A COMBINATORIAL DESCRIPTION OF THE LOSS LEGENDRIAN KNOT INVARIANT

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Abstract. In this note, we observe that the hat version of the Heegaard Floer invariant of Legendrian knots in contact three-manifolds defined by Lisca-Ozsváth-Stipsicz-Szabó can be combinatorially computed. We rely on Plamenevskaya's combinatorial description of the Heegaard Floer contact invariant.

1. Introduction

Since the construction of Heegaard Floer invariants introduced by Ozsváth and Szabó [OS04b], many mathematicians have been studying its computational aspects. Although the definition of Heegaard Floer homology involves analytic counts of pseudo-holomorphic disks in the symmetric product of a surface, certain Heegaard Floer homologies admit a purely combinatorial description. Sarkar and Wang [SW10] showed that any closed, oriented three-manifold admits a nice Heegaard diagram, in which the holomorphic disks in the symmetric product can be counted combinatorially. Using grid diagrams to represent knots in S^3, Manolescu, Ozsváth, and Sarkar gave a combinatorial description of knot Floer homology [MOS09]. Given a connected, oriented four-dimensional cobordism between two three-manifolds, under additional topological assumptions, Lipshitz, Manolescu, and Wang [LMW08] present a procedure for combinatorially determining the rank of the induced Heegaard Floer map on the hat version.

For three-manifolds Y equipped with a contact structure ξ, Ozsváth and Szabó [OS05] associate a homology class $\mathbf{c}(\xi) \in \widehat{HF}(-Y)$ in the Heegaard Floer homology which is an invariant of the contact manifold. In [Pla07], Plamenevskaya provides a combinatorial description of the Heegaard Floer contact invariant [OS05], by applying the Sarkar-Wang algorithm to the Honda-Kazez-Matić description [HKM09b] of the contact invariant. Throughout this paper, we use Floer homology with coefficients in $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

Theorem 1.1 ([Pla07])**.** Given a contact 3-manifold (Y, ξ), the Heegaard Floer contact invariant $\mathbf{c}(\xi) \in \widehat{HF}(-Y)$ can be computed combinatorially.

A Legendrian knot L in a contact 3-manifold (Y, ξ) is a knot which is everywhere trangent to the contact plane field ξ. Lisca, Ozsváth, Stipsicz, and Szabó define invariants for Legendrian knots inside a contact 3-manifold [LOSS09], colloquially referred to as the LOSS invariants. The definition uses open book decompositions and extend the Honda-Kazez-Matić interpretation of the contact invariant. The LOSS invariants are the homology class of a special cycle in the knot Floer homology

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of the Legendrian knot (see Theorem 2.2 for details). It is natural to ask whether the LOSS invariants admit a combinatorial description.

In the case of Legendrian knots inside the standard tight contact 3-sphere, Ozsváth, Szabó, and Thurston define the GRID invariants [OST08], which are special classes in the knot Floer homology of the Legendrian. The GRID invariants are inherently combinatorial for all versions of Heegaard Floer homology (plus, hat, and minus), since they are defined using grid diagrams. In comparison, by relying on open book decompositions, the LOSS invariants are defined for Legendrian knots in arbitrary contact 3-manifolds, encompassing much greater generality than their GRID counterparts. On the other hand, in this paper we only prove that the hat version for the LOSS Legendrian knot invariant is combinatorial.

Baldwin, Vela-Vick, and Vertesi [BVVV13] prove that the Legendrian invariants and the GRID invariants in knot Floer homology agree for Legendrian knots in the tight contact 3-sphere. They introduce an invariant, called BRAID, for transverse knots and Legendrian knots L inside a tight contact 3-sphere $(S^3, \xi_{\mathrm{std}})$, which recovers both the LOSS and GRID invariants.

For Legendrian and transverse links in universally tight lens spaces, Tovstopyat-Nelip [TN19] uses grid diagrams to define invariants which generalize the GRID invariants of [OST08] and agree with the BRAID and LOSS invariants defined in [BVVV13] and [LOSS09]. These invariants are combinatorial, but the contact three-manifolds are limited to universally tight lens spaces.

We study the question of whether the Legendrian and transverse link invariants in arbitrary contact 3-manifolds of [LOSS09] admit a combinatorial description. In the case of the hat version of the LOSS invariant, we adapt Plamenevskaya's [Pla07] proof of Theorem 1.1 to obtain a combinatorial description of $\widehat{\mathfrak{L}}(L)$.

