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ORIGINAL ARTICLE

Upper Bound of Critical Sets of Solutions of Elliptic Equations in the Plane

Jiuyi Zhu[1]

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Abstract

In this note, we investigate the measure of singular sets and critical sets of real-valued solutions of elliptic equations in two dimensions. These singular sets and critical sets are finitely many points in the plane. Adapting the Carleman estimates involving polynomial functions at singularities by Donnelly and Fefferman (J. Amer. Math. Soc. 3, 333–353, 1990), we obtain the upper bounds of singular points and critical points.

Keywords Critical sets · Singular sets · Carleman estimates

Mathematics Subject Classification (2010) 35A02 · 35B27 · 35J15

1 Introduction

We consider the upper bounds of singular sets for real-valued solutions of elliptic equations

$$\operatorname{div}(A(x)\nabla u) + b(x)\cdot\nabla u + c(x)u = 0 \qquad \text{in } \mathbb{B}_5 \tag{1.1}$$

and critical sets for real-valued solutions of elliptic equations

$$\operatorname{div}(A(x)\nabla u) + b(x)\cdot\nabla u = 0 \qquad \text{in } \mathbb{B}_5, \tag{1.2}$$

where $A(x) = (a_{ij}(x))_{2\times 2}$ is real-valued Lipschitz continuous, $b(x) = (b_1(x), b_2(x))$, $c(x)$ are bounded functions in the plane and $\mathbb{B}_5$ is the ball centered at origin with radius 5. Especially, we assume that $A(x)$ satisfies the uniform ellipticity conditions

$$\Lambda_1|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda_2|\xi|^2 \tag{1.3}$$

and the Lipschitz continuity conditions

$$|a_{ij}(x) - a_{ij}(y)| \le \Lambda_0|x-y| \quad \text{for any } x, y \in \mathbb{B}_5. \tag{1.4}$$

Dedicated to Professor Carlos E. Kenig on the Occasion of His 70th Birthday.

✉ Jiuyi Zhu
zhu@math.lsu.edu

[1] Department of Mathematics, Louisiana State University, 70803 Baton Rouge, LA, USA

The functions $b(x)$ and $c(x)$ are bounded as

$$\|b\|_{L^\infty(\mathbb{B}_5)} \le M_0, \quad \|c\|_{L^\infty(\mathbb{B}_5)} \le M_1. \tag{1.5}$$

The singular sets are given by $\mathcal{S} = \{x \in \mathbb{B}_2 | u(x) = |\nabla u(x)| = 0\}$. The critical sets are defined as $\mathcal{C} = \{x \in \mathbb{B}_2 || \nabla u(x)| = 0\}$. It is known that the singular sets and critical sets are finitely many discrete points for (1.1) and (1.2) in the plane, see e.g. the implicit bound of singular sets [14] and critical sets [15] for elliptic equations for any dimensions using compactness arguments. Around each singular point, the nodal sets consist of finitely many curves intersecting at this point with equal angles. The set of singular points is a subset of critical points. There are two types of points for critical points. One are the singular points, i.e. $\nabla u(x) = 0$ and $u(x) = 0$. The other are the non-sigular points, i.e. $\nabla u(x) = 0$, but $u(x) \neq 0$.

For real-valued harmonic functions, that is, $A(x) = \delta_{ij}$, $b(x) = 0$ and $c(x) = 0$ in (1.2), Han [13] showed that $H^0(\mathcal{S}) \le C\mathcal{N}(4)$, where $\mathcal{N}(r)$ is the frequency function given by

$$\mathcal{N}(r) = \frac{r \int_{\mathbb{B}_r} |\nabla u|^2}{\int_{\partial \mathbb{B}_r} u^2}. \tag{1.6}$$

Such upper bound can be also obtained by the analyticity of harmonic functions in [19]. For complexification of real-valued function u, say $\tilde{u}$, the upper bound of singular sets $H^0(\{z \in \mathbb{D}_1 | \tilde{u}(z) = \tilde{u}_{z_1}(z) = \tilde{u}_{z_2}(z) = 0\}) \le C\mathcal{N}^2(4)$ is obtained in [13]. Especially, some example is constructed in [13] to indicate that the real-valued property of solution is necessary to have the upper bound $H^0(\mathcal{S}) \le C\mathcal{N}(4)$. For the upper bound of singular sets of solutions in (1.1) for any dimension $n \ge 3$, an important conjecture

$$H^{n-1}(\mathcal{S}) \le C\mathcal{N}^2(4)$$

was raised by Lin in [19]. Naber and Valtorta [22] obtained an exponential upper bound of volume estimates for effective singular sets of (1.1) and effective critical sets of (1.2) using the new arguments of almost cone splitting in [8] and the covering lemma in any dimensions. Especially, the exponential upper bound holds for singular points $H^0(\mathcal{S}) \le e^{C\mathcal{N}(4)}$ and for critical points $H^0(\mathcal{C}) \le e^{C\hat{\mathcal{N}}(4)}$ in [22] in the plane, where

$$\hat{\mathcal{N}}(r) = \frac{r \int_{\mathbb{B}_r} |\nabla u|^2}{\int_{\partial \mathbb{B}_r} (u - u(0))^2}$$

is the modified version of the frequency function (1.6) for the study of critical sets. Note that we have considered the rescaled version in the aforementioned results. It is also interesting to study the bounds of singular sets and critical sets of eigenfunctions. For singular sets, see [10, 11] for the upper bound of singular sets of Laplace eigenfunctions on surfaces, and [23] for the upper bound of singular sets of Steklov eigenfunctions on surfaces. For critical sets, see [18] for bounded number of critical points and [9] for unbounded number of critical points of Laplace eigenfunctions with some given Riemannian metrics on two dimensional torus $\mathbb{T}^2$, and [24] for discussions of the upper bound of critical sets for Dirichlet eigenfunctions.

