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GEOMETRY OF NONTRANSITIVE GRAPHS

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We study nontransitive graphs and prove a number of results when they satisfy a coarse version of transitivity. Also, for each finitely generated group G, we produce continuum many pairwise non-quasiisometric regular graphs that have the same growth rate, number of ends, and asymptotic dimension as G.

1. Introduction

Woess [1991] asked the following natural question: does every transitive graph "look like" a Cayley graph? To be more precise, is every connected, locally finite, vertex-transitive graph quasiisometric to a Cayley graph of some finitely generated group? Diestel and Leader [2001] constructed infinite, vertex-transitive graphs of exponential growth, denoted $DL(m, n)$, and conjectured that these graphs would provide a negative answer to Woess' question. Eskin, Fisher, and Whyte [2012; 2013] confirmed this by demonstrating that $DL(m, n)$ is not quasiisometric to any finitely generated group when $m \neq n$. They also constructed a class of nonunimodular, three-dimensional, solvable, nonnilpotent Lie groups that do not admit a nonpositively curved, left-invariant metric, and showed that these groups are not quasiisometric to any finitely generated group.

The spaces in both of the above collections all have exponential volume growth, which leads us to ask: Does there exist a simply connected nilpotent Lie group or a vertex-transitive graph of polynomial growth that is not quasiisometric to any finitely generated group? The answer is yes for simply connected nilpotent Lie groups. Recall that there are uncountably many quasiisometry classes of simply connected nilpotent Lie groups. (In particular, there are uncountably many quasiisometry classes of Carnot groups.) Note that a finitely generated group which is quasiisometric to a simply connected nilpotent Lie group is virtually nilpotent by Gromov's polynomial growth theorem [Gromov 1981], and hence is quasiisometric to a finitely generated nilpotent group. Now, all finitely generated nilpotent groups are finitely presented. So whenever a simply connected nilpotent Lie group is quasiisometric

to a finitely generated group, it must lie in the quasiisometry class of some finitely presented group. Since there are uncountably many quasiisometry classes of simply connected nilpotent Lie groups, and only countably many finitely presented groups, there must be uncountably many simply connected nilpotent Lie groups which are not quasiisometric to any finitely generated group. On the other hand, the answer is no for locally finite vertex-transitive graphs of polynomial growth. Trofimov [1984] proved that such graphs are quasiisometric to finitely generated nilpotent groups.

Given the above discussion, we choose to look beyond the world of vertex-transitive graphs. Since the class of non-vertex-transitive graphs is so large, one expects that there are many graphs with geometric properties that are not shared by Cayley graphs of finitely generated groups. Thus, we aim to find a class of non-vertex-transitive graphs that are as close as possible to being quasiisometric to finitely generated groups. We start our discussion by considering a class of graphs that satisfy a coarse notion of vertex-transitivity. To this end, we introduce the following definition. We say that a graph X is coarsely transitive if there exists a constant $K \geq 1$ such that for any two vertices x and y, there exists a (K, K)-quasiisometry $X \to X$ which maps x to within K-distance of y. One can see that every vertex-transitive graph is coarsely transitive, but conversely, one can construct a coarsely transitive graph which is not vertex-transitive by starting with any vertex-transitive graph and attaching a new vertex to it. Thus, one may ask what properties of vertex-transitive graphs pass to coarsely transitive graphs. With this in mind, we come to our first result.

Corollary 5. *Let X be a connected, locally finite graph. If X is coarsely transitive and has two ends, then it is quasiisometric to $\mathbb{Z}$.*

One may view this result as the coarsely transitive generalization of the well known fact, due to Freudenthal and Hopf, that a finitely generated group with two ends is necessarily a finite extension of $\mathbb{Z}$. Freudenthal and Hopf also proved that finitely generated groups can only have zero, one, two, or infinitely many ends. Our next theorem shows that this phenomenon generalizes to coarsely transitive graphs.

Theorem 7. *Let X be a connected, locally finite graph. If X is coarsely transitive, then it has zero, one, two, or infinitely many ends.*

Although every locally finite vertex-transitive graph of polynomial growth is quasiisometric to a finitely generated nilpotent group, we do find continuum many locally finite regular graphs with integral degree of polynomial growth, one or two ends, and finite asymptotic dimension, which are not quasiisometric to any finitely generated group.

Theorem 13. *Given an infinite, locally finite, connected, vertex-transitive graph X, there exist continuum many pairwise non-quasiisometric 3-regular graphs that have the same growth rate, number of ends, and asymptotic dimension as X.*

In particular, for any infinite, finitely generated nilpotent group G, there exist continuum many pairwise non-quasiisometric 3-regular graphs that have the same degree of polynomial growth, number of ends, and asymptotic dimension as G.

Taking G to be the discrete Heisenberg group, Theorem 13 implies that there are continuum many pairwise non-quasiisometric 3-regular graphs with one end, polynomial growth rate of order four, and asymptotic dimension equal to three.

The proof of Theorem 13 proceeds by attaching line segments to our base Cayley graph X along an infinite geodesic ray in the following way. After fixing a base point and a parameter $\alpha \in (0, 1]$, we attach a segment of length $\lceil \log(n)^\alpha \rceil$ to the vertex on the ray at distance n^2 from the ray's endpoint. Calling this graph X_α, we then demonstrate that the image of any quasiisometric embedding of X into X_α lies in a bounded neighborhood of $X \subset X_\alpha$. Since the attached segments along the ray grow without bound, it then follows that X_α and X are not quasiisometric. Moreover, the parameter α controls the growth rate of the attached line segments in such a way that X_α and X_β are not quasiisometric for distinct α, β in $(0, 1]$. On the other hand, since the attached segments are sparse and grow slowly in length, the graphs X_α share several large-scale geometric properties with X.

2. Notation and basic definitions

For a metric space X, we use $d(x, y)$ to denote the distance between x and y. We denote the r-ball about x by $B_X(x, r)$ and the r-sphere about x in X by $S_X(x, r)$. When the metric space X is clear from context, we simply write $B(x, r)$ and $S(x, r)$.

Let $f : (X, d_X) \to (Y, d_Y)$ be a map of metric spaces. We say that f is an (L, A)-**quasiisometric embedding** if there are constants $L \geq 1$, $A \geq 0$ such that for every $a, b \in X$,

$$\tfrac{1}{L} d_X(a, b) - A \leq d_Y(f(a), f(b)) \leq L d_X(a, b) + A.$$

An (L, A)-quasiisometric embedding f is an (L, A)-**quasiisometry** if there is an (L', A')-quasiisometric embedding $g : Y \to X$ such that $d_X(g \circ f, \mathrm{Id}_X) < \infty$ and $d_Y(f \circ g, \mathrm{Id}_Y) < \infty$, and we call g a **quasiinverse** of f. Equivalently, an (L, A)-quasiisometric embedding f is an (L, A)-quasiisometry if it is **coarsely surjective**, that is, if there is a $C \geq 0$ such that the image of f is C-dense in Y. A map $f : X \to Y$ is a **quasiisometry** between X and Y if it is an (L, A)-quasiisometry for some $L \geq 1$, $A \geq 0$. Two metric spaces X and Y are **quasiisometric** if there exists a quasiisometry between them.

