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Optimal Dividends Under Model Uncertainty*

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Abstract. We consider a diffusive model for optimally distributing dividends, while allowing for Knightian model ambiguity concerning the drift of the surplus process. We show that the value function is the unique solution of a nonlinear Hamilton–Jacobi–Bellman variational inequality. In addition, this value function embodies a unique optimal threshold strategy for the insurer's surplus, thereby making it the smooth pasting of a nonlinear and a linear part at the location of the threshold. Furthermore, we obtain continuity and monotonicity of the value function in addition to continuity of the threshold strategy with respect to the parameter that measures ambiguity of our model.

Key words. optimal dividend strategy, model uncertainty, threshold strategy, stochastic game

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1. Introduction.

The optimal dividend payment has been a classical problem in insurance mathematics since the seminal work of De Finetti [12]. Gerber [14] showed that for the classical Cramér–Lundberg risk model, the optimal dividend strategy is a band strategy. Since those early papers, many insurance economists have studied the optimal dividend problem. For example, Asmussen and Taksar [1] considered a risk-neutral optimal dividend problem in the diffusive setup. They exploited the linear structure of the Hamilton–Jacobi–Bellman variational inequality (HJB-VI) and provided explicit solutions for the value function and the optimal threshold. Azcue and Muler [2] studied an optimal control problem of dividend payments and investment of the surplus in a Black–Scholes market. In their case, the uncontrolled reserve follows the Cramér–Lundberg risk model, which leads to a jump-diffusion problem. They used methods of viscosity solutions to characterize the value function and to show that the optimal dividend control has a band structure. Cohen and Young [11] determined the degree to which the diffusion approximation serves as a valid approximation for the Cramér–Lundberg model.

In the classical models considered above, it is assumed that the insurer has complete information about the dynamics. However, in reality it is rare that a decision maker has

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complete information about the parameters of the model. We model this uncertainty by including an adverse player who chooses a worst-case scenario. This ambiguity is one example of what is called in the literature *Knightian uncertainty*. For further research that involves such uncertainty, we refer the reader to Maenhout [21], Hansen et al. [16], Hansen and Sargent [15], Bayraktar and Zhang [3], Neufeld and Nutz [22], Lam [19], Cohen [7, 6], Cohen and Saha [10], and Cohen, Hening, and Sun [9].

In this paper, we formulate and analyze a problem of optimal dividends with model uncertainty. We assume that there is a *reference probability measure* $\mathbb{P}$, under which the dynamics of the surplus process before paying dividends is

$$X_t = x + mt + \sigma W_t, \qquad t \geq 0,$$

in which $(W_t)_{t\geq 0}$ is a $\mathbb{P}$-standard Brownian motion. To account for uncertainty in the value of m, the insurer considers a class $\mathcal{Q}(x)$ of probability measures that are equivalent to $\mathbb{P}$, satisfying further conditions, and chooses a dividend payment process D that maximizes the following payoff criteria:

$$\inf_{\mathbb{Q}\in\mathcal{Q}(x)} \mathbb{E}^{\mathbb{Q}}\Big[\int_0^\tau e^{-\varrho t} f(X_t)dt + \int_0^\tau e^{-\varrho t} dD_t + \frac{1}{\kappa} L^\varrho(\mathbb{Q}\|\mathbb{P})\Big].$$

We look at this robust problem as a game between the insurer (maximizer) and an adverse player (minimizer). The infimum is taken over the class of measures $\mathcal{Q}(x)$. The parameter $\varrho > 0$ is the discount factor and τ is the (random) time of ruin. The first integral is a running reward, and the second one represents the dividend payments. Finally, the last term penalizes the adverse player for deviating from the reference measure: $L^\varrho(\mathbb{Q}\|\mathbb{P})$ is the Kullback–Leibler divergence that measures how much $\mathbb{Q}$ deviates from $\mathbb{P}$, and $\kappa > 0$ measures the level of ambiguity, with increasing values of κ corresponding to increasing ambiguity.

We characterize the value function and the Stackelberg equilibrium of this game, which consists of an optimal dividend strategy and the optimal response of the adverse player. Specifically, we show that the value function is the unique smooth solution of the relevant HJB-VI equation and that an optimal dividend payment strategy is a threshold strategy, in which the threshold is determined by the HJB-VI as well. In particular, we show that for large values of κ, the threshold is 0, that is, dividends are paid immediately. We summarize these properties in the main theorem of the paper (Theorem 3.7). For this, we set up the HJB-VI equation and transform it into a free-boundary problem by hypothesizing that an optimal strategy is a threshold strategy. The penalty term from the payoff function translates to a quadratic term in the HJB-VI equation, which breaks the linearity of the differential equation. As a consequence, an explicit solution is out of reach, and it is not clear if there is a smooth solution to the HJB-VI. To show that, indeed, a smooth solution exists, we use the *shooting method*. On a high level, the shooting method solves boundary-value problems by using a class of parameterized initial-value problems.

