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2-Cartesian Fibrations I: A Model for ∞-Bicategories Fibred in ∞-Bicategories

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Abstract
In this paper, we provide a notion of ∞-bicategories fibred in ∞-bicategories which we call *2-Cartesian fibrations*. Our definition is formulated using the language of marked biscaled simplicial sets: Those are scaled simplicial sets equipped with an additional collection of triangles containing the scaled 2-simplices, which we call *lean triangles*, in addition to a collection of edges containing all degenerate 1-simplices. We prove the existence of a left proper combinatorial simplicial model category whose fibrant objects are precisely the 2-Cartesian fibrations over a chosen scaled simplicial set S. Over the terminal scaled simplicial set, this provides a new model structure modeling ∞-bicategories, which we show is Quillen equivalent to Lurie's scaled simplicial set model. We conclude by providing a characterization of 2-Cartesian fibrations over an ∞-bicategory. This characterization then allows us to identify those 2-Cartesian fibrations arising as the coherent nerve of a fibration of $\mathrm{Set}_{\Delta}^{+}$-enriched categories, thus showing that our definition recovers the preexisting notions of fibred 2-categories.

Keywords Infinity bicategory · Model structure · 2-Cartesian fibration · Scaled simplicial set

1 Introduction

Of Grothendieck's many insights, the construction of a fibred category from a functor may be one of the most influential notions in the development of (higher) category theory.

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Right/left fibrations, (co)Cartesian fibrations, and the associated Grothendieck constructions have become integral parts of the ∞-categorical toolbox.

In our previous works [2, 3], and [1], we have made extensive and free use of this toolbox in our exploration of cofinality for $(\infty, 2)$-categories. This paper can be seen as a necessary stepping-stone to the final phase of this exploration: providing a model structure for $(\infty, 2)$-categories fibred in $(\infty, 2)$-categories. The appropriate notion of (co)limits in $(\infty, 2)$-categories has already been explored in [9], and the connection to the various forms of the Grothendieck construction are made exceptionally clear there. In the exposition in [9], however, the authors cleverly sidestep the need for additional technology to handle fibred ∞-bicategories, opting instead to work with Set_Δ^+-enriched categories.

To generalize our cofinality criteria from [2] and [1], however, there is an unavoidable need for such a theory of fibrations, as well as an associated Grothendieck construction. As stated in [1] the cofinality criterion for ordinary $(\infty, 1)$-categorical colimits can be understood in term of weighted colimits by proving an equivalence of weight functors with values in the category of spaces. One essential ingredient of the proof of this fact is the theory of Cartesian fibrations, i.e. functors with values in ∞-categories. In [4] we will show that such proof will generalise in a straightforward manner once the necessary categorified theory of Cartesian fibrations is developed.

This paper is the first in a 2-part sequence which will provide this technology. We here define and develop a notion of *2-Cartesian fibration*, using the language of simplicial sets equipped with a marking and two scalings. We then show the existence of a model structure whose fibrant objects are precisely these 2-Cartesian fibrations. In the second paper , we will develop the corresponding Grothendieck construction and establish the full $(\infty, 2)$-categorical version of our results in [2] and [1]. Needless to say, we expect the theory of 2-Cartesian fibrations to have a wide range of applications outside our cofinality framework in much the same way as Cartesian fibrations now occupy a central position in the study of ∞-categories.

We would like to point out after this paper originally appeared as a preprint several new works have addressed similar topics [10, 14, 15]. Among these works the approach taken [10] is particular close to ours: The authors define notions of $(\infty, 2)$-categorical fibrations over scaled simplicial sets which model all possible variances (see next section for more details) of 2-dimensional fibrations. However, the authors do not give the corresponding model structures for these kinds of fibrations. As expected, our definitions coincide with those of [10] in the specific case we study in this paper.

1.1 Defining 2-Cartesian Fibrations

As one climbs up the ladder of categorification, the higher dimensionality manifests itself most obviously in the number of new variances that a functor can have. What seems like an innocent increase of complexity in the strict 2-categorical setting turns out to play a much more central role when working with $(\infty, 2)$-categories. In particular, there should be *four* sensible notions of ∞-bicategories fibred in ∞-bicategories. Loosely speaking, these correspond to a functor $f : \mathbb{B} \to \mathbb{C}$ having

1. Cartesian lifts of 1-morphisms and coCartesian lifts of 2-morphisms;
2. Cartesian lifts of 1-morphisms and Cartesian lifts of 2-morphisms;
3. coCartesian lifts of 1-morphisms and Cartesian lifts of 2-morphisms;
4. coCartesian lifts of 1-morphisms and coCartesian lifts of 2-morphisms.

We will not explore all four of these notions here, instead focusing on case (1). Note that this will also immediately address case (4), since this dualization can be achieved by taking the opposite simplicial sets. We adopt the terminology employed by the authors in [7] and denote the cases (1) and (4) as *outer* 2-Cartesian (resp. 2-coCartesian) fibrations and similarly the cases (2) and (3) as *inner* 2-Cartesian (resp. 2-coCartesian) fibrations. The reasons for our particular choice of variance (which could seem arbitrary to the reader) are related to the kind of cofinality we will discuss in our upcoming paper [4]. For ease of reading, we call fibrations with our chosen variance simply *2-Cartesian fibrations*, trusting that this terminology will be replaced in writings where it becomes unclear.

There are several different kinds of clues in the literature which can help explain what 2-Cartesian fibrations should look like. Of particular interest is the work [5] of Buckley, which provides explicit versions of these definitions—one in strict 2-categories, and one in bicategories. Unwinding Buckley's definitions (and dualizing appropriately), we find that they amount to the following conditions on a functor $F : B \to C$:

- The induced functors
$$F_{a,b} : B(a, b) \to C(a, b).$$
are coCartesian fibrations. We call the coCartesian morphisms of these fibrations *coCartesian 2-morphisms*.
- The horizontal composite of coCartesian 2-morphisms is coCartesian. Here we are noting that the vertical composites of coCartesian morphisms are automatically coCartesian.
- For any 1-morphism $g : c \to F(b)$ in C, there is a Cartesian morphism $\tilde{g} : \tilde{c} \to b$ in B lifting g.

For the rest of this discussion we will choose to focus on coCartesian 2-morphisms. We hope that the reader will trust us in believing that there is not much novelty in defining Cartesian 1-morphisms since the definition generalises in an straightforward manner.

One distinctly evident difficulty comes from the fact that a priori it seems that the definition of a coCartesian 2-morphism is dependent on the choice of a particular model for the mapping ∞-category. To justify our definition we will draw an analogy with the already well understood notion of an invertible 2-morphism, i.e. a thin 2-simplex. Let us suppose we are given a map of ∞-bicategories

$$p : \mathbb{B} \to \mathbb{C}$$

having the right lifting property against the class of scaled anodyne maps. We will call a 2-simplex σ in $\mathbb{B}$ *left-degenerate* if $\sigma|_{\Delta^{\{0,1\}}}$ is degenerate. One readily verifies that a left-degenerate triangle σ is thin if and only if $p(\sigma)$ is thin in $\mathbb{C}$ and every lifting problem of the form

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "f"] \arrow[d] & \mathbb{B} \arrow[d, "p"] \\ \Delta^n \arrow[r] & \mathbb{C} \end{tikzcd}$$

such that $p|_{\Delta^{\{0,1,n\}}} = \sigma$ admits a solution. To illuminate the previous claim, let us consider for every pair of objects $x, y \in \mathbb{B}$ the mapping categories $\mathbb{B}(x, y)$ defined in [7, Section 2.3], whose simplices are maps $\Delta^{n+1} \to \mathbb{B}$ sending the terminal vertex to y, and all other vertices to x. Then interpreting σ as an edge in $\mathbb{B}(x, y)$ we can see that the previous claim is essentially saying that an edge is an equivalence if and only if it is coCartesian and its image in $\mathbb{C}(p(x), p(y))$ is an equivalence. Guided by this intuition we will say that a left-degenerate

2-simplex is *p-coCartesian* (or simply coCartesian if no confusion should arise) if we can produce the dotted arrow below

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "f"] \arrow[d] & \mathbb{B} \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathbb{C} \end{tikzcd}$$

provided $f|_{\Delta^{\{0,1,n\}}} = \sigma$. Our definition will be completed once it can be extended to an arbitrary 2-simplex. To this end we then note that, for any 2-simplex γ in $\mathbb{B}$, we can define a 3-simplex η

$$\begin{tikzcd} & \gamma(2) & \\ \gamma(0) \arrow[ur, "\gamma_{02}"] \arrow[rr, equal] \arrow[dr, "\gamma_{01}"'] & & \gamma(0) \arrow[ul, "g"'] \arrow[dl, "\gamma_{01}"] \\ & \gamma(1) \arrow[uu, "\gamma_{12}"] & \end{tikzcd}$$

where $d^0(\eta)$ and $d^3(\eta)$ are scaled, $d^1(\eta) = \gamma$, and $d^2(\eta)$ is left-degenerate. This data exhibits and equivalence of composite 2-morphisms $d^3(\eta) \circ \gamma = d^0(\eta) \circ d^2(\eta)$. In particular, since $d^0(\eta), d^3(\eta)$ are thin we see that γ and $d^2(\eta)$ represent the same 2-morphism in the mapping ∞-category of $\mathbb{B}$. We will call $d^2(\eta)$ the *left-degeneration* of γ. This allows us to make the following definition: A 2-simplex is coCartesian if and only if its left-degeneration is coCartesian.

Once we have established our definition of coCartesian triangles, the definition of Cartesian edges is relatively straightforward. We simply require that lifting problems

$$\begin{tikzcd} \Lambda^n_n \arrow[r, "f"] \arrow[d] & \mathbb{B} \arrow[d, "p"] \\ \Delta^n \arrow[r] & \mathbb{C} \end{tikzcd}$$

have solutions, provided that $f|_{\Delta^{\{0,n-1,n\}}}$ is a coCartesian triangle, and $f|_{\Delta^{\{n-1,n\}}}$ is p-Cartesian.

There are several additional technical conditions that come into play in our definition, but these two form the core idea. As we will discuss below, this intuitive approach becomes very close to the formal definition when the base of our fibration is a fibrant scaled simplicial set (∞-bicategory). We refer the reader to the third section of the paper for a systematical approach of this definition.

1.2 Decorations and Data

To handle the data of Cartesian morphisms in the theory of Cartesian fibrations of $(\infty, 1)$-categories, Lurie introduces a decoration on simplicial sets. In [11, Ch. 3], he defines a *marked simplicial set* to consist of a simplicial set X and a collection of edges $M_X \subset X_1$ which contains all degenerate edges. In this way, one can "hardcode" the Cartesian edges into a simplicial set.

We will follow a similar approach in our development of the 2-Cartesian model structure. Unfortunately, this entails rather a lot of decoration. A scaled simplicial set (X, T_X) already comes equipped with a collection T_X of 'thin' triangles, which are taken to represent invertible 2-morphisms. To this we will add a *second* collection of decorated 2-simplices, C_X, which we take to represent the *coCartesian* 2-simplices. Since any invertible 2-morphism should, in particular, be coCartesian, we require $T_X \subset C_X$. We will call the elements of C_X the *lean* triangles.

In addition to all of this, we then add a collection of marked 1-simplices. These we take to represent the Cartesian morphisms. All in all, the data we will have to consider to codify our intuition about 2-Cartesian fibrations consists of a simplicial set and *three* decorations. We will call a simplicial set with these three decorations a *marked biscaled simplicial set* (or **MB** *simplicial set* for short) and denote the category of such by $\mathrm{Set}_\Delta^{\mathbf{mb}}$

1.3 Main Results

As the title and preceding discussion already suggest, the main result of this paper is that there is a model structure on the category of marked biscaled simplicial sets over a scaled simplicial set S, whose fibrant objects are precisely the 2-Cartesian fibrations.

Theorem (3.42) *Let S be a scaled simplicial set. Then there exists a left proper combinatorial simplicial model structure on $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$, which is characterized uniquely by the following properties:*

C) A morphism $f : X \to Y$ in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ is a cofibration if and only if f induces a monomorphism between the underlying simplicial sets.
F) An object $p : X \to S$ in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ is fibrant if and only if p is a 2-Cartesian fibration.

The proof of this theorem is, by necessity, quite technical. The main ingredients—verification of a pushout-product axiom for an appropriate collection of anodyne maps and a characterization of weak equivalences by their properties on fibres—dominate a large part of the paper.

From this model structure, we immediately obtain a model structure over the terminal scaled simplicial set—equivalently a model structure on $\mathrm{Set}_\Delta^{\mathbf{mb}}$. As one would hope, this turns out to provide a new model for ∞-bicategories. This model is quite similar to the model of [16], which was shown to be equivalent to Lurie's model on $\mathrm{Set}_\Delta^{\mathbf{sc}}$ in [7]. In our new model, however, there is a significant amount of redundant data: a second scaling which, for fibrant objects, agrees with the first scaling.

Theorem (3.43) *There is a a Quillen equivalence*

$$L : \mathrm{Set}_\Delta^{\mathrm{sc}} \rightleftarrows \mathrm{Set}_\Delta^{\mathbf{mb}} : U$$

*between the model structure on **MB** simplicial sets over Δ^0 and the model structure on scaled simplicial sets of [13].*

Once these core results are established, we explore the cases where the base is a fibrant scaled simplicial set, i.e. an ∞-bicategory. In this setting, it is possible to give a much more intuitive characterization of the fibrant objects.

Theorem (4.27) *Let $\mathbb{B}$ be an ∞-bicategory and let $p : (X, M_X, T_X \subset C_X) \to \mathbb{B}$ be an element of $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/\mathbb{B}}$. Then $p : X \to \mathbb{B}$ is fibrant if and only if*

1. *p has the right lifting property against the generating scaled anodyne maps.*

2. *The collection C_X lean of triangles in X contains all left-degenerate coCartesian triangles.*
3. *The collection C_X is stable under composition along 1-morphisms.*
4. *The collection E_X consists of precisely the p-cartesian edges of X.*
5. *Every morphism in $\mathbb{B}$ admits a p-Cartesian lift.*
6. *Every 2-morphism in $\mathbb{B}$ admits a p-coCartesian lift.*

In addition to being a useful characterization of 2-Cartesian fibrations, this serves as a confirmation that our intuitive understanding cribbed from [5] was correct. Condition (1) is a formality in the strict 2-categorical case, and conditions (2)-(5) closely parallel those conditions which we extracted from [5]. We can, in fact, extract a further corollary from this characterization:

Corollary *Let $F : \mathbb{B} \to \mathbb{C}$ be a 2-fibration in the sense of [5], dualized to require coCartesian 2-morphisms rather than Cartesian 2-morphisms. Then the induced map*

$$N^{\mathbf{sc}}(F) : (N^{\mathbf{sc}}(\mathbb{B}), M_{\mathbb{B}}, T_{\mathbb{B}} \subset C_{\mathbb{B}}) \to N^{\mathbf{sc}}(\mathbb{C})$$

is a 2-Cartesian fibration, where $M_{\mathbb{B}}$ is the set of Cartesian edges, and $C_{\mathbb{B}}$ is the set of coCartesian triangles.

1.4 Structure of This Paper

In Sect. 2, we briefly introduce some notational conventions and recapitulate the generating scaled anodyne morphisms of [13]. The next section, Sect. 3 is entirely given over to the proof that the desired 2-Cartesian model structure exists. This proof has two lengthy technical components: in Sect. 3.1, we define our chosen class of anodyne morphisms—the **MB**-anodyne morphisms—and prove that they satisfy a pushout-product axiom; in Sect. 3.2, we provide a fibrewise characterization of the putative equivalences, and then deduce the existence of the model structure. The final section, Sect. 4, is devoted to providing a clear characterization of the fibrant objects over an ∞-bicategory. In particular, the characterizations provided in this section make clear the connection between our 2-Cartesian fibrations, the $\mathrm{Set}_{\Delta}^{+}$-enriched fibrations of [9], and the 2-fibrations of [5].

2 Preliminaries

Recapitulating even the basics of the theory of quasi-categories, ∞-bicategories, and the various types of fibrations between them would take more space than the rest of the paper. Consequently, we here confine ourselves to fixing some notational conventions, and establishing definitions for later reference. Where possible, we will follow the notational conventions established in [11] and expanded in [13] and [12]. In referring to the works of Gagna, Harpaz, and Lanari [7–9], we will endeavor to either follow their notation, or explain where our conventions differ.

Definition 2.1 The simplex category Δ is the full subcategory of Cat, the ordinary 1-category of (small) categories, spanned by the posets

$$[n] = \{0 < 1 < \cdots < n\}, \quad \text{for } n \geq 0$$

The category of simplicial sets Set_{Δ} is defined to be the presheaf category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$.

Definition 2.2 The inclusion of $\Delta \subset \mathrm{Cat}$ defines a functor $\mathrm{N} : \mathrm{Cat} \to \mathrm{Set}_\Delta$ which sends a category C to the simplicial set $\mathrm{N}(C)$ whose n-simplices are given by functors $\sigma : [n] \to C$.

Definition 2.3 We say that a 2-simplex $\sigma : \Delta^2 \to X$ is *left degenerate* if its restriction $\sigma|_{\Delta^{\{0,1\}}}$ is a degenerate simplex in X.

Definition 2.4 Let $n \geq 0$ given $0 \leq i \leq j \leq n$ we denote by $\Delta^{[i,j]}$ the nerve of the subposet of $[n]$ consisting in those objects ℓ such that $i \leq \ell \leq j$.

Definition 2.5 A *marked simplicial set* is given by a pair (X, E_X) where X is a simplicial set and $E_X \subset X_1$ is a subset of the set of 1-simplices containing every degenerate 1-simplex. We refer to the elements of E_X as *marked edges*. A morphism of marked simplicial setes $(X, E_X) \to (Y, E_Y)$ is a morphism of the underlying simplicial sets such that $f(E_X) \subseteq f(E_Y)$. We denote by Set_Δ^+ the category of marked simplicial sets.

Remark 2.6 We will sometimes denote marked simplicial sets using a superscript notation $X^\dagger = (X, E_X)$. We will denote by $X^\flat := (X, \flat)$ the marked simplicial set whose marked edges are precisely the degenerate ones. Similarly we will denote by $X^\sharp := (X, \sharp)$ the marked simplicial set where *all* edges are marked.

Definition 2.7 A *scaled simplicial set* is given by a pair (X, T_X) where X is a simplicial set and $T_X \subseteq X_2$ is a subset of the set of 2-simplices containing every degenerate 2-simplex. We refer to the elements of T_X as *thin triangles* or *scaled triangles*. A morphism of scaled simplicial sets $(X, T_X) \to (Y, T_Y)$ is a morphism of the underlying simplicial sets such that $f(T_X) \subseteq f(T_Y)$. We denote by $\mathrm{Set}_\Delta^{\mathbf{sc}}$ the category of scaled simplicial sets.

Notation Given a simplicial set X we denote by $X_\flat := (X, \flat)$ the scaled simplicial set whose thin triangles are precisely the degenerate 2-simplices. We similarly denote $X_\sharp := (X, \sharp)$ the scaled simplicial set where *all* triangles are thin.

Remark 2.8 By 2-category we mean a category enriched over the symmetric monoidal category of categories. Similarly the notion of 2-functor will refer to an enriched functor. We denote by $2\,\mathrm{Cat}$ the ordinary 1-category of strict 2-categories.

Definition 2.9 We define a functor $\underline{\mathrm{N}} : 2\,\mathrm{Cat} \to \mathrm{Cat}_{\mathrm{Set}_\Delta^+}$ with values in the category of Set_Δ^+-enriched categories which sends an strict 2-category $\mathbb{D}$ to the Set_Δ^+-enriched category $\underline{\mathrm{N}}(\mathbb{D})$ defined as follows:

- The objects of $\underline{\mathrm{N}}(\mathbb{D})$ are given by the objects of $\mathbb{D}$.
- Given a pair of objects $x, y \in \mathbb{D}$ we define a marked simplicial set $\underline{\mathrm{N}}(\mathbb{D})(x, y)$ with underlying simplicial set given by $\mathrm{N}(\mathbb{D}(x, y))$ (see Definition 2.2), where the marking is given by the equivalences in $\mathbb{D}(x, y)$.

We call $\underline{\mathrm{N}}$ the Hom-wise nerve functor. Note that since the functor N is fully faithful it follows that the Hom-wise nerve $\underline{\mathrm{N}}$, is also fully faithful.

Definition 2.10 Let $n \geq 0$ and define a 2-category $\mathbb{O}^n$ as follows:

- Objects are given by the elements of the poset $[n]$.
- For every $i, j \in [n]$ the category $\mathbb{O}^n(i, j)$ is either empty if $i > j$ or given by the poset of subsets $S \subseteq [n]$ such that $\min(S) = i$ and $\max(S) = j$ ordered by inclusion. The non-trivial composition functors for $i \leq j \leq k$ are induced by union of subsets

$$\mathbb{O}^n(i, j) \times \mathbb{O}^n(j, k) \to \mathbb{O}^n(i, k);\ (S, T) \mapsto S \cup T.$$

  The action on morphisms of the composition functors is the obvious one since union preserves our given order.

This definition extends to a functor $\mathbb{O}^\bullet : \Delta \to 2\,\mathrm{Cat} \to \mathrm{Cat}_{\mathrm{Set}_\Delta^+}$ where the last functor is given by the Hom-wise nerve.