Let us introduce the double index for u as

$$N(u, r) = \log_2 \frac{\|u\|_{L^\infty(\mathbb{B}_{2r})}}{\|u\|_{L^\infty(\mathbb{B}_r)}}.$$

Frequency function $\mathcal{N}(r)$ in (1.6) characterizes the growth rate of the solutions. It implies that the bounds of double index $N(u, r)$. It is well-known that the frequency function $\mathcal{N}(r)$ is almost monotone, i.e. $e^{Cr}\mathcal{N}(r)$ is monotone for $0 \le r \le r_0$, where C and r_0 depend on the coefficients in (1.1). Based on the monotonicity of the frequency function, $\mathcal{N}(r)$ and $N(u,r)$ are comparable in the sense that

$$C_1\mathcal{N}(r) - C \le N(u,r) \le C_2\mathcal{N}(3r) + C \tag{1.7}$$

for $0 < r \le R_1 \le \frac{r_0}{3}$, where $0 < C_1 < 1$, $C_2 > 1$, C and R_1 depend on the coefficients in (1.1). Furthermore, we can get that the almost monotonicity of double index,

$$N(u,r) \le C_3 N(u,tr) \tag{1.8}$$

for $t > 2$ and $0 < r < R_0 \le R_1$, where R_0 depends on the coefficients in (1.1) (see e.g. [12, 21]). Assume that $N = N(u, \frac{R_0}{2}) \ge 1$ is large. Then it follows from (1.7) that $\mathcal{N}(\frac{3R_0}{2})$ is large. Our first result is to show the following upper bound of singular points.

Theorem 1.1 *Assume that u satisfies the equation* (1.1)*. Then it holds that*

$$H^0(\{\mathcal{S} \cap \mathbb{B}_{\frac{R_0}{25}}\}) \le C\mathcal{N}\left(\frac{3R_0}{2}\right),$$

where R_0 depends on Λ_0, Λ_1, Λ_2, M_0 and M_1.

To study the critical points in (1.2), we introduce the double index for ∇u as

$$\hat{N}(\nabla u, r) = \log_2 \frac{\|\nabla u\|_{L^\infty(\mathbb{B}_{2r})}}{\|\nabla u\|_{L^\infty(\mathbb{B}_r)}}.$$

The frequency function $\hat{\mathcal{N}}(r)$ is almost monotone, i.e. $e^{Cr}\hat{\mathcal{N}}(r)$ is monotone for $0 \le r \le r_0$, where C and r_0 depend on the coefficients in (1.2). This monotonicity of the frequency function implies that $\hat{\mathcal{N}}(r)$ and $\hat{N}(\nabla u, r)$ are comparable in the sense that

$$C_1\hat{\mathcal{N}}(r) - C \le \hat{N}(\nabla u, r) \le C_2\hat{\mathcal{N}}(3r) + C \tag{1.9}$$

for $0 < r \le R_1 \le \frac{r_0}{3}$, where $0 < C_1 < 1$, $C_2 > 1$, C and R_1 depend on the coefficients in (1.2). Moreover, the almost the monotonicity of double index holds,

$$\hat{N}(\nabla u, r) \le C_3 \hat{N}(\nabla u, tr) \tag{1.10}$$

for $t > 2$ and $0 < r < R_0 \le R_1$, where R_0 depends on the coefficients in (1.2) (see e.g. [20, 22]). Assume that $\hat{N} = \hat{N}(\nabla u, \frac{R_0}{2}) \ge 1$ is large. Our second result is to show that

Theorem 1.2 *Assume that u satisfies the equation* (1.2)*. Then it holds that*

$$H^0(\{\mathcal{C} \cap \mathbb{B}_{\frac{R_0}{25}}\}) \le C\hat{\mathcal{N}}\left(\frac{3R_0}{2}\right),$$

where R_0 depends on Λ_0, Λ_1, Λ_2 and M_0.

This note is organized as follows. In Section 2, we reduce the second order elliptic operators with Lipschitz leading coefficients to the Euclidean Laplace operators. Then, we present the Carleman estimates involving polynomial functions at singularities in the plane. Section 3 is devoted to the derivation of upper bounds of singular sets and critical sets in Theorems 1.1 and 1.2. The letter C and C_i denote some generic positive constants and do not depend on u. It may vary in different lines and sections.