A **graph** is a pair of sets $X = (V, E)$ where $E \subset V \times V$. We call V the set of **vertices** and E the set of **edges**. We denote the vertices of a graph X as $V(X)$ and the edges of a graph as $E(X)$. Given an edge $\{x, y\}$, we call x and y the endpoints of $\{x, y\}$ and we say that x and y are adjacent. A **graph isomorphism**

between graphs X and Y is a bijection $f : V(X) \to V(Y)$ such that x and y are adjacent in $V(X)$ if and only if $f(x)$ and $f(y)$ are adjacent in $V(Y)$. A **graph automorphism** of a graph X is a graph isomorphism from X to itself. A graph is **vertex-transitive** (or simply **transitive**) if its automorphism group acts transitively on its vertices.

A graph is **connected** if any two vertices can be connected by a path. For any connected graph X, a natural metric is induced on the set of vertices by defining the distance between two vertices as the length of a shortest path between them. Since we are mainly interested in viewing graphs as metric spaces, we use the symbol X to denote both the graph and the corresponding metric space. If $S \subset V(X)$, then the **subgraph of X induced by** S is the graph whose vertex set is S and whose edge set is the subset of edges in $E(X)$ that have both endpoints in S. We reuse the symbol S to denote this induced subgraph, and we use $X \setminus S$ to denote the subgraph of X induced by $V(X) \setminus S$. For the entirety of this note, unless stated otherwise, we assume graphs are connected, locally finite, and unbounded as metric spaces.

Next we recall the definitions of some large-scale geometric properties of graphs. For a graph X and subgraph S, let $U(X, S)$ denote the set of unbounded connected components of $X \setminus S$. Letting X be a connected graph, we define the **number of ends of X** to be

$$e(X) = \sup\{|U(X, B)| : B \text{ is a bounded subgraph of } X\}.$$

Note that in particular, a graph has zero ends if and only if it is bounded. We also give an equivalent definition in terms of rays. A **ray** in X is a semiinfinite simple path; that is, it is an infinite sequence of vertices $v_0, v_1, \ldots$ such that each vertex appears at most once in the sequence and every two consecutive vertices are adjacent. Two rays r_1 and r_2 are said to be equivalent if there is a ray r_3 that contains infinitely many of the vertices in each of r_1 and r_2. This defines an equivalence relation on the set of rays in X. Then the **ends** of X are defined to be the equivalence classes of rays in X, and $e(X)$ is equal to the cardinality of the set of ends of X.

Let X be a metric space, and let $n \geq 0$ be an integer. We say that $\mathrm{asdim}(X) \leq n$ if for every $R \geq 1$ there is a uniformly bounded cover $\mathcal{U}$ of X such that every ball in X of radius R intersects at most $n + 1$ elements of $\mathcal{U}$ (here $\mathcal{U}$ is uniformly bounded if $\sup_{U \in \mathcal{U}} \mathrm{diam}(U) < \infty$). Then the **asymptotic dimension** of X, denoted by $\mathrm{asdim}(X)$, is the smallest integer $n \geq 0$ such that $\mathrm{asdim}(X) \leq n$. If no such n exists, we define $\mathrm{asdim}(X) = \infty$.

Let $f, g : \mathbb{N} \to \mathbb{N}$ be increasing functions. We write $f \preceq g$ if there is a $c \in \mathbb{N}$ such that $f(n) \leq cg(cn + c)$ for all $n \in \mathbb{N}$. If $f \preceq g$ and $g \preceq f$, then we write $f \approx g$ and say that f and g are (asymptotically) equivalent. Note that $\approx$ defines an equivalence relation on the set of increasing functions $\mathbb{N} \to \mathbb{N}$. Suppose that X is an unbounded, locally finite graph, and fix a vertex $x_0 \in X$. Let $f_{X,x_0} : \mathbb{N} \to \mathbb{N}$ be defined by

$f_{X,x_0}(n) = |B_X(x_0, n)|$. Observe that if x_1 is another vertex and $c = d(x_0, x_1)$, then $B(x_1, n) \subset B(x_0, n+c)$. So $f_{X,x_1}(n) \leq f_{X,x_0}(n+c)$, which means $f_{X,x_1} \preceq f_{X,x_0}$. By symmetry we get $f_{X,x_0} \approx f_{X,x_1}$. Hence the equivalence class of f_{X,x_0}, which we define to be the **growth rate** of X, does not depend on the choice of x_0. Thus we can talk about the growth function f_X of X, which is well-defined up to equivalence. In particular, two graphs X and Y are said to have the same growth rate if their growth functions f_X and f_Y are equivalent, that is, $f_X \approx f_Y$. If $f_X \preceq n^d$ for some integer $d \geq 0$, then we say that X has a **polynomial growth rate**. In this case, the smallest d for which $f_X \preceq n^d$ is called the **order of polynomial growth**.

Recall that growth rate, number of ends, and asymptotic dimension are invariant under quasiisometry.

3. Coarsely transitive graphs

Let $K \geq 1$. A graph X is said to be K-**coarsely transitive** if for any pair of vertices x, y in X, there exists a (K, K)-quasiisometry $f : X \to X$ such that $d(f(x), y) \leq K$. A graph is said to be **coarsely transitive** if it is K-coarsely transitive for some $K \geq 1$. Note that all transitive graphs are coarsely transitive.

Suppose X is K-coarsely transitive, and take vertices x and y in X. By definition, there is a (K, K)-quasiisometry $f : X \to X$ with $d(f(x), y) \leq K$. Define $f' : X \to X$ by $f'(x) = y$ and $f'(z) = f(z)$ for all $z \neq x$. Then f' is a $(K, 2K)$-quasiisometry with $f'(x) = y$. Hence, we obtain the following lemma.

Lemma 1. *A metric space X is coarsely transitive if and only if there is some $K \geq 1$ such that for any pair of vertices x, y in X, there is a (K, K)-quasiisometry $f : X \to X$ with $f(x) = y$.*

Moving forward, we take K-coarsely transitive to mean this equivalent condition.