Remark 2.11 We will abuse notation and denote by $\mathbb{O}^n$ the 2-category defined in Definition 2.10 together with its image under the Hom-wise nerve.

Definition 2.12 Let $\mathcal{C}$ be a Set_Δ^+-enriched category. We define a scaled simplicial set $N^{\mathbf{sc}}(\mathcal{C})$ whose n-simplices are given by functors of Set_Δ^+-enriched categories $\mathbb{O}^n \to \mathcal{C}$. A 2-simplex $\mathbb{O}^2 \to \mathcal{C}$ is thin if and only if it factors through $\mathbb{O}^2_\sharp \to \mathcal{C}$ where $\mathbb{O}^2_\sharp$ denotes the Set_Δ^+-enriched category obtained from $\mathbb{O}^2$ by maximally marking all mapping spaces.

The definition extends to a functor $N^{\mathbf{sc}} : \mathrm{Cat}_{\mathrm{Set}_\Delta^+} \to \mathrm{Set}_\Delta^{\mathbf{sc}}$ which has as left adjoint which we denote by $\mathfrak{C}^{\mathbf{sc}} : \mathrm{Set}_\Delta^{\mathbf{sc}} \to \mathrm{Cat}_{\mathrm{Set}_\Delta^+}$. It follows from [13, Theorem 4.2.7] that the adjunction

$$\mathfrak{C}^{\mathbf{sc}} : \mathrm{Set}_\Delta^{\mathbf{sc}} \rightleftarrows \mathrm{Cat}_{\mathrm{Set}_\Delta^+} : N^{\mathbf{sc}}$$

is a Quillen equivalence between the model structure of scaled simplicial sets and the Bergner model structure on $\mathrm{Cat}_{\mathrm{Set}_\Delta^+}$.

Definition 2.13 The set of *generating scaled anodyne maps* $\mathbf{S}$ is the set of maps of scaled simplicial sets consisting of:

(i) the inner horns inclusions

$$\left(\Lambda^n_i, \{\Delta^{\{i-1,i,i+1\}}\}\right) \to \left(\Delta^n, \{\Delta^{\{i-1,i,i+1\}}\}\right) \quad , \quad n \geq 2 \quad , \quad 0 < i < n;$$

(ii) the map

$$(\Delta^4, T) \to (\Delta^4, T \cup \{\Delta^{\{0,3,4\}},\ \Delta^{\{0,1,4\}}\}),$$

where we define

$$T \overset{\text{def}}{=} \{\Delta^{\{0,2,4\}},\ \Delta^{\{1,2,3\}},\ \Delta^{\{0,1,3\}},\ \Delta^{\{1,3,4\}},\ \Delta^{\{0,1,2\}}\};$$

(iii) the set of maps

$$\left(\Lambda^n_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \{\Delta^{\{0,1,n\}}\}\right) \to \left(\Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0, \{\Delta^{\{0,1,n\}}\}\right) \quad , \quad n \geq 3.$$

A general map of scaled simplicial set is said to be *scaled anodyne* if it belongs to the weakly saturated closure of $\mathbf{S}$.

Definition 2.14 We say that a map of scaled simplicial sets $p : X \to S$ is a weak $\mathbf{S}$-fibration if it has the right lifting property with respect to the class of scaled anodyne maps.

Definition 2.15 We say that a scaled simplicial set $X := (X, C_X)$ is a ∞-bicategory if the unique map $X \to \Delta^0$ is a weak $\mathbf{S}$-fibration.

Example 2.16 For every 2-category $\mathbb{D}$ the scaled nerve functor yields a ∞-bicategory $N^{\mathbf{sc}}(\mathbb{D})$.

In general, we will denote fibrant objects in $\mathrm{Set}_\Delta^{\mathbf{sc}}$ using blackboard characters, e.g. $\mathbb{D}$. We will use undecorated roman majescules, e.g. X, to denote objects of any category, adding explicit decorations as necessary for clarity.

Definition 2.17 Consider the cosimplicial object

$$Q : \Delta \to \mathrm{Set}_{\Delta}^{\mathrm{sc}};$$
$$[n] \mapsto \Delta^0 \coprod_{\Delta^n} (\Delta^n \star \Delta^0),$$

equipped with the minimal scaling. Given an ∞-bicategory $X \in \mathrm{Set}_{\Delta}^{\mathrm{sc}}$, for any $a, b \in X$, we define a simplicial set $X(a, b)$ whose n-simplices are maps $Q_n \to X$ which send the first vertex to a and the second to b. It was shown in [7, Proposition 2.33] that $X(a, b)$ is a model for the mapping ∞-category from a to b in X.

3 The Model Structure

3.1 Marked Biscaled Simplicial Sets and MB-Anodyne Morphisms

Definition 3.1 A *marked biscaled* simplicial set (**MB** simplicial set) is given by the following data

- A simplicial set X.
- A collection of edges $E_X \subseteq X_1$ containing all degenerate edges. We will refer to the elements of this collection as *marked edges*
- A collection of triangles $T_X \subseteq X_2$ containing all degenerate triangles. We will refer to the elements of this collection as *thin triangles*.
- A collection of triangles $C_X \subseteq X_2$ such that $T_X \subseteq C_X$. We will refer to the elements of this collection as *lean triangles*.

We will denote such objects as triples $(X, E_X, T_X \subseteq C_X)$. A map $(X, E_X, T_X \subseteq C_X) \to (Y, E_Y, T_Y \subseteq C_Y)$ is given by a map of simplicial sets $f : X \to Y$ which maps marked edges in X (resp. thin triangles, resp. lean triangles) to marked edges in Y (resp. thin triangles, resp. lean triangles). We denote by $\mathrm{Set}_{\Delta}^{\mathbf{mb}}$ the category of **MB** simplicial sets.

Notation Let $(X, E_X, T_X \subseteq C_X)$ be a **MB** simplicial set. If the collection E_X consist only of degenerate edges then we will use the notation $(X, E_X, T_X \subseteq C_X) = (X, \flat, T_X \subseteq E_X)$ and do similarly for the collection T_X. If C_X consists only of degenerate triangles we fix the notation $(X, E_X, T_X \subseteq C_X) = (X, E_X, \flat)$. In an analogous fashion we use the symbol "$\sharp$" to denote a collection containing all edges (resp. all triangles). Finally, we will employ the notation (X, E_X, T_X) whenever we have $T_X = C_X$.

Remark 3.2 We will often abuse notation when defining the collections E_X (resp. T_X, resp. C_X) and just specified its non-degenerate edges (resp. triangles).

Definition 3.3 We define a category $\Delta_{\mathbf{MB}}$ by appending to the simplex category Δ three objects $[1]_+$, $[2]_t$ and $[2]_l$ and morphisms

$$[1] \xrightarrow{i_+} [1]_+, \qquad [2] \xrightarrow{i_l} [2]_l \xrightarrow{i_l} [2]_t,$$
$$s_0^+ : [1]_+ \to [0], \qquad s_i^t : [2]_t \to [1], \quad \text{for } i = 0, 1$$

such that $s_0^+ \circ i_+ = s_0$ and such that $s_i^t \circ i_t \circ i_l = s_i$. We can produce a functor $R : \mathrm{Set}_{\Delta}^{\mathbf{mb}} \to \mathrm{Fun}(\Delta_{\mathbf{MB}}^{\mathrm{op}}, \mathrm{Set})$ which sends a **MB** simplicial set $(X, E_X, T_X \subseteq C_X)$ to the functor $R(X)$ which maps $[1]_+$ to the collection of marked edges, $[2]_l$ to the collection of lean 2-simplices and $[2]_t$ to the collection of thin triangles. The functor $R(X)$ maps the new morphisms to the obvious inclusions

$$E_X \subseteq X_1, \qquad T_X \subseteq C_X \subseteq X_2$$

between the collections and to the inclusion of degenerate edges (resp. triangles) into the marked edges (resp. thin simplices).

Remark 3.4 It follows by direct inspection that the functor $R : \mathrm{Set}_\Delta^{\mathbf{mb}} \to \mathrm{Fun}(\Delta_{\mathbf{MB}}^{\mathrm{op}}, \mathrm{Set})$ is fully faithful with essential image those presheaves mapping the morphisms i_+, i_l and i_t to monomorphisms in Set. It is also straightforward to verify that R has a left adjoint L. This implies that the category of **MB** simplicial sets is a reflective subcategory of a presheaf category and thus *locally presentable* (see [6, Definition 1.17])

Remark 3.5 Let $X, Y \in \mathrm{Set}_\Delta^{\mathbf{mb}}$ the product $X \times Y \in \mathrm{Set}_\Delta^{\mathbf{mb}}$ is given by the underlying product of the simplicial sets equipped with the following decorations:

- An edge $\Delta^1 \to X \times Y$ (resp. triangle) is declared marked (resp. thin resp. lean) if and only if its image in X and its image in Y is marked (resp. thin resp. lean).

Remark 3.6 Observe that we have a functor, $L : \mathrm{Set}_\Delta^{\mathbf{sc}} \longrightarrow \mathrm{Set}_\Delta^{\mathbf{mb}}$ sending a scaled simplicial set (X, T_X) to $(X, \flat, T_X)$ which is left adjoint to the forgetful functor U sending $(X, E_X, T_X \subseteq C_X)$ to (X, T_X).

Definition 3.7 The set of *generating marked-biscaled anodyne maps* **MB** is the set of maps of **MB** simplicial sets consisting of:

(A1) The inner horn inclusions

$$\left(\Lambda_i^n, \flat, \{\Delta^{\{i-1,i,i+1\}}\}\right) \to \left(\Delta^n, \flat, \{\Delta^{\{i-1,i,i+1\}}\}\right) \quad , \quad n \geq 2 \quad , \quad 0 < i < n;$$

which are a direct generalization of the inner-horn right-lifting property of ∞-categories. For $n = 2$ these morphisms guarantee the existence of composites of 1-morphisms.

(A2) The map

$$(\Delta^4, \flat, T) \to (\Delta^4, \flat, T \cup \{\Delta^{\{0,3,4\}},\ \Delta^{\{0,1,4\}}\}),$$

where we define

$$T \overset{\text{def}}{=} \{\Delta^{\{0,2,4\}},\ \Delta^{\{1,2,3\}},\ \Delta^{\{0,1,3\}},\ \Delta^{\{1,3,4\}},\ \Delta^{\{0,1,2\}}\};$$

These morphisms encode a general 2-out-of-3 property for thin triangles.

(A3) The set of maps

$$\left(\Lambda_0^n \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \flat \subset \{\Delta^{\{0,1,n\}}\}\right) \to \left(\Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \flat \subset \{\Delta^{\{0,1,n\}}\}\right) \quad , \quad n \geq 2.$$

These maps force left-degenerate (Definition 2.3) lean-scaled triangles to represent coCartesian edges of the mapping category. For $n = 2$ this requires the existence of p-coCartesian lifts of edges in the mapping category of the base to exist.

(A4) The set of maps

$$\left(\Lambda_n^n, \{\Delta^{\{n-1,n\}}\}, \flat \subset \{\Delta^{\{0,n-1,n\}}\}\right) \to \left(\Delta^n, \{\Delta^{\{n-1,n\}}\}, \flat \subset \{\Delta^{\{0,n-1,n\}}\}\right) \quad , \quad n \geq 2.$$

This forces the marked morphisms to be p-Cartesian with respect to the given thin and lean triangles.

(A5) The inclusion of the terminal vertex

$$\left(\Delta^0, \sharp, \sharp\right) \to \left(\Delta^1, \sharp, \sharp\right).$$

This requires p-Cartesian lifts of morphisms in the base to exist.

(S1) The map

$$\left(\Delta^2, \{\Delta^{\{0,1\}}, \Delta^{\{1,2\}}\}, \sharp\right) \to \left(\Delta^2, \sharp, \sharp\right),$$

requiring that p-Cartesian morphisms compose across thin triangles.

(S2) The map

$$\left(\Delta^2, \flat, \flat \subset \sharp\right) \to \left(\Delta^2, \flat, \sharp\right),$$

which requires that lean triangles over thin triangles are, themselves, thin.

(S3) The map

$$\left(\Delta^3, \flat, \{\Delta^{\{i-1,i,i+1\}}\} \subset U_i\right) \to \left(\Delta^3, \flat, \{\Delta^{\{i-1,i,i+1\}}\} \subset \sharp\right) \quad , \quad 0 < i < 3$$

where U_i is the collection of all triangles except i-th face. This and the next two generators serve to establish composability and limited 2-out-of-3 properties for lean triangles.

(S4) The map

$$\left(\Delta^3 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \flat \subset U_0\right) \to \left(\Delta^3 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \flat \subset \sharp\right)$$

where U_0 is the collection of all triangles except the 0-th face.

(S5) The map

$$\left(\Delta^3, \{\Delta^{\{2,3\}}\}, \flat \subset U_3\right) \to \left(\Delta^3, \{\Delta^{\{2,3\}}\}, \flat \subset \sharp\right)$$

where U_3 is the collections of all triangles except the 3-rd face.

(E) For every Kan complex K, the map

$$\left(K, \flat, \sharp\right) \to \left(K, \sharp, \sharp\right).$$

Which requires that every equivalence is a marked morphism.

A map of **MB** simplicial sets is said to be **MB**-anodyne if it belongs to the weakly saturated closure of **MB**.

Remark 3.8 We would like to point out that a priory the collection (E) is not a set. This issue can be solved by allowing K to range over a set of representatives for all isomorphism classes of Kan complexes with only countably many simplices as explained in [11, Remark 3.1.1.3].

Definition 3.9 Let $f : (X, E_X, T_X \subseteq C_X) \to (Y, E_Y, T_Y \subseteq C_Y)$ be a map of **MB** simplicial sets. We say that f is a **MB**-fibration if it has the right lifting property against the class of **MB**-anodyne morphisms.

Lemma 3.10 *Let $f : (X, E_X, T_X \subseteq C_X) \to (Y, E_Y, T_Y \subseteq C_Y)$ be a **MB**-fibration and denote by X_y the fibre of f over $y \in Y$. Then X_y is an ∞-bicategory with precisely the equivalences marked.*

Proof Observe that it follows from (S2) that the thin triangles and the lean triangles of X_y must coincide. Since X_y has the right lifting property against maps (A1)-(A3) it follows that X_y is an ∞-bicategory. It follows from (E) that all equivalences must be marked.

Let $u : a \to b$ be a marked edge in X_y and let $s : \Lambda^2_2 \to X_y$ be the map that sends the edge $1 \to 2$ to u and the edge $0 \to 2$ to the identity morphism on b. It follows that we can provide an extension of s to a thin 2-simplex $\sigma : \Delta^2 \to X_y$ which provides with a morphism $v : b \to a$ such that $u \circ v \simeq \mathrm{id}$. To finish the proof we construct a morphism $(\Lambda^3_3, \Delta^{\{2,3\}}, \sharp) \to X_y$ as depicted by the diagram below

$$\begin{tikzcd}
& b & \\
a \arrow[ur, "u"] \arrow[rr, "u"] \arrow[dr, "\mathrm{id}"'] & & b \arrow[ul, "\mathrm{id}"'] \arrow[dl, "v"] \\
& a \arrow[uu, "u"] &
\end{tikzcd}$$

where the only non-degenerate triangle is given by the 0-th face which is precisely σ. An extension of this map to $(\Delta^3, \Delta^{\{2,3\}}, \sharp)$ where we scale the face missing the vertex 3 using a morphism of type (S5) yields a a thin 2-simplex exhibiting $v \circ u \simeq \mathrm{id}$. □

Lemma 3.11 *The morphism*

$$\theta : \left(\Delta^2, \{\Delta^{\{1,2\}}, \Delta^{\{0,2\}}\}, \sharp\right) \to \left(\Delta^2, \sharp, \sharp\right)$$

*is **MB**-anodyne.*

Proof We first note that, given a **MB**-fibration $f : (X, E_X, T_X \subseteq C_X) \to (S, \sharp, T_S)$, we can find a lift of θ as follows. Suppose we have a lifting problem

$$\begin{tikzcd}
\left(\Delta^2, \{\Delta^{\{1,2\}}, \Delta^{\{0,2\}}\}, \sharp\right) \arrow[r, "\sigma"] \arrow[d, "\theta"'] & X \arrow[d, "f"] \\
\left(\Delta^2, \sharp, \sharp\right) \arrow[r] & S
\end{tikzcd}$$

Where the top arrow corresponds to the thin 2-simplex

$$\sigma : \quad \begin{tikzcd}
& b \arrow[dr, "u"] & \\
a \arrow[ur, "v"] \arrow[rr, "w"'] & & c
\end{tikzcd}$$

Since $f : X \to S$ is a **MB**-fibration, we can choose a marked lift $\hat{v} : \hat{a} \to b$ of $f(v)$. Using a lift of type (A1) to compose u and $\hat{v}$ and a lift of type (A4) to obtain a morphism from a to $\hat{a}$, we can obtain a Λ^3_2-horn, all of whose sides are thin-scaled. We can fill this to a maximally thin-scaled 3-simplex using a pushout of type (A1) and a pushout of type (A2). This three-simplex has the form

$$\begin{tikzcd}
& c & \\
a \arrow[ur, "w"] \arrow[rr, "p"'] \arrow[dr, "v"'] & & \hat{a} \arrow[ul, "q"'] \arrow[dl, "\hat{v}"] \\
& b \arrow[uu, "u"] &
\end{tikzcd}$$

Since every triangle is scaled, we can apply lifts of type (S1) to show that q is marked. This implies that p is and equivalence in the fibre over $f(a)$, and so p is marked. Thus, a lift of type (S1) shows that v is marked as desired.

To finish the proof we use the small object argument to produce a factorization

$$\left(\Delta^2, \{\Delta^{\{1,2\}}, \Delta^{\{0,2\}}\}, \sharp\right) \xrightarrow{\alpha} X \xrightarrow{\beta} \left(\Delta^2, \sharp, \sharp\right)$$

where the first morphism is **MB**-anodyne and the second morphism is a **MB**-fibration. The first part of the proof then implies that there exists a section $\gamma : (\Delta^2, \sharp, \sharp) \to X$ such that $\gamma \circ \theta = \alpha$ and such that $\mathrm{id} = \beta \circ \gamma$. This shows that θ as a retract of a **MB**-anodyne morphism and thus the claim holds. □

Definition 3.12 We say that a map $f : (X, E_X, T_X \subseteq C_X) \to (Y, E_Y, T_Y \subseteq C_Y)$ of **MB** simplicial sets is a cofibration if its underlying map of simplicial sets is a cofibration. Equivalently, a cofibration of **MB** simplicial sets is given by a monomorphism in the category $\mathrm{Set}_{\Delta}^{\mathbf{mb}}$.

Remark 3.13 The generators of the class of cofibrations are given by

(C1) $\left(\partial\Delta^n, \flat, \flat\right) \to \left(\Delta^n, \flat, \flat\right)$ for $n \geq 0$ where $\partial\Delta^0 = \varnothing$.
(C2) $\left(\Delta^1, \flat, \flat\right) \to \left(\Delta^1, \sharp, \flat\right)$.
(C3) $\left(\Delta^2, \flat, \flat\right) \to \left(\Delta^2, \flat, \flat \subset \sharp\right)$.
(C4) $\left(\Delta^2, \flat, \flat \subset \sharp\right) \to \left(\Delta^2, \flat, \sharp\right)$.

Note that (C4) and (S2) are the same morphism.

Proposition 3.14 *Let $f : (X, E_X, T_X \subseteq C_X) \to (Y, E_Y, T_Y \subseteq C_Y)$ be a cofibration and let $g : (A, E_A, T_A \subseteq C_A) \to (B, E_B, T_B \subseteq C_B)$ be a **MB**-anodyne morphism. Then the pushout-product*

$$f \wedge g : X \times B \coprod_{X \times A} Y \times A \to Y \times B$$

*is **MB**-anodyne.*[1]

Before embarking on our proof of the pushout-product, we will tackle one particularly recalcitrant case by itself. As it so happens, a case nearly precisely dual to this one also occurs in checking the pushout-product. To save paper (and the reader's eyesight), we will only provide the proof of one of these cases, trusting that it will be apparent how to dualize the argument.

We we first prove two quick lemmata, which will somewhat ease the coming proof.

Construction 1 Let $m \geq 2$ and consider a list of vertices $\vec{i} = \{i_1, \ldots i_{k+1}\}$ of Δ^m with $k < m$. We denote by $\Lambda^m_{\vec{i}}$ the simplicial subset of Δ^m whose non-degenerate simplices are given by subsets $J \subset [n]$ satisfying the following property

- There exists $\ell \in [m]$ such that $\ell \notin J$ and $\ell \notin \vec{i}$.

[1] Note that this proposition is about the pushout-product of marked biscaled simplicial sets. For readability, we have omitted the marking and biscaling from the notation in the conclusion.

Lemma 3.15 *Let $\vec{i} = \{i_1, \ldots i_{k+1}\}$ be a list of non-consecutive vertices of Δ^m which does not contain $0, m$. We define a biscaling $T_{\vec{i}}$ on Δ^m by declaring that $\Delta^{\{i-1,i,i+1\}}$ is thin for every $i \in \vec{i}$. Then the morphism*

$$(\Lambda^m_{\vec{i}}, \flat, T_{\vec{i}}) \to (\Delta^m, \flat, T_{\vec{i}})$$

is in the weakly saturated hull of morphisms of type (A1) for $m \geq 2$.