2 Carleman Estimates for Euclidean Laplace

In this section, we first construct Lipschitz metrics from the Lipschitz leading coefficient $A(x)$ in (1.1) and (1.2). The arguments are adapted from [4]. We present the details for the convenience of the readers. Then we reduce the study on Laplace–Beltrami operator to Euclidean Laplace by isothermal coordinates. At last, we present the Carleman estimates involving polynomials for Euclidean Laplace. Without loss of generality, we consider the construction of geodesic coordinates at origin. We introduce a “radial" coordinate and a conformal change metric g_{ij} in $\mathbb{B}_{r_0}$,

$$r = r(x) = (a^{ij}(0)x_i x_j)^{\frac{1}{2}} \tag{2.1}$$

and

$$g_{ij}(x) = a^{ij}(x)\hat{\psi}(x),$$

where

$$\hat{\psi}(x) = a_{kl}(x)\frac{\partial r}{\partial x^k}\frac{\partial r}{\partial x^l}$$

for $x \neq 0$ and $(a^{ij}) = (a_{ij})^{-1}$ is the inverse matrix. In the whole paper, we adopt the Einstein notation. The summation of index is understood. From the assumption of (1.3), $\hat{\psi}$ is bounded above and below as

$$\frac{\Lambda_1}{\Lambda_2} \leq \hat{\psi} \leq \frac{\Lambda_2}{\Lambda_1}.$$

It is easy to see that $\hat{\psi}$ is Lipschitz continuous. With these auxiliary quantities, the following replacement of geodesic polar coordinates are constructed in [4]. In the geodesic ball $\hat{\mathbb{B}}_{\hat{r}_0} = \{x \in \mathbb{B}_{r_0} | r(x) \leq \hat{r}_0\}$, the following properties hold:

(i) $g_{ij}(x)$ is Lipschitz continuous;

(ii) $g_{ij}(x)$ is uniformly elliptic with $\frac{\Lambda_1}{\Lambda_2^2}\|\xi\|^2 \leq g_{ij}(x)\xi_i\xi_j \leq \frac{\Lambda_2}{\Lambda_1^2}\|\xi\|^2$.

(iii) Let $\Sigma = \partial\hat{\mathbb{B}}_{\hat{r}_0}$. We can parametrize $\hat{\mathbb{B}}_{\hat{r}_0}\backslash\{0\}$ by the polar coordinate r and θ, with r defined by (2.1) and θ be the local coordinates on Σ. In these polar coordinates, the metric can be written as

$$g_{ij}(x)dx^i dx^j = dr^2 + r^2\gamma d\theta d\theta$$

with $\gamma = \frac{1}{r^2}g_{kl}(x)\frac{\partial x^k}{\partial\theta}\frac{\partial x^l}{\partial\theta}$.

The existence of the coordinates (r, θ) allows us to pass to “geodesic polar coordinates". In particular, $r(x) = (a^{ij}(0)x_i x_j)^{\frac{1}{2}}$ is the geodesic distance to the origin in the metric g_{ij}. Thus, we may identify $\hat{\mathbb{B}}_{\hat{r}_0}$ as the Euclidean ball $\mathbb{B}_{\hat{r}_0}$. The Laplace–Beltrami operator is given as

$$\triangle_g = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_i}\left(g^{ij}\sqrt{g}\frac{\partial}{\partial x_j}\right),$$

where $g = \det(g_{ij})$. If u is a solution of (1.1), in the new metric g_{ij}, then u is locally the solution of the equation

$$\triangle_g u + \hat{b}(x)\cdot\nabla_g u + \hat{c}(x)u = 0 \quad \text{in } \mathbb{B}_{\hat{r}_0}, \tag{2.2}$$

where

$$\begin{cases} \hat{b}_i = -\frac{1}{2\tilde{a}\hat{\psi}}\frac{\partial\tilde{a}}{\partial x_i} + \frac{1}{\hat{\psi}}b_i, \\ \hat{c}(x) = \frac{c(x)}{\hat{\psi}}, \end{cases}$$

and $\tilde{a} = \det(a^{ij})$. By the Lipschitz continuity of $A(x)$, then $\tilde{a}$ is Lipschitz continuous. Hence $\hat{b} = (\hat{b}_1, \hat{b}_2)$ is bounded. By the properties of $\hat{\psi}$, and the conditions (1.5) on b and c, we still write the conditions for $\hat{b}$ and $\hat{c}$ as

$$\begin{cases} \|\hat{b}\|_{L^\infty(\mathbb{B}_{\hat{r}_0})} \leq CM_0, \\ \|\hat{c}\|_{L^\infty(\mathbb{B}_{\hat{r}_0})} \leq CM_1, \end{cases} \tag{2.3}$$

where C depends on Λ_1 and Λ_2. By the same construction of Lipschitz metric g, the solution u in the equation (1.2) satisfies

$$\triangle_g u + \hat{b}(x) \cdot \nabla_g u = 0 \quad \text{in } \mathbb{B}_{\hat{r}_0}, \tag{2.4}$$

with $\hat{b} = -\frac{1}{2\tilde{a}\hat{\psi}} \frac{\partial \tilde{a}}{\partial x_i} + \frac{1}{\hat{\psi}} b_i$ and $\|\hat{b}\|_{L^\infty(\mathbb{B}_{\hat{r}_0})} \leq CM_0$.

