathbb{Q}$ satisfy $t - \varepsilon < q < t$. Then, there is a unique maximizer $\hat{t}$ of $X(s)$ over $s \in [q, \infty)$, so $X(\hat{t}) > X(s)$ for all $s \in [q, \infty) \setminus \{\hat{t}\}$. Specifically, $\hat{t} \in M^U$ and $X(\hat{t}) > X(t)$, so since t maximizes $X(s)$ over $s \in [t, \infty)$, $\hat{t}$ cannot be greater than or equal to t. Therefore, $t - \varepsilon < q < \hat{t} < t$.

Part (iii): The first statement follows immediately from Part (ii). We now prove the second statement. We start by showing that for each $\omega \in \Omega_{\mathbb{Q}}$, there does not exist $t \in \mathbb{R}$ such that $X(s)$ has three maximizers over $s \in [t, \infty)$. If, on the contrary, such a value of t exists, at least two of the maximizers must be greater than t, and therefore, these two maximizers are also maximizers of $X(s)$ over $s \in [q, \infty)$ for some $q \in \mathbb{Q}$. Thus, on $\Omega_{\mathbb{Q}}$, for each $t \in M^N$, there exists a unique $\hat{t} \in M^U$ such that $\hat{t} > t$ and $(t, \hat{t}) \cap M = \varnothing$.

Part (iv): The set M is nonempty because, for any $s \in \mathbb{R}$, any maximizer t of $X(u)$ over $u \in [s, \infty)$ lies in M. Then, by the first statement of Part (iii), the set M^N is infinite. By the second statement of Part (iii), M^N is countable. $\square$

Remark B.5. Originally discovered by Taylor [57], it is well known that the zero set of Brownian motion almost surely has Hausdorff dimension $\frac{1}{2}$. Parts (i) and (ii) of Theorem B.4 imply that M is the supporting set of the Lebesgue–Stieltjes measure defined by the nondecreasing function $t \mapsto -\sup_{t \leq s < \infty} X(s)$. Using this fact, Taylor's proof (see also Theorem 4.24 in [42]) can be modified to show that the set M almost surely has Hausdorff dimension $\frac{1}{2}$.

Appendix C: The Brownian queue and stationary last-passage process

This section discusses the Brownian queue in the formulation of [45]. Let A and S be two independent, two-sided Brownian motions, and let $\lambda > 0$. For $s < t$, $A(s, t)$ represents the arrivals to the queue in the time interval $(s, t]$, and $\lambda(t - s) - S(s, t)$ is the amount of service available in $(s, t]$. For $t \in \mathbb{R}$, set

$$q(t) = \sup_{-\infty < s \leq t} \{A(s, t) + S(s, t) - \lambda(t - s)\} \quad \text{and} \quad d(t) = A(t) + q(0) - q(t).$$

In queuing terms, $q(t)$ is the length of the queue at time t, and for $s < t$, $d(s, t)$ is the number of departures from the queue in the interval $(s, t]$. These processes are not integer-valued, but are viewed as heavy-traffic limits. We also define $e(t) = S(t) + q(0) - q(t)$. The following is due to O'Connell and Yor [45]. Without the statements for the process e, the theorem is a special case of a more general result previously shown by Harrison and Williams [34].

Theorem C.1 ([45], Theorem 4). *The processes d and e are independent, two-sided Brownian motions. Furthermore, for each $t \in \mathbb{R}$, $\{d(s, t), e(s, t) : -\infty < s \leq t\}$ is independent of $\{q(u) : u \geq t\}$.*

We reformulate Theorem C.1 in terms of the queuing mappings of (3.3)–(3.5) and (3.6)–(3.8).

Theorem C.2. *Let Y be a two-sided Brownian motion with drift $\lambda > 0$, independent of the two-sided Brownian motion C (with no drift). Then, $\overleftarrow{D}(Y, C)$ is a two-sided Brownian motion with drift λ, independent of the two-sided Brownian motion $\overleftarrow{R}(Y, C)$. Furthermore, for all $t \in \mathbb{R}$, $\{(\overleftarrow{D}(Y, C)(s, t), \overleftarrow{R}(Y, C)(s, t)) : -\infty < s \leq t\}$ is independent of $\{\overleftarrow{Q}(Y, C)(u) : u \geq t\}$.*