Proof We proceed by induction on the length of $\vec{i}$. When length$(\vec{i}) = 1$, this is simply a morphism of type (A1).

Now suppose that this holds for length$(\vec{i}) < k + 1$ and let $i_1, \ldots, i_{k+1}$ be a $k + 1$-tuple satisfying the hypotheses above. Define $\vec{j} = \vec{i} \setminus \{i_1\}$, and consider the $m - 1$-simplex

$$\sigma : \Delta^{m-1} \to \Delta^m$$

given by the i_1-th face map. Then $\sigma \cap \Lambda^m_{\vec{i}} = \Lambda^{m-1}_{\vec{j}}$, and so, by the inductive hypothesis, we can fill this simplex to obtain a new simplicial subset

$$\Lambda^m_{\vec{i}} \subset X \subset \Delta^m.$$

We then see that X will consist of precisely those subsimplices of Δ^m which either (a) skip i_1 or (b) skip a vertex j not belonging to $\vec{i}$. More simply put, precisely those simplices which skip a vertex not contained in $\{i_2, \ldots, i_{k+1}\}$. Consequently,

$$X = (\Lambda^m_{\vec{i}\setminus\{i_1\}}, \flat, T_{\vec{i}})$$

and so, by the inductive hypothesis, this map is in the weakly saturated hull of morphisms of type (A1). □

Lemma 3.16 *Let $\vec{i} := \{0, i_1, \ldots, i_{k+1}\}$ be a set of distinct vertices of $\Delta^m \coprod_{\Delta^{\{0,1\}}} \Delta^0$ with $m \geq 2$ such that*

- $1 < i_1 \leq i_2 \leq \cdots \leq i_{k+1} < n$
- *The simplex $\{0, 1, m\}$ is lean-scaled.*

Then the map

$$(\Lambda^m_{\vec{i}}) \coprod_{\Delta^{\{0,1\}}} \Delta^0 \to \Delta^m \coprod_{\Delta^{\{0,1\}}} \Delta^0$$

*is **MB**-anodyne.*

Proof We once again proceed by induction on the length of $\vec{i}$. If $\vec{i} = \{0\}$, then this is a morphism of type (A3). If $\vec{i} = \{0, i_1\}$, then we can fill the simplex obtained by deleting i_1 using a pushout of type (A3), the resulting inclusion is again an inclusion of type (A3).

We now assume, inductively, that the statement holds for any $\vec{i}$ of length less than $k + 2$, and let $\vec{i} = \{0, i_1, \ldots, i_{k+1}\}$. Consider the simplex $\sigma : \Delta^{m-1} \to \Delta^m$ obtained by deleting i_1. Then we see that

$$(\Lambda^m_{\vec{i}} \coprod_{\Delta^{\{0,1\}}} \Delta^0) \cap \sigma = \Lambda^{m-1}_{\vec{i}\setminus i_1} \coprod_{\Delta^{\{0,1\}}} \Delta^0$$

so that, by the inductive hypothesis, we can fill σ using an **MB**-anodyne morphism. The resulting simplicial subset X in

$$(\Lambda^m_{\vec{i}}) \coprod_{\Delta^{\{0,1\}}} \Delta^0 \to X \to \Delta^m \coprod_{\Delta^{\{0,1\}}} \Delta^0$$

consists of precisely those subsimplices of Δ^m which skip i_1 or which skip an element not in $\vec{i}$. More precisely

$$X = \Lambda^m_{\vec{i}\setminus\{i_1\}} \coprod_{\Delta^{\{0,1\}}} \Delta^0$$

and thus, by the inductive hypothesis,

$$X \to \Delta^m \coprod_{\Delta^{\{0,1\}}} \Delta^0$$

is **MB**-anodyne, completing the proof. $\square$

Proposition 3.17 *Denote by*

$$f : (\partial\Delta^n, \flat, \flat) \to (\Delta^n, \flat, \flat)$$

a morphism of type (C1), and by

$$g : \left(\Lambda^m_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \flat \subset \{\Delta^{\{0,1,m\}}\}\right) \to \left(\Delta^m \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \flat \subset \{\Delta^{\{0,1,m\}}\}\right)$$

*a morphism of type (A3). Then the pushout-product $f \wedge g$ is **MB**-anodyne.*

Before beginning the proof, we create a diagram for reference. We visualize the product of the targets as a grid, with some simplices which get collapsed.

$$\begin{tikzcd}
00 \arrow[r] \arrow[d] & 01 \arrow[r] \arrow[d] & 02 \arrow[r] \arrow[d] & 03 \arrow[d] \\
10 \arrow[r] \arrow[d] & 11 \arrow[r] \arrow[d] & 12 \arrow[r] \arrow[d] & 13 \arrow[d] \\
20 \arrow[r] \arrow[d] & 21 \arrow[r] \arrow[d] & 22 \arrow[r] \arrow[d] & 23 \arrow[d] \\
30 \arrow[r] \arrow[d] & 31 \arrow[r] \arrow[d] & 32 \arrow[r] \arrow[d] & 33 \arrow[d] \\
40 \arrow[r] & 41 \arrow[r] & 42 \arrow[r] & 43
\end{tikzcd}$$

In the diagram above, we are looking at $\Delta^4 \times \Delta^3$, and the 1-simplices in red are those which get collapsed.

Proof Since the case $n = 0$ is simply the original type (A3) morphism, we may, without loss of generality, assume $n \geq 1$. To prove the claim we will provide a filtration

$$X_0 \to X_1 \to \cdots \to X_{k-1} \to X_k = \Delta^n \times \left(\Delta^m \coprod_{\Delta^{\{0,1\}}} \Delta^0 \right)$$

$$X_0 = \partial\Delta^n \times \left(\Delta^m \coprod_{\Delta^{\{0,1\}}} \Delta^0 \right) \coprod_{\partial\Delta^n \times (\Lambda^m_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0)} \Delta^n \times \left(\Lambda^m_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0 \right)$$

and show that each step $X_\alpha \to X_{\alpha+1}$ is **MB**-anodyne. Let us remind the reader that the marking and biscaling on $\Delta^n \times \Delta^m$ is determined by the universal property of the product as discussed in Remark 3.5 and each object in the filtration carries the inherited marking and biscaling from $\Delta^n \times \Delta^m$.

We begin by fixing some notation for the $n+m$ simplices in $\Delta^n \times \Delta^m$. We denote the objects of $(a, b) \in \Delta^n \times \Delta^m$ simply as ab according to the diagram above. A non-degenerate simplex $\sigma : \Delta^k \to \Delta^n \times \Delta^m$ is specified by a sequence of vertices $\{a_i b_i\}_{i=0}^k$ such that $a_i < a_{i+1}$ or $b_i < b_{i+1}$. The non-degenerate simplices of maximal dimension are precisely those such that either $a_{i+1} = a_i$ and $b_{i+1} = b_i + 1$ or $a_{i+1} = a_i + 1$ and $b_{i+1} = b_i$.

Let $\sigma : \Delta^k \to \Delta^n \times \Delta^m$ with vertex sequence given by $\{a_i b_i\}_{i=0}^k$. Then σ factors through X_0 if *at least one* of the following conditions is satisfied:

- There exists $j \in [n]$ such that $a_i \neq j$ for $0 \leq i \leq k$. In other words, the path in our grid determined by the vertex sequence skips the j-th row.
- There exists $j \in [m]$ such that $j \neq 0$ and $b_i \neq j$ for $0 \leq i \leq k$. As before, this means that the path determined by the vertex sequence skips the j-th column.

The next step in our proof is to define a total order on the set of non-degenerate simplices of maximal dimension. Once this order is provided $\{\sigma_1 < \sigma_1 < \cdots < \sigma_k\}$ we will define X_ℓ as the subsimplicial set of $\Delta^n \times \Delta^m$ containing the non-degenerate simplices θ of maximal dimension such that $\theta \leq \sigma_\ell$. Let $\theta, \sigma : \Delta^{m+n} \to \Delta^n \times \Delta^m$ be two distinct simplices of maximal dimension with associated vertex sequences $\{a_i b_i\}_{i=0}^{m+n}$ and $\{c_i d_i\}_{i=0}^{m+n}$. By maximality it follows that $a_0 b_0 = c_0 d_0 = 00$. Let $1 \leq \nu < m + n$ be the first index such that $a_\nu b_\nu \neq c_\nu d_\nu$. Then we say that $\theta < \sigma$ if $b_\nu < d_\nu$.

We observe that the decorations of $\Delta^n \times \Delta^m$ are already contained in X_0 unless $n = 1$ and $m = 2$. We will deal with this case separately. Let us suppose that $n = 1$ and $m = 2$ then it follows that every triangle in $\Delta^1 \times \Delta^2$ is lean. The filtration in this case is given by

$$X_0 \to X_1 \to X_2 \to \Delta^1 \times \left(\Delta^2 \coprod_{\Delta^{\{0,1\}}} \Delta^0 \right)$$

Let $\sigma_1 : \Delta^3 \to X_1$ be simplex specified by $00 \to 10 \to 11 \to 12$. We observe that the restriction of σ_1 to X_0 is given by $(\Lambda^3_1)^\dagger := (\Lambda^3_1, \Delta^{\{1,2\}}, \Delta^{\{0,1,2\}} \subset \sharp)$. We observe that the morphism

$$(\Lambda^3_1)^\dagger := (\Lambda^3_1, \Delta^{\{1,2\}}, \Delta^{\{0,1,2\}} \subset \sharp) \to (\Delta^3, \Delta^{\{1,2\}}, \Delta^{\{0,1,2\}} \subset \sharp) = (\Delta^3)^\dagger$$

is **MB**-anodyne since can be obtained via pushouts from a morphism of type (A1) and a morphism of type (S3). It follows that we have a pushout diagram

$$\begin{tikzcd} (\Lambda^3_1)^\dagger \arrow[r] \arrow[d] & (\Delta^3)^\dagger \arrow[d] \\ X_0 \arrow[r] & X_1 \end{tikzcd}$$

which shows that the first step is **MB**-anodyne. Now we consider the simplex $\sigma_2 : 00 \to 01 \to 11 \to 12$ in X_2. The restriction of σ_2 is given by $(P, \Delta^{\{0,1\}}, \Delta^{\{0,1,2\}} \subset \sharp)$ where P is the union inside of Δ^3 of the face that skips 1 and the face that skips 3. We can add the 0-th face using a pushout along a morphism of type (A1) thus yielding

$$(P, \Delta^{\{0,1\}}, \Delta^{\{0,1,2\}} \subset \sharp) \to (\Lambda^3_2, \Delta^{\{0,1\}}, V \subset \sharp) \to (\Delta^3, \Delta^{\{0,1\}}, V \subset \sharp)$$

where $V = \{\Delta^{\{0,1,2\}}, \Delta^{\{1,2,3\}}\}$. The first map is in the weakly saturated hull of morphisms of type (A1) and the second is in the weakly saturated hull of morphisms of type (A1) and (S3). It follows by an analogous reasoning that $X_1 \to X_2$ is **MB**-anodyne.

The last 3-simplex to add is given by $\sigma_3 = 00 \to 01 \to 02 \to 12$ which we view as a map

$$\sigma_3 : \Delta^3 \coprod_{\Delta^{\{0,1\}}} \Delta^0 \to \Delta^1 \times \left(\Delta^2 \coprod_{\Delta^{\{0,1\}}} \Delta^0\right).$$

As before we compute the restriction of σ_3 to X_2 which is precisely given by $A^\diamond = (\Lambda^3_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \flat \subset \sharp)$. We define $B^\diamond = \left(\Delta^3 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \Delta^{\{1,2,3\}} \subset \sharp\right)$. It follows by direct inspection that we have a pushout square

$$\begin{tikzcd} A^\diamond \arrow[r] \arrow[d] & (B)^\diamond \arrow[d] \\ X_2 \arrow[r] & \left(\Delta^2 \coprod_{\Delta^{\{0,1\}}} \Delta^0\right) \end{tikzcd}$$

so it will suffice to show that the top horizontal morphism is **MB**-anodyne. We construct the following factorization

$$\left(\Lambda^3_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \flat \subset \sharp\right) \to \left(\Delta^3 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \flat \subset \sharp\right) \to \left(\Delta^3 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \Delta^{\{1,2,3\}} \subset \sharp\right)$$

where we note that the first map is in the weakly saturated hull of morphisms of type (A3) and (S4). The second morphism is in the weakly saturated hull of morphisms of type (S2) and so the claim holds.

From this point on we will assume that X_0 contains all the decorations. We proceed by cases. First we will assume that $\sigma_\alpha : \Delta^{n+m} \to X_\alpha$ satisfies $\sigma_\alpha(0 \to 1) = 00 \to 10$. Just as we did before we will compute the restriction of σ_α to $X_{\alpha-1}$. Let $\{a_i b_i\}_{i=0}^{n+m}$ be the vertex sequence associated to σ_α. We define $\vec{i} = \{0 < i < n+m \mid a_{i-1} < a_i,\ a_i = a_{i+1}\}$ and observe that the restriction of σ_α to $X_{\alpha-1}$ is precisely given by $\Lambda^{n+m}_{\vec{i}}$ as in Construction 1. It follows by construction that for every $j \in \vec{i}$ the triangle $\{i-1, i, i+1\}$ is thin. Consequently we can apply Lemma 3.15 to show that $X_{\alpha-1} \to X_\alpha$ is **MB**-anodyne.

To finish the proof we consider a morphism $\sigma_\alpha : \Delta^{m+n} \coprod_{\Delta^{\{0,1\}}} \Delta^0 \to X_\alpha$ such that $\sigma_\alpha(0 \to 1) = 00 \to 01$. Now we define $\vec{i} = \{0 < i < n+m \mid a_{i-1} < a_i,\ a_i = a_{i+1}\} \cup \{0\}$

$$\begin{tikzcd}
00 \arrow[r] \arrow[d] & 01 \arrow[r] \arrow[d] & 02 \arrow[r] \arrow[d] & 03 \arrow[d] \\
10 \arrow[r] \arrow[d] & 11 \arrow[r] \arrow[d] & 12 \arrow[r] \arrow[d] & 13 \arrow[d] \\
20 \arrow[r] \arrow[d] & 21 \arrow[r] \arrow[d] & 22 \arrow[r] \arrow[d] & 23 \arrow[d] \\
30 \arrow[r] \arrow[d] & 31 \arrow[r] \arrow[d] & 32 \arrow[r] \arrow[d] & 33 \arrow[d] \\
40 \arrow[r] & 41 \arrow[r] & 42 \arrow[r] & 43
\end{tikzcd}$$

Fig. 1 Above, we depict in blue and in red two simplices of maximal dimension. Note that in our ordering the simplex depicted by the red path is smaller than the simplex in blue

and observe that $1 \notin \vec{i}$. It follows that the restriction of σ_α to $X_{\alpha-1}$ is given by $\Lambda^{n+m}_{\vec{i}}$ and that the conditions of Lemma 3.16 apply. Therefore we see that the morphism $X_{\alpha-1} \to X_\alpha$ is **MB**-anodyne and thus the proof is finished. □

While a significant majority of the cases of the pushout-product remain, all of the remaining cases involve far less difficulty than this one. We can now turn to the main event.

Proof of Proposition 3.14 The proof will consist of the usual rigmarole — checking on pairs of generators. While there are 44 cases in all, the vast majority of these turn out to be trivial or extremely simple. The two cases dealt with by the preceding propositions are by far the most complicated cases.

We will label our cases first by the generating cofibration, and then by the generating **MB**-anodyne morphism.

(C1) The cofibration is of the form $\left(\partial\Delta^n, \flat, \flat\right) \to \left(\Delta^n, \flat, \flat\right)$.

- (A1) Since the marking is trivial, and the thin and lean scalings agree, we can consider only the thin scalings. Case (1A) from 3.1.8 in [13] then shows that this can be obtained as a pushout of morphisms of type (A1) and morphisms for the type from remark 3.1.4 in [13].
- (A2) This is precisely case (1B) from 3.1.8 [13]
- (A3) This is Proposition 3.17.
- (A4) The dual of the argument given for Proposition 3.17 suffices once we have replaced "degenerate 1-simplices" with "marked 1-simplices".
- (A5) We note that the map of underlying simplicial sets is

$$Y_0 := (\Delta^n \times \{1\}) \coprod_{\partial\Delta^n \times \{1\}} (\partial\Delta^n \times \Delta^1) \to \Delta^n \times \Delta^1$$

We can define a sequence of $n+1$ simplices in $\Delta^n \times \Delta^1$ via the maps

$$\sigma_k : [n+1] \to [n] \times [1];$$
$$i \mapsto \begin{cases} (i,0) & i \leq k \\ (i-1,1) & i > k \end{cases}$$

We then define Y_i inductively as $Y_{i-1} \cup \sigma_{i-1}$ (Following [11, 2.1.2.6]). We see that the morphism $Y_{i-1} \to Y_i$ is a pushout with a Λ^{n+1}_{i+1}-horn. It will thus suffice for us to note two things:

- When $i < n$, the 2-simplex $\sigma_i|_{\Delta^{\{i-1,i,i+1\}}}$ is the simplex
$$(i,0) \to (i,1) \to (i+1,1)$$
in $\Delta^{\{i-1,i\}} \times \Delta^1$, and thus is necessarily thin-scaled. We thus obtain a pushout of type (A1).
- when $i = n$, the 2-simplex $\sigma_n|_{\Delta^{\{0,n-1,n\}}}$ is the simplex
$$(0,0) \to (n,0) \to (n,1)$$
in $\Delta^{\{0,n\}} \times \Delta^1$, and thus is necessarily thin-scaled. Moreover, the morphism $\sigma_{n+1}|_{\Delta^{\{n-1,n\}}}$ is
$$(n,0) \to (n,1)$$
and thus is marked. Hence, we obtain a pushout of type (A4).

(S1) This is an isomorphism when $n \geq 1$, and is a morphism of type (S1) when $n = 0$.

(S2) This is an isomorphism on underlying marked lean scaled simplicial sets, and thus in the saturated hull of morphisms of type (S2).

(S3) We will treat the case $i = 2$—the case $i = 1$ follows virtually identically. When $n > 2$ this is an isomorphism and when $n = 0$, this is a morphism of type (S3). This means that we may consider the following two cases:

- If $n = 2$, we note that this is an isomorphism on the underlying marked simplicial sets, and indeed differs only in the lean-scaling. The only missing lean-scaled simplex is $00 \to 11 \to 23$ in $\Delta^2 \times \Delta^3$. We may expand this to a 3-simplex $00 \to 11 \to 12 \to 23$. It is easily checked that this 3-simplex gives us a pushout of type (S3) (with $i = 1$), showing that the morphism is **MB**-anodyne.
- If $n = 1$, we again have that the source and target differ only in their lean-scaling. It is easy to check that the missing simplices are the simplices $00 \to 11 \to 13$ and $00 \to 01 \to 13$ in $\Delta^1 \times \Delta^3$. In the former case, we can extend to the 3-simplex $00 \to 11 \to 12 \to 13$ and scale the desired 2-simplex with a pushout of type (S3), and in the latter case we can extend to the 3-simplex $00 \to 01 \to 02 \to 13$ and scaled the desired 2-simplex with a pushout of type (S3).

(S4) This case is almost dual to the next one and left as an exercise.

(S5) When $n \geq 2$, this is an isomorphism. When $n = 0$, this is a morphism of type (S5). When $n = 1$, we get the identity on underlying marked simplicial sets

$$(\Delta^3)^\dagger \times (\Delta^1)^\flat \to (\Delta^3)^\dagger \times (\Delta^1)^\flat, \ (\Delta^3)^\dagger = (\Delta^3, \Delta^{\{2,3\}})$$

The lean scaling on the target is maximal. The missing scaled simplices in the source are $00 \to 10 \to 21$, $00 \to 11 \to 21$. One can then note that the 3-simplex $00 \to 11 \to 21 \to 31$ is of type (S5), and can thus be filled. Similarly, the 3-simplex $00 \to 10 \to 21 \to 31$ is of type (S5), and can be filled.

(E) If $n \geq 1$, this is an isomorphism. If $n = 0$, this is again a morphism of type (E).

(C2) The cofibration is of the form $\left(\Delta^1, \flat, \flat\right) \to \left(\Delta^1, \sharp, \flat\right)$.

(A1) This is isomorphism on underlying marked, lean-scaled simplicial sets, and thus **MB**-anodyne.
(A2) This is an isomorphism.
(A3) This is an isomorphism.
(A4) This is an isomorphism.
(A5) This gives us the inclusion

$$(\Delta^1 \times \Delta^1, E_\dagger, \sharp) \to (\Delta^1 \times \Delta^1, \sharp, \sharp)$$

Where $E_\dagger$ is the marking containing $\Delta^1 \times \{0\}$, $\Delta^1 \times \{1\}$, and $\{1\} \times \Delta^1$. A pushout of type (S1) marks the diagonal, and a pushout by the morphism

$$\left(\Delta^2, \{\Delta^{\{1,2\}}, \Delta^{\{0,2\}}\}, \sharp\right) \to \left(\Delta^2, \sharp, \sharp\right)$$

marks the remaining edge. By Lemma 3.11, this is **MB**-anodyne.
(S1) This is the identity on $(\Delta^2 \times \Delta^1)$ the underlying simplicial sets. Moreover, every triangle in both simplicial sets is thin scaled. The only 1-simplex which is not marked in the source is $00 \to 21$, and the target is maximally marked. We can add the remaining marked edge using a pushout of type (S1).
(S2) This is an isomorphism.
(S3) This is an isomorphism.
(S4) This is an isomorphism.
(S5) This is an isomorphism.
(E) The source and target of the pushout-product differ only in their marking. However, every edge which is marked in the target by not in the source will be the product of a non-degenerate edge in K and the non-degenerate edge in Δ^1. Consequently, it will be the diagonal in a square $\Delta^1 \times \Delta^1 \subset \Delta^1 \times K$. Since every other 1-simplex of this square will be marked, the diagonal can be marked with a pushout of type (S1).