First, we observe a basic obstruction to coarse transitivity. By an abuse of notation, we use $\mathbb{Z}^d$ to denote the Cayley graph of $\mathbb{Z}^d$ with respect to the standard symmetric generating set. Let $B_n = B(0, n)$ be the subgraph of $\mathbb{Z}$ induced by the vertex set $\{k \in \mathbb{Z} : |k| \leq n\}$.

Lemma 2. *Let $L \geq 1$ and $A \geq 0$ be given. For all sufficiently large n, if $f : B_n \to \mathbb{Z}$ is an (L, A)-quasiisometric embedding, then either $f(-n) < f(0) < f(n)$ or $f(n) < f(0) < f(-n)$.*

Proof. Letting $n > L^2 + 2LA$, we have that for each $k = -n, -n+1, \ldots, 0$,

$$d(f(n), f(k)) \geq \tfrac{1}{L}(n-k) - A \geq \tfrac{1}{L}n - A > L + A.$$

Similarly, $d(f(-n), f(k)) > L + A$ for each $k = 0, 1, \ldots, n$. On the other hand, for each $k = -n, -n+1, \ldots, n-1$, we have

$$d(f(k), f(k+1)) \leq L + A.$$

First, suppose that $f(-n) < f(0)$. If $m \leq f(-n)$, then

$$d(m, f(0)) \geq d(f(-n), f(0)) > L + A.$$

Since $d(f(1), f(0)) \leq L + A$, we must have $f(-n) < f(1)$. By induction we get $f(-n) < f(n)$. Now, if $f(n) < f(0)$, then by a similar argument, we would get $f(n) < f(-n)$ which contradicts $f(-n) < f(n)$. Thus, $f(0) < f(n)$, and we have $f(-n) < f(0) < f(n)$, as desired.

Now suppose that $f(0) < f(-n)$. Then by a symmetric argument, we get $f(n) < f(0) < f(-n)$. $\qquad \square$

Let T_n be the subgraph of $\mathbb{Z}^2$ induced by the vertex set $\{(k, 0) : |k| \leq n\} \cup \{(0, k) : 0 \leq k \leq n\}$. Then T_n can be thought of as a tripod with legs of length n.

Proposition 3. *Let $L \geq 1$ and $A \geq 0$ be given. Then for all sufficiently large n, there does not exist an (L, A)-quasiisometric embedding $T_n \to \mathbb{Z}$.*

Proof. The union of any two legs of T_n is isometric to B_n. Therefore, T_n contains three distinct subgraphs, S_1, S_2, and S_3, each of which is isometric to B_n. Suppose for contradiction that there is an (L, A)-quasiisometric embedding $f : T_n \to \mathbb{Z}$. Then for $i = 1, 2, 3$, the restriction $f|_{S_i}$ is an (L, A)-quasiisometric embedding of $S_i \cong B_n$ into $\mathbb{Z}$. Using the previous lemma, we may assume without loss of generality that $f(-n, 0) < f(0, 0) < f(n, 0)$. Then $f(-n, 0) < f(0, 0)$ implies $f(0, 0) < f(0, n)$, and $f(0, 0) < f(n, 0)$ implies $f(0, n) < f(0, 0)$. This is impossible, so no such f exists. $\qquad \square$

Hence, a graph which has arbitrarily large parts which coarsely look like T_n and B_n cannot be coarsely transitive. For example, the subgraph of $\mathbb{Z}^2$ induced by the vertex set $\{(x, y) : |y| \leq |x|\}$ is not coarsely transitive, because for each $K \geq 1$, there is an $n \gg K$ such that no (K, K)-quasiisometry which maps $(n, 0)$ to $(0, 0)$ exists.

It is known that connected transitive graphs which are unbounded have either one, two, or infinitely many ends [Diestel et al. 1993], and moreover that two-ended transitive graphs are quasiisometric to $\mathbb{Z}$ [Miraftab and Rühmann 2018]. We show that these two properties extend to coarsely transitive graphs.

Proposition 4. *Let X be a coarsely transitive graph with at least two ends, and let $B_0 = B(x_0, r)$ be a ball with $|U(X, B_0)| \geq 2$. Then there is an $R > 0$ such that for any ball B of radius R, we have $|U(X, B)| \geq |U(X, B_0)|$.*

Proof. Suppose X is K-coarsely transitive. Set $R = Kr + 3K^2$, and let $B_1 = B(x_1, R)$ where x_1 is arbitrary. Since X is K-coarsely transitive, there is a (K, K)-quasiisometry $f : X \to X$ with $f(x_1) = x_0$. We will show that f induces a surjection $U(X, B_1) \to U(X, B_0)$. First, we observe that

$$(1) \qquad y \notin B_1 \implies d(x_0, f(y)) = d(f(x_1), f(y)) \geq \tfrac{1}{K} d(x_1, y) - K$$
$$> \tfrac{1}{K} R - K = r + 2K.$$
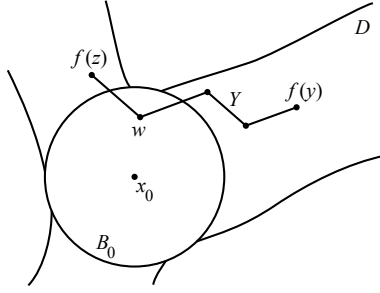
Figure 1. γ must intersect B_0.

That is, f maps the complement of B_1 to the complement of $B(x_0, r + 2K)$. Let $C \in U(X, B_1)$, and pick any $y \in C$. Then $f(y) \notin B_0$. Hence, let D be the connected component of $X \setminus B_0$ which contains $f(y)$. We now show that $f(C) \subset D$. Let $z \in C$ be adjacent to y. Then

$$d(f(y), f(z)) \leq Kd(y, z) + K = 2K.$$

Let γ be a path of minimal length in X between $f(y)$ and $f(z)$. If $f(z) \notin D$, then there must exist a point $w \in \gamma \cap B_0$ (see Figure 1).