Applying the isothermal coordinates for the surfaces with Lipschitz Riemannian metrics in [7] or [17] (or the so called pseudo-analyticity used on p. 79 in [5]), we have $\triangle_g = \phi(x)^{-1}\triangle$, where $\phi(x) > 0$ is continuous. Therefore, we can write (2.2) as

$$\triangle u + \tilde{b}(x) \cdot \nabla u + \tilde{c}(x) u = 0 \quad \text{in } \mathbb{B}_{\hat{r}_0} \tag{2.5}$$

and (2.4) as

$$\triangle u + \tilde{b}(x) \cdot \nabla u = 0 \quad \text{in } \mathbb{B}_{\hat{r}_0}, \tag{2.6}$$

where $\tilde{b}(x)$ and $\tilde{c}(x)$ satisfy the same conditions as (2.3).

Next, we will establish Carleman estimates involving polynomial functions at singularities for differential operators with the Euclidean Laplace as the leading term. It is directly from the Carleman estimates from [11]. We present the details to show the role of real-valued functions and the double index N in the Carleman estimates. Let

$$\overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right) \quad \text{and} \quad \partial = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right).$$

Note that $\overline{\partial}\partial = \frac{1}{4}\triangle$. Let $P(z) = \prod(z - z_i)^{d_i}$ for some $d_i \geq 0$, where $z = x_1 + ix_2$ and $z_i = x_1^i + ix_2^i$ are in the complex plane. It is shown in [11] that

$$\int_{\mathbb{D}_5} |\overline{\partial} F|^2 |P|^{-2} e^{C\alpha|z|^2} \geq C\alpha \int_{\mathbb{D}_5} |F|^2 |P|^{-2} e^{C\alpha|z|^2} \tag{2.7}$$

for any smooth (possibly complex-valued) function $F \in C_0^\infty(\mathbb{D}_5 \setminus \cup \mathbb{D}_i(z_i))$ and positive constant α. Here $\mathbb{D}_5$ is a ball in the complex plane with radius 5 and $\mathbb{D}_i(z_i)$ are some small pairwise disjoint balls centered at z_i with radius δ. Let $f \in C_0^\infty(\mathbb{D}_5 \setminus \cup \mathbb{D}_i(z_i))$ be a real-valued function. We will show the following Carleman estimates hold

$$\int_{\mathbb{B}_5} |\triangle f + \tilde{b}(x) \cdot \nabla f + \tilde{c}(x) f|^2 |P|^{-2} e^{CN|z|^2} \geq CN^2 \int_{\mathbb{B}_5} |f|^2 |P|^{-2} e^{CN|z|^2}. \tag{2.8}$$

Choosing $F = \partial f$ and $\alpha = N$ in (2.7), we obtain

$$\int_{\mathbb{D}_5} |\triangle f|^2 |P|^{-2} e^{CN|z|^2} \geq CN \int_{\mathbb{D}_5} |\partial f|^2 |P|^{-2} e^{CN|z|^2}. \tag{2.9}$$

Since f is a real-valued function, it holds that

$$|\overline{\partial} f| = |\partial f| = \frac{1}{2}|\nabla f|.$$

Let $F = f$ in (2.7) and $\alpha = N$. We have

$$\int_{\mathbb{D}_5} |\nabla f|^2 |P|^{-2} e^{CN|z|^2} \geq CN \int_{\mathbb{D}_5} |f|^2 |P|^{-2} e^{CN|z|^2}. \tag{2.10}$$

Furthermore, it follows from (2.7) and (2.9) that

$$\int_{\mathbb{D}_5} |\triangle f|^2 |P|^{-2} e^{CN|z|^2} \geq CN^2 \int_{\mathbb{D}_5} |f|^2 |P|^{-2} e^{CN|z|^2}. \tag{2.11}$$

In order to consider the equation (1.1), we need to take the first order term and zero order term into considerations. Assume that $N \geq CM_0$ and $N \geq CM_1$. We identify $\mathbb{D}_5$ as $\mathbb{B}_5$, and $z_i = (x_1^i, x_2^i)$ in $\mathbb{R}^2$. Thus, the inequalities (2.10) and (2.11) hold for $f \in C_0^\infty(\mathbb{B}_5 \backslash \cup D_i(z_i))$, where $D_i(z_i)$ are small pairwise disjoint balls centered at z_i with radius δ in $\mathbb{B}_5$. By the triangle inequality, (2.3), (2.10) and (2.11), we can obtain the following Carleman estimates,

$$\begin{aligned}
&\int_{\mathbb{B}_5} |\triangle f + \tilde{b}(x) \cdot \nabla f + \tilde{c}(x) f|^2 |P|^{-2} e^{CN|z|^2} \\
&\geq \int_{\mathbb{B}_5} |\triangle f|^2 |P|^{-2} e^{CN|z|^2} - CM_1 \int_{\mathbb{B}_5} |\nabla f|^2 |P|^{-2} e^{CN|z|^2} - CM_0 \int_{\mathbb{B}_5} |f|^2 |P|^{-2} e^{CN|z|^2} \\
&\geq CN^2 \int_{\mathbb{B}_5} |f|^2 |P|^{-2} e^{CN|z|^2} - CM_1 N \int_{\mathbb{B}_5} |f|^2 |P|^{-2} e^{CN|z|^2} \\
&\geq CN^2 \int_{\mathbb{B}_5} |f|^2 |P|^{-2} e^{CN|z|^2}.
\end{aligned}$$

Thus, the Carleman estimates (2.8) are arrived.