(C3) The cofibration is of the form $\left(\Delta^2, \flat, \flat\right) \to \left(\Delta^2, \flat, \flat \subset \sharp\right)$.

(A1) When $n > 2$, this is an isomorphism. If $n = 2$, this is an isomorphism on the underlying marked thin-scaled simplicial sets, so we can consider only the lean scaling.
The target is maximally lean scaled. In the source, there are precisely three 2-simplices which are not lean scaled:

$$00 \to 12 \to 22 \tag{1}$$
$$00 \to 11 \to 22 \tag{2}$$
$$00 \to 10 \to 22 \tag{3}$$

For the first, we can extend to the 3-simplex $00 \to 02 \to 12 \to 22$, and obtain obtain a pushout of type (S3) with $i = 1$. For the third, we can extend to the 3-simplex $00 \to 10 \to 20 \to 22$, and obtain a pushout of type (S3) with $i = 2$. For the second, we can then extend to the 3-simplex $00 \to 10 \to 11 \to 22$, and obtain a pushout of type (S3) (with $i = 1$).
(A2) The pushout-product is an isomorphism on underlying marked thin-scaled simplicial sets, so once again we consider the lean triangles. The underlying simplicial

sets are both $\Delta^2 \times \Delta^4$. There are two triangles which are lean in the target, but not the source, namely:

$$00 \to 13 \to 24 \tag{4}$$

$$00 \to 11 \to 24 \tag{5}$$

For (4), if we extend to the 3-simplex $00 \to 03 \to 13 \to 24$, we obtain a pushout of type (S3) with $i = 1$. For (5), if we extend to the 3-simplex $00 \to 11 \to 21 \to 24$, we obtain a pushout of type (S3) with $i = 2$.

(A3) This is an isomorphism when $n > 2$. When $n = 2$, we first note that we can neglect the thin scaling and the marking. Since this is the case, we consider the corresponding inclusion of lean-scaled simplicial sets. The underlying map is

$$\mathrm{id} : \Delta^2 \times \left(\Delta^2 \coprod \Delta^0\right) \to \Delta^2 \times \left(\Delta^2 \coprod \Delta^0\right)$$

and the target carries a maximal scaling. The only unscaled simplex in the source is

$$00 \to 11 \to 22$$

We can then consider the simplex

$$00 \to 01 \to 11 \to 22$$

Since $00 \to 01$ is degenerate, we can scale the remaining simplex via a pushout of type (S4).

(A4) This is an isomorphism when $n > 2$. When $n = 2$, we again note that is sufficient only to consider the marking and the lean scaling since the source of our morphism already contains every thin triangle. In this case, we obtain an isomorphism on the underlying simplicial set $\Delta^2 \times \Delta^2$. The markings are identical on the source and target, so we are again left to consider only the lean scaling. The target is maximally scaled, and the only unscaled simplex in the source is $00 \to 11 \to 22$. Considering the 3-simplex

$$00 \to 11 \to 21 \to 22,$$

we note that $21 \to 22$ is marked. Thus, a pushout of type (S5) suffices.

(A5) The underlying map of simplicial sets is the identity on $\Delta^2 \times \Delta^1$. It is, as above, and isomorphism on the marking and thin-scaling. There are precisely three simplices which we need to lean-scale:

$$00 \to 11 \to 21 \tag{6}$$

$$00 \to 10 \to 21 \tag{7}$$

$$00 \to 10 \to 20 \tag{8}$$

For (6), we can extend to the 3-simplex $00 \to 01 \to 11 \to 21$, and then obtain the desired scaling via a pushout of type (S3) with $i = 1$. For (7), we can extend to the 3-simplex $00 \to 10 \to 11 \to 21$ and obtain the desired scaling via a pushout of type (S3) with $i = 2$. Finally, for (8), we can extend to the 3-simplex $00 \to 10 \to 20 \to 21$, and obtain a pushout of type (S5) (since the morphism $20 \to 21$ is marked).

(S1) This is an isomorphism.

(S2) This is an isomorphism on the underlying marked lean-scaled simplicial sets, and thus a sequence of pushouts of type (S2).

(S3) In both cases, the underlying map of simplicial sets is the identity on $\Delta^2 \times \Delta^3$, and in both cases, there is only one 2-simplex we need to lean scale.

- When $i = 2$, the missing scaling is on $00 \to 11 \to 23$. We can extend to the 3-simplex $00 \to 11 \to 21 \to 23$, and scale the missing 2-simplex using a pushout of type (S3) with $i = 2$.
- When $i = 1$, the missing scaling is on $00 \to 12 \to 23$. We can extend to the 3-simplex $00 \to 02 \to 12 \to 23$, and scale the missing 2-simplex using a pushout of type (S3) with $i = 1$.

(S4) This is effectively dual to the next case.

(S5) On the underlying marked simplicial sets, this is the identity on the marked simplicial set

$$(\Delta^3, \{\Delta^{\{2,3\}}\}) \times (\Delta^2)^\flat.$$

The only simplex which is lean-scaled in the target but not the source is $00 \to 11 \to 22$. However, if we consider the 3-simplex

$$00 \to 11 \to 22 \to 32$$

in $\Delta^3 \times \Delta^2$ whose edge $22 \to 32$ is marked, we obtain a pushout of type (S5) giving the desired scaling.

(E) This is an isomorphism.

(C4) The cofibration is of the form $\left(\Delta^2, \flat, \flat \subset \sharp\right) \to \left(\Delta^2, \flat, \sharp\right)$.

(A1)-(E) All of these are, necessarily, isomorphisms on the underlying marked lean-scaled simplicial sets (since, forgetting about thin simplices, the morphisms of type (C4) are isomorphisms of marked lean-scaled simplicial sets), since every thin triangle in the target is lean scaled in the source we see that the morphisms are **MB**-anodyne.

□

Though the preceding arguments may seem an abuse of the reader's patience, now that the pushout-product is established, we can freely use it without directly working with these technicalities. In particular, we gain access to well-behaved mapping spaces, mapping categories, and mapping bicategories for $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$—a key convenience in the work to come.

Definition 3.18 Given two **MB** simplicial sets $(K, E_K, T_K \subseteq C_K)$, $(X, E_X, T_X \subseteq C_X)$ we define another **MB** simplicial set denoted by $\mathrm{Fun}^{\mathbf{mb}}(K, X)$ and characterized by the following universal property

$$\mathrm{Hom}_{\mathrm{Set}_\Delta^{\mathbf{mb}}}\left(A, \mathrm{Fun}^{\mathbf{mb}}(K, X)\right) \simeq \mathrm{Hom}_{\mathrm{Set}_\Delta^{\mathbf{mb}}}\left(A \times K, X\right).$$

As a direct consequence of Proposition 3.14 we obtain the following corollary.

Corollary 3.19 *Let $f : (X, E_X, T_X \subseteq C_X) \to (Y, E_Y, T_Y \subseteq C_Y)$ be a **MB**-fibration. Then for every $K \in \mathrm{Set}_\Delta^{\mathbf{mb}}$ the induced morphism $\mathrm{Fun}^{\mathbf{mb}}(K, X) \to \mathrm{Fun}^{\mathbf{mb}}(K, Y)$ is a **MB**-fibration.*

Definition 3.20 Let $f : X \to Y$ be a **MB**-fibration and consider another map of **MB** simplicial sets $g : K \to Y$. The previous corollary and Lemma 3.10 allows us to define an ∞-bicategory $\mathrm{Map}_Y(K, X)$ by means of the pullback square

$$\begin{tikzcd} \mathrm{Map}_Y(K,X) \arrow[r] \arrow[d] & \mathrm{Fun}^{\mathbf{mb}}(K,X) \arrow[d] \\ \Delta^0 \arrow[r, "g"] & \mathrm{Fun}^{\mathbf{mb}}(K,Y) \end{tikzcd}$$

Proposition 3.21 *Let $f : X \to Y$ be a **MB**-fibration. Suppose that we are given morphisms of **MB** simplicial sets*

$$L \to [h]K \to [g]Y$$

*such that h is a cofibration (resp. **MB**-anodyne). Then the induced morphism*

$$h^* : \mathrm{Map}_Y(K, X) \to \mathrm{Map}_Y(L, X)$$

is a fibration of scaled simplicial sets (resp. trivial fibration).

Proof Suppose that h is a cofibration and let $A \to B$ be a **MB**-anodyne morphism. To show that h^* has the right lifting property against the class of scaled anodyne maps we consider the adjoint lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathrm{Map}_Y(K,X) \arrow[d, "h^*"] \\ B \arrow[r] & \mathrm{Map}_Y(L,X) \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} L \times B \coprod_{L\times A} K \times A \arrow[r] \arrow[d] & X \arrow[d] \\ K \times B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

and conclude that the dotted arrow exists due to Proposition 3.14.

Note that according to Lemma 3.10 the marking on both ∞-bicategories is precisely given by equivalences. Therefore using (A5) in Definition 3.7 we see that h^* is an isofibration. We can conclude from the construction of the model structure on $\mathrm{Set}_\Delta^{\mathbf{sc}}$ as a Cisinski model structure in [7] that h^* is a fibration of ∞-bicategories. The case where h is a **MB**-anodyne follows immediately from Proposition 3.14. □

3.2 The Model Structure

Let $S \in \mathrm{Set}_\Delta^{\mathrm{sc}}$ for the rest of the section we will denote $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ the category of **MB** simplicial set over $(S, \sharp, T_S \subset \sharp)$. In this section we will establish the existence of model structure on $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ using a refinement of Jeff Smith's theorem due to Lurie [11, Prop. A.2.6.13].

Definition 3.22 We say that an object $\pi : X \to S$ in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ is an *outer* 2-*Cartesian* fibration if it is a **MB**-fibration.

Remark 3.23 We will frequently abuse notation and refer to outer 2-Cartesian as *2-Cartesian fibrations*.

Remark 3.24 Given a scaled simplicial set (S, T_S) we will frequently abuse notation and denote the **MB** simplicial set $(S, \sharp, T_S \subset \sharp)$ simply by S.

Definition 3.25 Let $\pi : X \to S$ be a morphism of **MB** simplicial sets. Given an object $K \to S$, we define $\mathrm{Map}^{\mathrm{th}}_S(K, X)$ to be the **MB** simplicial subset of $\mathrm{Map}_S(K, X)$ consisting only of the thin triangles. Note that if π is a 2-Cartesian fibration this is precisely the underlying ∞-category of $\mathrm{Map}_S(K, X)$.

We similarly denote by $\mathrm{Map}^{\widetilde{\simeq}}_S(K, X)$ the **MB** simplicial subset consisting of thin triangles and marked edges. As before, we note that if π is a 2-Cartesian fibration, the simplicial set $\mathrm{Map}^{\widetilde{\simeq}}_S(K, X)$ can be identified with the maximal Kan complex in $\mathrm{Map}_S(K, X)$.

Definition 3.26 We define a functor $I : \mathrm{Set}^+_\Delta \to \mathrm{Set}^{\mathbf{mb}}_\Delta$ mapping a marked simplicial set (K, E_K) to the **MB** simplicial set $(K, E_K, \sharp)$. If K is maximally marked we adopt the notation $I(K^\sharp) = K^\sharp_\sharp$

Remark 3.27 Note that we can endow the $(\mathrm{Set}^{\mathbf{mb}}_\Delta)_{/S}$ with the structure of a Set^+_Δ-enriched category by means of $\mathrm{Map}^{\mathrm{th}}_S(-,-)$. In addition given $K \in \mathrm{Set}^+_\Delta$ and $\pi : X \to S$ we define $K \otimes X := I(K) \times X$ equipped with a map to S given by first projecting to X and then composing with π. This construction shows that $(\mathrm{Set}^{\mathbf{mb}}_\Delta)_{/S}$ is tensored over Set^+_Δ. One can easily show that $(\mathrm{Set}^{\mathbf{mb}}_\Delta)_{/S}$ is also cotensored over Set^+_Δ.

In a similar way one can use $\mathrm{Map}^{\widetilde{\simeq}}_S(-,-)$ to endow $(\mathrm{Set}^{\mathbf{mb}}_\Delta)_{/S}$ with the structure of a Set_Δ-enriched category. In this case the cotensor is given by $K \otimes X = I(K^\sharp) \times X$.

Definition 3.28 Let $L \to [h]K \to [p]S$ be a morphism in $(\mathrm{Set}^{\mathbf{mb}}_\Delta)_{/S}$. We say that h is a cofibration when it is a monomorphism of **MB** simplicial sets. We will call h a weak equivalence if for every 2-Cartesian fibration $\pi : X \to S$ the induced morphism

$$h^* : \mathrm{Map}_S(K, X) \to \mathrm{Map}_S(L, X)$$

is a bicategorical equivalence.

Definition 3.29 Given two **MB** simplicial sets $p : X \to S$ and $q : Y \to S$ over S, we call a morphism

$$\begin{tikzcd}
(\Delta^1, \sharp, \sharp) \times X \arrow[rr, "h"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\
& S &
\end{tikzcd}$$

a *marked homotopy over* S from $h|_{\{0\}\times X}$ to $h|_{\{1\}\times X}$. We say that a morphism $f : X \to Y$ is a *marked homotopy equivalence* if there is a morphism $g : Y \to X$ over S and marked homotopies from $f \circ g$ to id_Y and from $g \circ f$ to id_X.

Proposition 3.30 *Suppose we are given a pushout diagram in* $(\mathrm{Set}^{\mathbf{mb}}_\Delta)_{/S}$

$$\begin{tikzcd}
L \arrow[r, "v"] \arrow[d, "u"] & K \arrow[d] \\
R \arrow[r, "w"] & P
\end{tikzcd}$$

where u is a cofibration and v is a weak equivalence. Then w is also a weak equivalence.

Proof Let $\pi : X \to S$ be a 2-Cartesian fibration. Then it follows that we have a pullback diagram of fibrant scaled simplicial sets

$$\begin{tikzcd}
\mathrm{Map}_S(P, X) \arrow[r, "w^*"] \arrow[d] & \mathrm{Map}_S(R, X) \arrow[d, "u^*"] \\
\mathrm{Map}_S(K, X) \arrow[r, "v^*"] & \mathrm{Map}_S(L, X)
\end{tikzcd}$$

where u^* is a fibration according to Proposition 3.21 and v^* is a bicategorical equivalence. Since this pullback already represents the homotopy pullback it follows that w^* is also a bicategorical equivalence. □

Proposition 3.31 *Let $L \to [h]K \to [p]S$ be a morphism in $(\mathrm{Set}_{\Delta}^{\mathbf{mb}})_{/S}$. Then the following are equivalent*

(i) The map $h : L \to K$ is a weak equivalence.
(ii) For every 2-Cartesian fibration $\pi : X \to S$ the induced morphism

$$\mathrm{Map}_S^{\mathrm{th}}(K, X) \to [\simeq]\,\mathrm{Map}_S^{\mathrm{th}}(L, X)$$

is an equivalence of ∞-categories.
(iii) For every 2-Cartesian fibration $\pi : X \to S$ the induced morphism

$$\mathrm{Map}_S^{\simeq}(K, X) \to [\simeq]\,\mathrm{Map}_S^{\simeq}(L, X)$$

is a homotopy equivalence of Kan complexes.

Proof The implications (i) $\implies$ (ii) $\implies$ (iii) are obvious. To show (iii) $\implies$ (i) we apply the small object argument to factor the morphism p (resp. $q = p \circ h$)

$$K \to F_K \to S$$

where the first morphism is **MB**-anodyne and the second has the right lifting property against the class of **MB**-anodyne morphisms and similarly for q. In particular we obtain 2-Cartesian fibrations $\pi_K : F_K \to S$ and $\pi_L : F_L \to S$. The functoriality of the small object argument implies the existence of a commutative diagram over S

$$\begin{tikzcd} L \arrow[r] \arrow[d] & F_L \arrow[d, "\varphi"] \\ K \arrow[r] & F_K \end{tikzcd}$$

Using Proposition 3.21 we obtain for every 2-Cartesian fibration $\pi : X \to S$ a commutative diagram

$$\begin{tikzcd} \mathrm{Map}_S(F_K, X) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Map}_S(F_L, X) \arrow[d, "\simeq"] \\ \mathrm{Map}_S(K, X) \arrow[r] & \mathrm{Map}_S(L, X) \end{tikzcd}$$

where the horizontal morphisms are trivial fibrations of ∞-bicategories. This shows that the map $F_L \to F_K$ satisfies condition (iii). It will therefore suffice to show that $F_L \to F_K$ is a weak equivalence.

We observe that we have an equivalence of Kan complexes

$$\mathrm{Map}_S^{\simeq}(F_K, F_L) \to [\simeq]\,\mathrm{Map}_S^{\simeq}(F_L, F_L)$$

It follows that we have a morphism $\gamma : F_K \to F_L$ over S and a homotopy (again over S) expressing $\gamma \circ \varphi \sim \mathrm{id}_{\pi_L}$. Observe that both $\varphi \circ \gamma$ and id_{π_K} get mapped under

$$\mathrm{Map}_S^{\simeq}(F_K, F_K) \to [\simeq]\,\mathrm{Map}_S^{\simeq}(F_L, F_K)$$

to equivalent objects. Using our hypothesis it follows that $\varphi \circ \gamma \sim \mathrm{id}_{\pi_K}$. To finish the proof we observe that given a 2-Cartesian fibration $X \to S$ we can use the morphism γ to construct an inverse up to marked homotopy for the map

$$\mathrm{Map}_S(F_K, X) \to \mathrm{Map}_S(F_L, K)$$

thus concluding the proof. □

Lemma 3.32 *Let $L \to [h]K \to [p]S$ be a morphism in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ such that $p : K \to S$ and $p \circ h : L \to S$ are 2-Cartesian fibrations. Then the conditions (i)–(iii) in Proposition 3.31 are additionally equivalent to*

(iv) The morphism f is a marked homotopy equivalence over S.

Proof The equivalence of (iv) and (iii) is purely formal, so the result follows from Proposition 3.31 □

Definition 3.33 We say that a morphism $L \to [h]K \to S$ is a trivial fibration it it has the right lifting property against the class of cofibrations.

Remark 3.34 Observe that every trivial fibration is in particular a weak equivalence. Indeed, if $h : L \to K$ has the right lifting property against all cofibrations we can produce a section $s : K \to L$ (over S) and an a marked homotopy $L \times (\Delta^1)^\sharp \to L$ between the identity on $L \to S$ and $s \circ h$. This provides us with a deformation retract on the mapping ∞-bicategories.

Definition 3.35 Suppose we have a morphism

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & \Delta^n_\flat & \end{tikzcd}$$

of 2-Cartesian fibrations over $\Delta^n_\flat$, for $n \geq 1$, and a commutative diagram

$$\begin{tikzcd} (\partial\Delta^m, \flat, \flat) \arrow[r, "\alpha"] \arrow[d] & X \arrow[d, "f"] \\ (\Delta^m, \flat, \flat) \arrow[r, "\beta"] & Y \end{tikzcd}$$

such that $r = q \circ \beta : \Delta^m \to \Delta^n$ is surjective.

We define $j_\beta \in [m]$ to be the largest element such that $r(j_\beta) < r(m)$. We additionally define a simplicial subset $S^{m+1}_{j_\beta} \subset \Delta^{m+1}$ to be the union of:

- all m-simplices of Δ^{m+1} *other than* the faces missing $j_\beta + 2$ or $j_\beta + 1$;
- the $(m-1)$ simplex which misses *both* $j_\beta + 2$ and $j_\beta + 1$.

See Fig. 2 for a geometric interpretation. We equipp Δ^{m+1} with a marking and biscaling as follows:

- The only non-degenerate marked edge is given by $j_\beta + 1 \to j_\beta + 2$.
- A 2-simplex is lean if it contains the edge $j_\beta + 1 \to j_\beta + 2$.
- A 2-simplex is thin if it is lean and its image in $\Delta^n_\flat$ under the morphism $r \circ s_{j_\beta}$ is degenerate where s_{j_β} denotes the j_β-th degeneracy map.

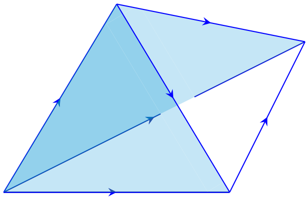

Fig. 2 The simplicial subset $S_1^3 \subset \Delta^3$

We denote the resulting **MB** simplicial set by $(\Delta^{m+1}, E_\beta, T_\beta \subseteq C_\beta)$ and view it as an object of $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/\Delta^n_\flat}$ by means of the map $r \circ s_{j_\beta}$. We similarly denote $(S^{m+1}_{j_\beta}, E^S_\beta, T^S_\beta \subseteq C^S_\beta)$ the **MB** simplicial set obtained from the inherited decorations.