We also analyze the dependence of the game on the ambiguity parameter κ. Namely, we show continuity and monotonicity properties of the value function and the optimal dividend-threshold with respect to κ. Finally, we show that when the ambiguity parameter $\kappa \to 0^+$, the problem converges to the classical risk-neutral optimal dividend problem studied by Asmussen and Taksar [1]. Also, we compute the first-order correction terms for the value function and optimal threshold for small values of κ.

In section 5, we include numerical experiments. Specifically, we use the algorithm that we proposed to establish the theoretical existence of a solution to the HJB-VI (namely, the shooting method) to numerically solve our problem for various values of κ. This emphasizes the advantage of the shooting method as both a theoretical and a numerical tool, as compared with other techniques, such as showing that the value function is a viscosity solution of the relevant HJB equation and then establishing its regularity; see, for example, [3].

In summary, our main contributions are as follows. We

- formulate a (diffusive) dividend problem with model uncertainty;
- show that the value function solves a nonlinear HJB-VI (Theorem 3.7);
- show that there is an optimal threshold strategy (Theorem 3.7);
- analyze the dependence of the value function and the optimal threshold strategy on the ambiguity parameter (Theorems 4.1 and 4.2).

The paper is organized as follows. In section 2, we motivate and present the stochastic differential game. Next, in section 3, we provide the HJB-VI for the value function and prove that the value function is the unique smooth solution of the HJB-VI with boundary condition. Moreover, we show that the maximizer has an optimal unique threshold strategy. In section 4, we study the dependence of the solution on the ambiguity parameter. Finally, section 5 includes numerical experiments for the value function and the threshold over varying values of κ.

2. The stochastic model.

In this section, we present the ingredients for the optimal dividend problem under model uncertainty.

2.1. Stochastic game.

Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space that supports a one-dimensional standard Brownian motion W. We consider the following time-homogeneous dynamics for an insurer's uncontrolled surplus process $\hat{X}$:

$$d\hat{X}_t = m dt + \sigma dW_t, \quad t \geq 0,$$

with $\hat{X}_{0^-} = x \geq 0$. The insurer chooses its dividend strategy to maximize a robust payoff functional that accounts for the uncertainty about the underlying model.

Definition 2.1 (admissible strategies). *An* admissible strategy for the maximizer *for any initial state $x \in \mathbb{R}_+$ is an $\mathbb{F}$-adapted, nondecreasing process D taking values in $\mathbb{R}_+$ with right-continuous with left limits (RCLL) sample paths, with*

$$dX_t = m dt + \sigma dW_t - dD_t, \quad t \geq 0, \tag{2.1}$$

and $X_{0^-} = x \geq 0$, and with $D_t - D_{t^-} \leq X_{t^-}$. Let $\mathcal{A}(x)$ denote the collection of admissible strategies D with initial condition $x \geq 0$.

An admissible strategy for the minimizer (*which we also call the* adverse player) *is a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}, \mathbb{F})$, which is defined by*

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(t) = \exp\left\{\int_0^t \xi_s dW_s - \frac{1}{2}\int_0^t \xi_s^2 ds\right\}, \quad t \in \mathbb{R}_+, \tag{2.2}$$

for some $\mathbb{F}$*-progressively measurable process* ξ *satisfying*

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^\infty e^{-\varrho s}\xi_s^2\,ds\right]<\infty \quad \text{and} \quad \mathbb{E}^{\mathbb{P}}\left[e^{\frac{1}{2}\int_0^t \xi_s^2\,ds}\right]<\infty, \quad t\in\mathbb{R}_+. \tag{2.3}$$

We call ξ *the* Girsanov kernel *of* $\mathbb{Q}$*. Let* $\mathcal{Q}(x)$ *denote the collection of admissible strategies* Q *with initial condition* $x\geq 0$*.*

Remark 2.2. The conditions in (2.3) ensure that (2.2) is a uniformly integrable martingale and that the discounted Kullback–Leibler divergence (or relative entropy) between $\mathbb{Q}$ and $\mathbb{P}$ is well-defined in what follows.