Proof. Let $A(t) = \lambda t - Y(t)$ and $S(t) = C(t)$, two independent two-sided Brownian motions. Then

$$q(t) = \sup_{-\infty < s \leq t} \{A(s, t) + S(s, t) - \lambda(t - s)\} = \sup_{-\infty < s \leq t} \{C(s, t) - Y(s, t)\} = \overleftarrow{Q}(Y, C)(t).$$

Next observe that

$$\overleftarrow{D}(Y, C)(t) = Y(t) + \overleftarrow{Q}(Y, C)(t) - \overleftarrow{Q}(Y, C)(0) = -A(t) + \lambda t + q(t) - q(0) = \lambda t - d(t),$$

$$\text{and} \quad \overleftarrow{R}(Y, C)(t) = C(t) + \overleftarrow{Q}(Y, C)(0) - \overleftarrow{Q}(Y, C)(t) = S(t) + q(0) - q(t) = e(t).$$

The result follows from Theorem C.1. $\square$

Fix a parameter $\lambda > 0$. Given an environment of Brownian motions $\mathbf{B} = \{B_m\}_{m \in \mathbb{Z}}$, set
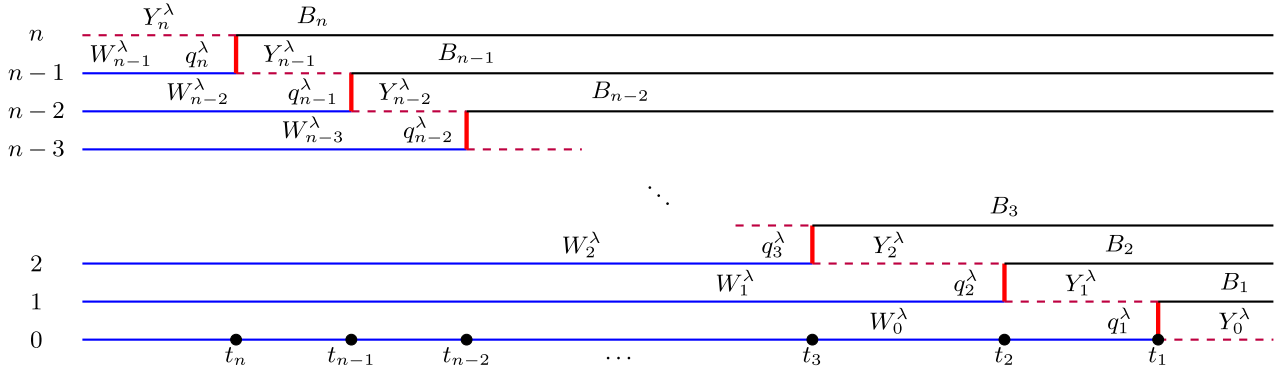
$$Y_0^\lambda(t) = -B_0(t) + \lambda t.$$

Fig. 17. Independence structure for stationary BLPP. Each process Y_r^λ is associated to the (purple) dashed segment on level r, processes B_r and W_r^λ cover the remaining portions of horizontal level r, and the process q_r^λ is associated to the (red) vertical edge from level $r-1$ to r at time point t_r.

For $m > 0$, recalling definitions (3.6)–(3.8), set

(C.1) $q_m^\lambda(t) := \overleftarrow{Q}\left(Y_{m-1}^\lambda, B_m\right)(t), \qquad Y_m^\lambda(t) := \overleftarrow{D}\left(Y_{m-1}^\lambda, B_m\right)(t), \quad \text{and} \quad W_{m-1}^\lambda(t) := \overleftarrow{R}\left(Y_{m-1}^\lambda, B_m\right)(t).$

The increment-stationary BLPP is constructed as follows: For $(n, t) \in \mathbb{Z}_{>0} \times \mathbb{R}$, set

$$L_{(n,t)}^\lambda = \sup_{-\infty < s \leq t} \left\{Y_0^\lambda(s) + L_{(1,s),(n,t)}(\mathbf{B})\right\}.$$

One can check inductively, as in Section 4 of [45], that for $m \geq 1$ and $s, t \in \mathbb{R}$,

$$q_m^\lambda(t) = L_{(m,t)}^\lambda - L_{(m-1,t)}^\lambda, \quad \text{and} \quad Y_m^\lambda(s, t) = L_{(m,t)}^\lambda - L_{(m,s)}^\lambda.$$

The stationarity of these increments comes from the next theorem that follows from Theorem C.2 and induction. It is the zero-temperature analogue of Theorem 3.3 in [56] and Theorem 2.11 in [2]. Figure 17 demonstrates the independence structure, and we give credit to [2] for a very similar picture.