Lemma 3.36 *Let $n \geq 1$. Suppose we are given a morphism $f : X \to Y$ of 2-Cartesian fibrations over $\Delta^n_\flat$ and a lifting problem*

$$\begin{tikzcd} (\partial\Delta^m, \flat, \flat) \arrow[r, "\alpha"] \arrow[d] & X \arrow[d, "f"] \\ (\Delta^m, \flat, \flat) \arrow[r, "\beta"] & Y \end{tikzcd}$$

as in Definition 3.35. Suppose further that f satisfies condition (ii) from 3.37.

Then there exists a commutative diagram

$$\begin{tikzcd} (S^{m+1}_{j_\beta}, E^S_\beta, T^S_\beta \subseteq C^S_\beta) \arrow[r, "\varepsilon"] \arrow[d] & X \arrow[d, "f"] \\ (\Delta^{m+1}, E_\beta, T_\beta \subseteq C_\beta) \arrow[r, "\theta"] & Y \end{tikzcd}$$

such that the following conditions hold:

1. *The restriction of θ to be face missing $j_\beta + 1$ equals β and similarly, the restriction of ε to face missing $j_\beta + 1$ equals α.*
2. *Let ξ denote the restriction of θ to the face missing $j_\beta + 2$. Then either $j_\xi = j_\beta + 1$ if $j_\beta < m - 1$ or ξ factors through Δ^{n-1} and similarly for ε.*

Proof We start the proof by fixing the notation $\alpha(i) = x_i$ (resp. $\beta(i) = y_i$). Let us pick a marked morphism $e : \hat{x}_{j_\beta} \to x_{j_\beta+1}$. To ease notation, let us just denote j_β simply by j. We define **MB** simplicial sets

$$B^m_j = (\Delta^m, \flat, \flat) \coprod_{\Delta^{\{j+1\}}} (\Delta^1, \sharp, \sharp) \quad , \quad \partial B^m_j = (\partial\Delta^m, \flat, \flat) \coprod_{\Delta^{\{j+1\}}} (\Delta^1, \sharp, \sharp).$$

For the rest of the proof we will omit the marking and biscalings to ease the notation. Note that we have commutative diagrams

$$\begin{tikzcd} B^m_j \arrow[r] \arrow[d, "\gamma^m_j"'] & Y \arrow[d, "q"] \\ \Delta^{m+1} \arrow[r, "r\circ s_j"] & \Delta^n \end{tikzcd} \qquad \begin{tikzcd} \partial B^m_j \arrow[r] \arrow[d, "\iota^m_j"'] & X \arrow[d, "p"] \\ S^{m+1}_j \arrow[r] & \Delta^n \end{tikzcd}$$

where bottom horizontal map in the second diagram is the restriction of $r \circ s_j$ to S^{m+1}_j. We claim that the left vertical maps in both diagrams are **MB**-anodyne. Once this is proven, we let θ be a solution to the left-most commutative square. Note that we can form another diagram

$$\begin{tikzcd} \partial B_j^m \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ S_j^{m+1} \arrow[r] & Y \end{tikzcd}$$

where bottom horizontal map is the composite $S_j^{m+1} \to \Delta^{m+1} \to [\theta]Y$. Since f has the right lifting property against **MB**-anodyne morphisms our result follows.

First we will prove the family of cases where $j = m-1$ by using induction on m. The case $m = 1$ is obviously true. Suppose that our claim holds for $m-1$ and let us prove the case m. Let $W_{-1}^m = B_{m-1}^m$ and define for $0 \leq i \leq m-1$ a **MB** simplicial subset $W_i^m \subset \Delta^{m+1}$ (with the decorations defined in Definition 3.35) consisting in those simplices that are either in W_{i-1}^m or are contained in the i-th face for $0 \leq i \leq m-1$. This yields a filtration

$$W_{-1}^m \to W_0^m \to \cdots \to W_{m-1}^m = \Lambda_{m+1}^{m+1}$$

We similarly set $\partial W_{-1}^m = \partial B_{m-1}^m$ and produce an analogous filtration by adding step-wise the faces $0 \leq i \leq m-1$

$$\partial W_{-1}^m \to \partial W_0^m \to \cdots \to \partial W_{m-1}^m = S_j^{m+1}$$

It will then suffice to show that each step in both filtrations is **MB**-anodyne. Let $0 \leq i \leq m-1$ then we can produce a pushout squares

$$\begin{tikzcd} W_{i-1}^{m-1} \arrow[r] \arrow[d] & \Delta^m \arrow[d] \\ W_{i-1}^m \arrow[r] & W_i^m \end{tikzcd} \qquad \begin{tikzcd} W_{i-1}^{m-1} \arrow[r] \arrow[d] & \Delta^m \arrow[d] \\ \partial W_{i-1}^m \arrow[r] & \partial W_i^m \end{tikzcd}$$

where the morphism $W_{i-1}^{m-1} \to W_{i-1}^m$ is given by the restriction of the inclusion of the i-th face $\Delta^m \to W_i^m$ to W_{i-1}^m and similarly for the other diagram. The claim now follows from the inductive hypothesis.

The general proof will employ induction on j and each case will be proved using induction on m. Note that given $j \geq 0$ the ground case for the induction on m is given by $m = j+1$. In particular we have proved all the ground cases already. Now we will deal with ground case of the induction on j, namely $j = 0$. Assume the claim to hold for $m-1 \geq 1$ and let us prove the case m. Let $Z_{m+2}^m = B_0^m$ and define for every $3 \leq i \leq m+1$ a **MB** subsimplicial set $Z_{i-1}^m \subset \Delta^{m+1}$ consisting in those simplices that are either contained in Z_i^m or are contained in the $(i-1)$-th face of Δ^{m+1}. We similarly denote $\partial Z_{m+2}^m = \partial B_0^m$ and consider a pair of filtrations

$$Z_{m+2}^m \to Z_{m+1}^m \to \cdots \to Z_3^m \to \Lambda_2^{m+1}$$
$$\partial Z_{m+2}^m \to \partial Z_{m+1}^m \to \cdots \to \partial Z_3^m \to S_j^m$$

where the last step in both filtrations is given by attaching the face missing 0. A similar argument as above shows that the claim follows from the inductive hypothesis for the every step except the last one. To prove that the last map in both filtrations is **MB**-anodyne we consider a pushout diagram

$$\begin{array}{ccc} \Lambda_1^m & \longrightarrow & \Delta^m \\ \downarrow & & \downarrow \\ Z_3^m & \longrightarrow & \Lambda_2^{m+1} \end{array}$$

where the morphism $\Lambda_1^{m-1} \to Z_3^m$ is the restriction to Z_3^m of the inclusion of the 0-th face into Λ_2^{m+1}. Note that the triangle $\{0, 1, 2\}$ must be already be thin if $m > 3$ or it can be chosen to be thin since it lies above a degenerate triangle in Δ^n. The analogous conclusion also holds for ∂Z_3^m. Finally let us assume the claim holds for $j - 1 \geq 0$. The proof of this final inductive hypothesis is a mix of both previous cases. We will give a sketch here and leave the details for the interested reader. The idea is to add stepwise to B_j^n (resp. ∂B_j^n) the faces missing i for $n \leq i \leq j + 3$. One can check that at each step this result map is **MB**-anodyne using the induction hypothesis. Then we add the faces missing ℓ for $0 \leq \ell \leq j$ and again we find that each step in this process is **MB**-anodyne. In the case of ∂B_j^m we have already reached S_j^{m+1}. For B_j^m after this process we reach Λ_{j+2}^{m+1} where the triangle $\{j + 1, j + 2, j + 3\}$ must be thin since it is lean by construction and lies over a thin triangle. The conclusion now follows. □

Proposition 3.37 *Given a diagram of the form*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & S & \end{array}$$

where both p and q are 2-Cartesian fibrations. Then the following statements are equivalent:

(i) The map f is a trivial fibration.
*(ii) The map f has the right lifting property against **MB**-anodyne maps and for every $s \in S$ the induced map on fibres $f_s : X_s \to [\simeq]Y_s$ is a bicategorical equivalence.*

Proof The implication (i) $\Longrightarrow$ (ii) is clear. Now suppose that (ii) holds. Then we immediately see that for every $s \in S$ the map f_s is a trivial fibration of scaled simplicial sets. First we will show that we can lift the maps

$$\left(\partial\Delta^m, \flat, \flat\right) \to \left(\Delta^m, \flat, \flat\right) \quad , \quad m \geq 0.$$

Suppose we are given a lifting problem of the form

$$\begin{array}{ccc} \partial\Delta^m & \xrightarrow{\alpha} & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \Delta^m & \xrightarrow{\beta} & Y \end{array}$$

and let κ_β be the smallest integer such that $q \circ \beta : \Delta^m \to \Delta^{\kappa_\beta} \to S$. We will use induction on κ_β. Note that when $\kappa_\beta = 0$ the lifting problem occurs in one of the fibres and thus the solution exists. Suppose the claim holds for $0 < \kappa_\beta - 1 \leq m - 1$. We will assume without loss of generality that $S = \Delta^{\kappa_\beta}$. Let us remark that by construction the map $r = q \circ \beta : \Delta^m \to \Delta^{\kappa_\beta}$ must be surjective. Let $j_\beta \in [m]$ be the biggest element such that $r(j_\beta) < r(m) = \kappa_\beta$. We can now use Lemma 3.36 to produce a commutative diagram

$$\begin{tikzcd} S^{m+1}_{j_\beta} \arrow[r, "\varepsilon"] \arrow[d] & X \arrow[d, "f"] \\ \Delta^{m+1} \arrow[r, "\theta"] & Y \end{tikzcd}$$

satisfying the conditions of the lemma. It follows from the proof Lemma 3.36 that the triangle $\theta(\{j_\beta, j_\beta+1, j_\beta+2\})$ must be scaled. Restricting this diagram along the face missing $j_\beta+2$ yields another commutative square

$$\begin{tikzcd} \partial\Delta^m \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta^m \arrow[r, "\xi"] & Y \end{tikzcd}$$

We claim that our original lifting problem admits a solution if this later lifting problem admits a solution. Indeed, given a solution of this later lifting problem we can produce a commutative diagram

$$\begin{tikzcd} S^{m+1}_{j_\beta} \arrow[r] \arrow[d] & \Lambda^{m+1}_{j_\beta+1} \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^{m+1} \arrow[r, equal] & \Delta^{m+1} \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

where the dotted arrow exists since the triangle $\theta(\{j_\beta, j_\beta+1, j_\beta+2\})$ is scaled. It follows from Lemma 3.36 that the restriction of this solution to the face missing $j_\beta+1$ is a solution for our original lifting problem.

We can further see that if $j_\beta = n-1$ then ξ must factor through $\Delta^{\kappa_\beta - 1}$ and the existence of the solution follows from the inductive hypothesis. If $j_\beta < m-1$ it follows that $q \circ \xi$ must be surjective and that $j_\xi > j_\beta$ so we can keep applying Lemma 3.36 until we obtain the solution. The inductive step is proved and the claim holds.

To finish the proof we must show that f detects marked edges and lean (resp. thin) triangles. Let $e : \Delta^1 \to X$ be such that $f(e)$ is marked. Let us denote $e(i) = x_i$ for $i \in \{0,1\}$ and similarly denote $f(x_i) = y_i$. Pick a marked lift $\widetilde{e} : \hat{x}_0 \to x_1$ and observe that we can produce a 2-simplex $\sigma : \Delta^2 \to X$ such that $\sigma|_{\Delta^{\{1,2\}}} = \widetilde{e}$ and $\sigma|_{\Delta^{\{0,2\}}} = e$. It follows from Lemma 3.11 that $f(\sigma)$ is fully marked and since its restriction to $\Delta^{\{0,1\}}$ lies in $Y_{q(y_0)}$ that particular edge must be an equivalence. However f detects equivalences in the fibres so it follows that $\sigma|_{\Delta^{\{0,1\}}}$ is marked in X. The claim follows from Definition 3.7 (S1).

Suppose we are given $\varphi : \Delta^2 \to X$ such that $f(\varphi)$ is lean-scaled in Y. As usual we will assume without loss of generality that $S = \Delta^2_\flat$ a minimally scaled 2-simplex. We can additionally assume that φ is not contained in some X_i for $i \in [2]$, otherwise the claim follows immediately. Let $s : \Delta^2 \xrightarrow{\varphi} X \xrightarrow{p} S = \Delta^2_\flat$ and define define j_φ as the biggest integer such that $s(j_\varphi) < s(2)$. Then a totally analogous argument to that of Lemma 3.36 shows that we can produce a 3-simplex $T : \Delta^3 \to X$ such that:

- The restriction of T to the face missing $j_\varphi + 1$ equals φ.
- The edge $j_\varphi + 1 \to j_\varphi + 2$ is marked.
- Every triangle of T containing the edge $j_\varphi + 1 \to j_\varphi + 2$ is lean.

We claim that by construction $f(T)$ must be fully lean-scaled in Y. There are two cases to study: $j_\varphi = 0$ and $j_\varphi = 1$. If $j_\varphi = 0$ then it follows that every triangle in $f(T)$ is lean except the 2-nd face. However the triangle given by the vertices $\{1, 2, 3\}$ is lean by construction and lies over an edge. Since lean triangles lying over thin triangles are themselves thin it follows that we can lean-scale the missing face using a morphism of type (S3). If $j_\varphi = 1$ then it follows that every triangle in $f(T)$ is lean except the 3-rd face. We can lean-scale this face using a morphism of type (S5).

We proceed now by cases:

(a) The map s is given by $a \to a \to b$. Note that in this case we have $j_\varphi = 1$ and let us consider $T : \Delta^3 \to X$ as before. We see that the face missing 3 is contained in X_a and since its image in Y is lean (it is in fact thin) it follows that it must be lean in X. It follows that we can scale $\varphi : \Delta^2 \to X$ using a morphism of type (S3) since the triangle $\{1, 2, 3\}$ gets mapped under T to a thin 2-simplex.

(b) The map s is given by $a \to b \to b$. Now we see that we can scale the face missing 2 in $T : \Delta^3 \to X$ using the previous case. We can scale $\varphi : \Delta^2 \to X$ using a morphism of type (S3) since the triangle $\{0, 1, 2\}$ gets mapped under T to a thin 2-simplex.

(c) The map s is given by $a \to b \to c$. It follows that we can scale the face missing 3 in $T : \Delta^3 \to X$ using case b). Since the triangle $\{1, 2, 3\}$ gets mapped under T to a thin 2-simplex we can scale $\varphi : \Delta^2 \to X$ using a morphism of type (S3).

To prove that f detects thin triangles we only need to observe that if the image of a 2-simplex $\varphi : \Delta^2 \to X$ gets mapped under f to a thin triangle then by the discussion for lean-triangles it follows that φ is a lean in X. We can then thin-scale φ using a morphism of type (S2). □

Proposition 3.38 *Suppose we are given a morphism of 2-Cartesian fibrations*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

Then the following are equivalent

(i) The map f is a weak equivalence.

(ii) For every $s \in S$ the induced morphism $f_s : X_s \to Y_s$ is an equivalence of scaled simplicial sets.

Proof The implication (i) $\implies$ (ii) is clear since we can construct an inverse up to homotopy for f as we did in the proof of Proposition 3.31. To prove the converse we will apply the small object argument and obtain a factorization of f

$$X \to [u]L \to [v]Y$$

where the map u is **MB**-anodyne and v has the right lifting property against the class of **MB**-anodyne maps. It follows from Proposition 3.21 that u must be a weak equivalence. Now we observe that $L \to S$ must be a 2-Cartesian fibration. It follows from 2-out-of-3 that the induced morphism on fibres $L_s \to Y_s$ must be a bicategorical equivalence for every $s \in S$. We can now apply Proposition 3.37 to obtain that v must be a trivial fibration. This finishes the proof. □

Definition 3.39 Recall from [11, A.2.6.10] that a class of morphisms W in a presentable category $\mathcal{A}$ is *perfect* if it satisfies the following conditions

1. Every isomorphism belongs to W.

2. Given a pair of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ if any two of the morphisms f, g and $g \circ f$ belong to W, then so does the third.
3. The class W is stable under filtered colimits.
4. There exists a (small) subset $W_0 \subseteq W$ such that every morphism belonging to W can be obtained as a filtered colimit of morphisms belonging to W_0.

Lemma 3.40 *The class of weak equivalences in* $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ *is perfect in the sense of Definition 3.39.*

Proof Using the small object argument we produce a functor (which preserves filtered colimits by our assumptions as seen in [11, Proposition A.1.2.5] and Remark 3.8)

$$T : (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S} \to (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$$

equipped with a natural transformation $\mathrm{id} \Rightarrow T$ such that for every $K \in (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ the map $K \to T(K)$ is **MB**-anodyne and $T(K)$ is a 2-Cartesian fibration. It follows that a morphism $h : K \to L$ is a weak equivalence if and only if $T(h)$ is a weak equivalence. We finally consider the composite

$$W_S : (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S} \to [T](\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S} \to \prod_{s \in S} \mathrm{Set}_\Delta^{\mathbf{mb}} \to \prod_{s \in S} \mathrm{Set}_\Delta^{\mathrm{sc}}$$

where the second functor is given by taking pullback along each fibre and the second functor is a product of forgetful functors. It follows from a simple inspection that W_S preserves filtered colimits. Let $\mathcal{E}_S = \prod_{s \in S} \mathcal{E}$ where $\mathcal{E}$ denotes the collection of weak equivalences in $\mathrm{Set}_\Delta^{\mathrm{sc}}$. Since $\mathcal{E}$ is perfect then so is $\mathcal{E}_S$. We claim that the collection of weak equivalences in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ is precisely given by $W_S^{-1}(\mathcal{E}_S)$. Once this is proved the result will follow from [11, A.2.6.12].

Let $\mathbb{E}$ denote the collection of weak equivalences in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ and let $\alpha : X \to Y$ be a morphism. Let us suppose that $\alpha \in W_S^{-1}(\mathcal{E}_S)$ then it follows from Proposition 3.38 that $T(\alpha) \in \mathbb{E}$. Since $\alpha \in \mathbb{E}$ if and only if $T(\alpha) \in \mathbb{E}$ it follows that $W_S^{-1}(\mathcal{E}_S) \subseteq \mathbb{E}$. The converse follows easily. □

Lemma 3.41 *Let* $p : X \to S$ *and* $n \geq 0$. *Then the morphism* $r : X \times (\Delta^n)^\sharp_\sharp \to X$ *given by projection to* X *is a weak equivalence.*

Proof Note that the inclusion of the terminal object $t_n : (\Delta^0)^\sharp_\sharp \to (\Delta^n)^\sharp_\sharp$ induces a section $s : X \to X \times (\Delta^n)^\sharp_\sharp$. Since our class of weak equivalences satisfies 2-out-of-3 it follows that it is enough to show that s is a weak equivalence. We will show that the map t_n is **MB**-anodyne. Then the claim will follow from Proposition 3.14.

We prove that t_n is **MB**-anodyne using induction on n. If $n = 1$ then t_1 is the generator (A5). We define for $0 \leq i \leq n-1$ a **MB** subsimplicial set $A_i \subset (\Delta^n, \sharp, \sharp)$ consisting in those simplices that are contained in the j-th face fo $j \leq i$. This produces a filtration

$$\Delta^0 \to A_0 \to \cdots \to A_{n-2} \to A_{n-1} = (\Lambda^n_n, \sharp, \sharp) \to (\Delta^n, \sharp, \sharp).$$

It is easy to verify that each step in this filtration is **MB**-anodyne. □

Theorem 3.42 *Let S be a scaled simplicial set. Then there exists a left proper combinatorial simplicial model structure on* $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$, *which is characterized uniquely by the following properties:*

C) A morphism $f : X \to Y$ in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ is a cofibration if and only if f induces a monomorphism betwee the underlying simplicial sets.
F) An object $X \in (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ is fibrant if and only if X is a 2-Cartesian fibration.

Proof We will use [11, Prop. A.2.6.13] to deduce the existence of a left proper combinatorial model structure in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$. Lemma 3.40 shows that the class of weak equivalences is perfect. We proved in Proposition 3.30 that weak equivalences are stable under pushouts along cofibrations. It is also immediate to see that trivial fibrations are in particular weak equivalences so the conditions of [11, Prop. A.2.6.13] apply. Now we wish to show that this model structure is compatible with the simplicial structure. This follows from [11, Prop. A.3.1.7] coupled with Lemma 3.41.