The payoff function. Define the time of ruin τ by

$$\tau := \inf\{t\geq 0: X_t = 0\}$$

for $X_{0^-}=x\geq 0$. The payoff associated with the initial condition x and the strategy profile $(D,\mathbb{Q})$ is given by

$$J(x,D,\mathbb{Q};\kappa) := \mathbb{E}^{\mathbb{Q}}\left[\int_0^\tau e^{-\varrho t}(f(X_t)dt + dD_t)\right] + \frac{1}{\kappa}L^\varrho(\mathbb{Q}\|\mathbb{P}), \tag{2.4}$$

in which L^ϱ is the so-called (discounted) *Kullback–Leibler divergence*,

$$L^\varrho(\mathbb{Q}\|\mathbb{P}) := \mathbb{E}^{\mathbb{Q}}\left[\int_0^\infty \varrho e^{-\varrho t}\ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}}(t)\right)dt\right], \tag{2.5}$$

and $\kappa\geq 0$ measures the insurer's degree of ambiguity concerning $\mathbb{P}$. Increasing κ corresponds to increasing uncertainty. In (2.4), f is a nondecreasing running-reward function.

We rewrite $L^\varrho(\mathbb{Q}\|\mathbb{P})$ in (2.5) as

$$\begin{aligned}
L^\varrho(\mathbb{Q}\|\mathbb{P}) &= \mathbb{E}^{\mathbb{Q}}\left[\int_0^\infty \varrho e^{-\varrho t}\ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}}(t)\right)dt\right]\\
&= \mathbb{E}^{\mathbb{Q}}\left[\int_0^\infty \varrho e^{-\varrho t}\left\{\int_0^t \xi_s dW_s - \frac{1}{2}\int_0^t \xi_s^2\,ds\right\}dt\right]\\
&= \mathbb{E}^{\mathbb{Q}}\left[\int_0^\infty \varrho e^{-\varrho t}\left\{\int_0^t \xi_s(dW_s-\xi_s ds) + \frac{1}{2}\int_0^t \xi_s^2\,ds\right\}dt\right]\\
&= \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{2}\int_0^\infty \varrho e^{-\varrho t}\int_0^t \xi_s^2\,ds\,dt\right]\\
&= \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{2}\int_0^\infty\left(\int_s^\infty \varrho e^{-\varrho t}dt\right)\xi_s^2\,ds\right]\\
&= \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{2}\int_0^\infty e^{-\varrho t}\xi_t^2\,dt\right],
\end{aligned}$$

in which the fourth line follows because

$$W_t^{\mathbb{Q}} := W_t - \int_0^t \xi_s ds, \qquad t\geq 0, \tag{2.6}$$

is a $\mathbb{Q}$-Brownian motion. The value function is defined by

$$V(x;\kappa) = \sup_{D\in\mathcal{A}(x)} \inf_{\mathbb{Q}\in\mathcal{Q}(x)} J(x,D,\mathbb{Q};\kappa). \tag{2.7}$$

Remark 2.3. Solving the above problem is equivalent to solving the constrained problem:

$$\sup_{D\in\mathcal{A}(x)} \inf_{\mathbb{Q}\in\mathcal{Q}(x)} \mathbb{E}^{\mathbb{Q}}\left[\int_0^\tau e^{-\varrho t}(f(X_t)dt + dD_t)\right],$$
$$\text{subject to} \qquad L^{\varrho}(\mathbb{Q}\|\mathbb{P}) \le \eta,$$

for an appropriate choice of $\eta > 0$. The constant $1/\kappa > 0$ may be seen as the Lagrange multiplier for the Kullback–Leibler constraint. The parameter η increases with κ, so that, when η is small, we have confidence in the reference model and so κ is small, which implies a high penalization for deviating from the reference model. For further discussion, see, for example, [17, equations (14)–(15)] and Remark 2.4 below.

Note that under $\mathbb{Q}$, X follows the process

$$dX_t = (m+\sigma\xi_t)dt + \sigma dW_t^{\mathbb{Q}} - dD_t, \quad t\ge 0. \tag{2.8}$$

Given an admissible strategy of the adverse player $\mathbb{Q}$ with Girsanov's kernel ξ, define the admissible strategy $\mathbb{Q}^\tau$ with the Girsanov's kernel ξ^τ, satisfying $\xi_t^\tau = \xi_t$ for $t\in[0,\tau]$, and $\xi_t^\tau = 0$ for $t>\tau$. Then, the distribution of X and D until time τ is the same under both measures $\mathbb{Q}$ and $\mathbb{Q}^\tau$. On the other hand,

$$L^{\varrho}(\mathbb{Q}\|\mathbb{P}) \ge L^{\varrho}(\mathbb{Q}^\tau\|\mathbb{P}) = \frac{1}{2}\mathbb{E}^{\mathbb{Q}}\left[\int_0^\tau e^{-\varrho t}(\xi_t^\tau)^2\,dt\right].$$