3 Upper Bounds of Singular Points and Critical Points

In this section, we first study the upper bound of singular points for (1.1) by adapting the arguments in [11]. We may choose $R_0 \leq \frac{\hat{r}_0}{2}$. By rescaling, we set $R_0 = 5$ and $\hat{r}_0 = 10$. Suppose that $|z_i| \leq \frac{1}{5}$. We first consider the singular points with the vanishing order more than two. Assume that u vanishes at z_i with order $n_i + 1$, where $n_i = d_i + 1$ with $d_i \geq 1$. If the vanishing order is two, i.e. $n_i = 1$, we will consider it with some special argument later on. Near the singular point z_i, $u(x)$ can be approximated by a homogeneous polynomial with degree $d_i + 2$ in $D_i(z_i)$, where $D_i(z_i)$ are the pairwise disjoint small disks with radius δ_i centered at z_i, see e.g. [6]. Since singular points are discrete and finite, such small δ_i exist. We choose the smallest δ_i such that $\delta = \min \delta_i$. Then we can assume that the $D_i(z_i)$ are small disjoint disks with radius δ. We choose the polynomial $P(z) = \prod (z - z_i)^{d_i}$. Let $f \in C_0^\infty(\mathbb{B}_5 \backslash \cup_i D_i(z_i))$ be a real-valued function.

Based on the above preparations, we are ready to show the proof of Theorem 1.1.

Proof of Theorem 1.1 As discussed above, we first consider the case $d_i \geq 1$. We choose a cut off function $\psi \in C_0^\infty(\mathbb{B}_1 \backslash \cup_i D_i(z_i))$ with the following properties:

(1) $\psi(z) = 1$ if $|z| \leq \frac{1}{2}$ and $|z - z_i| \geq 2\delta$,
(2) $|\nabla \psi| \leq C$ and $|\triangle \psi| \leq C$ if $|z| \geq \frac{1}{2}$,
(3) $|\nabla \psi| \leq C\delta^{-1}$ and $|\triangle \psi| \leq C\delta^{-2}$ if $|z - z_i| \leq 2\delta$.

We substitute $f = \psi u$ into the Carleman estimates (2.8). Direct calculations show that

$$\begin{aligned}\triangle f + \tilde{b}(x)\cdot\nabla f + \tilde{c}(x) f &= \triangle\psi u + 2\nabla\psi\cdot\nabla u + \psi\triangle u + \tilde{b}(x)\cdot\nabla\psi u\\ &\quad + \tilde{b}(x)\cdot\nabla u\psi + \tilde{c}(x)u\psi\\ &= \triangle\psi u + 2\nabla\psi\cdot\nabla u + \tilde{b}(x)\cdot\nabla\psi u,\end{aligned}$$

where we have used the equation (2.5). In the neighborhood $|z - z_i| \le 2\delta$, by the vanishing order of u at z_i, we can check that

$$|\nabla\psi u| \le C\delta^{d_i}, \quad |\triangle\psi u| \le C\delta^{d_i}, \quad \text{and} \quad |\nabla u\cdot\nabla\psi| \le C\delta^{d_i}.$$

Near the neighborhood $|z - z_i| \le 2\delta$, it holds that

$$|P|^{-2}|\triangle(\psi u) + \tilde{b}(x)\cdot\nabla(\psi u) + \tilde{c}(x)\psi u|^2 \le C\delta^{-2d_i}\delta^{2d_i} \le C.$$

Thus, $|P|^{-2}|\triangle(\psi u)+\tilde{b}(x)\cdot\nabla(\psi u)+\tilde{c}(x)u|^2$ is uniformly integrable near the singular points z_i as $\delta \to 0$. If $|z| \ge \frac{1}{2}$, from the assumption of ψ, we can see that

$$\begin{aligned}|\triangle f + \tilde{b}\cdot\nabla f + \tilde{c}(x) f| &= |\triangle\psi u + 2\nabla\psi\cdot\nabla u + \tilde{b}\cdot\nabla\psi u|\\ &\le C(|u| + |\nabla u|).\end{aligned}$$

Substituting $f = \psi u$ in the Carleman estimates (2.8) and applying the Lebegue dominated convergence theorem as $\delta \to 0$, we have

$$\int_{\frac{1}{2}\le|z|\le 1} (|u|^2 + |\nabla u|^2)|P|^{-2}e^{CN|z|^2} \ge CN^2 \int_{|z|\le\frac{1}{3}} |u|^2|P|^{-2}e^{CN|z|^2}.$$

We take the maximum and minimum of $|P|$ out of the integrations. It holds that

$$e^{CN}\max_{\frac{1}{2}\le|z|\le 1}|P|^{-2}\int_{\frac{1}{2}\le|z|\le 1}(|u|^2 + |\nabla u|^2) \ge CN^2 \min_{|z|\le\frac{1}{3}}|P|^{-2}\int_{|z|\le\frac{1}{3}}|u|^2.$$

By standard elliptic estimates, we get that

$$e^{CN}\max_{\frac{1}{2}\le|z|\le 1}|P|^{-2}\int_{\frac{2}{5}\le|z|\le\frac{6}{5}}|u|^2 \ge CN^2 \min_{|z|\le\frac{1}{3}}|P|^{-2}\int_{|z|\le\frac{1}{3}}|u|^2. \tag{3.1}$$