Theorem C.3. *Let Y_m^λ, q_m^λ, W_m^λ be as constructed in (C.1). Then, the following hold.*

(i) *For all $m > 0$ and $t \in \mathbb{R}$, $q_m^\lambda(t) \sim \mathrm{Exp}(\lambda)$. $\{W_m^\lambda\}_{m \geq 0}$ is a field of independent two-sided Brownian motions. $\{Y_m^\lambda\}_{m \geq 0}$ is a field of (non-independent) two-sided Brownian motions with drift λ.*

(ii) *Let $-\infty < t_n \leq t_{n-1} \leq \cdots \leq t_1 < \infty$. Then, the following are mutually independent:*

$$\left\{W_0^\lambda(u, t_1) : u \leq t_1\right\}, \qquad q_1^\lambda(t_1), \qquad \left\{Y_0^\lambda(t_1, u) : u \geq t_1\right\},$$

$$\left\{W_r^\lambda(u, t_{r+1}) : u \leq t_{r+1}\right\}, \qquad q_{r+1}^\lambda(t_{r+1}), \qquad \left\{Y_r^\lambda(t_{r+1}, u) : t_{r+1} \leq u \leq t_r\right\},$$

$$\text{and} \quad \left\{B_r(t_r, u) : u \geq t_r\right\}, \quad \text{for } 1 \leq r \leq n-1, \quad \left\{Y_n^\lambda(u, t_n) : u \leq t_n\right\}, \quad \text{and} \quad \left\{B_n(t_n, u) : u \geq t_n\right\}.$$

Proof. To prove Part (i), begin with the assumption:

$$\left(Y_0^\lambda, B_1, \ldots, B_n\right) \quad \text{are mutually independent.}$$

By Theorem C.2, $Y_1^\lambda = \overleftarrow{D}(Y_0^\lambda, B_1)$ is a two-sided Brownian motion with drift λ, independent of the two-sided Brownian motion $W_0^\lambda = \overleftarrow{R}(Y_0^\lambda, B_1)$. Hence,

$$\left(W_0^\lambda, Y_1^\lambda, B_2, \ldots, B_n\right) \quad \text{are mutually independent.}$$

Inductively, assume that Y_r^λ is a two-sided Brownian motion with drift λ and

$$\left(W_0^\lambda, \ldots, W_{r-1}^\lambda, Y_r^\lambda, B_{r+1}, \ldots, B_n\right) \quad \text{are mutually independent.}$$

Apply Theorem C.2 to $Y_{r+1}^\lambda = \overleftarrow{D}(Y_r^\lambda, B_{r+1})$ and $W_r^\lambda = \overleftarrow{R}(Y_r^\lambda, B_{r+1})$ to continue the induction. Part (i) is proved. The proof of Part (ii) is the proof of Theorem 3.3 in [56].   □

The proof of Theorem 2.3 is achieved through the coupling of q_r^λ, Y_r^λ, and W_r^λ. The following theorem is the key.

Theorem C.4 (Zero-temperature analogue of Theorem 3.1 in [2]). *Fix real numbers $t > s$, $\lambda > 0$, and $\gamma > \lambda^{-2} > \delta > 0$. Then, with probability one,*

(C.2)
$$\limsup_{n\to\infty}\left[L_{(0,t),(n,n\delta)}(\mathbf{W}^\lambda) - L_{(0,s),(n,n\delta)}(\mathbf{W}^\lambda)\right] \le B_0(s,t) - \lambda(t-s) = -Y_0^\lambda(s,t)$$
$$\le \liminf_{n\to\infty}\left[L_{(0,t),(n,n\gamma)}(\mathbf{W}^\lambda) - L_{(0,s),(n,n\gamma)}(\mathbf{W}^\lambda)\right],$$

and

(C.3)
$$\limsup_{n\to\infty}\left[L_{(0,t),(n,n\delta)}(\mathbf{W}^\lambda) - L_{(1,t),(n,n\delta)}(\mathbf{W}^\lambda)\right]$$
$$\le q_1^\lambda(t) \le \liminf_{n\to\infty}\left[L_{(0,t),(n,n\gamma)}(\mathbf{W}^\lambda) - L_{(1,t),(n,n\gamma)}(\mathbf{W}^\lambda)\right].$$