Therefore, the adverse player would prefer to use $\mathbb{Q}^\tau$ over $\mathbb{Q}$. As a result, we define

$$J(x,D,\mathbb{Q};\kappa) = \mathbb{E}^{\mathbb{Q}}\left[\int_0^\tau e^{-\varrho t}\left\{\left(f(X_t) + \frac{1}{2\kappa}\xi_t^2\right)dt + dD_t\right\}\right]. \tag{2.9}$$

We use the notation $J(x,D;0)$ and $V(x;0)$ to denote, respectively, the payoff and the value function for the risk-neutral problem. That is,

$$J(x,D;0) = \mathbb{E}^{\mathbb{P}}\left[\int_0^\tau e^{-\varrho t}\left\{f(X_t)dt + dD_t\right\}\right] = J(x,D,\mathbb{P};\kappa),$$

and

$$V(x;0) = \sup_{D\in\mathcal{A}(x)} J(x,D;0). \tag{2.10}$$

Remark 2.4.

- For $\kappa \sim 0^+$, the penalty for deviating from the reference measure is very large. As we will see in the main theorem, the adverse player's optimal strategy is (stochastic and) of order κ. Therefore, we have convergence to the risk-neutral problem as $\kappa\to 0^+$. In addition, when $\kappa\to\infty$, the minimizer can choose that the process ξ_t goes to infinity at a rate much slower than that of κ, such that the time of ruin τ for the process X_t in (2.1) converges to 0 under $\mathbb{Q}$. Thus, as $\kappa\to\infty$, $V(x,\kappa)$ heuristically should converge to x. We prove both of these results rigorously in what follows.

- The value function is set to be the lower value of the game, meaning sup inf. Hence, we refer to equilibrium as Stackelberg equilibrium. This setup is consistent with the traditional setup of the Knightian uncertainty; see, for example, [21, 16, 15, 3, 22, 17, 4, 19, 7, 10]. This structure emerges from the theory of risk-sensitive control (see Fleming and Soner [13, Theorem XI.7.2] with cost instead of a reward) and the variational representation due to Boué and Dupuis [5]. The latter is also known as the duality presentation; see, for example, Cohen and Dolinsky [8]. We follow the above references and analyze the dividends problem. The technical problem of whether the upper value and the lower value coincide is not addressed in this paper.

3. Solution of the stochastic game.

3.1. Threshold strategies. We show that an optimal strategy of the maximizer is a *threshold strategy*. To rigorously define such a strategy, we use the *Skorokhod map on an interval*; see [18], [20], and [24]. Fix $\beta \in \mathbb{R}_+$; then, for any RCLL function $\eta : \mathbb{R}_+ \to \mathbb{R}$, there exist RCLL functions $\chi, \zeta : \mathbb{R}_+ \to \mathbb{R}$ that satisfy the following properties:

(i) For every $t \geq 0$, $\chi_t = \eta_t - \zeta_t \leq \beta$.
(ii) ζ is nondecreasing, with $\zeta_{0^-} = 0$ and

$$\int_0^\infty 1_{(-\infty,\beta)}(\chi_t) d\zeta_t = 0.$$

Given $\beta \geq 0$ and RCLL $\eta : \mathbb{R}_+ \to \mathbb{R}$, the pair (χ, ζ) is unique on $\mathbb{R}_+$. Let $\Gamma_\beta(\eta) = (\Gamma^1_\beta, \Gamma^2_\beta)(\eta)$ denote the ordered pair (χ, ζ).

The following continuity property is well known; see, for example, Kruk et al. [18].

Lemma 3.1. *There exists a constant $c_S > 0$ such that for every $t \geq 0$, $\beta \in \mathbb{R}_+$, and RCLL functions $\eta, \tilde{\eta} : \mathbb{R}_+ \to \mathbb{R}$,*

$$\sup_{s\in[0,t]} \left\{ \left| \Gamma^1_\beta(\eta_s) - \Gamma^1_\beta(\tilde{\eta}_s) \right| + \left| \Gamma^2_\beta(\eta_s) - \Gamma^2_\beta(\tilde{\eta}_s) \right| \right\} \leq c_S \sup_{s\in[0,t]} |\eta_s - \tilde{\eta}_s|.$$

Definition 3.2. *Fix $x, \beta \geq 0$. The strategy D^β is called a β-threshold strategy if $(X_t, D^\beta_t) = \Gamma_\beta(x + m \cdot + \sigma W_\cdot)_t$, for all $t \geq 0$, with $X_{0^-} = x$.*

One can easily verify that any β-threshold strategy D^β is admissible. Essentially, D^β pays all surplus in excess of β as dividends.