In this case,

$$d(x_0, f(y)) \leq d(x_0, w) + d(w, f(y)) \leq r + \text{length}(\gamma) = r + d(f(y), f(z)) \leq r + 2K,$$

contrary to (1). Therefore, $f(z) \in D$. Since C is connected, it follows that $f(C) \subset D$. Since C is unbounded and f is a quasiisometry, D must be unbounded. Therefore, $D \in U(X, B_0)$, and we let $F(C) = D$. Thus we get a well-defined map $F : U(X, B_1) \to U(X, B_0)$ which we will show is surjective. Let $D \in U(X, B_0)$, and let g be a quasiinverse of f. Without loss of generality we may assume that $g(x_0) = x_1$. Like before, take a ball $B_2 = B(x_0, R')$ of sufficiently large radius R' such that

$$x \notin B_2 \implies d(x_1, g(x)) > R + 2K.$$

Then by similar reasoning used before, g maps each element of $U(X, B_2)$ into an element of $U(X, B_1)$. Since D is an unbounded component of $X \setminus B_0$, and B_2 is just a bounded neighborhood of B_0, there must be a $D' \in U(X, B_2)$ with $D' \subset D$. Then like before, $g(D') \subset C$ for some $C \in U(X, B_1)$. Since f and g are quasiinverses and D' is unbounded, there is a point in C that f maps into $D' \subset D$. Hence $f(C) \subset D$, and therefore $F(C) = D$. Thus, F is surjective, and $|U(X, B_1)| \geq |U(X, B_0)|$. Since B_1 was an arbitrary ball of radius R, we are done. $\qquad \square$

When X is two-ended, we get the following corollary.

Corollary 5. *If X is a coarsely transitive graph with $e(X) = 2$, then there is an $R > 0$ such that every ball B of radius R satisfies $|U(X, B)| = 2$.*
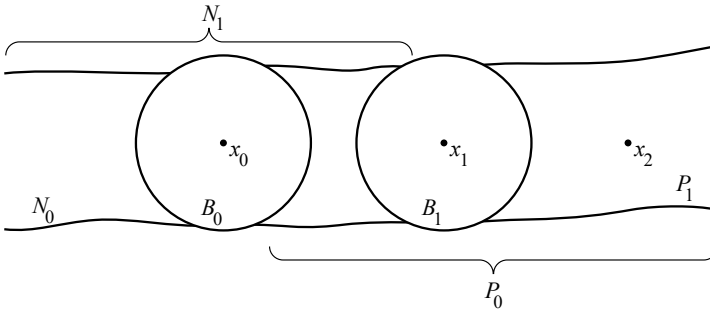
Figure 2. Constructing a sequence of balls.

Proof. Since $e(X) = 2$, there is a ball B_0 with $|U(X, B_0)| = 2$. Thus, Proposition 4 implies that there is an $R > 0$ such that every ball B of radius R has $|U(X, B)| \geq |U(X, B_0)| = 2$. On the other hand, since $e(X) = 2$, every such ball B has $|U(X, B)| \leq 2$. □

We first show that a two-ended, coarsely transitive graph X is quasiisometric to $\mathbb{Z}$. By Corollary 5, any ball of sufficiently large radius (independent of the center point) will roughly separate X into two unbounded components. Thus, we may construct a biinfinite, pairwise-disjoint sequence of such balls, and this sequence of balls will look like the integers when viewing X from afar. Indeed, the map which sends the integers to the centers of the balls will be the desired quasiisometry.

Theorem 6. *Let X be a coarsely transitive graph. If $e(X) = 2$, then X is quasiisometric to $\mathbb{Z}$.*

Proof. Using Corollary 5, let $r > 0$ such that any ball B of radius r satisfies $|U(X, B)| = 2$. Fix a vertex x_0 and let $B_0 = B(x_0, r)$. Let P_0 and N_0 denote the two elements of $U(X, B_0)$. Pick a vertex $x_1 \in P_0$ with $d(x_0, x_1) = 2r + 1$. Let $B_1 = B(x_1, r)$ and note that $B_1 \subset P_0$. Then $N_0 \cup B_0$ is an unbounded connected subgraph of $X \setminus B_1$ and thus must be contained in one of the two elements of $U(X, B_1)$. Let $N_1 \in U(X, B_1)$ denote the component containing $N_0 \cup B_0$, and let $P_1 \in U(X, B_1)$ denote the other element. Since $N_0 \cup B_0 \subset N_1$, it follows that $P_1 \subset P_0$. Then pick a vertex $x_2 \in P_1$ with $d(x_1, x_2) = 2r + 1$, and similarly define B_2, P_2, and N_2 (see Figure 2).

We continue this process, as well as a symmetric process in the direction of N_0 instead of P_0, to construct x_k, B_k, P_k, and N_k for all $k \in \mathbb{Z}$, such that

- $d(x_k, x_{k+1}) = 2r + 1$,

- $B_{k+1} \cup P_{k+1} \subset P_k$,

- $N_k \cup B_k \subset N_{k+1}$.
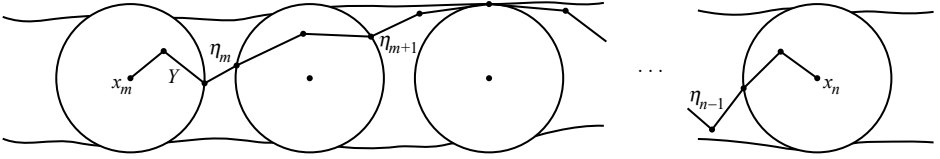
Figure 3. The subpaths η_j.

Now, consider the map $\mathbb{Z} \to X$ given by $k \mapsto x_k$. We show that this is a bi-Lipschitz embedding. Let $m, n \in \mathbb{Z}$ with $m < n$. By the triangle inequality,

$$d(x_m, x_n) \leq d(x_m, x_{m+1}) + \cdots + d(x_{n-1}, x_n)$$
$$= (2r + 1) + \cdots + (2r + 1) = (2r + 1)(n - m).$$

For the other inequality, let γ be a path between x_m and x_n of minimal length. Since $B_{k+1} \cup P_{k+1} \subset P_k$ and $N_k \cup B_k \subset N_{k+1}$, we have $x_n \in P_j$ and $x_m \in N_j$ for each $j = m + 1, \ldots, n - 1$. Since γ is a path between N_j and P_j, it must intersect B_j. Hence, for each $j = m, \ldots, n - 1$, γ contains a subpath η_j between B_j and B_{j+1}, and since γ has minimal length, the η_j have nonoverlapping edges (see Figure 3).

Since the B_k are pairwise disjoint, length$(\eta_j) \geq 1$. Hence

$$d(x_m, x_n) = \text{length}(\gamma) \geq \sum_{j=m}^{n-1} \text{length}(\eta_j) \geq \sum_{j=m}^{n-1} 1 = n - m.$$

It remains to show that our given map is coarsely surjective. For each $k \in \mathbb{Z}$, let γ_k be a path between x_k and x_{k+1} of length $d(x_k, x_{k+1}) = 2r + 1$, and define L to be the union of the γ_k.