It is clear that every **MB**-anodyne morphism is a trivial cofibration which implies that every fibrant object is a 2-Cartesian fibration. To show that every 2-Cartesian fibration defines a fibrant object we consider a lifting problem

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{\beta} & S \end{array}$$

where i is a general trivial cofibration and $p : X \to S$ is a 2-Cartesian fibration. We consider the induced morphism of mapping ∞-bicategories

$$i^* : \mathrm{Map}_S(B, X) \to \mathrm{Map}_S(A, X)$$

and observe that due to Proposition 3.21 the induced morphism is simultaneosly a bicategorical equivalence and a fibration. Therefore i^* is trivial fibration of ∞-bicategories. The solution to our lifting problem is obtained by taking a preimage of the object $\alpha \in \mathrm{Map}_S(A, X)$. □

Theorem 3.43 *The adjunction presented in Remark 3.6*

$$L : \mathrm{Set}_\Delta^{\mathrm{sc}} \rightleftarrows \mathrm{Set}_\Delta^{\mathbf{mb}} : U$$

*is a Quillen equivalence where the right-hand side is equipped with the model structure of **MB** simplicial sets over the point constructed in Theorem 3.42.*

Proof First we will show that L preserves cofibrations and trivial cofibrations. The case of cofibrations is immediate. Now let us suppose that $(A, T_A) \to (B, T_B)$ is a trivial cofibration of scaled simplicial sets. Let $\mathbb{D}$ be a fibrant object in $\mathrm{Set}_\Delta^{\mathbf{mb}}$ and note that as stated before $\mathbb{D}$ is an ∞-bicategory with all the equivalences marked. It is immediate that the morphism

$$\mathrm{Fun}^{\mathbf{mb}}(L(B), \mathbb{D}) \to \mathrm{Fun}^{\mathbf{mb}}(L(A), \mathbb{D}))$$

can be identified with the analogous morphism

$$\mathrm{Fun}^{\mathbf{sc}}(B, U(\mathbb{D})) \to \mathrm{Fun}^{\mathbf{sc}}(A, U(\mathbb{D}))$$

between the underlying scaled simplicial sets. It follows that $L \dashv U$ is a Quillen adjunction. Note that $U \circ L = \mathrm{id}$. To conclude the proof suppose that $\mathbb{B}$ is a fibrant **MB** simiplicial set. In particular, we need to show that the map

$$(\mathbb{B}, \flat, T_{\mathbb{B}}) \to (\mathbb{B}, E_{\mathbb{B}}, T_{\mathbb{B}})$$

is a weak equivalence. However the above morphism is a pushout of a morphism of type (E) in Definition 3.7. □

4 2-Cartesian Fibrations Over a Fibrant Base

The goal of this section is to give a characterization of 2-Cartesian fibrations in the specific case where $S \in \mathrm{Set}_\Delta^{\mathrm{sc}}$ is an ∞-bicategory. For the rest of this section we will fix a functor of ∞-bicategories $p : X \to S$.

Definition 4.1 Let $p : X \to S$ be a weak **S**-fibration (Definition 2.14). We call a left-degenerate (Definition 2.3) 2-simplex $\sigma : \Delta^2 \to X$, *p-coCartesian* if there exists a solution for any lifting problem of the form

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "f"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] & S \end{tikzcd}$$

provided $f|_{\Delta^{\{0,1,n\}}} = \sigma$.

Remark 4.2 Recall the definition of the mapping ∞-category $X(a, b)$ described in Definition 2.17. Let $\sigma : \Delta^2 \to X$ be a p-coCartesian simplex such that $\sigma(0) = a$ and $\sigma(2) = b$. Since σ is left-degenerate it can be viewed as an edge in $X(a, b)$. We can further observe that by definition the dotted arrow in the diagram

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "\rho"] \arrow[d] & X(a,b) \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

exists provided the restriction of ρ to $\Delta^{\{0,1\}}$ is precisely σ. This shows that p-coCartesian triangles define coCartesian edges in the mapping space. We wish to show that this property precisely characterizes coCartesian triangles. The proof of this later fact will involve a little bit of work.

Lemma 4.3 *Let X be an ∞-bicategory and consider an n-simplex $\sigma : \Delta^n \to X$ with $n \geq 2$. Suppose that there exists some $0 < k < n$ such that the restriction of σ to $\Delta^{[0,k]}$, see Definition 2.4, is degenerate on $\sigma(0) = a$. Then there exists a morphism*

$$\hat{\sigma} : \Delta^{n+1} \to X$$

with the following properties:

- *The restriction of $\hat{\sigma}$ to its $(k+1)$-face equals σ.*
- *The restriction of $\hat{\sigma}$ to $\Delta^{[0,k+1]}$ is degenerate on a.*
- *For every $k+2 \leq j \leq n+1$ the 2-simplex $\Delta^{\{k+1,k+2,j\}}$ is thin in X.*

Proof Our first observation is that if $k = n-1$ then we can define $\hat{\sigma} = s_{n-1}(\sigma)$ and this provides the desired solution. We will assume for the rest of the proof that $n-k > 1$. We define a simplicial subset $\iota : R^n_k \to \Delta^{n+1}$ consisting precisely of those simplices $\theta : \Delta^k \to \Delta^{n+1}$ satisfying *at least one* of the following conditions

(a) The simplex θ skips the vertex $k+1$.
(b) The simplex θ skips the vertex $n+1$.
(c) The simplex θ is one of the triangles $\Delta^{\{k+1,k+2,j\}}$ for $k+2 < j \leq n+1$

We endow Δ^{n+1} with a scaling by scaling those triangles contained in $\Delta^{[0,k+1]}$ in addition to the triangles $\Delta^{\{k+1,k+2,j\}}$ for $k+2 < j \leq n+1$. The proof will be performed in two steps: First we will show that ι is an scaled anodyne morphism. Finally, we will produce an extension of σ to R_k^n.

We inductively define scaled simplicial subsets $A^n_{(k,i)} \subset \Delta^{n+1}$ (where Δ^{n+1} carries the scaling defined above) consisting in those simplices that either belong to $A^n_{(k,i-1)}$ or are contained in the face missing i for $1 \leq i \leq k$ and where we are using the convention $A^n_{(k,0)} = R^n_k$. Let $B^n_{(k,n)} \subset \Delta^{n+1}$ be the simplicial subset whose simplices either belong to $A^n_{(k,k)}$ or factor through the n-th face. We inductively define $B^n_{(k,j)}$ from $B^n_{(k,j+1)}$ by adding the face missing j for $k+3 \leq j \leq n$ with the convention $A^n_{(k,k)} = B^n_{(k,n+1)}$. This yields a filtration

$$R_k^n \to A^n_{(k,1)} \to \cdots \to A^n_{(k,k)} \to B^n_{(k,n)} \to \cdots \to B^n_{(k,k+3)} \to \Delta^{n+1}$$

We wish to show that each step in the filtration is given by a scaled anodyne morphism. Note that $B^n_{(k,k+3)}$ contains all faces except the face missing 0 and the face missing $k+2$. Since the triangle $\Delta^{\{k+1,k+2,k+3\}}$ is thin it is easy to verify that the last step in our filtration is scaled anodyne. We observe that we can produce pushout diagrams

$$\begin{tikzcd} A^{n-1}_{(k-1,i)} \arrow[r] \arrow[d] & \Delta^n \arrow[d, "d_{i+1}"] \\ A^n_{(k,i)} \arrow[r] & A^n_{(k,i+1)} \end{tikzcd} \qquad \begin{tikzcd} B^{n-1}_{(k,j-1)} \arrow[r] \arrow[d] & \Delta^n \arrow[d, "d_{j+1}"] \\ B^n_{(k,j)} \arrow[r] & B^n_{(k,j-1)} \end{tikzcd}$$

where the morphism $A^{n-1}_{(k-1,i)} \to A^n_{(k,i)}$ (resp.$B^{n-1}_{(k,j-1)} \to B^n_{(k,j)}$) is the restriction of the inclusion of the i-th face (resp $(j-1)$-th face) into $A^n_{(k,i-1)}$ (resp. $B^n_{(k,j)}$). Suppose that each step in our filtration is scaled anodyne for $\kappa \leq n-1$. Then it follows that each $A^{n-1}_{(k,i)} \to \Delta^n$ is scaled anodyne. Therefore we can use the pushout diagrams above to show that each step in the filtration is scaled anodyne for $\kappa = n$. The ground case we need to show is $n = 3$ and $k = 1$. In this setting the filtration is of the form

$$R_2^3 \to A^3_{(1,1)} \to \Delta^4.$$

Note that in this case $k+3 = n+1$ so the filtration terminates at $A^3_{(1,1)}$. In particular, the morphism $B^3_{(1,4)} = A^3_{(1,1)} \to \Delta^4$ is scaled anodyne. To verify that the first morphism is scaled anodyne we add to R_2^3 the face that misses the vertices 0 and 1 by taking a pushout along the morphism $(\Lambda^2_1, \sharp) \to (\Delta^2, \sharp)$ obtaining a factorization

$$R_2^3 \to Q \to A^3_{(1,1)}$$

It follows that the restriction of the face missing 1 to Q is given by a horn Λ^3_2 where the triangle $\{1,2,3\}$ is thin. The ground case now follows.

To finish the proof we need to produce the extension from $\sigma : \Delta^n \to X$ to a map $\rho : R_k^n \to X$. We define L_k^n as the subsimplicial of R_k^n consisting in those simplices satisfying conditions a) or c). We define $\rho(k+1 \to k+2) = \sigma(k \to k+1)$ and extend σ to L_k^n by picking the obvious composites of morphisms. Note that if $n-k = 2$ then we can produce the desired extension by just setting $d_{n+1}(\rho) = s_k(d_n(\sigma))$. Therefore will assume that L_k^n already contains those simplices that factor through $\Delta^{[0,k+2]}$. To finish the proof we will show that $L_k^n \to R_k^n$ is scaled anodyne. We consider morphisms

$$\alpha_{k+j} : \Delta^{[0,k+j]} \to \Delta^{[0,n]} \subset R_k^n \quad , \quad \text{for } 3 \leq j \leq n-k.$$

Let us set $C^n_{(k,2)} = L^n_k$. We define inductively $C^n_{(k,j)}$ by attaching the simplices α_{k+j} to $C^n_{(k,j-1)}$. We obtain our final filtration

$$L^n_k \to C^n_{(k,3)} \to \cdots \to C^n_{(k,n-k)} = R^n_k.$$

Note that we have pushout diagrams

$$\begin{tikzcd} R_k^{[0,k+j]} \arrow[r] \arrow[d] & \Delta^{[0,k+j]} \arrow[d, "\alpha_k"] \\ C^n_{(k,j)} \arrow[r] & C^n_{(k,j+1)} \end{tikzcd}$$

where the top horizontal morphism is scaled anodyne by the first part of this proof. The result follows. □

Proposition 4.4 *Let $p : X \to S$ be a weak $\mathbf{S}$-fibration. Then a left-degenerate triangle $\sigma : \Delta^2 \to X$ with $\sigma(0) = a$ and $\sigma(2) = b$ is coCartesian if and only if it defines as coCartesian edge in the mapping space $X(a, b)$.*

Proof It is immediate that if σ is coCartesian then it defines a coCartesian edge in the corresponding mapping space. For the converse let $n \geq 3$ and consider a lifting problem

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "f"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "\alpha"] & S \end{tikzcd}$$

such that $f|_{\Delta^{\{0,1,n\}}} = \sigma$. We define $1 \leq k \leq n-1$ to be the biggest integer such that the restriction of f to $\Delta^{[0,k]}$ is degenerate on a. Note that if $n - k = 1$ then the lifting problem takes place in the mapping space $X(a, b)$ and the solution is guaranteed. We define a subsimplicial set $P^n_k \subset \Delta^{n+1}$ consisting of those simplices $\rho : \Delta^k \to \Delta^{n+1}$ satisfying *at least one* of the following conditions

(a) The simplex ρ skips the vertex $n + 1$.
(b) The simplex ρ skips a pair of vertices $(k + 1, i)$ with $i \neq 0$.
(c) The simplex ρ factors through $\Delta^{\{k+1,k+2,j\}}$ with $k + 2 < j \leq n + 1$.

Now we can apply Lemma 4.3 to the simplex α to obtain a map $\hat{\alpha} : \Delta^{n+1} \to S$ satisfying the conditions stated in the lemma. Our first goal is to produce a commutative diagram

$$\begin{tikzcd} \Lambda^n_0 \arrow[r] \arrow[d] \arrow[rr, bend left, "f"] & P^n_k \arrow[r, "\hat{f}"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "d_{k+1}"] & \Delta^{n+1} \arrow[r, "\hat{\alpha}"] \arrow[ur, dotted, "\varepsilon"] & S \end{tikzcd}$$

since any dotted arrow as above will provide a solution to the original lifting problem. We define $\hat{f}$ as follows:

- On simplices satisfying condition b) the value of $\hat{f}$ is completely determined by f.
- We want to define the map $\hat{f}$ on simplices satisfying condition a). We consider a simplex

$$\sigma_{k+1} : \Delta^{[0,k+1]} \to \Lambda^n_0 \to X$$

We define the image $\Delta^{[0,k+2]} \to P_k^n$ in X to be the value of the k-th degeneracy operator on σ_{k+1}. This is compatible with the morphism $\hat{\alpha}$ as seen in the proof of Lemma 4.3. Moreover, if $n-k=2$ this completes the definition of $\hat{f}$ on simplices satisfying a). Let us suppose that $n-k>2$ and let $M_k^n \subset P_k^n$ be the simplicial subset consisting in those simplices satisfying b) in addition to those simplices contained in $\Delta^{[0,k+2]}$. It follows from the previous discussion that we have a commutative diagram

$$\begin{tikzcd} \Lambda_0^n \arrow[r] \arrow[d] \arrow[rr, bend left, "f"] & M_k^n \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "d_{k+1}"] & \Delta^{n+1} \arrow[r, "\hat{\alpha}"] & S \end{tikzcd}$$

To finally construct $\hat{f}$ it will be enough to show that $M_k^n \to P_k^n$ is scaled anodyne.

We define $Q_k^n \subset P_k^n$ as the simplicial subset consisting in those simplices satisfying condition a) and b). We will show that each step in the factorization

$$M_k^n \to Q_k^n \to P_k^n$$

is scaled anodyne. It is easy to see that $Q_k^n \to P_k^n$ is scaled anodyne since we can add the missing triangles by taking the adequate composites. To show the claim for $M_k^n \to Q_k^n$ we proceed in an almost identical way as in the proof of Lemma 4.3 we produce a filtration by inductively adding to to M_k^n the simplex $\Delta^{[0,k+j]}$ for $3 \leq j \leq n-k$. We leave the standard verification that each step in this filtration is scaled anodyne to the interested reader.It follows that the desired extension $\hat{f} : P_k^n \to X$ exists.

To finish the proof we will construct the dotted arrow ε above. Let S_k^n be the subsimplicial subset of Δ^{n+1} consisting in those simplices belonging to P_k^n in addition to the faces that skip the vertices i for $1 \leq i \leq k$. A totally analogous argument as that for Lemma 4.3 shows that the map inclusion $P_k^n \to S_k^n$ is scaled anodyne. We can now add the faces that skip the vertices $k+2 \leq j \leq n$ to obtain a new simplicial set T_k^n. We observe that T_k^n only misses the $(k+1)$-face and the 0-face since the triangle $\Delta^{\{k,k+1,k+2\}}$ must be thin. It is easy to see that $T_k^n \to \Delta^{n+1}$ is scaled anodyne. To finish the proof provide a solution to the lifting problem

$$\begin{tikzcd} S_k^n \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ T_k^n \arrow[r] \arrow[ur, dotted, "\varphi"] & S \end{tikzcd}$$

We define $D_{(k,n)}^n$ by adding to S_k^n the face missing the vertex n. We define $D_{(k,j-1)}^n$ by adding to $D_{(k,j)}^n$ the face missing j for $k+2 \leq j \leq n$. This produces a filtration

$$S_k^n \to D_{(k,n)}^n \to \cdots D_{(k,k+2)}^n = T_k^n$$

We will show how to produce the solution by extending the map stepwise. As usual, we produce a pushout diagram

$$\begin{tikzcd} D_{(k,j-1)}^{n-1} \arrow[r] \arrow[d] & \Delta^n \arrow[d] \\ D_{(k,j)}^n \arrow[r] & D_{(k,j-1)}^n \end{tikzcd}$$

Now we observe that if $n-k=2$ then original filtration is of the form

$$S^n_{n-2} \to D^n_{(n-2,n)} = T^n_{n-2}$$

the previously depicted pushout diagram particularizes now to

$$\begin{tikzcd} \Lambda^n_0 \arrow[r] \arrow[d] & \Delta^n \arrow[d] \\ S^n_{n-2} \arrow[r] & T^n_{n-2} \end{tikzcd}$$

where the left-most Λ^n_0 represents an 0-horn in the mapping space and thus the existence of the extension is guaranteed. An inductive argument shows that we can produce the map φ and the proof is concluded. □

Definition 4.5 We say that $p : X \to S$ is *locally fibred* if it satisfies the conditions

(i) The map $p : X \to S$ is a weak **S**-fibration.
(ii) For every left-degenerate $\widetilde{\sigma} : \Delta^2 \to S$ together with $\tau : \Delta^1 \to X$ such that $\widetilde{\sigma}|_{\Delta^{\{0,2\}}} = p(\tau)$, then there exists a left-degenerate simplex $\sigma : \Delta^2 \to X$ such that σ is coCartesian and $p(\sigma) = \widetilde{\sigma}$.

The following proposition follows immediately from our definitions.

Proposition 4.6 *Let $p : X \to S$ be locally fibred. The given $a, b \in X$ a pair of objects it follows that the induced morphism on mapping spaces*

$$p_{a,b} : X(a, b) \to S(p(a), p(b))$$

is a coCartesian fibration of ∞-categories.

Definition 4.7 Let $\sigma, \tau : \Delta^2 \to X$ be a pair of 2-simplices such that τ is left-degenerate. We say τ is the *left-degeneration* of σ if there exists a 3-simplex $\rho : \Delta^3 \to X$ with the following properties:

- The face $d_3(\rho)$ equals $s_0(d_2(\sigma))$.
- The face $d_2(\rho)$ equals τ.
- The face $d_1(\rho)$ equals σ.
- The face $d_0(\rho)$ is thin in X.

See Fig. 3 for a pictorial interpretation.

Remark 4.8 We remark that if X is an ∞-bicategory then the left-degeneration of a 2-simplex always exist. It is trivial to see that every left-degenerate triangle is its own left-degeneration.

Definition 4.9 We say that a triangle $\sigma : \Delta^2 \to X$ is *coCartesian* if its left-degeneration is coCartesian. We denote the collection of coCartesian triangles by C_X.

Lemma 4.10 *Let $p : X \to S$ be locally fibred. Suppose that we are given a 2-simplex $\sigma : \Delta^2 \to X$ such that σ is p-coCartesian and its image under p is thin in S. Then σ is a thin simplex of X.*

Proof If σ is left-degenerate the claim follows immediately from Proposition 4.6 since σ represents a coCartesian edge in the mapping space $X(a, b)$ whose image in $S(p(a), p(b))$ is an equivalence. To show the general case we let τ be the left-degeneration of ρ witnessed by a 3-simplex $\rho : \Delta^3 \to X$. Note that since S is an ∞-bicategory it follows that $p(\tau)$ is thin in S since every face of $p(\rho)$ is thin except possibly the 2-face. Using the first part of the proof we see that τ must be thin in X. It follows that we can scale σ in X. □

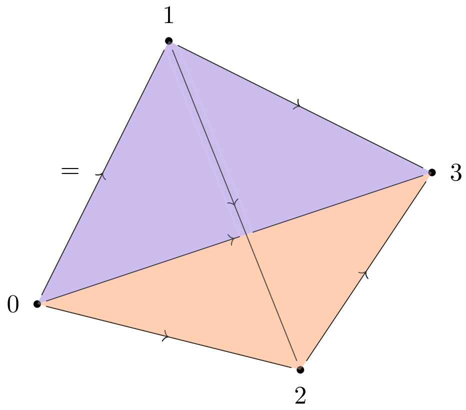

Fig. 3 A 3-simplex ρ displaying τ (blue) as the left-degeneration of σ (red). Here, the front left face $\Delta^{\{0,1,2\}}$ is degenerate, and the front-right face $\Delta^{\{1,2,3\}}$ is thin

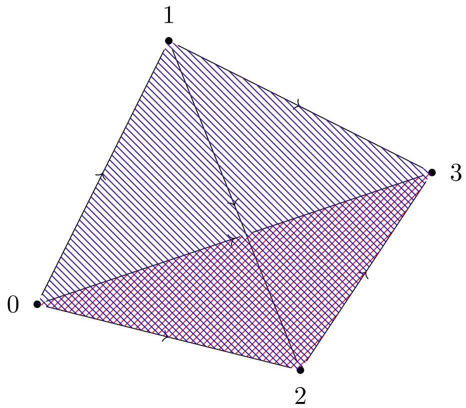

Fig. 4 A 3-simplex, as in Definition 4.11 with i=1.The three blue-hatched triangles are coCartesian, and the triangle $\Delta^{\{0,1,2\}}$ is thin. When C_X is a functorial family, we can conclude that the red-hatched triangle is also coCartesian

Definition 4.11 Let $p : X \to S$ be a weak **S**-fibration. We say that the collection of coCartesian triangles C_X, is a *functorial family* if the following holds:

- Let $0 < i < 3$ and suppose we are given a three simplex $\rho : \Delta^3 \to X$ such that the face $\Delta^{\{i-1,i,i+1\}}$ is thin and all of the faces of ρ are coCartesian except possibly the face missing i. Then the image of ρ only consists in coCartesian triangles.

Definition 4.12 Let $p : X \to S$ be a locally fibred morphism. We say that p is *functorially fibred* if the collection of coCartesian triangles is functorial.

Lemma 4.13 *Let $p : X \to S$ be a functorially fibred map. Given a left-degenerate three simplex $\rho : \Delta^3 \to X$ such that all of its faces except possibly the 0-face belong to C_X then it follows that the 0-face must also belong to C_X.*

Proof Let us first suppose that restriction of ρ to $\Delta^{[0,2]}$ is degenerate of $\rho(0)$. Then ρ defines a 2-simplex in the mapping space $X(a, b)$ where all edges are coCartesian except the edge $1 \to 2$. By the limited 2-out-of-3 property of coCartesian edges it follows that $1 \to 2$ is also coCartesian. Then the result follows from Proposition 4.4.