3.2. The HJB variational inequality and the value function. For $\kappa > 0$, we anticipate that V in (2.7) solves the following HJB-VI with boundary condition:

$$\begin{cases} \left[\inf_{\xi\in\mathbb{R}} \left\{ \frac{1}{2}\sigma^2\phi''(x) + (m + \sigma\xi)\phi'(x) - \varrho\phi(x) + f(x) + \frac{1}{2\kappa}\xi^2 \right\} \right] \\ \qquad\qquad\qquad\qquad \vee [1 - \phi'(x)] = 0, \qquad x > 0, \\ \phi(0) = 0. \end{cases} \tag{3.1}$$

The coefficient of $\phi'(x)$ arises from the $\mathbb{Q}$-drift of X in (2.8); recall that V is a $\mathbb{Q}$-expectation. By substituting the optimal solution from the $\inf_{\xi\in\mathbb{R}}$, namely, $\xi^* = -\kappa\sigma\phi'(x)$, (3.1) becomes

$$\text{(HJB}(\kappa)) \qquad \begin{cases} [\phi''(x) + H(x,\phi(x),\phi'(x))] \vee [1-\phi'(x)] = 0, & x \in (0,\infty), \\ \phi(0) = 0, \end{cases}$$

in which

$$H(x,y,z) := \frac{2}{\sigma^2}\left(mz - \frac{1}{2}\sigma^2\kappa z^2 - \varrho y + f(x)\right). \tag{3.2}$$

Moreover, (HJB(κ)) makes sense when $\kappa = 0$. As we prove below, (HJB(κ)) admits a unique solution in $\mathcal{C}^2(\mathbb{R}_+)$ for all $\kappa \geq 0$, and it solves

$$\begin{cases} \phi_\beta''(x) + H(x,\phi_\beta(x),\phi_\beta'(x)) = 0, & 0 \leq x \leq \beta, \\ 1 - \phi_\beta'(x) = 0, & \beta \leq x, \\ \phi_\beta(0) = 0, \end{cases} \tag{3.3}$$

for some $\beta \in \mathbb{R}_+$, which (together with a verification result) implies that an optimal dividend strategy is a threshold strategy.

Remark 3.3. Later, we will identify β via the smooth pasting (free-boundary) condition $\phi_\beta''(\beta) = 0$, which is consistent with ϕ being twice continuously differentiable at $x = \beta$.

Remark 3.4. Note that, when $\kappa = 0$ and $f \equiv 0$, (HJB(κ)) coincides with the HJB-VI given in Asmussen and Taksar [1, equations (3.10)–(3.11)].

Consider an admissible control D, and let X be given by (2.1). From the minimization in (3.1), we define a candidate strategy for the adverse player: For any $\phi \in \mathcal{C}^2(\mathbb{R}_+)$ and $t \in \mathbb{R}_+$, set

$$\begin{aligned} \xi_t^\phi &:= \arg\min_{\xi\in\mathbb{R}}\left\{\frac{1}{2}\sigma^2\phi''(X_t) + (m+\sigma\xi)\phi'(X_t) - \varrho\phi(X_t) + f(X_t) + \frac{1}{2\kappa}\xi^2\right\} \\ &= -\kappa\sigma\phi'(X_t). \end{aligned} \tag{3.4}$$

$\xi^\phi = \{\xi_t^\phi\}_{t\geq 0}$ is an $\mathbb{F}$-progressively measurable process because, from Definition 2.1, D and W are.

Notation 3.5. Fix an admissible control D. If ξ^ϕ satisfies the conditions (2.3) for the controlled process X under the control D given in (2.1), we denote by $\mathbb{Q}^\phi(D)$ the measure associated with the Girsanov kernel ξ^ϕ. When $\phi = V(\cdot;\kappa)$, we denote for simplicity $\mathbb{Q}^\kappa(D) := \mathbb{Q}^{V(\cdot;\kappa)}(D)$ to be the measure associated with the Girsanov kernel $\xi^{V(\cdot;\kappa)}$.

In what follows, we also need some mild constraints on the running-reward function f. Specifically, we impose the following assumption for the rest of this section.