Let $x \in X$ be arbitrary and let $B = B(x, r)$. Let $\delta = d(x, x_0)$, and let $N \geq 3r + \delta + 1$ be such that $x_N \notin B$. Let C be the component of $X \setminus B$ which contains x_N. Suppose for contradiction that $\{x_k\}_{k \geq N}$ is not entirely contained in C, and let $n > N$ be the smallest index with $x_n \notin C$. Thus, $x_{n-1} \in C$, and we let γ be a path of minimal length between x_{n-1} and x_n. Since $x_n \notin C$, γ must intersect B at a point, say, w. Then

$$d(x_n, x_0) \leq d(x_n, w) + d(w, x) + d(x, x_0) \leq (2r + 1) + r + \delta = 3r + \delta + 1,$$

and

$$d(x_n, x_0) \geq n - 0 > N \geq 3r + \delta + 1,$$

which is a contradiction. Thus, $\{x_k\}_{k \geq N} \subset C$ which means that $C \in U(X, B)$. Similarly, we may assume (by taking possibly larger N) that $\{x_k\}_{k \leq -N}$ is contained in an element of $U(X, B)$. Let $m > N$ be such that $B_m \subset C$. If $\{x_k\}_{k \leq -N} \subset C$, then we would have $|U(X, B \cup B_m)| \geq 3$ (see Figure 4).

This would contradict $e(X) = 2$; therefore $\{x_k\}_{k \leq -N}$ must be contained in the other unbounded component of $X \setminus B$. Since x_{-N} and x_N are in separate components
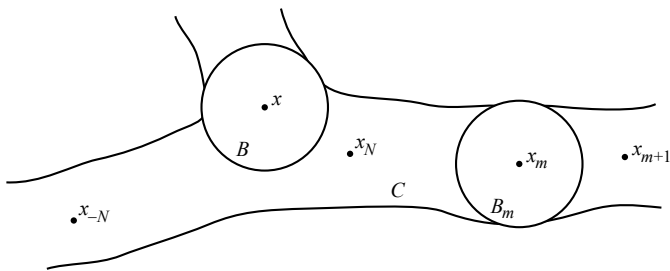
Figure 4. x_{-N} cannot be in the same component as x_N.
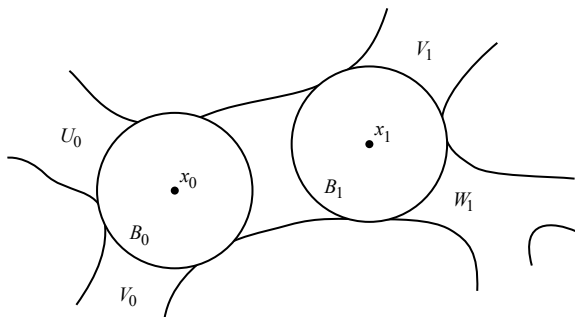
Figure 5. An infinite branching.

of $X \setminus B$, every path between them must intersect B. In particular, L must intersect B at some point v. Then $v \in \gamma_k$ for some $k \in \mathbb{Z}$, and therefore

$$d(x_k, x) \le d(x_k, v) + d(v, x) \le (2r + 1) + r = 3r + 1.$$

Since x was chosen arbitrarily, coarse surjectivity follows. $\square$

We finish this section with the coarsely transitive generalization of the classification of ends of transitive graphs.

Theorem 7. *A coarsely transitive graph has either zero, one, two, or infinitely many ends.*

Proof. Let X be a coarsely transitive graph with more than two ends. By Proposition 4, there is an $r > 0$ such that any ball B of radius r satisfies $|U(X, B)| \ge 3$. Fix a vertex $x_0 \in X$, and let $B_0 = B(x_0, r)$. Then $U(X, B_0)$ has at least three elements, which we call U_0, V_0, W_0. Pick a vertex $x_1 \in W_0$ with $d(x_0, x_1) = 2r + 1$. Let $B_1 = B(x_1, r)$ and note that $B_1 \subset W_0$. Then $B_0 \cup U_0 \cup V_0$ is an unbounded connected subgraph of $X \setminus B_1$, and hence is contained in an element, say U_1, of $U(X, B_1)$. Let V_1 and W_1 denote two other elements of $U(X, B_1)$. Then $U_0, V_0, V_1, W_1 \in U(X, B_0 \cup B_1)$ which implies that $|U(X, B_0 \cup B_1)| \ge 4$ (Figure 5).

Continuing in this way, we pick a vertex $x_2 \in W_1$, and consider $B_2 = B(x_2, r)$. We then find that $|U(X, B_0 \cup B_1 \cup B_2)| \geq 6$. Hence, we construct pairwise-disjoint balls B_k for all $k \in \mathbb{N}$ so that for any $M > 0$, there is an $n \in \mathbb{N}$ for which $|U(X, \bigcup_{k=0}^n B_k)| > M$. Thus, $e(X) = \infty$. $\qquad\square$

4. Quasiisometry classes of graphs of polynomial growth

Our goal for this section is to show that given an infinite, locally finite, connected, transitive graph X, there exist continuum many 3-regular graphs that are pairwise non-quasiisometric and yet share several large-scale geometric properties. We define a **geodesic** P in X to be a biinfinite path such that for any two vertices x, y on P, P contains a shortest-length path between x and y. In other words, $d(x, y) = d_P(x, y)$ where d is the path metric on X and d_P is the restriction of d to P. By Theorem 4.1 in [Watkins 1986], every vertex in X lies on a geodesic. Thus, take any geodesic P in X and label its vertices by $\{x_n\}_{n \in \mathbb{Z}}$ such that $d(x_m, x_n) = |m - n|$ for all $m, n \in \mathbb{Z}$. We construct a family of graphs from X as follows. For each $0 < \alpha \leq 1$, define $g_\alpha : (0, \infty) \to \mathbb{N}$ by $g_\alpha(x) = \lceil \log(x)^\alpha \rceil$ and note that $g_\alpha(n) \leq n$ for all $n \in \mathbb{N}$. For each positive integer n, let S_n^α be the subgraph of $\mathbb{Z}$ induced by the vertex set $\{k : 0 \leq k \leq g_\alpha(n)\}$. Define

$$X_\alpha := \left(X \sqcup \bigsqcup_{n > 0} S_n^\alpha \right) / \sim,$$

where for each positive integer n, we identify $x_{n^2} \in X$ with $0 \in S_n^\alpha$. From now on, we denote the vertex $k \in S_n^\alpha \subset X_\alpha$ by k_n. For example, 0_n and x_{n^2} denote the same vertex in X_α, and $g_\alpha(n)_n$ denotes the "tip" of the segment S_n^α in X_α. To reduce clutter, we define $t_n^\alpha = g_\alpha(n)_n$ for all $n > 0$ (see Figure 6).

For the remainder of this section, X will denote an infinite, locally finite, connected, transitive graph, and if $0 < \alpha \leq 1$, then X_α will denote the graph we constructed from X above.