Theorem 1.2. Given a contact 3-manifold (Y, ξ) and a null-homologous Legendrian knot $L \subset Y$, the LOSS invariant $\widehat{\mathfrak{L}}(L)$ can be computed combinatorially.

Proof. By the proof of Theorem 2.3, there exists a doubly-pointed Heegaard diagram for the Legendrian knot $(-Y, L)$ which is a nice diagram in the sense of Sarkar-Wang [SW10]. Nice diagrams are Heegaard diagrams (S, α, β, w) in which any region of $S \backslash (\alpha \cup \beta)$ not containing the basepoint w is either a bigon or a square. Nice diagrams have the property that the counts of pseudo-holomorphic disks which appear in the Heegaard Floer differential maps are combinatorially determined. Theorem 1.2 then follows. $\square$

It is known that the LOSS invariants also define transverse knot invariants via Legendrian approximation. Thus, it follows from Theorem 1.2 that the hat version of the LOSS invariant of transverse knots can also be computed combinatorially.

The proof of Theorem 1.2 relies on the LOSS invariant definition via an open book decomposition compatible with the Legendrian. By comparison, the theorem can also follow from a strikingly different perspective, by using sutured Heegaard Floer homology. Indeed, given a Legendrian L in a contact 3-manifold (Y, ξ), Honda, Kazez, and Matić [HKM09a] define the Legendrian invariant $EH(L)$ as an element in the sutured Floer homology of the knot complement. Stipsicz and Vértesi [SV09] identify the LOSS invariant $\widehat{\mathfrak{L}}(L)$ with the image of $EH(L)$ under a HKM contact gluing map. Leigon and Salmoiraghi provide a combinatorial description of the

HKM gluing map [LS20, Corollary 1.5], which altogether yields an alternate proof of Theorem 1.2. The proof presented in this note does not rely on this alternate view of the LOSS invariant $\widehat{\mathfrak{L}}(L)$ in terms of sutured Floer homology.

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2. The Legendrian LOSS invariants

Suppose that $L \subset (Y, \xi)$ is a Legendrian knot in a contact three-manifold. Consider an open book decomposition (Σ, ϕ) compatible with ξ containing L on a page. Recall that Honda-Kazez-Matić construct a Heegaard diagram for $-Y$ associated to (Σ, ϕ). This Heegaard diagram gives rise to an explicit description of the Heegaard Floer contact invariant $c(Y, \xi) \in \widehat{HF}(-Y)$ originally defined in [OS05], by identifying the contact invariant with the homology class of the cycle $\mathbf{c} = \{c_1, \ldots, c_{2g}\}$, a $2g$-tuple of intersection points in the Heegaard diagram.

Lisca-Ozsváth-Stipsicz-Szabó build on the Honda-Kazez-Matic algorithm to give a doubly-pointed Heegaard diagram (S, β, α, w, z) for $(-Y, \xi, L)$. By [LOSS09, Lemma 3.1], there is a basis of arcs $\{a_1, \ldots, a_{2g}\}$ for Σ adapted to L, meaning that $L \cap a_i = \emptyset$ for $i \geq 2$ and L intersects the arc a_1 in a unique transverse point. That the set of arcs $\{a_1, \ldots, a_{2g}\}$ forms a basis for Σ means that the arcs are disjoint properly embedded arcs on Σ such that $\Sigma \setminus \cup_{i=1}^{n} a_i$ is a disk. We obtain another basis $\{b_1, \ldots, b_{2g}\}$ for Σ, where each arc $b_i \subset \Sigma$ is obtained from a_i by a small isotopy in the direction given by the boundary orientation, and b_i intersects a_i in exactly one transverse intersection point $x_i \in \Sigma$, as in Figure 1.

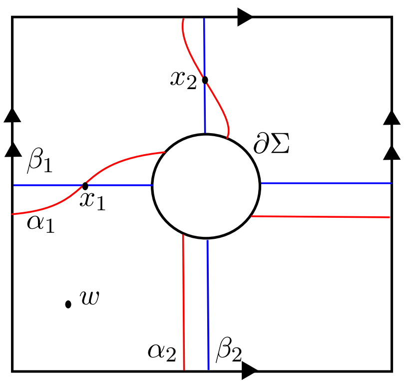

Figure 1. An example of $(\Sigma, \alpha, \beta, w)$, where Σ is a surface of genus 1 with one boundary component, $\alpha = \{\alpha_1, \alpha_2\}$ consists of the two red arcs, and $\beta = \{\beta_1, \beta_2\}$ consists of small isotopic translates of the α arcs.