Assumption 3.6. The function f is nonnegative, nondecreasing, and Lipschitz with corresponding coefficient $\varrho - \delta$ for some $\delta > 0$. That is,

$$|f(x) - f(y)| \leq (\varrho - \delta)|x - y|.$$

In addition, we set $f(0) = 0$ to account for zero reward when there is no surplus.

Under this assumption, we have the following theorem.

Theorem 3.7. *For any $\kappa \in [0,\infty)$, the following hold:*

(1) *The value function given by (2.7) is the unique $\mathcal{C}^2(\mathbb{R}_+)$ solution of (HJB(κ)). Moreover, V is concave and solves (3.3) for some $\beta \geq 0$.*

(2) *Let β_κ denote the value of $\beta \geq 0$ associated with κ. Then, β_κ is unique. Furthermore, for $2m \leq \sigma^2\kappa$, we have $\beta_\kappa = 0$, and for $2m > \sigma^2\kappa$, we have $0 < \beta_\kappa \leq \frac{m}{\delta}$.*

(3) *Let $\xi^{V(\cdot;\kappa)}$ be given by (3.4) with $\phi = V(\cdot;\kappa)$. Then, $\xi^{V(\cdot;\kappa)}$ is bounded; hence, the conditions in (2.3) hold. Let D^{β_κ} denote the β_κ-threshold strategy. Then, the couple $(D^{\beta_\kappa}, \mathbb{Q}^\kappa)$ forms a Stackelberg equilibrium in the following sense:*

$$
\begin{aligned}
V(x;\kappa) &= J(x, D^{\beta_\kappa}, \mathbb{Q}^\kappa(D^{\beta_\kappa});\kappa) \\
&= \inf_{\mathbb{Q}\in\mathcal{Q}(x)} J(x, D^{\beta_\kappa}, \mathbb{Q};\kappa) \\
&= \sup_{D\in\mathcal{A}(x)} J(x, D, \mathbb{Q}^\kappa(D);\kappa),
\end{aligned}
\tag{3.5}
$$

in which $\mathbb{Q}^\kappa$ is defined[1] *in Notation 3.5.*

The proof of Theorem 3.7 follows from the next four propositions; see section 3.3 for their proofs.

Proposition 3.8. *Suppose $\phi \in \mathcal{C}^2(\mathbb{R}_+)$ solves (HJB(κ)) with ϕ' bounded. Then, for $\kappa > 0$,*

$$
\phi(x) \geq \sup_{D\in\mathcal{A}(x)} J(x, D, \mathbb{Q}^\phi;\kappa), \qquad x \in [0,\infty), \tag{3.6}
$$

and for $\kappa = 0$,

$$
\phi(x) \geq \sup_{D\in\mathcal{A}(x)} J(x, D;\kappa), \qquad x \in [0,\infty). \tag{3.7}
$$

As a consequence, $\phi \geq V$.

Proposition 3.9. *Suppose $\phi_\beta \in \mathcal{C}^2(\mathbb{R}_+)$ solves (3.3) for some $\beta \geq 0$. Let D^β be the corresponding β-threshold strategy. Then, for $\kappa > 0$,*

$$
\phi_\beta(x) \leq \inf_{\mathbb{Q}\in\mathcal{Q}(x)} J(x, D^\beta, \mathbb{Q};\kappa), \qquad x \in [0,\infty), \tag{3.8}
$$

and for $\kappa = 0$,

$$
\phi_\beta(x) \leq J(x, D^\beta;0), \qquad x \in [0,\infty). \tag{3.9}
$$

As a consequence, $\phi_\beta \leq V$.

Proposition 3.10. *For every $\kappa \geq 0$, (HJB(κ)) admits a unique $\mathcal{C}^2(\mathbb{R}_+)$ solution with a bounded derivative, uniformly in κ. Moreover, the solution also solves (3.3). Let β_κ denote the value of $\beta \geq 0$ associated with κ. Then, for $2m \leq \sigma^2\kappa$, we have $\beta_\kappa = 0$, and for $2m > \sigma^2\kappa$, we have $0 < \beta_\kappa \leq \frac{m}{\delta}$.*

[1]In what follows, when there is no ambiguity about the control D, we abuse notation by removing the argument D from $\mathbb{Q}^\phi(D)$ and writing $J(x, D, \mathbb{Q}^\phi;\kappa)$ for $J(x, D, \mathbb{Q}^\phi(D);\kappa)$.