Here, $L_{\mathbf{x},\mathbf{y}}(\mathbf{W}^\lambda)$ is the last passage time between points $\mathbf{x} \le \mathbf{y} \in \mathbb{Z}_{>0} \times \mathbb{R}$ when $\mathbf{W}^\lambda := \{W_m^\lambda\}_{m\ge 0}$ is the random environment. (W_m^λ need not be defined for $m < 0$, since all points $(m,t) \in \mathbb{Z} \times \mathbb{R}$ in the above expression satisfy $m \ge 0$.) While Theorem C.4 is not explicitly stated in [2], all necessary lemmas are provided in Section 4 of that paper.

Lemma C.5. *Let $\mathcal{A} \subseteq \mathbb{R}$ be any countable set. Consider the space $\mathbb{R}^{\mathcal{A}}$ of functions $\mathcal{A} \to \mathbb{R}$, equipped with the Borel product σ-algebra. Fix $\theta > 0$. Then,*

$$\{h_0^\theta(t) : t \in \mathcal{A}\} \overset{d}{=} \left\{-B_0(t) + \frac{1}{\sqrt{\theta}}t : t \in \mathcal{A}\right\} \quad and \quad \{v_1^\theta(t) : t \in \mathcal{A}\} \overset{d}{=} \{q_1^{\frac{1}{\sqrt{\theta}}}(t) : t \in \mathcal{A}\}.$$

Proof. This proof is a multivariate extension of the "Proof of Theorem 2.5, assuming Theorem 3.1" on page 1937 of [2]. This method originated in the setting of the log-gamma polymer in [25]. From definition (2.6) of h_0^θ, $h_0^\theta(0) = 0 = B_0(0)$. Hence, to prove the first statement, it is sufficient to show that

$$\{h_0^\theta(s,t) : s, t \in \mathcal{A}, s < t\} \overset{d}{=} \left\{-B_0(s,t) + \frac{1}{\sqrt{\theta}}(t-s) : s, t \in \mathcal{A}, s < t\right\}.$$

Theorem 2.3 establishes that, for fixed $s, t \in \mathbb{R}$,

$$h_0^\theta(s,t) = \lim_{n\to\infty}\left[L_{(0,s),(n,n\theta)}(\mathbf{B}) - L_{(0,t),(n,n\theta)}(\mathbf{B})\right] \quad \text{a.s.}$$

By Theorem C.3, $\{W_m^\lambda\}_{m\ge 0}$ is a field of independent, two-sided Brownian motions. By equality of distribution between $\{W_m^\lambda\}_{m\ge 0}$ and $\{B_m\}_{m\ge 0}$, the first inequality of (C.2) implies that if $\{s_1, \ldots, s_k\}$ and $\{t_1, \ldots, t_k\}$ are finite subsets of $\mathcal{A}$ with $s_i < t_i$ for all i, and $x_1, \ldots, x_k \in \overline{\mathbb{R}}$, then whenever $\theta < \lambda^{-2}$, or equivalently, $\lambda < \frac{1}{\sqrt{\theta}}$,

$$\mathbb{P}\left(-h_0^\theta(s_i,t_i) \ge x_i, 1 \le i \le k\right) \le \mathbb{P}\left(B_0(s_i,t_i) - \lambda(t_i-s_i) \ge x_i, 1 \le i \le k\right).$$

Since $\lambda > 0$ and $s_i < t_i$ for all i, the right-hand side decreases as λ increases, so taking $\lambda \nearrow \frac{1}{\sqrt{\theta}}$ gives us

$$\mathbb{P}\left(-h_0^\theta(s_i,t_i) \ge x_i, 1 \le i \le k\right) \le \mathbb{P}\left(B_0(s_i,t_i) - \frac{1}{\sqrt{\theta}}(t_i-s_i) \ge x_i, 1 \le i \le k\right).$$

The inequality on the right of (C.2) with $\lambda \searrow \frac{1}{\sqrt{\theta}}$ establishes the reverse inequality.