Proposition 8. *The graphs X and X_α have the same number of ends, asymptotic dimension, and growth rates.*
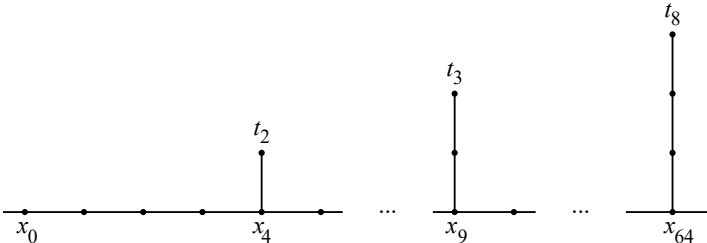
Figure 6. The graph X_α when $X = \mathbb{Z}$ and $\alpha = 1$.

Proof. Recall that the ends of a graph are given by equivalence classes of rays (semiinfinite paths with no self-intersection). Let r be an arbitrary ray in X_α. Since rays are infinite and do not have repeating vertices, by possibly removing an initial finite subpath of r we obtain a ray r' which intersects each S_n^α at most once at $0_n = x_{n^2} \in X$. So r' is a ray in $X \subset X_\alpha$. Hence, each ray in X_α is equivalent to a ray in X, and thus, the ends of X_α are identified with the ends of X. Therefore, $e(X_\alpha) = e(X)$.

Let P be the geodesic in X with respect to which X_α is defined, and consider the subgraph of X_α defined by

$$Y := P \cup \bigcup_{n>0} S_n^\alpha.$$

For example, if $X = \mathbb{Z}$, then Y is all of X_α. We claim that Y isometrically embeds into the 3-regular tree T. Indeed, let γ_0 be a geodesic in T with vertices $\{v_k\}_{k\in\mathbb{Z}}$, and map P isometrically onto γ_0. Then for each $n \in \mathbb{N}$, since T is a 3-regular tree, we may take a (necessarily geodesic) ray γ_n, which emanates from v_n and does not intersect γ_0 elsewhere. Note that since T is a tree, the rays γ_n are pairwise disjoint. Then we may isometrically embed S_n^α into γ_n with the condition that $0 \in S_n^\alpha$ is mapped to v_n. The result is an isometric embedding of Y into T. Since trees have asymptotic dimension 1, we have $\mathrm{asdim}(Y) \le 1$. Then

$$\mathrm{asdim}(X_\alpha) = \mathrm{asdim}(X \cup Y) \le \max\{\mathrm{asdim}(X), \mathrm{asdim}(Y)\} = \mathrm{asdim}(X).$$

On the other hand, $\mathrm{asdim}(X) \le \mathrm{asdim}(X_\alpha)$ because $X \subset X_\alpha$. Thus $\mathrm{asdim}(X_\alpha) = \mathrm{asdim}(X)$.

Lastly, we show that X and X_α have the same rate of growth. First, note that since $X \subset X_\alpha$, we have $|B_X(x_0, n)| \le |B_{X_\alpha}(x_0, n)|$. Next, we show that $S_{X_\alpha}(x_0, n)$ has at most one more element than $S_X(x_0, n)$, in which case,

$$|B_{X_\alpha}(x_0, n)| = 1 + \sum_{i=1}^n |S_{X_\alpha}(x_0, i)| \le 1 + \sum_{i=1}^n (|S_X(x_0, i)| + 1)$$

$$= |B_X(x_0, n)| + n \le 2|B_X(x_0, n)|,$$

where $n \le |B_X(x_0, n)|$ because $\{x_1, \ldots, x_n\} \subset B_X(x_0, n)$. First note that $d(x_0, j_m) < d(x_0, k_m)$ for all $m > 0$ and $j < k \le g_\alpha(m)$. Moreover,

$$d(x_0, t_m^\alpha) = m^2 + g_\alpha(m) \le m^2 + m = m(m+1) < (m+1)^2 = d(x_0, 0_{m+1}).$$

Since $|S_{X_\alpha}(x_0, 1)| = |S_X(x_0, 1)|$, we let $n > 1$, and let m^2 be the largest square such that $m^2 \le n$. Note that the intersection of $S_{X_\alpha}(x_0, n)$ with $X \subset X_\alpha$ is $S_X(x_0, n)$. Since $d(x_0, 0_m) \le n$, it follows from the above observations that $S_{X_\alpha}(x_0, n)$ does not intersect S_k^α for any $k < m$. Also, since $d(x_0, 0_{m+1}) > n$, $S_{X_\alpha}(x_0, n)$ does not intersect S_k^α for any $k > m$. Finally, $S_{X_\alpha}(x_0, n)$ intersects S_m^α once if $m^2 \le n \le$

$m^2 + g_\alpha(m)$, and otherwise, the intersection is empty. Hence, $|S_{X_\alpha}(x_0, n)| \leq |S_X(x_0, n)| + 1$. Thus

$$|B_X(x_0, n)| \leq |B_{X_\alpha}(x_0, n)| \leq 2|B_X(x_0, n)|,$$

which implies that X and X_α have the same rate of growth. □

While X and X_α share some large-scale geometric properties, it turns out by the following proposition that they are not quasiisometric to each other.

Proposition 9. *If* $f : X \to X_\alpha$ *is a quasiisometric embedding, then*

$$\sup_{x \in X} d(f(x), X) < \infty.$$

Proof. Suppose $f : X \to X_\alpha$ is an (L, A)-quasiisometric embedding. Assume for contradiction that for some $x \in X$,

$$d(f(x), X) > L^3 + 2L^2A + A.$$

Let $P = \{x_n\}_{n \in \mathbb{Z}}$ be the geodesic with respect to which X_α is defined. Since X is transitive, we may assume without loss of generality that $x = x_0$. Then $f(x_0) \in S_k^\alpha$ for some k, but since S_k^α is bounded, there must be an $m < 0$ such that $f(x_m) \notin S_k^\alpha$ and $f(x_{m+1}) \in S_k^\alpha$, and an $n > 0$ such that $f(x_{n-1}) \in S_k^\alpha$ and $f(x_n) \notin S_k^\alpha$ (see Figure 7).

We then have that

$$\begin{aligned}
d(f(x_m), f(x_n)) &\leq d(f(x_m), 0_k) + d(0_k, f(x_n)) \\
&\leq d(f(x_m), f(x_{m+1})) + d(f(x_{n-1}), f(x_n)) \\
&\leq (L + A) + (L + A) = 2L + 2A.
\end{aligned}$$
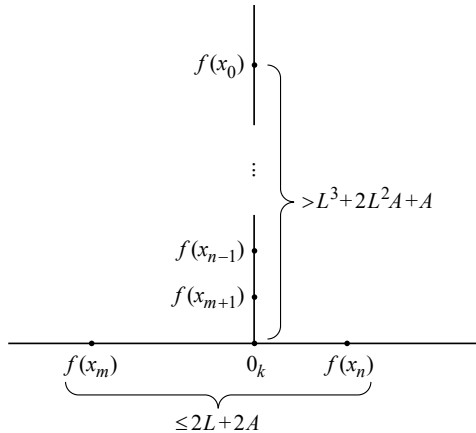
Figure 7. X cannot be pulled too high up a single S_k^α.