Proposition 3.11. *For every $\kappa \geq 0$, there is a unique β-threshold optimal strategy.*

Proposition 3.12. *The value function given by (2.7) is concave, that is, for any $\kappa \geq 0$,*

$$V''(x;\kappa) \leq 0, \quad x \in [0,\infty).$$

Proof of Theorem 3.7. From Proposition 3.10, for every $\kappa \geq 0$, (HJB(κ)) admits a unique $\mathcal{C}^2(\mathbb{R}_+)$ solution that also solves (3.3) for some $\beta_\kappa \geq 0$, which is unique by Proposition 3.11. Let g_{β_κ} denote this solution. From Proposition 3.9, we have

$$g_{\beta_\kappa}(\cdot) \leq \inf_{\mathbb{Q}\in\mathcal{Q}(x)} J(\cdot, D^{\beta_\kappa}, \mathbb{Q};\kappa) \leq V(\cdot;\kappa).$$

On the other hand, from Proposition 3.8,

$$g_{\beta_\kappa}(\cdot) \geq \sup_{D\in\mathcal{A}(x)} J(\cdot, D, \mathbb{Q}^{\beta_\kappa};\kappa) \geq V(\cdot;\kappa).$$

By combining the above two inequalities, we deduce $V(\cdot;\kappa) = g_{\beta_\kappa}(\cdot)$, as well as the equilibrium relation in (3.5). We prove concavity of V below in Proposition 3.12. From Proposition 3.10, we also have $\beta_\kappa = 0$ if $2m \leq \sigma^2\kappa$. ∎

Remark 3.13. Note that when the insurer has large enough ambiguity aversion, namely, $\kappa \geq 2m/\sigma^2$, then it is optimal to pay out all the surplus as dividends, and $V(x;\kappa) = x$ for all $x \geq 0$ in this case.

3.3. Proofs of Propositions 3.8–3.12.

Proof of Proposition 3.8. Suppose $\phi \in \mathcal{C}^2(\mathbb{R}_+)$ solves (HJB(κ)) with bounded first derivative. For arbitrary $D \in \mathcal{A}(x)$ and $\mathbb{Q} \in \mathcal{Q}(x)$, Itô's lemma applied to $e^{-\varrho(\tau\wedge t)}\phi(X_{\tau\wedge t})$ and $\mathbb{Q}$-expectation give us

$$\begin{aligned} (3.10) \qquad & \mathbb{E}^{\mathbb{Q}}\left[e^{-\varrho(\tau\wedge t)}\phi(X_{\tau\wedge t})\right] \\ &= \phi(x) + \mathbb{E}^{\mathbb{Q}}\left[\int_0^{\tau\wedge t} e^{-\varrho s}\left(\frac{1}{2}\sigma^2\phi''(X_s) + (m+\sigma\xi_s)\phi'(X_s) - \varrho\phi(X_s)\right)ds\right] \\ &\quad - \mathbb{E}^{\mathbb{Q}}\left[\int_0^{\tau\wedge t} e^{-\varrho s}\phi'(X_s)dD_u^{\mathsf{C}}\right] + \mathbb{E}^{\mathbb{Q}}\left[\sum_{0\leq s\leq(\tau\wedge t)} e^{-\varrho s}\Delta\phi(X)_s\right], \end{aligned}$$

in which ξ is the Girsanov kernel of $\mathbb{Q}$, and the process D^{C} denotes the continuous part of D. Consider (3.10) with $\xi = \xi^\phi$ defined in (3.4) and with the associated measure $\mathbb{Q} = \mathbb{Q}^\phi$. Because ϕ solves (HJB(κ)) (and specifically, $\phi'(x) \geq 1$), we obtain

$$\begin{aligned} (3.11) \qquad & \mathbb{E}^{\mathbb{Q}^\phi}\left[e^{-\varrho(\tau\wedge t)}\phi(X_{\tau\wedge t})\right] \\ &\leq \phi(x) - \mathbb{E}^{\mathbb{Q}^\phi}\left[\int_0^{\tau\wedge t} e^{-\varrho s}\left\{\left(f(X_s) + \frac{1}{2\kappa}(\xi_s^\phi)^2\mathbb{1}_{\{\kappa>0\}}\right)ds + dD_s\right\}\right] \\ &\quad + \mathbb{E}^{\mathbb{Q}^\phi}\left[\sum_{0\leq s\leq(\tau\wedge t)} e^{-\varrho s}(\Delta\phi(X)_s + \Delta D_s)\right]. \end{aligned}$$

Because $\Delta X_s = -\Delta D_s$, we have

$$\Delta\phi(X)_s + \Delta D_s = \phi(X_s - \Delta D_s) - \phi(X_s) + \Delta D_s = \int_{X_s-\Delta D_s}^{X_s} (1-\phi'(u))du \le 0. \tag{3.12}$$