The same argument works for v_1^θ, utilizing this monotonicity for $\gamma < \delta$:

$$q_1^\gamma(t) = \overleftarrow{Q}(Y, B_1)(t) = \sup_{-\infty < s \le t}\{B_1(s,t) + B_0(s,t) - \gamma(t-s)\}$$

$$\ge \sup_{-\infty < s \le t}\{B_1(s,t) + B_0(s,t) - \delta(t-s)\} \ge q_1^\delta(t). \qquad \square$$

Appendix D: Time reversal

Theorem D.1. *Let* Z, B, Y, C *be continuous functions satisfying* $Z(0) = B(0) = Y(0) = C(0) = 0$ *and*

$$\lim_{t \to \pm\infty} \big(B(t) - Z(t)\big) = \lim_{t \to \pm\infty} \big(C(t) - Y(t)\big) = \mp\infty.$$

Then,

$$Y = D(Z, B) \quad and \quad C = R(Z, B) \quad if \ and \ only \ if \quad Z = \overleftarrow{D}(Y, C) \quad and \quad B = \overleftarrow{R}(Y, C),$$

The equalities above denote equality as functions of t. *If either of the above equivalent conditions are satisfied, then also* $Q(Z, B) = \overleftarrow{Q}(Y, C)$.

   The following lemmas will help to prove Theorem D.1.

Lemma D.2. *Let* $Z, B : \mathbb{R} \to \mathbb{R}$ *be continuous functions satisfying* $Z(0) = B(0) = 0$ *and*

$$\lim_{t \to \pm\infty} \big(B(t) - Z(t)\big) = \mp\infty.$$

For a function $f : \mathbb{R} \to \mathbb{R}$, *recall that we define* $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ *by* $\widetilde{f}(t) = -f(-t)$. *Then, for all* $t \in \mathbb{R}$,

$$Q(Z, B)(-t) = \overleftarrow{Q}(\widetilde{Z}, \widetilde{B})(t), \qquad -D(Z, B)(-t) = \overleftarrow{D}(\widetilde{Z}, \widetilde{B})(t), \quad and \quad -R(Z, B)(-t) = \overleftarrow{R}(\widetilde{Z}, \widetilde{B})(t).$$

Proof. This is a routine check, using the definitions.   □

   The following is a well-known fact, but is often stated without proof, so we include full justification for the sake of completeness. For example, it appears as Equation (1.4) in [48] and Equation (13) in [45].

Lemma D.3. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that*

$$\lim_{t \to \pm\infty} f(t) = \pm\infty.$$

Set $F(t) = \sup_{-\infty < s \leq t} f(s)$. *Then,*

$$(D.1) \qquad\qquad\qquad \inf_{t \leq s < \infty} \big(2F(s) - f(s)\big) = F(t).$$

Proof. The left-hand side of (D.1) is

$$\inf_{t \leq s < \infty} \big(2F(s) - f(s)\big) = \inf_{t \leq s < \infty} \Big(2 \sup_{-\infty < u \leq s} f(u) - f(s)\Big).$$

For all $s \geq t$, $\sup_{\infty < u \leq s} f(u)$ is greater than or equal to both $f(s)$ and $\sup_{-\infty < u \leq t} f(u)$. Therefore,

$$2 \sup_{-\infty < u \leq s} f(u) - f(s) \geq \sup_{-\infty < u \leq t} f(u) + f(s) - f(s) = F(t).$$

This establishes one direction of Equation (D.1). To show the other direction, we show that there exists $s \geq t$ such that

$$2F(s) - f(s) = F(t).$$

Note that $f(t) \leq F(t)$ and that $\lim_{s \to \infty} f(s) = \infty$ by assumption. Hence, by continuity of f, $f(s) = F(t)$ for some $s \geq t$. Let

$$s^* = \inf\big\{s \geq t : f(s) = F(t)\big\}.$$

Then, $F(s^*) = F(t)$. Therefore,

$$2F\big(s^\star\big) - f\big(s^\star\big) = 2F(t) - F(t) = F(t). \qquad\qquad\qquad □$$

Proof of Theorem D.1. Assume that Z, B, Y, C satisfy the conditions of the Theorem. First, assume that $Z = \overleftarrow{D}(Y, C)$ and $B = \overleftarrow{R}(Y, C)$. By definitions (3.6)–(3.8),

$$Z(s) = Y(s) + \sup_{-\infty < u \leq s} \big\{ C(u, s) - Y(u, s) \big\} - \sup_{-\infty < u \leq 0} \big\{ C(u, 0) - Y(u, 0) \big\}$$

$$= C(s) + \sup_{-\infty < u \leq s} \big\{ Y(u) - C(u) \big\} - \sup_{-\infty < u \leq 0} \big\{ Y(u) - C(u) \big\}, \quad \text{and}$$

$$B(s) = C(s) + \sup_{-\infty < u \leq 0} \big\{ C(u, 0) - Y(u, 0) \big\} - \sup_{-\infty < u \leq s} \big\{ C(u, s) - Y(u, s) \big\}$$

$$= Y(s) + \sup_{-\infty < u \leq 0} \big\{ Y(u) - C(u) \big\} - \sup_{-\infty < u \leq s} \big\{ Y(u) - C(u) \big\}.$$