We suppose now that $\Delta^{[0,2]}$ is not degenerate on $\rho(0)$. We apply Lemma 4.3 to obtain a simplex $\Xi : \Delta^4 \to X$. Note that 4-th face of Ξ can be chosen to be $s_1(d_3(\rho))$. It follows that every triangle in the 4-th face of Ξ is coCartesian. We further note that the triangle $\Delta^{\{1,2,4\}}$

is the left-degeneration of $d_0(\rho)$. We claim that every triangle in $d_1(\Xi) = \sigma$ is coCartesian: First we observe that $d_0(\sigma)$ is thin and that $d_1(\sigma) = d_1(\rho)$. Since every triangle of $d_4(\Xi)$ is coCartesian we see that $d_3(\sigma)$ is coCartesian. It follows that every triangle of σ is coCartesian except possibly the 2-nd face. Since the map is functorially fibred the claim follows.

To finish the proof we consider $d_3(\Xi) = \theta$ and observe that the restriction of θ to $\Delta^{[0,2]}$ is degenerate of $\theta(0) = \rho(0)$. Moreover it follows that $d_1(\theta) = d_2(\sigma)$. We see that every triangle of θ is coCartesian except possible the 0-th face. We can apply now the first part of the proof to conclude.

Definition 4.14 Let $p : X \to S$ be a functorially fibred map. We say that an edge $e : \Delta^1 \to X$ is *strongly* p-cartesian (resp. p-Cartesian) if every lifting problem

$$\begin{tikzcd} \Lambda^n_n \arrow[r, "f"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "\sigma"] \arrow[ur, "\hat{f}"] & S \end{tikzcd}$$

admits a solution for $n \geq 3$ provided the following conditions are satisfied:

(i) $f|_{\Delta^{\{n-1,n\}}} = e$.
(ii) $f|_{\Delta^{\{0,n-1,n\}}}$ is coCartesian (resp. thin).

In the case $n = 2$, we distinguish two cases:

- If $f|_{\Delta^{\{1,2\}}} = e$ is *strongly* p-Cartesian the solution of the lifting problem $\hat{f}$ exists and defines a coCartesian triangle in X.
- If $f|_{\Delta^{\{1,2\}}} = e$ is p-Cartesian and $\sigma : \Delta^2 \to S$ is thin then the solution of the lifting problem $\hat{f}$ exists and defines a thin triangle in X.

Definition 4.15 We say that a functorially fibred map $p : X \to S$ is an outer 2-fibration or **O2**-fibration, if every degenerate edge is *strongly* p-Cartesian.

Remark 4.16 Recall from [9, Definition 2.1.1] that a map of scaled simplicial sets $p : X \to S$ is a *weak fibration* if it has the right lifting property against the following types of maps

i) The inner horn inclusions

$$\left(\Lambda^n_i, \{\Delta^{\{i-1,i,i+1\}}\}\right) \to \left(\Delta^n, \{\Delta^{\{i-1,i,i+1\}}\}\right) \quad , \quad n \geq 2 \quad , \quad 0 < i < n;$$

ii) The left-horn inclusions

$$\left(\Lambda^n_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \{\Delta^{\{0,1,n\}}\}\right) \to \left(\Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0, \{\Delta^{\{0,1,n\}}\}\right) \quad , \quad n \geq 2.$$

iii) The right-horn inclusions

$$\left(\Lambda^n_n \coprod_{\Delta^{\{n-1,n\}}} \Delta^0, \{\Delta^{\{0,n-1,n\}}\}\right) \to \left(\Delta^n \coprod_{\Delta^{\{n-1,n\}}} \Delta^0, \{\Delta^{\{0,n-1,n\}}\}\right) \quad , \quad n \geq 2.$$

Observe that an **O2**-fibration is a *weak fibration* in the terminology of [9].

Lemma 4.17 *Let $p : X \to S$ be a* **O2***-fibration. Given a 3-simplex $\rho : \Delta^3 \to X$ such that*

- *The restriction $\rho|_{\Delta^{\{2,3\}}}$ is a p-Cartesian edge.*
- *Every face of ρ belongs to C_X except possibly the face missing 3.*

Then every face of ρ belongs to C_X.

Proof Let us fix some notation before diving into the proof. We denote $a = \rho(0)$, $c = \rho(2)$, $d = \rho(3)$ and $\rho|_{\Delta^{\{2,3\}}} = \alpha$. First let us assume that $\rho|_{\Delta^{\{0,1\}}}$ is degenerate on a. Using Proposition 2.3.3 in [9] we obtain a homotopy pullback diagram

$$\begin{tikzcd} X(a,c) \arrow[r, "\alpha\circ-"] \arrow[d] & X(a,d) \arrow[d] \\ S(p(a),p(c)) \arrow[r, "p(\alpha)\circ-"] & S(p(a),p(d)) \end{tikzcd}$$

Let $\varepsilon \in X(a,c)$ denote the morphism represented by the 3-face in ρ. We claim that our hypothesis imply that its image under postcomposition with α must be coCartesian. Let $\rho(0 \to 2) = u$ and $\rho(1 \to 2) = v$ and pick composites $\alpha \circ u$ and $\alpha \circ v$ represented by the corresponding thin 2-simplices. We construct a 3-simplex $\tau : \Delta^3 \to X$ such that $d_3(\tau) = d_3(\rho)$, $d_0(\tau) = \alpha \circ v$ and $d_1(\tau) = \alpha \circ u$. This definitions gives a $\Lambda^3_2 \to X$ such that the triangle $\{1, 2, 3\}$ is thin and thus we can pick an extension to Δ^3 to yield the desired τ. It is clear that $d_2(\tau) = \gamma$ is the image of ε under post-composition with α.

We now apply Lemma 4.3 to ρ to obtain a 4-simplex $\nu : \Delta^4 \to X$. Observe that the 0-th face of $d_3(\nu)$ is the left-degeneration of $d_0(\rho)$ which implies that it must be coCartesian. We see that every face of $d_3(\nu)$ must be coCartesian except possibly the face missing 1. Since the triangle $\{0, 1, 2\}$ is thin it follows every face of $d_3(\nu)$ is coCartesian.

To finish the proof of the claim we construct a map $\overline{\kappa} : \Lambda^4_3 \to X$ as follows:

- The 4-th face is given by $s_0(d_3(\rho))$.
- The 1-st face is given by $d_1(\nu)$.
- The 0-th face is given by τ.
- The 2-nd face is given by picking a lift of the morphism $\Lambda^3_2 \to X$ which sends the 0-th face to $\alpha \circ u$, the 1-st face to $d_1(\rho)$ and the 3-rd face to $s_0(u)$. Let us note that the 2-nd face of any extension must be coCartesian.

Since the triangle $\{2, 3, 4\}$ is thin we can pick an extension $\kappa : \Delta^4 \to X$. It follows that every face of $d_3(\kappa)$ is coCartesian except possibly the face missing 0 which is precisely γ. The claim now follows from Lemma 4.13.

Since $X(a,c)$ can be expressed as a homotopy pullback it follows that ε must be coCartesian as an edge in $X(a,c)$. The claim now follows from Proposition 4.4

To prove the general version of the lemma we will reduce it to the previous case. We will fix once and for all the notation regarding ρ by means of the diagram below

$$\begin{tikzcd} a \arrow[r, "u"] \arrow[rr, bend left, "w"] \arrow[dr, "f"'] & b \arrow[r, "v"] \arrow[d, "g"] & c \arrow[dl, "h"] \\ & d & \end{tikzcd}$$

Let us consider Λ^2_1 sitting inside the 3-face of ρ and another such horn sitting inside the 2-face of ρ. Let σ_3 (resp. σ_2) denote the corresponding thin 2-simplices obtained by extending the horns. We denote the 1-face of these thin simplices by $v \circ u$ (resp. $g \circ u$). We define a morphism

$$\Lambda^3_2 \to X$$

by sending the 0-face to σ_3, the 1-face to $d_3(\rho)$, the 3-face to $s_0(u)$. Since X is an ∞-bicategory we can produce a lift to a 3-simplex that we call θ_4. Observe that if $d_2(\theta_4)$ belongs to C_X then every face of θ_4 is coCartesian except possibly the 1-face. Since $\{0, 1, 2\}$ is thin (in fact degenerate) it follows that $d_3(\rho) \in C_X$. We define a morphism

$$\Lambda^3_1 \to X$$

by sending the 0-face to $d_0(\rho)$, the 2-face to σ_2 and the 3-face to σ_3. We extend this horn to a 3-simplex that we call θ_0. By construction it follows that every face of θ_0 is coCartesian except possibly the face missing 1. Since the triangle $\{0, 1, 2\}$ is thin by definition we see that every face of θ_0 belongs to C_X. Finally, let us define

$$\Lambda^3_2 \to X$$

by sending the 0-face to σ_2, the 1-face to $d_2(\rho)$ and the 3-face to $s_0(u)$. We call θ_3 the extension of this horn to a 3-simplex. We observe that every face of θ_3 belongs to C_X.

Let $\theta_1 = \rho$ and observe that the 3-simplices θ_i for $i \in [4]$, $i \neq 2$ assemble into a Λ^4_2 and that the face $\Delta^{\{1,2,3\}}$ is thin by construction. We take our final extension $\theta : \Delta^4 \to X$ and observe that $d_2(\theta)$ satisfies the conditions of the lemma and its first edge is degenerate. We finish the proof by noting that $d_3 d_2(\theta)$ is coCartesian if and only if $d_3(\rho)$ is. □

Given an **O2**-fibration $p : X \to S$, the condition that an edge e of X be strongly p-Cartesian edge is *prima facie* stronger than the condition that e be p-Cartesian. It turns out, however, that these two notions coincide. To prove this, we must first establish a purely technical result (Corollary 4.19). The reader interested only in the characteristics of p-Cartesian edges may safely skip to Proposition 4.20.

Lemma 4.18 *Let S be an ∞-bicategory. There is a map*

$$E : \coprod_{n\geq 2} S_n \longrightarrow \coprod_{n\geq 3} S_n$$

which raises the dimension of each simplex by 1, and such that, for $\sigma : \Delta^n \to S$, the map $E(\sigma) : \Delta^{n+1} \to S$ has the following properties.

- *Every triangle in Δ^{n+1} which contains the edge $(n-1) \to n$ is mapped to a thin triangle in S.*
- *The n^{th} face of $E(\sigma)$ is σ.*
- *$E(\sigma)$ sends the triangle $\Delta^{n-1,n,n+1}$ to $s_1(\sigma|_{\Delta^{\{n,n+1\}}})$.*
- *When the dimension of σ is greater than 2, the following identities hold:*

$$d_i E(\sigma) = \begin{cases} E(d_i(\sigma)) & i \leq n-2, \quad n > 2 \\ \sigma & i = n \end{cases}$$

Proof We will prove the lemma by induction on the dimension of a simplex σ. For simplicity, we denote the last edge in the spine of an n-simplex σ by $e_\sigma = \sigma|_{\Delta^{\{n,n+1\}}}$.

We begin by defining E on simplices of dimension 2. Consider the restriction

$$\Xi : \Lambda^2_1 \to \Delta^2 \to [\sigma]S$$

and pick an extension of Ξ to a thin 2-simplex $\hat{\sigma}$. We fix the notation $\hat{h} = d_1(\hat{\sigma})$ and $h = d_1(\sigma)$. We construct a morphism $\Lambda^3_1 \to S$ as follows:

- The face missing the vertex 0 equals $s_1(e_\sigma)$.
- The face missing the vertex 2 equals σ.

- The face missing the vertex 3 equals $\hat{\sigma}$

Since the triangle $\Delta^{\{0,1,2\}} = \hat{\sigma}$ is thin by construction we can extend this inner horn to a 3-simplex $E(\sigma) : \Delta^3 \to S$.

We then proceed by induction. Suppose we have defined E for $k < n$, and let $\sigma : \Delta^n \to S$. Define a simplicial subset $A_n \subset \Delta^{n+1}$ which consists of all of the n-dimensional faces except two. The face which skips the vertex $(n-1)$, and the face which skips the vertex $n+1$. The final condition on E requires that $E(\sigma)$ restrict to a map

$$\alpha : A_n \longrightarrow S$$

such that

$$d_i E(\sigma) = \begin{cases} E(d_i(\sigma)) & i \leq n-2, \quad n > 2 \\ \sigma & i = n. \end{cases}$$

To assure that this definition is valid, we must show that it agrees on shared $(n-1)$-dimensional faces. We must check the case $n = 3$ separately, since we do not have recourse to the above identities for 2-simplices.

- **The case** $n = 3$. We consider the intersection of the i^{th} and j^{th} faces, where $i < j$. We will abusively denote our presumptive definition for the i^{th} face of α by $d_i(\alpha)$
  First suppose that $i = 0$ and $j = 1$. Then

  $$d_0(d_1(\alpha)) = d_0(E(d_1(\sigma)))$$

  By construction, the latter is $s_1(e_{d_1(\sigma)}) = s_1(e(\sigma))$. On the other hand, we have that

  $$d_0(d_0(\alpha)) = d_0(E(d_0(\sigma))) = s_1(e_{d_0(\sigma)}) = s_1(\sigma).$$

  so that the simplices agree on the overlap.
  We then consider $i \leq 1$ and $j = 3$. On the one hand,

  $$d_i(d_3(\alpha)) = d_i(\sigma).$$

  On the other hand,

  $$d_2(d_i(\alpha)) = d_2(E(d_i(\sigma))) = d_i(\sigma)$$

  by construction. We thus see that the map $\alpha : A_3 \to S$ is well-defined.
- **The case** $n > 3$. We once again consider $i < j$, and abusively denote the presumptive definition of the i^{th} face of α by $d_i(\alpha)$.
  First suppose $i < j \leq n-2$. We then compute

  $$\begin{aligned} d_i(d_j(\alpha)) &= d_i(E(d_j(\sigma)) = E(d_i(d_j(\sigma))) \\ &= E(d_{j-1}(d_i(\sigma))) = d_{j-1}(E(d_i(\sigma))) = d_{j-1}(d_i(\alpha)) \end{aligned}$$

  so that the definitions of α agree on the intersection of the i^{th} and j^{th} faces. Notice that we have used the final defining identity of $E(\sigma)$ twice, thus necessitating the hypothesis that $n > 3$.
  Finally, suppose $i \leq n-2$ and $j = n$. Then

  $$d_i(d_n(\alpha)) = d_i(\sigma) = d_{n-1}(E(d_i(\sigma))) = d_{n-1}(d_i(\alpha))$$

  as desired. Thus, the map $\alpha : A_n \to S$ is well-defined.

We can then complete the inductive argument. It is easy to see that we have pullback diagram

$$\begin{tikzcd} \Lambda^n_{n-1} \arrow[r] \arrow[d] & \Delta^n \arrow[d, "d_{n+1}"] \\ A_n \arrow[r] & \Delta^{n+1} \end{tikzcd}$$

Note that the triangle $\Delta^{\{n-2,n-1,n\}}$ in A_n gets mapped to a thin triangle in S by the inductive hypothesis. In particular we can extend α to a morphism $\Lambda^{n+1}_{n-1} \to S$. We finish the proof of the lemma by choosing an extension to $E(\sigma) : \Delta^{n+1} \to S$. □

Corollary 4.19 *Let $p : X \to S$ be an* **O2**-*fibration. Then for each simplex $\theta : \Delta^n \to X$, the simplices $E(\theta)$ and $E(p(\theta))$ from Lemma 4.18 can be chosen so that the diagram*

$$\begin{tikzcd} \Delta^{n+1} \arrow[r, "E(\theta)"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^{n+1} \arrow[r, "E(p(\theta))"] & S \end{tikzcd}$$

commutes and the simplex $E(\theta)$ satisfies the following properties:

(i) The map $E(\theta)$ sends every triangle containing the edge $n-1 \to n$ to a thin triangle.
(ii) The map $E(\theta)$ sends the triangle $\Delta^{\{n-1,n,n+1\}}$ to $s_1(e)$, where e is the final edge of θ.
(iii) The n^{th} face of $E(\theta)$ equals θ.
(iv) If the triangle $\Delta^{\{0,n-1,n\}}$ gets mapped under θ to an element of C_X then $E(\theta)$ sends the triangle $\Delta^{\{0,n,n+1\}}$ to an element of C_X.

Proof This is virtually identical to the proof of Lemma 4.18. One simply performs each step of the argument there relative to the fibration p. Property (iv) holds precisely because C_X is a functorial family.

Proposition 4.20 *Let $p : X \to S$ be an* **O2**-*fibration. Then an edge $e : \Delta^1 \to X$ is strongly p-Cartesian if and only if it is p-Cartesian.*

Proof The 'only if' direction is definitional. To show the other direction, let us suppose that e is p-Cartesian and consider a lifting problem

$$\begin{tikzcd} \Lambda^n_n \arrow[r, "f"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "\sigma"] \arrow[ur, dotted, "\hat{f}"] & S \end{tikzcd}$$

such that $f|_{\Delta^{\{n-1,n\}}} = e$ and $f|_{\Delta^{\{0,n-1,n\}}}$ belongs to C_X. Fix a choice of maps E guaranteed by Corollary 4.19.

Define a simplicial subset $B_n \subset \Delta^{n+1}$ to be the subset containing the i^{th} face for $0 \leq i \leq n-2$, as well as the face which skips *both* the vertices n and $n-1$. We construct a commutative diagram

$$\begin{tikzcd} B_n \arrow[r, "\beta"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^{n+1} \arrow[r, "E(\sigma)"'] & S \end{tikzcd}$$

as follows:

- The map β sends the i^{th} face to $E(d_i(f))$ (as constructed above) for $0 \leq i \leq n-2$.
- The map β sends the face skipping the vertices n and $n-1$ to $d_{n-1}(f)$.

We then consider the pullback diagram

$$\begin{tikzcd} \Lambda^n_n \arrow[r] \arrow[d] & \Delta^n \arrow[d, "d_{n-1}"] \\ B_n \arrow[r] & \Delta^{n+1} \end{tikzcd}$$

and observe that by construction the restriction of β along the composite

$$\Lambda^n_n \longrightarrow B_n \xrightarrow{\beta} X$$

maps the final edge of Λ^n_n to an identity morphism and the triangle $\Delta^{\{0,n-1,n\}}$ to a coCartesian triangle.

Let $\widehat{B}_n \subset \Delta^{n+1}$ be the simplicial subset obtained from B_n by adding the face that skips the vertex $n-1$. Since p is an $\mathbf{O2}$-fibration we can extend β to a morphism $\gamma : \widehat{B}_n \to X$. We thus obtain a commutative diagram

$$\begin{tikzcd} \widehat{B}_n \arrow[r, "\gamma"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^{n+1} \arrow[r, "E(\sigma)"'] & S \end{tikzcd}$$

We can now consider the pullback diagram diagram

$$\begin{tikzcd} \Lambda^n_n \arrow[r] \arrow[d] & \Delta^n \arrow[d, "d_{n+1}"] \\ \widehat{B}_n \arrow[r] & \Delta^{n+1} \end{tikzcd}$$

and observe that, since γ maps the final edge of Λ^n_n to e and maps the triangle $\Delta^{\{0,n-1,n\}}$ to a thin triangle, it follows from the fact that e is p-Cartesian that we have a commutative diagram

$$\begin{tikzcd} \Lambda^{n+1}_n \arrow[r, "\varepsilon"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^{n+1} \arrow[r, "E(\sigma)"'] \arrow[ur, dotted] & S \end{tikzcd}$$

Since ε maps $\Delta^{\{n-1,n,n+1\}}$ to a thin triangle, the dotted arrow in the diagram exists. This arrow is the desired morphism $E(f)$, completing the proof. □

Corollary 4.21 *Let $p : X \to S$ be a $\mathbf{O2}$-fibration and let $\sigma : \Delta^2 \to X$ be a thin 2-simplex as pictured below*

$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "h"] & & c \end{tikzcd}$$

Suppose that g is strongly p-Cartesian. Then f is strongly p-Cartesian if and only if h is strongly p-Cartesian.

Proof By Proposition 4.20 it will suffice to prove the claim replacing strongly p-Cartesian with simply p-Cartesian. This is shown in Lemma 2.3.8 and Lemma 2.3.9 in [9], which provide some of the expected limited 2-out-of-3 properties for Cartesian edges. □

Corollary 4.22 *Let $p : X \to S$ be a* **O2***-fibration. Then an edge $e : b \to c$ in X is strongly p-Cartesian if and only if for every object $a \in X$ post-composition with e induces a homotopy pullback diagram*

$$\begin{tikzcd} X(a,b) \arrow[r, "e\circ-"] \arrow[d] & X(a,c) \arrow[d] \\ S(p(a),p(b)) \arrow[r, "p(e)\circ-"] & S(p(a),p(c)) \end{tikzcd}$$

Proof Combine Proposition 4.20 with [9, Prop. 2.3.3]. □

Proposition 4.23 *Let $p : X \to S$ be an* **O2***-fibration. Given a pair of objects $a, b \in X$ and an (strongly) p-Cartesian edge $e : a' \to b$ such that $p(a) = p(a')$ we have a pullback diagram in $\mathcal{C}\mathrm{at}_\infty$*

$$\begin{tikzcd} X_{p(a)}(a,a') \arrow[r, "e\circ-"] \arrow[d] & X(a,b) \arrow[d] \\ \Delta^0 \arrow[r, "p(e)"] & S(p(a),p(b)) \end{tikzcd}$$

Proof Let $X_{p(e)} \to \Delta^1$ denote the pullback of $X \to S$ along the map selecting the edge $p(e)$. We claim that we have a pullback diagram of simplicial sets

$$\begin{tikzcd} X_{p(e)}(a,b) \arrow[r] \arrow[d] & X(a,b) \arrow[d] \\ \Delta^0 \arrow[r, "p(e)"] & S(p(a),p(b)) \end{tikzcd}$$

Let $\sigma : \Delta^n \to X_{p(e)}(a,b)$ with associated $(n+1)$-simplex $\overline{\sigma} : \Delta^{n+1} \to X_{p(e)}$. We note that the composite

$$\kappa : \Delta^{n+1} \xrightarrow{\overline{\sigma}} X_{p(e)} \to \Delta^1 \xrightarrow{p(e)} S$$

defines an degenerate $(n+1)$-simplex in S. We can further see that κ represents a simplex $\Delta^n \to S(p(a), p(b))$ which is degenerate on the object $p(e)$. This proves the existence of the commutative diagram above. It is immediate to see that every simplex $\Delta^n \to X(a,b)$ whose image on $S(p(a), p(b))$ is degenerate on $p(e)$ factors through $X_{p(e)}(a,b)$ which implies that the diagram in question is in fact, a pullback diagram.