For all $y \in X$ with $d(x_0, y) \leq L^2 + 2LA$, it follows that

$$d(f(x_0), f(y)) \leq L(L^2 + 2LA) + A = L^3 + 2L^2A + A.$$

Since $f(x_m), f(x_n) \notin S_k^\alpha$ and $d(f(x_0), X) > L^3 + 2L^2A + A$, it follows that $|m|, n > L^2 + 2LA$, and hence $n - m > 2L^2 + 4LA$. Thus

$$d(f(x_m), f(x_n)) > \tfrac{1}{L}(2L^2 + 4LA) - A = 2L + 3A,$$

which is a contradiction. Therefore, $\sup_{x \in X} d(f(x), X) \leq L^3 + 2L^2A + A$.   □

Corollary 10. *X and X_α are not quasiisometric for any $0 < \alpha \leq 1$.*

Proof. By Proposition 9, the image of any quasiisometric embedding $X \to X_\alpha$ lies in a bounded neighborhood of the base graph $X \subset X_\alpha$. But in X_α, since the lengths of the segments S_n^α increase without bound, the distance from the tips t_n^α of those segments to the base $X \subset X_\alpha$ grow arbitrarily large. Thus, any quasiisometric embedding $X \to X_\alpha$ cannot be coarsely surjective.   □

Our next result states that furthermore, X_α and X_β are not quasiisometric if $\alpha \neq \beta$. The proof uses a similar argument. We show that any quasiisometric embedding $X_\alpha \to X_\beta$, for $\alpha < \beta$, fails to be coarsely surjective. If $f : X_\alpha \to X_\beta$ is a quasiisometric embedding, then Proposition 9 implies that f maps the base graph $X \subset X_\alpha$ to a neighborhood of the base graph $X \subset X_\beta$. Hence, the segments S_n^α in X_α must be coarsely mapped to the segments S_n^β in X_β. Now, the lengths of S_n^β grow faster than the lengths of S_n^α, but since the consecutive distances between the S_n^α grow quadratically, and f distorts distances up to a fixed linear factor, we will see that the distances between the tips t_n^β in X_β and $f(X_\alpha)$ grow arbitrarily large.

For the proof, we first need a lemma.

Lemma 11. *Let $0 < \alpha < \beta$. If $T : \mathbb{R} \to \mathbb{R}$ is affine and $p \in \mathbb{R}[x]$ is a polynomial such that $p > 0$ on $(0, \infty)$, then*

$$\lim_{x \to \infty} \frac{T\big(g_\alpha(p(x))\big)}{g_\beta(x)} = 0.$$

Proof. Firstly, $\log(x)^{\alpha - \beta} \to 0$ as $x \to \infty$; thus $g_\alpha(x)/g_\beta(x) \to 0$ as $x \to \infty$. Let $d = \deg p$. Since $\lim_{x \to \infty} \log(p(x))/\log(x) \to d$, we have that $g_\alpha(p(x))/g_\alpha(x) \to d^\alpha$ as $x \to \infty$. Hence

$$\lim_{x \to \infty} \frac{g_\alpha(p(x))}{g_\beta(x)} = \lim_{x \to \infty} \left(\frac{g_\alpha(p(x))}{g_\alpha(x)} \cdot \frac{g_\alpha(x)}{g_\beta(x)} \right)$$

$$= \lim_{x \to \infty} \frac{g_\alpha(p(x))}{g_\alpha(x)} \cdot \lim_{x \to \infty} \frac{g_\alpha(x)}{g_\beta(x)} = d^\alpha \cdot 0 = 0.$$

Since T is affine and $g_\beta(x) \to \infty$ as $x \to \infty$, we obtain the desired equality.   □

Proposition 12. *For $0 < \alpha, \beta \leq 1$, if $\alpha \neq \beta$, then X_α and X_β are not quasiisometric.*

Proof. Assume $\alpha < \beta$, and let $f : X_\alpha \to X_\beta$ be an (L, A)-quasiisometric embedding. Since f is arbitrary, we are done if we show that f is not coarsely surjective. Recall that for $0 < \gamma \leq 1$, we denote the "tip" of the segment $S_n^\gamma \subset X_\gamma$ by t_n^γ. Let $M > 0$. We show that if n is sufficiently large, then $d(f(X_\alpha), t_n^\beta) > M$.

Since $f|_X$ is a quasiisometric embedding $X \to X_\beta$, [Proposition 9](#) implies there is a $D > 0$ such that $\sup_{x \in X} d(f(x), X) \leq D$. By [Lemma 11](#), we have

$$\lim_{n \to \infty} \frac{Lg_\alpha(L(n + 2n^2 + A)) + A}{g_\beta(n)} \to 0.$$

In particular, there is an $N > 0$ such that for all $n \geq N$, we have

$$g_\beta(n) > Lg_\alpha(L(n + 2n^2 + A)) + A + (M + D).$$

Fix an integer $n \geq N$ large enough so that $d(f(x_0), 0_n) \leq 2n^2$, and set

$$R = L(n + d(f(x_0), 0_n) + A).$$

Then for all $x \in X_\alpha$ with $d(x_0, x) > R$, we have that

$$d(f(x_0), f(x)) \geq \tfrac{1}{L} d(x_0, x) - A > n + d(f(x_0), 0_n)$$
$$\geq g_\beta(n) + d(f(x_0), 0_n) \geq d(f(x_0), t_n^\beta),$$

where the last inequality is strict when $f(x_0) \in S_n^\beta \setminus \{0_n\}$; otherwise equality holds. We now claim that $d(f(x), t_n^\beta) > M$. Suppose for contradiction that $d(f(x), t_n^\beta) \leq M$. Since $d(t_n^\beta, X) = g_\beta(n) > M + D$ and $d(f(x_0), X) \leq D$, we must have that $d(f(x_0), t_n^\beta) > M$. It follows that $f(x)$ lies on S_n^β and in particular, $f(x)$ lies on every path from $f(x_0)$ to t_n^β. This implies $d(f(x_0), f(x)) \leq d(f(x_0), t_n^\beta)$ which is a contradiction.