Then,

$$D(Z, B)(t) = Z(t) + Q(Z, B)(0) - Q(Z, B)(t)$$

$$= Z(t) + \sup_{0 \leq s < \infty} \big\{ B(0, s) - Z(0, s) \big\} - \sup_{t \leq s < \infty} \big\{ B(t, s) - Z(t, s) \big\}$$

$$= B(t) + \sup_{0 \leq s < \infty} \big\{ B(s) - Z(s) \big\} - \sup_{t \leq s < \infty} \big\{ B(s) - Z(s) \big\}$$

$$= Y(t) + \sup_{-\infty < s \leq 0} \big\{ Y(s) - C(s) \big\} - \sup_{-\infty < s \leq t} \big\{ Y(s) - C(s) \big\}$$

$$+ \sup_{0 \leq s < \infty} \Big\{ Y(s) - C(s) - 2 \sup_{-\infty < u \leq s} \big\{ Y(u) - C(u) \big\} \Big\}$$

$$- \sup_{t \leq s < \infty} \Big\{ Y(s) - C(s) - 2 \sup_{-\infty < u \leq s} \big\{ Y(u) - C(u) \big\} \Big\}.$$

To show that this equals $Y(t)$, it is therefore sufficient to show that for $t \in \mathbb{R}$,

$$\sup_{-\infty < s \leq t} \big\{ Y(s) - C(s) \big\} = \inf_{t \leq s < \infty} \Big\{ 2 \sup_{-\infty < u \leq s} \big\{ Y(u) - C(u) \big\} - \big(Y(s) - C(s) \big) \Big\},$$

which follows from Lemma D.3. The proof that $R(Z, B)(t) = C(t)$ follows by the same reasoning. The converse then follows by the previous case and Lemma D.2: if $Y = D(Z, B)$ and $C = R(Z, B)$, $\widetilde{Y} = \overleftarrow{D}(\widetilde{Z}, \widetilde{B})$ and $\widetilde{C} = \overleftarrow{R}(\widetilde{Z}, \widetilde{B})$, so $\widetilde{Z} = D(\widetilde{Y}, \widetilde{C})$ and $\widetilde{B} = R(\widetilde{Y}, \widetilde{C})$. Hence $Z = \overleftarrow{D}(Y, C)$ and $B = \overleftarrow{R}(Y, C)$.

We finish by showing that $Q(Z, B) = \overleftarrow{Q}(Y, C)$ whenever the two equivalent conditions of the theorem are met. For all $t \in \mathbb{R}$,

$$Y(t) = D(Z, B)(t) = Z(t) + Q(Z, B)(0) - Q(Z, B)(t), \quad \text{and}$$

$$Z(t) = \overleftarrow{D}(Y, C) = Y(t) + \overleftarrow{Q}(Y, C)(t) - \overleftarrow{Q}(Y, C)(0).$$

Putting these two equations together,

$$Q(Z, B)(t) - \overleftarrow{Q}(Y, C)(t) = Q(Z, B)(0) - \overleftarrow{Q}(Z, B)(0).$$

This is true for all $t \in \mathbb{R}$, so $Q(Z, B)(t) - \overleftarrow{Q}(Y, C)(t)$ must be equal to some constant. Recall that

$$Q(Z, B)(t) = \sup_{t \leq s < \infty} \big\{ B(t, s) - Z(t, s) \big\}, \quad \text{and} \quad \overleftarrow{Q}(Y, C)(t) = \sup_{-\infty < s \leq t} \big\{ C(s, t) - Y(s, t) \big\}.$$

By the limit conditions of the theorem, maximizers exist for each of the supremums above, so $Q(Z, B)$ and $\overleftarrow{Q}(Y, C)$ both achieve a minimum value of 0. Hence, the constant must be 0. $\qquad\square$

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