We observe that it follows from Proposition 4.6 that the right-most vertical map is a coCartesian fibration. This in turn implies that this diagram is a pullback diagram in $\mathcal{C}\mathrm{at}_\infty$. Therefore to show our claim we need to verify that the induced morphism

$$X_{p(a)}(a,a') \to X_{p(e)}(a,b)$$

is an equivalence of ∞-categories. It is immediate to check that $X_{p(a)}(a,a') = X_{p(e)}(a,a')$. The claim now follows from Corollary 4.22. □

Definition 4.24 Let $p : X \to S$ be an **O2**-fibration. We say that p is an **O2C**-fibration if for every edge $e : s \to p(x)$ in S there exists a p-Cartesian lift $\hat{e} : \Delta^1 \to X$ such that $p(\hat{e}) = e$.

Remark 4.25 The terminology **O2C**-fibration is reminiscent to the alreay defined notion of outer 2-Cartesian fibration. We will show that both definitions are equivalent whenever S is an ∞-bicategory in Theorem 4.27.

Corollary 4.26 *Let $p : X \to S$ be an* **O2C***-fibration and let $p^c : X^c \to S$ denote the restriction to p to the simplicial subset X^c consisting only in simplices whose triangles are in C_X. Then p^c is an* outer Cartesian fibration *in the sense of [9]. In particular, p is an outer Cartesian fibration if and only if all of its triangles belong to C_X.*

Theorem 4.27 *Let $p : X \to S$ be a locally fibred map equipped with a coCartesian family of triangles C_X. Let E_X denote the collection of p-cartesian edges. Then p is a* **O2C***-fibration if and only if the map $(X, E_X, T_X \subseteq C_X) \to (S, \sharp, T_S \subset \sharp)$ is a 2-Cartesian fibration in the sense of Definition 3.22.*

Proof Let us suppose that p is a **O2C**-fibration. We need to show that p has the right lifting property with respects to the maps of Definition 3.7. The only cases that are not hardcoded into the definitions are: (S1), (S2), (S4), (S5), and (E). (S1) follows from Corollary 4.21, (S2) follows from Lemma 4.10, (S4) follows from Lemma 4.13, (S5) follows from Lemma 4.17 and finally (E) follows from Corollary 4.22. The converse is clear. □

Proposition 4.28 *Suppose we are given a morphism of 2-Cartesian fibrations*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

Then the following statements are equivalent:

i) For every $s \in S$ the map $f_s : X_s \to Y_s$ is a bicategorical equivalence.
ii) The map f is a bicategorical equivalence.

Proof The implication (i) $\implies$ (ii) is a direct consequence of Proposition 3.38. To prove the converse let us $u, v \in X$ such that $p(u) = p(v) = s$. Then it follows from Proposition 4.23 that we can identify the morphism

$$X_s(u, v) \to Y_s(f(u), f(v))$$

with the the fibre over id_s of the map

$$\begin{tikzcd} X(u, v) \arrow[rr, "f_{uv}"] \arrow[dr] & & Y(f(u), f(v)) \arrow[dl] \\ & S(s, s) & \end{tikzcd}$$

Since f is a bicategorical equivalence it follows (see [13, Theorem 4.2.2]) that f_{uv} is a categorical equivalence and we can use [11, Prop. 3.3.1.5] to show that the map f_s is fully faithful. To finish the proof we will show that f_s is essentially surjective. Let $y \in Y_s$ and pick $x \in X$ together with an equivalence $\alpha : f(x) \to y$. Let us pick an inverse to $p(\alpha)$ namely $\gamma : s \to p(x)$ and a p-Cartesian lift of γ which we call $\beta : \hat{x} \to x$. It is easy to see that β must be an equivalence. To finish the proof we can assemble $f(\beta)$ and α into a Λ^2_1 and construct a extension to $\sigma : \Delta^2 \to X$ such that the edge $\Delta^{\{0,2\}}$ belongs to Y_s. □

4.1 Fibrations of Simplicially Enriched Categories

Definition 4.29 We say that a Set_Δ^+-enriched category $\mathcal{C}$ is a $\mathcal{C}\mathrm{at}_\infty$-category if it is a fibrant object in the model structure of Set_Δ^+-enriched categories.

Proposition 4.30 *Let $f : \mathcal{C} \to \mathcal{D}$ be a fibration of $\mathcal{C}\mathrm{at}_\infty$-categories and recall the functor $N^{\mathbf{sc}} : \mathrm{Cat}_{\mathrm{Set}_\Delta^+} \to \mathrm{Set}_\Delta^{\mathbf{sc}}$ from Definition 2.12. Then the map*

$$N^{\mathbf{sc}}(f) : N^{\mathbf{sc}}(\mathcal{C}) \to N^{\mathbf{sc}}(\mathcal{D})$$

is a functorially fibred morphism if and only if the following hold:

i) For every $x, y \in \mathcal{C}$ the map $\mathcal{C}(x, y) \to \mathcal{D}(f(x), f(y))$ is a coCartesian fibration of ∞-categories.

ii) Let $x, y, z \in \mathcal{C}$ and consider a pair of coCartesian edges $e_1 : \Delta^1 \to \mathcal{C}(x, y)$ and $e_2 : \Delta^1 \to \mathcal{C}(y, z)$. Then the composite

$$\Delta^1 \xrightarrow{e_1 \times e_2} \mathcal{C}(x, y) \times \mathcal{C}(y, z) \longrightarrow \mathcal{C}(x, y)$$

defines a coCartesian edge in the target.

Proof Observe that since $N^{\mathbf{sc}}$ is a right Quillen functor it follows it follows that $N^{\mathbf{sc}}(f)$ is a fibration in the model structure on scaled simplicial sets. In particular, it is a weak $\mathbf{S}$-fibration. We will show that condition (i) is satisfied if and only if $N^{\mathbf{sc}}(f)$ is locally fibred and that condition ii) is satisfied if and only if the collection of coCartesian triangles is functorial.

Let us suppose that f is functorially fibred and recall the Set_Δ^+-categories $\mathbb{O}^n$ for $n \geq 0$ defined Definition 2.10. Given a simplex $\Delta^n \to \mathcal{C}(x, y)$, we define a Set_Δ^+-category by means of the pushout

$$\begin{array}{ccc} \mathbb{O}^n & \longrightarrow & \mathbb{O}^0 \\ \downarrow & & \downarrow \\ \mathbb{O}^{n+1} & \longrightarrow & \mathbb{Q}^n \end{array}$$

where the left-most horizontal morphism is induced by the map of posets $d_{n+1} : [n] \to [n+1]$. We construct a morphism $\hat{l}_\sigma : \mathbb{O}^{n+1} \to \mathcal{C}$ as follows:

- On objects we set $\hat{l}_\sigma(i) = x$ if $0 \leq i \leq n$ and $\hat{l}_\sigma(n+1) = y$.
- Given $0 \leq i \leq j \leq n$ the morphism $\mathbb{O}^{n+1}(i, j) \to \mathcal{C}(x, x)$ is constant on the identity on x and similarly for $\mathbb{O}^{n+1}(n+1, n+1) \to \mathcal{C}(y, y)$.
- The morphism $\mathbb{O}^{n+1}(i, n+1) \to \mathcal{C}(x, y)$ for $0 \leq i \leq n$ factors through

$$\mathbb{O}^n(i, n+1) \to \Delta^n \xrightarrow{\sigma} \mathcal{C}(x, y)$$

  where the first morphism sends $S \subseteq [n+1]$ to $\max(S \setminus \{n+1\}) \in \Delta^n$.

It is easy to see that our definition of $\hat{l}_\sigma$ factors through the pushout producing a morphism $l_\sigma : \mathbb{Q}^n \to \mathcal{C}$. Since $\mathfrak{C}^{\mathbf{sc}}$ is a left adjoint it follows that $\mathbb{Q}^n \simeq \mathfrak{C}^{\mathbf{sc}}[\Delta^{n+1} \coprod_{\Delta^n} \Delta^0]$.

Suppose we are given a lifting problem

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "u"] \arrow[d] & \mathcal{C}(x, y) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathcal{D}(f(x), f(y)) \end{tikzcd}$$

and let $e : \Delta^1 \to \mathcal{C}(x, y)$ denote the restriction of u to $\Delta^{\{0,1\}} \subset \Lambda^n_0$. Let us further suppose that the morphism $Q^2 \xrightarrow{l_e} \mathcal{C}$ corresponds to a left-degenerate coCartesian triangle in $N^{\mathbf{sc}}(\mathcal{C})$. We will show that we can construct the dotted arrow in the diagram. We define $\mathbb{Q}^n_0 = \mathfrak{C}^{\mathbf{sc}}[\Lambda^{n+1}_0 \coprod_{\Delta^n} \Delta^0]$ and observe that we can construct another commutative diagram

$$\begin{tikzcd} \mathbb{Q}^n_0 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "f"] \\ \mathbb{Q}^n \arrow[r] \arrow[ur, dotted, "\omega"] & \mathcal{D} \end{tikzcd}$$

which admits a solution since its adjoint lifting problem admits one. The definition of the top horizontal morphism is induced by our construction l_σ applied to the left-horn. We provide a solution to the original lifting problem by considering the simplex

$$\Delta^n \xrightarrow{\iota} \mathbb{Q}^n(0, n+1) \xrightarrow{\omega} \mathcal{C}(x, y)$$

where ι sends the vertex i to the subset $[0, i]$. An analogous argument as before shows that we can produce coCartesian lifts of morphisms in the base. We conclude that $i)$ holds.

Let $\sigma : \Delta^2 \to N^{\mathbf{sc}}(\mathcal{C})$ be a left degenerate simplex whose adjoint morphism $\mathbb{Q}^2 \to \mathcal{C}$ defines a coCartesian edge. Let $n \geq 3$ and consider a lifting problem

$$\begin{tikzcd} \Lambda^n_0 \coprod\limits_{\Delta^{\{0,1\}}} \Delta^0 \arrow[r] \arrow[d] & N^{\mathbf{sc}}(\mathcal{C}) \arrow[d] \\ \Delta^n \coprod\limits_{\Delta^{\{0,1\}}} \Delta^0 \arrow[r] & N^{\mathbf{sc}}(\mathcal{D}) \end{tikzcd}$$

It follows from unraveling the definitions that we only need to solve the adjoint lifting problem

$$\begin{tikzcd} \mathbb{P}^n_0 = \mathfrak{C}^{\mathbf{sc}}[\Lambda^n_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0](*, n) \arrow[r] \arrow[d] & \mathcal{C}(x, y) \arrow[d] \\ \mathbb{P}^n = \mathfrak{C}^{\mathbf{sc}}[\Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0](*, n) \arrow[r] & \mathcal{D}(f(x), f(y)) \end{tikzcd}$$

where $*$ denotes the collapsed vertex where the vertices 0, 1 get mapped onto. We identify $\mathbb{P}^n$ with the nerve of the poset of subsets $S \subseteq [n]$ such that $\min(S) = 0$ and $\max(S) = n$ ordered by inclusion. It follows that $\mathbb{P}^n_0 \subset \mathbb{P}^n$ is the simplicial subset consisting in those simplices $\sigma : \Delta^k \to \mathbb{P}^n$ represented by a chain of inclusions $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_k$ satisfying at least one of the the following conditions:

- There exists $1 < j \leq n-1$ such that $j \in S_i$ for $0 \leq i \leq k$.
- There exists $1 \leq j \leq n-1$ such that $j \notin S_i$ for $0 \leq i \leq k$.

Note that we can view $\mathbb{P}^n$ geometrically as a $(n-1)$-dimensional cube. Then it follows that $\mathbb{P}^n_0$ is the union of all of the $(n-2)$-dimensional faces of $\mathbb{P}^n$ except the face consisting in subsets S such that $1 \in S$. We will further equipp both simplicial sets with a marking given by the edge $0n \to 01n$. Since the image of that particular edge is a coCartesian edge in $\mathcal{C}(x, y)$ it will sufice to show that the inclusion $\mathbb{P}^n_0 \to \mathbb{P}^n$ is an anodyne morphism in the coCartesian model structure.

Let $\sigma_i : S^i_0 \subset S^i_1 \subset \cdots \subset S^i_{n-1}$ for $i = 1, 2$ be a pair of *distinct* non-degenerate simplices of maximal dimension. Observe that by maximality $S^i_0 = 0n$ for $i = 1, 2$. Let

$0 \leq \nu \leq n-2$ be the first index such that $S^1_\nu \neq S^2_\nu$. We say $\sigma_1 < \sigma_2$ if and only if $\max(S^1_\nu \setminus \{n\}) < \max(S^2_\nu \setminus \{n\})$. Let us considered the totally ordered set of non-degenerate simplices of maximal dimension $\{\sigma_1 < \sigma_2 \cdots \sigma_{n!}\}$. We can now produce a filtration

$$\mathbb{P}^n_0 \to X_{n!} \to X_{n!-1} \to \cdots X_2 \to X_1 = \mathbb{P}^n$$

where $X_j \subset \mathbb{P}^n$ consists in those simplices ρ that either factor through $\mathbb{P}^n_0$ or are contained in a non-degenerate simplex of maximal dimension $\rho \subset \sigma_\ell$ for $\ell \geq j$. The proof is by now routine and left as an exercise to the reader.

To finish the proof we will show that condition (ii) holds if and only if the collection of coCartesian triangles is functorial. Let us suppose that (ii) holds and consider a simplex $\rho_i : \Delta^3 \to N^{\mathbf{sc}}(\mathcal{C})$ such that the image triangle $\{i-1, i, i+1\}$ is thin in $N^{\mathbf{sc}}(\mathcal{C})$. Let us assume that every triangle of Δ^3 except the i-th face corresponds via the adjoint map $\alpha_i : \mathbb{O}^3 \to \mathcal{C}$ to a coCartesian edge in the mapping space of $\mathcal{C}$. We consider a pair of commutative diagrams

$$\begin{tikzcd} 013 \arrow[r] & 0123 \\ 03 \arrow[u] \arrow[r] & 023 \arrow[u, "\simeq"'] \end{tikzcd} \qquad \begin{tikzcd} 013 \arrow[r, "\simeq"] & 0123 \\ 03 \arrow[u] \arrow[r] & 023 \arrow[u] \end{tikzcd}$$

that we interpret as the image of the morphism $\mathbb{O}^3(0,3) \to \mathcal{C}(\alpha_i(0), \alpha_i(3))$ for $i = 1, 2$. We have circled in both diagrams the coCartesian edges and denoted by "$\simeq$" the equivalences associated to the thin triangle $\{i-1, i, i+1\}$. Note that in the first diagram the edge $013 \to 0123$ can be obtained from the coCartesian edge $13 \to 123$ via precomposition with a degenerate edge. Our assumptions then imply that $013 \to 0123$ is coCartesian and thus the whole diagram must consist of coCartesian edges. Since the edge $03 \to 023$ corresponds to the face missing 1 of ρ_1 the claim holds. The argument for the second diagram is totally analogous.

To finish the proof let us suppose that the collection of coCartesian triangles is functorial. Let $x, y, z \in \mathcal{C}$, we claim that in order to show that the map

$$\gamma_{x,y} : \mathcal{C}(x, y) \times \mathcal{C}(y, z) \to \mathcal{C}(x, z)$$

preserves coCartesian edges it suffices to prove the particular cases where one of the two morphisms we want to compose is degenerate. Indeed, given $e : \Delta^1 \to \mathcal{C}(x, y) \times \mathcal{C}(y, z)$ determined by a pair of edges $(f \to g, u \to v)$ we can produce a 2-simplex

$$\theta : \Delta^2 \to \mathcal{C}(x, z),\ \theta : u \circ f \to v \circ f \to v \circ g$$

such that $d_1(\theta) = \gamma_{x,y}(e)$ and where $d_i(\theta)$ is given by a composition with a degenerate edge for $i = 0, 2$.

Let $f \to g$ be a coCartesian edge in $\mathcal{C}(x, y)$ and let u be an object of $\mathcal{C}(y, z)$. We consider a map $\tau : \mathbb{O}^3 \to \mathcal{C}$ defined as follows:

- We have $\tau(0) = \tau(1) = x$, $\tau(2) = y$ and $\tau(3) = z$.
- The map $\mathbb{O}^3(0, 1) \to \mathcal{C}(x, x)$ is degenerate on the identity morphism.
- The map $\mathbb{O}^3(0, 2) \to \mathcal{C}(x, y)$ selects the morphism $f \to g$
- The map $\mathbb{O}^3(1, 2) \to \mathcal{C}(x, y)$ selects the object g.
- The map $\mathbb{O}^3(2, 3) \to \mathcal{C}(y, z)$ selects the object u.
- The map $\mathbb{O}^3(1, 3) \to \mathcal{C}(x, z)$ selects the degenerate edge on $u \circ g$.
- The map $\mathbb{O}^3(0, 3) \to \mathcal{C}(x, z)$ factors as $\mathbb{O}^3(0, 3) \xrightarrow{\pi} \Delta^1 \to \mathcal{C}(x, y)$ where the second morphism selects the edge $u \circ f \to u \circ g$ and the first morphism is determined by $\pi(03) = \pi(023) = 0$ and $\pi(013) = \pi(0123) = 1$.

It follows that the adjoint map $\kappa : \Delta^3 \to N^{\mathbf{sc}}(\mathcal{C})$ maps the triangle $\{1, 2, 3\}$ to a thin triangle in $N^{\mathbf{sc}}(\mathcal{C})$ and that every triangle of κ gets mapped to a coCartesian triangle except possible the face missing 2. Since the collection of coCartesian triangles is functorial it follows that $\{0, 1, 3\}$ is also a coCartesian triangle. This shows that $u \circ f \to u \circ g$ must be a coCartesian edge in $\mathcal{C}(x, z)$. We leave the completely analogous verification that precomposition with degenerate edges preserves coCartesian edges as an exercise for the reader.

Definition 4.31 Let $f : \mathcal{C} \to \mathcal{D}$ be a map of $\mathcal{C}\mathrm{at}_\infty$-categories. An edge $e : x \to y$ is said to be f-Cartesian if for every $z \in \mathcal{C}$ the following diagram

$$\begin{tikzcd} \mathcal{C}(z, x) \arrow[r] \arrow[d] & \mathcal{C}(z, y) \arrow[d] \\ \mathcal{D}(f(z), f(x)) \arrow[r] & \mathcal{D}(f(z), f(y)) \end{tikzcd}$$

is a homotopy pullback square in Set_Δ^+.

The next theorem follows readily from Proposition 4.30.

Theorem 4.32 *Let $f : \mathcal{C} \to \mathcal{D}$ be a fibration of $\mathcal{C}\mathrm{at}_\infty$-categories. Then $N^{\mathbf{sc}}(f)$ is a 2-Cartesian fibration if and only if the following conditions hold:*

i) For every $x, y \in \mathcal{C}$ the map $\mathcal{C}(x, y) \to \mathcal{D}(f(x), f(y))$ is a coCartesian fibration of ∞-categories.

ii) Let $x, y, z \in \mathcal{C}$ and consider a pair of coCartesian edges $e_1 : \Delta^1 \to \mathcal{C}(x, y)$ and $e_2 : \Delta^1 \to \mathcal{C}(y, z)$. Then the composite

$$\Delta^1 \xrightarrow{e_1 \times e_2} \mathcal{C}(x, y) \times \mathcal{C}(y, z) \longrightarrow \mathcal{C}(x, y)$$

defines a coCartesian edge in the target.

iii) For every morphism $e : d \to f(y)$ in $\mathcal{D}$. There exists an f-Cartesian lift $\hat{e} : \hat{d} \to y$ with $f(\hat{e}) = e$.

Remark 4.33 We say that a functor of 2-categories $f : \mathbb{C} \to \mathbb{D}$ is a 2-Cartesian fibration if and only if $\underline{\mathrm{N}}(f)$ (see Definition 2.9) satisfies the conditions of Theorem 4.32. It follows that this definition (after taking the pertinent duals) recovers the notion of 2-fibration presented in [5]. In particular, it follows from Theorem 4.32 that our definition generalises the classical notion of a 2-fibration to the realm of ∞-bicategories.

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