We claim that

$$e^{C\sum d_i} \le \frac{\min_{|z|\le\frac{1}{3}}|P|^{-2}}{\max_{\frac{1}{2}\le|z|\le 1}|P|^{-2}}. \tag{3.2}$$

To show (3.2), it is equivalent to prove

$$e^{C\sum d_i} \le \left(\frac{\min_{\frac{1}{2}\le|z|\le 1}|P|}{\max_{|z|\le\frac{1}{3}}|P|}\right)^2. \tag{3.3}$$

Since $|z_i| \le \frac{1}{5}$, we have

$$\min_{\frac{1}{2}\le|z|\le 1}|P| \ge \left(\frac{1}{2} - \frac{1}{5}\right)^{C\sum d_i} = \left(\frac{3}{10}\right)^{C\sum d_i} \tag{3.4}$$

and

$$\max_{|z|\le\frac{1}{3}}|P| \le \left(\frac{1}{3} - \frac{1}{5}\right)^{C\sum d_i} = \left(\frac{2}{15}\right)^{C\sum d_i}. \tag{3.5}$$

Together with (3.4) and (3.5), we arrive at (3.3), i.e. (3.2). It follows from (3.1) and (3.2) that

$$e^{C\sum d_i} \le \frac{e^{CN}\int_{\frac{2}{5}\le|z|\le\frac{6}{5}}|u|^2}{\int_{|z|\le\frac{1}{3}}|u|^2}.$$

By the almost monotonicity of the double index $N(u,r)$ in (1.8), it holds that

$$\frac{\int_{\frac{2}{5}\le|z|\le\frac{6}{5}}|u|^2}{\int_{|z|\le\frac{1}{3}}|u|^2} \le e^{CN}.$$

Thus, we have

$$\sum d_i \le CN. \tag{3.6}$$

Hence, we arrive at the conclusion in the theorem in the case $d_i \ge 1$.

Now, we treat the case for singular points with vanishing order two, i.e. $d_i = 0$. We consider the polynomial $P_1(z) = \prod(z - z_i)^{\frac{1}{2}}$. We want to replace $P(z)$ in the above arguments by $P_1(z)$. Near the singular point z_i, we can still show that $|P_1(z)|^{-2}|\triangle(\psi u) + \tilde{b}(x)\cdot\nabla(\psi u) + \tilde{c}(x)\psi u|^2$ is uniformly integrable as $\delta \to 0$. If $|z - z_i| \le 2\delta$, we can check

$$|\nabla\psi u| \le C, \quad |\triangle\psi u| \le C, \quad \text{and} \quad |\nabla u\cdot\nabla\psi| \le C.$$

Thus,

$$|P_1(z)|^{-2}|\triangle(\psi u) + \tilde{b}(x)\cdot\nabla(\psi u) + \tilde{c}(x)\psi u|^2 \le C\delta^{-1},$$

which is uniformly integral in $\mathbb{B}_5$. However, $P_1(z)$ is not defined as single-valued holomorphic function. As indicated in [11], we can pass to a finite branched cover of the disc $\mathbb{D}_5$ punctured at z_i. Since the Carleman estimates (2.7) are obtained by integration by parts, these Carleman estimates are arrived in a straightforward manner. The integrand in these estimates involves function such as f and $|P_1|$ which are independent of the sheet. Therefore, we still have the Carleman estimates (2.8) in the punctured disc. Following the arguments as we did to get (3.6) for $n_i \ge 2$, the conclusion $\sum_{z_i\in\mathbb{B}_{1/5}} 1 \le CN$ will still be arrived for $n_i = 1$. Recall that we have done a rescaling argument to have $R_0 = 5$. Thus, the estimate (3.6) implies that

$$H^0(\{\mathcal{S}\cap\mathbb{B}_{\frac{R_0}{25}}\}) \le CN.$$

It follows from (1.7) that $N \le C\mathcal{N}(\frac{3R_0}{2})$ for some large N. Therefore, the proof of the theorem is arrived. □

Next, we show the upper bound of critical sets for (1.2). As before, we may choose $R_0 \le \frac{\hat{r}_0}{2}$ and set $R_0 = 5$ and $\hat{r}_0 = 10$ by rescaling. Suppose that the critical points $|z_i| \le \frac{1}{5}$. Near the critical point z_i, $u(z)$ can be approximated by the function $P_c + u(z_i)$ in $D_i(z_i)$, where P_c is some homogeneous polynomial with degree $d_i + 2$ and $D_i(z_i)$ is some small disk with radius δ_i. Furthermore, $\nabla u(z)$ can be approximated by ∇P_c in $D_i(z_i)$, see e.g. [16]. In particular, if $u(z_i) = 0$, then z_i is the singular point. Since critical points are finitely many discrete points, we choose the smallest δ_i such that $\delta = \min\delta_i$ and assume that the $\mathbb{D}_i(z_i) = D_i(z_i)$ are small disjoint disks with radius δ, where we have identified the complex plane with $\mathbb{R}^2$.