It remains to show that $d(f(x), t_n^\beta) > M$ for all $x \in X_\alpha$ with $d(x_0, x) \leq R$. Observe that

$$B_{X_\alpha}(x_0, R) \subset B_X(x_0, R) \cup \bigcup_{i=1}^{\lfloor \sqrt{R} \rfloor} S_i^\alpha.$$

If $x \in B_X(x_0, R)$, then we have $x \in X$ and thus, $d(f(x), X) \leq D$. That implies $d(f(x), t_n^\beta) > M$ because $g_\beta(n) > M + D$. On the other hand, if $x \in S_k^\alpha$, where $k^2 \leq R$, then

$$d(f(0_k), f(x)) \leq Ld(0_k, x) + A \leq Lg_\alpha(k) + A$$
$$\leq Lg_\alpha(R) + A \leq Lg_\alpha(L(n + 2n^2 + A)) + A.$$

Therefore

$$d(f(x), t_n^\beta) \geq d(f(0_k), t_n^\beta) - d(f(0_k), f(x))$$
$$\geq (g_\beta(n) - D) - (Lg_\alpha(L(n + 2n^2 + A)) + A) > M.$$
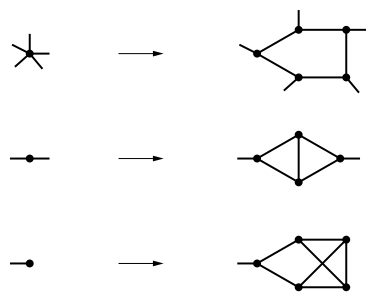
Figure 8. Replacements to obtain a 3-regular graph [Bermudo et al. 2013, p. 1582].

Thus, $d(f(x), t_n^\beta) > M$ for all $x \in X_\alpha$ with $d(x_0, x) \le R$. We have now shown that $d(f(x), t_n^\beta) > M$ for all $x \in X_\alpha$. Since M can be arbitrarily large, f is not coarsely surjective. $\square$

Theorem 13. *Given an infinite, locally finite, connected, transitive graph X, there exist continuum many pairwise non-quasiisometric 3-regular graphs that have the same growth rate, number of ends, and asymptotic dimension as X.*

In particular, given any infinite, finitely generated nilpotent group G, there exist continuum many pairwise non-quasiisometric 3-regular graphs that have the same degree of polynomial degree of growth, number of ends, and asymptotic dimension as G.

Proof. By Proposition 8, each element of the set $\{X_\alpha : 0 < \alpha \le 1\}$ has the same number of ends, asymptotic dimension, and growth rate as X. By Proposition 12, they are all in distinct quasiisometry classes. To show that X_α is quasiisometric to a 3-regular graph we mimic the argument used to prove Theorem 19 in [Bermudo et al. 2013]. The main idea of the proof is to replace vertices of degree $\neq 3$ with certain structures, as seen in Figure 8.

The graphs considered in [Bermudo et al. 2013] allow for edges of any length, so the replacing structures are defined with appropriate edge lengths in order to have bounded diameters. In our setting all edges have fixed length 1, but since X is transitive, the vertices of X_α have bounded degree. As a result, the replacing structures for us (with all edge lengths 1) do have bounded diameters. Thus we obtain a 3-regular graph that is quasiisometric to X_α. $\square$

5. Further questions

We finish this article by discussing some open questions concerning the geometry of coarsely transitive graphs. We start with the following question.

Question 1. Does there exist a coarsely transitive graph that is not quasiisometric to a transitive graph? If so, can we ensure that it is locally finite?

From the definition of vertex-transitivity, we have that the automorphism group of a transitive graph is always nontrivial. A similar property holds for coarsely transitive graphs in that the group of quasiisometries is always nontrivial. Thus, if one could construct a coarsely transitive graph where every graph in its quasiisometry class has a trivial automorphism group, one would have an answer to the above question. In particular, such a graph would have coarse symmetries but no actual symmetries.

Another question one may consider is whether there are examples of polynomially growing graphs, which fail to be quasiisometric to any finitely generated group, but have more symmetries than the ones found in Theorem 13. After all, one can see that by Proposition 3, the constructed graphs X_α are not even coarsely transitive. Therefore, we have the following question.

Question 2. Does there exist a locally finite, coarsely transitive graph of polynomial growth and finite asymptotic dimension that is not quasiisometric to any finitely generated group?

In light of Trofimov's result, a positive answer to Question 2 would provide a positive answer to Question 1. If such a graph exists, it would be the most symmetric one could hope to have for a locally finite graph of polynomial growth which fails to be quasiisometric to a finitely generated group.

References

[Bermudo et al. 2013] S. Bermudo, J. M. Rodríguez, J. M. Sigarreta, and J.-M. Vilaire, "Gromov hyperbolic graphs", *Discrete Math.* **313**:15 (2013), 1575–1585. MR Zbl

[Diestel and Leader 2001] R. Diestel and I. Leader, "A conjecture concerning a limit of non-Cayley graphs", *J. Algebraic Combin.* **14**:1 (2001), 17–25. MR Zbl

[Diestel et al. 1993] R. Diestel, H. A. Jung, and R. G. Möller, "On vertex transitive graphs of infinite degree", *Arch. Math.* (*Basel*) **60**:6 (1993), 591–600. MR Zbl

[Eskin et al. 2012] A. Eskin, D. Fisher, and K. Whyte, "Coarse differentiation of quasi-isometries, I: spaces not quasi-isometric to Cayley graphs", *Ann. of Math.* (2) **176**:1 (2012), 221–260. MR Zbl

[Eskin et al. 2013] A. Eskin, D. Fisher, and K. Whyte, "Coarse differentiation of quasi-isometries, II: rigidity for Sol and lamplighter groups", *Ann. of Math.* (2) **177**:3 (2013), 869–910. MR Zbl

[Gromov 1981] M. Gromov, "Groups of polynomial growth and expanding maps", *Publ. Math. de l'IHÉS* **53**:1 (1981), 53–78. Zbl

[Miraftab and Rühmann 2018] B. Miraftab and T. Rühmann, "Two-ended quasi-transitive graphs", 2018. Zbl arXiv 1812.04866

[Trofimov 1984] V. I. Trofimov, "Graphs with polynomial growth", *Mat. Sb.* (*N.S.*) **123(165)**:3 (1984), 407–421. In Russian; translated in *Sb. Math.* **51**:2 (1985), 405–417. MR Zbl

[Watkins 1986] M. E. Watkins, "Infinite paths that contain only shortest paths", *J. Combin. Theory Ser. B* **41**:3 (1986), 341–355. MR Zbl

[Woess 1991] W. Woess, "Topological groups and infinite graphs", *Discrete Math.* **95**:1-3 (1991), 373–384. MR Zbl

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