Let:

- $S = \Sigma \cup -\Sigma$, or the union of two copies of the fiber surface Σ glued along their boundary,
- $\alpha = \{\alpha_1, \ldots, \alpha_{2g}\}$, where $\alpha_i = a_i \cup \overline{a_i}$, where $\overline{a_i} \subset -\Sigma$ is a copy of a_i,
- $\beta = \{\beta_1, \ldots, \beta_{2g}\}$, where $\beta_i = b_i \cup \overline{\phi(\beta_i)}$, where $\overline{\phi(b_i)} \subset -\Sigma$ is a copy of $\phi(b_i)$,

- the basepoint $w \in \Sigma$ lies outside of the thin regions created by the isotopies between a_i and b_i,
- the basepoint $z \in \Sigma$ is placed in one of the two thin regions between a_1 and b_1, determined by the orientation of the Legendrian L. See Figure 2.

In particular, the (S, β, α, w) agrees with the Honda-Kazez-Matic singly-pointed Heegaard diagram for $-Y$.

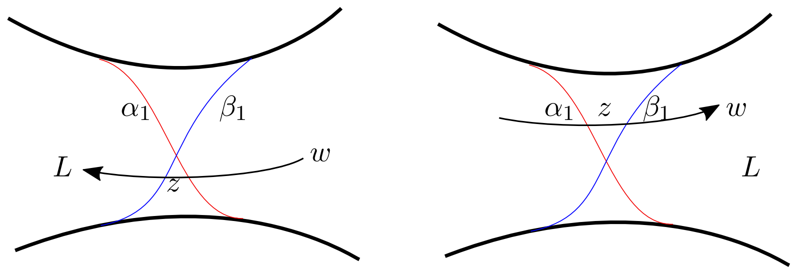

FIGURE 2. The basepoint is placed in one of the two regions of the small isotopy between the arcs α_1 and β_1, corresponding to the orientation of the Legendrian knot L.

The contact invariant $c(\xi) \in \widehat{HF}(-Y)$ is the homology class of the cycle $\mathbf{c} = \{x_1, \ldots, x_{2g}\}$ in $\widehat{HF}(S, \beta, \alpha, w)$. The combinatorial description of the contact invariant $c(\xi)$ stated in Theorem 1.1 is a corollary of Theorem 2.1.

Theorem 2.1 ([Pla07, Theorem 2.1])**.** Suppose (Σ, h) is an open book decomposition for the contact 3-manifold (Y, ξ). There exists an equivalent open book decomposition (Σ, h'') for (Y, ξ) such that singly-pointed Heegaard diagram described by Honda-Kazez-Matic for $-Y$ has only bigon and square regions (except for the one polygonal region D_w containing the basepoint w). The monodromy h'' differs from h by an isotopy; that is, $h'' = \psi \circ h$, where $\psi : \Sigma \to \Sigma$ is a diffeomorphism fixing the boundary and isotopic to the identity.

The LOSS invariant is an invariant of the Legendrian knot L determined by the $2g$-tuple of intersection points $\mathbf{x} = \{x_1, \ldots, x_{2g}\}$ which also defines a cycle in the $\mathbb{F}$-vector space $\widehat{HFK}(-Y, L, \mathbf{t}_\xi)$. Consider an equivalence relation on the set of pairs (M, m), where M is a vector space over $\mathbb{F}$ and $m \in M$, defined as follows: two pairs (M, m) and (N, n) are equivalent if there is a vector space isomorphism $f : M \to N$ such that $f(m) = n$. Let $[M, m]$ denote the equivalence class of (M, m).

Theorem 2.2 ([LOSS09])**.** Let L be an oriented, null-homologous Legendrian knot in the closed contact three-manifold (Y, ξ), and let $\mathbf{t}_\xi$ denote the spin^c structure on Y induced by ξ. Then $\widehat{\mathfrak{L}}(L)$ is an invariant of the Legendrian isotopy class of L, where $\widehat{\mathfrak{L}}(L)$ is defined as $[\widehat{HFK}(-Y, L, \mathbf{t}_\xi), [\mathbf{x}]]$, where $[\mathbf{x}]$ is the homology class of the cycle $\mathbf{x} = \{x_1, \ldots, x_{2g}\}$.

We give a combinatorial description of the hat version $\widehat{\mathfrak{L}}(L)$ of the LOSS invariant.