Proof of Theorem 1.2 We still consider the case $d_i \ge 1$ at the beginning. If $f \in C_0^\infty(\mathbb{D}_5\setminus\cup_i \mathbb{D}_i(z_i))$ and $P(z) = \prod(z - z_i)^{d_i}$, by choosing $\alpha = \hat{N}$ and $F = f$ in (2.7), it follows that the following Carleman estimates hold

$$\int_{\mathbb{D}_5}|\overline{\partial}f|^2|P|^{-2}e^{C\hat{N}|z|^2} \ge C\hat{N}\int_{\mathbb{D}_5}|f|^2|P|^{-2}e^{C\hat{N}|z|^2}. \tag{3.7}$$

We choose the same real-valued cut-off function ψ in the last theorem. That is, the cut-off function $\psi \in C_0^\infty(\mathbb{D}_1 \setminus \cup_i \mathbb{D}_i(z_i))$ satisfies the following properties:

(1) $\psi(z) = 1$ if $|z| \leq \frac{1}{2}$ and $|z - z_i| \geq 2\delta$,
(2) $|\nabla\psi| \leq C$ and $|\triangle\psi| \leq C$ if $|z| \geq \frac{1}{2}$,
(3) $|\nabla\psi| \leq C\delta^{-1}$ and $|\triangle\psi| \leq C\delta^{-2}$ if $|z - z_i| \leq 2\delta$.

Substituting $f = \psi\partial u$ in the Carleman estimates (3.7), we have

$$\int_{\mathbb{D}_5} |\bar{\partial}\psi\partial u|^2|P|^{-2}e^{C\hat{N}|z|^2} + \int_{\mathbb{D}_5} \psi^2|\triangle u|^2|P|^{-2}e^{C\hat{N}|z|^2} \geq C\hat{N}\int_{\mathbb{D}_5} \psi^2|\partial u|^2|P|^{-2}e^{C\hat{N}|z|^2}.$$

From the equation (2.6) and the fact that $\tilde{b}$ is bounded, we get

$$\int_{\mathbb{D}_5} |\bar{\partial}\psi\partial u|^2|P|^{-2}e^{C\hat{N}|z|^2} \geq C\hat{N}\int_{\mathbb{D}_5} \psi^2|\partial u|^2|P|^{-2}e^{C\hat{N}|z|^2}.$$

Thus, we have

$$\int_{\mathbb{B}_5} |\nabla\psi|^2|\nabla u|^2|P|^{-2}e^{C\hat{N}|z|^2} \geq C\hat{N}\int_{\mathbb{B}_3} \psi^2|\nabla u|^2|P|^{-2}e^{C\hat{N}|z|^2}. \tag{3.8}$$

Near the critical point z_i, for $|z - z_i| \leq 2\delta$, we can check that

$$|\nabla\psi \cdot \nabla u| \leq C\delta^{d_i}.$$

Thus,

$$|\nabla\psi|^2|\nabla u|^2|P|^{-2} \leq C\delta^{2d_i}\delta^{-2d_i} \leq C.$$

Then $|\nabla\psi|^2|\nabla u|^2|P|^{-2}$ is uniformly integrated as $\delta \to 0$. From the assumption of ψ, applying the dominated convergence theorem as $\delta \to 0$, we have

$$\int_{\frac{1}{2}\leq|z|\leq 1} |\nabla u|^2|P|^{-2}e^{C\hat{N}|z|^2} \geq C\hat{N}\int_{|z|\leq\frac{1}{3}} |\nabla u|^2|P|^{-2}e^{C\hat{N}|z|^2}.$$

By taking the maximum and minimum of $|P|$, we get

$$e^{C\hat{N}} \max_{\frac{1}{2}\leq|z|\leq 1} |P|^{-2}\int_{\frac{1}{2}\leq|z|\leq 1} |\nabla u|^2 \geq C\hat{N} \min_{|z|\leq\frac{1}{3}} |P|^{-2}\int_{|z|\leq\frac{1}{3}} |\nabla u|^2. \tag{3.9}$$

From (3.2), we have

$$e^{C\sum d_i} \leq \frac{\min_{|z|\leq\frac{1}{3}} |P|^{-2}}{\max_{\frac{1}{2}\leq|z|\leq 1} |P|^{-2}}. \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$e^{C\sum d_i} \leq \frac{e^{C\hat{N}}\int_{\frac{1}{2}\leq|z|\leq 1} |\nabla u|^2}{\int_{|z|\leq\frac{1}{3}} |\nabla u|^2}.$$

By the almost monotonicity of the double index of $\hat{N}(\nabla u, r)$ in (1.10), we show that

$$\frac{\int_{\frac{1}{2}\leq|z|\leq 1} |\nabla u|^2}{\int_{|z|\leq\frac{1}{3}} |\nabla u|^2} \leq e^{C\hat{N}}.$$

Hence, we arrive at

$$\sum d_i \leq C\hat{N}. \tag{3.11}$$

Therefore, in the case $d_i \geq 1$, the conclusion of the theorem follows.

Now, we deal with the case for critical points with vanishing order two, i.e. $d_i = 0$. We follow the arguments in the last theorem for singular sets with vanishing order two. We replace $P(z)$ in the above arguments by $P_1(z)$, where $P_1(z) = \prod(z - z_i)^{\frac{1}{2}}$. If $|z - z_i| \leq 2\delta$, we can check $|\nabla u \cdot \nabla\psi| \leq C$. Thus,

$$|P_1(z)|^{-2}|\nabla u \cdot \nabla\psi|^2 \leq C\delta^{-1},$$

which is uniformly integral in $\mathbb{B}_5$ as $\delta \to 0$. However, $P_1(z)$ is not defined as single-valued holomorphic function. We can pass to a finite branched cover of the disc $\mathbb{D}_5$ punctured at z_i. Since the Carleman estimates (3.7) are obtained by integration by parts, these Carleman estimates are arrived similarly. The integrand in these estimates involves functions such as f and $|P_1|$ which are independent of the sheet. Therefore, we still have the Carleman estimates (3.8) in the punctured disc. Following the arguments as we did to get (3.11) for $d_i \geq 1$, the conclusion $\sum_{z_i \in \mathbb{B}_{1/5}} 1 \leq C\hat{N}$ will be arrived for $d_i = 0$. Rescaling back to R_0, the estimate (3.11) yields that

$$H^0(\{\mathcal{C} \cap \mathbb{B}_{\frac{R_0}{25}}\}) \leq C\hat{N}.$$

It follows from (1.9) that $\hat{N} \leq C\hat{\mathcal{N}}(\frac{3R_0}{2})$ for some large $\hat{N}$. This completes the proof of the theorem. $\square$

The rest of the section is devoted to the discussion of upper bound of critical points with large coefficients in the equations in the plane. To study the local growth of gradient near each point, we introduce

$$\hat{\mathcal{N}}(x, r) = \frac{r\int_{\mathbb{B}_r(x)} |\nabla u|^2}{\int_{\partial\mathbb{B}_r(x)}(u - u(x))^2},$$

where $\mathbb{B}_r(x)$ is the ball centered at x with radius r. It is known that

$$\hat{\mathcal{N}}(x, r) \leq C\hat{\mathcal{N}}(2R_0) \tag{3.12}$$

for $x \in \mathbb{B}_{\frac{R_0}{4}}$ and $0 < r \leq \frac{3R_0}{2}$, see e.g. [22]. The arguments in Theorem 1.2 can be applied to study the upper bound of critical points for elliptic equations with a large drift term. We consider the elliptic equations

$$\operatorname{div}(A(x)\nabla u) + \lambda b(x) \cdot \nabla u = 0 \qquad \text{in } \mathbb{B}_5,$$

where $A(x) = (a_{ij}(x))_{2\times 2}$ satisfies the assumptions (1.3) and (1.4), $b(x)$ satisfies the condition (1.5), and possibly $\lambda \to \infty$. Let $\tilde{\mathcal{N}} = \hat{\mathcal{N}}(2R_0)$. If $\lambda \leq C\tilde{\mathcal{N}}$, we can perform the same argument in Theorem 1.2 directly. Thus, we will obtain the upper bound

$$H^0(\{\mathcal{C} \cap \mathbb{B}_{\frac{R_0}{25}}\}) \leq C\tilde{\mathcal{N}}.$$

If $\lambda \geq C\tilde{\mathcal{N}}$, we first do some rescaling arguments. Let $v(x) = u(\frac{\tilde{\mathcal{N}}}{\lambda}x + x_0)$ for $x_0 \in \mathbb{B}_{\frac{R_0}{4}}$. We consider the critical sets of u in $\mathbb{B}_{\frac{5\tilde{\mathcal{N}}}{\lambda}}(x_0)$. Thus, $v(x)$ satisfies the equation

$$\operatorname{div}(\bar{A}(x)\nabla v) + \tilde{\mathcal{N}}\bar{b}(x) \cdot \nabla v = 0 \qquad \text{in } \mathbb{B}_5,$$

where $\bar{A}(x) = (\bar{a}_{ij}(x))_{2\times 2} = (a_{ij}(\frac{\tilde{\mathcal{N}}}{\lambda}x))_{2\times 2}$ and $\bar{b}(x) = b(\frac{\tilde{\mathcal{N}}}{\lambda}x)$. By the arguments in the proof of Theorem 1.2, we can show that

$$H^0(\{\mathbb{B}_{\frac{R_0}{25}} \mid |\nabla v(x)| = 0\}) \leq C\hat{N}\left(\nabla v, \frac{R_0}{2}\right).$$

From (3.12) and (1.9), it holds that

$$\hat{N}\left(\nabla v, \frac{R_0}{2}\right) \leq \hat{\mathcal{N}}\left(x_0, \frac{3R_0\tilde{\mathcal{N}}}{2\lambda}\right) \leq C\tilde{\mathcal{N}}.$$

Thus, we have

$$H^0(\{\mathbb{B}_{\frac{\tilde{\mathcal{N}}R_0}{25\lambda}}(x_0)| \, |\nabla u(x)| = 0\}) \leq C\tilde{\mathcal{N}}.$$

Covering the ball $\mathbb{B}_{\frac{R_0}{25}}$ with $C\frac{\lambda^2}{\tilde{\mathcal{N}}^2}$ number of $\mathbb{B}_{\frac{\tilde{\mathcal{N}}R_0}{25\lambda}}(x_0)$ balls for $x_0 \in \mathbb{B}_{\frac{R_0}{25}}$, we obtain that

$$H^0(\{\mathbb{B}_{\frac{R_0}{25}}| \, |\nabla u(x)| = 0\}) \leq C\frac{\lambda^2}{\tilde{\mathcal{N}}}.$$

For the upper bound of singular points with a large first order term or zero order term, see [11, 23].

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