Theorem 2.3. Suppose (Σ, h) is an open book decomposition for the contact 3-manifold (Y, ξ), equipped with a basis $\{a_i\}$ of arcs in Σ adapted to the Legendrian knot $L \subset Y$. There exists an equivalent open book decomposition (Σ, h') for (Y, ξ)

and a basis $\{a''_i\}$ adapted to L such that the doubly-pointed Heegaard diagram for $(-Y, L)$, described by Lisca-Ozsváth-Stipsicz-Szabó, has only bigon and square regions (except for the one polygonal region D_w containing the basepoint w). The monodromy h'' differs from h by an isotopy; that is, $h' = \psi \circ h$, where $\psi : \Sigma \to \Sigma$ is a diffeomorphism fixing the boundary and isotopic to the identity.

Proof. Consider the open book decomposition (Σ, h) with basis $\{a_i\}$ adapted to $L \subset Y$ as above. Let $S = \Sigma \cup -\Sigma$. We follow the steps in the proof of Theorem 2.1 to obtain an equivalent open book decomposition (Σ, h') for Y such that the associated singly-pointed Heegaard diagram $\mathcal{H}' = (S, \beta', \alpha, w)$ described by Honda-Kazez-Matić for $-Y$ has only bigon and square regions (except for the one polygonal region D_w containing the basepoint w). The monodromy h' differs from h by an isotopy; that is, $h' = \phi \circ h$, where $\phi : \Sigma \to \Sigma$ is a diffeomorphism fixing the boundary and isotopic to the identity.

In more details, in the proof of Theorem 2.1, Plamenevskaya applies the Sarkar-Wang algorithm to modify the Heegaard diagram via a sequence of isotopies. In particular, all of these isotopies occur on the β curves and occur in the $-\Sigma$ (which we refer to as the monodromy side of S) part of the Heegaard surface S. Recall that the second basepoint z is placed in a region between a_1 and b_1 in Σ (which we refer to as the standard side of S). Thus, an isotopy of β_i occurs entirely in $-\Sigma$ and either:

(1) does not go through the region containing z.
(2) does go through the region containing z. If the isotopy is a finger move through the region containing z, then it divides this region into two pieces, one of which is entirely contained on the monodromy side $-\Sigma$; the basepoint z is chosen to remain in the region that intersects the standard side Σ.

We emphasize that these isotopies never cross the arc δ_α, respectively δ_β, connecting w and z in the complement of the α circles, respectively β circles, since the arcs δ_α and δ_β lie on the standard side Σ of the surface S. The resulting diagram $(S, \alpha, \beta', w, z)$ is a doubly-pointed nice diagram for the Legendrian L, since L is Legendrian isotopic to $\delta_\alpha \cup \delta_\beta$. Thus, the resulting filtered chain complex associated to the doubly-pointed Heegaard diagram $(S', \alpha', \beta', w, z)$ is filtered chain homotopic to the filtered chain complex of (S, α, β, w, z) by [OS04a, Theorem 3.1].

Therefore, we obtain a nice Heegaard diagram by performing this sequence of isotopies in $-\Sigma \subset S$. A composition of these isotopies gives a diffeomorphism $\psi : \Sigma \to \Sigma$ fixing the boundary and isotopic to the identity. The resulting open book decomposition $(\Sigma, h' = \psi \circ h)$ is equivalent to the original open book decomposition (Σ, h). $\square$

As described in Theorem 1.2, the existence of the nice diagram produced in Theorem 2.3 immediately implies that the LOSS Legendrian invariant $\widehat{\mathfrak{L}}(L)$ can be computed combinatorially.

It would be interesting to use the combinatorial description to compute the Legendrian and transverse invariant, especially if this approach were simpler than other methods. Plamenevskaya [Pla07, Section 3] uses the combinatorial approach to compute the Ozsváth-Szabó contact invariant for the standard tight contact structure on $S^1 \times S^2$, although this contact manifold is easy to understand without Floer theory. Etgü-Ozbagci [EO10] showed the contact invariant for this example can be computed more simply using a chain complex with fewer generators using a contact surgery diagram perspective. As is often the case that when working with invariants via combinatorial descriptions that arise using nice diagrams, the number of generators of the chain complex can be quite large, which makes most computations impractical (without relying on computer software). We point the reader to the literature for further computations: in [LOSS09, Section 6], the LOSS invariant is explicitly determined for several examples of Legendrian knots.

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