By combining (3.11)–(3.12), we get

$$\mathbb{E}^{\mathbb{Q}^\phi}\left[e^{-\varrho(\tau\wedge t)}\phi(X_t)\right] + \mathbb{E}^{\mathbb{Q}^\phi}\left[\int_0^{(\tau\wedge t)} e^{-\varrho s}\left\{\left(f(X_s) + \frac{1}{2\kappa}(\xi_s^\phi)^2 \mathbb{1}_{\{\kappa>0\}}\right)ds + dD_s\right\}\right] \le \phi(x). \tag{3.13}$$

Note that ϕ is nonnegative; indeed, the solution of (HJB(κ)) satisfies $\phi'(x) \ge 1$ with $\phi(0)=0$. Because ϕ is nonnegative, we can omit the leftmost term in (3.13) while maintaining the inequality. The nonnegativity of the terms within the integral together with the monotone convergence theorem (as $t\to\infty$) imply

$$J(x, D, \mathbb{Q}^\phi;\kappa) = \mathbb{E}^{\mathbb{Q}^\phi}\left[\int_0^\tau e^{-\varrho s}\left\{\left(f(X_s) + \frac{1}{2\kappa}(\xi_s^\phi)^2\mathbb{1}_{\{\kappa>0\}}\right)ds + dD_s\right\}\right] \le \phi(x), \tag{3.14}$$

from which we deduce inequality (3.6). Finally, because $\xi^\phi = 0$ when $\kappa=0$, we have $\mathbb{Q}^\phi = \mathbb{P}$ when $\kappa = 0$, and inequality (3.7) follows. ■

Proof of Proposition 3.9. We split the proof into two cases: $x\in[0,\beta]$ and $x\in(\beta,\infty)$.

Case 1. Fix $x\in[0,\beta]$, and choose an arbitrary $\mathbb{Q}\in\mathcal{Q}(x)$ with Girsanov kernel ξ. In the following, we combine the proofs for both the cases $\{\kappa=0\}$ and $\{\kappa>0\}$ by making use of the indicator function $\mathbb{1}_{\{\kappa>0\}}$, which is able to capture or ignore relevant terms depending on whether $\{\kappa>0\}$ is true or false. By basic properties of quadratic functions, for every $t\ge0$,

$$\begin{aligned}&\frac{1}{2}\sigma^2\phi_\beta''(X_t) + (m+\sigma\xi_t\mathbb{1}_{\{\kappa>0\}})\phi_\beta'(X_t) - \varrho\phi_\beta(X_t) + f(X_t) + \frac{1}{2\kappa}(\xi_t)^2\mathbb{1}_{\{\kappa>0\}}\\ &\quad\ge \frac{1}{2}\sigma^2\left(\phi_\beta''(X_t) + H(X_t,\phi_\beta(X_t),\phi_\beta'(X_t))\right) = 0,\end{aligned} \tag{3.15}$$

in which H is given in (3.2), and the equality follows from the first equation in (3.3).

Note that, because $x\in[0,\beta]$ and D^β is a β-threshold strategy, D^β has no jumps (that is, $(D^\beta)^{\mathsf{c}} = D^\beta$). From (3.10) and (3.15), we have

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[e^{-\varrho(\tau\wedge t)}\phi_\beta(X_{\tau\wedge t})\right] &\ge \phi_\beta(x) - \mathbb{E}^{\mathbb{Q}}\left[\int_0^{\tau\wedge t} e^{-\varrho s}\left(f(X_s) + \frac{1}{2\kappa}\xi_s^2\mathbb{1}_{\{\kappa>0\}}\right)ds\right]\\ &\quad - \mathbb{E}^{\mathbb{Q}}\left[\int_0^{\tau\wedge t} e^{-\varrho s}\phi_\beta'(X_s)dD_s^\beta\right].\end{aligned} \tag{3.16}$$

By using $\phi_\beta'(\beta)=1$ from (3.3) and by noting $dD_s^\beta \ne 0$ only when $X_s=\beta$, we obtain

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}\left[e^{-\varrho(\tau\wedge t)}\phi_\beta(X_{\tau\wedge t})\right]\\ &\quad\ge \phi_\beta(x) - \mathbb{E}^{\mathbb{Q}}\left[\int_0^{\tau\wedge t} e^{-\varrho s}\left\{\left(f(X_s) + \frac{1}{2\kappa}\xi_s^2\mathbb{1}_{\{\kappa>0\}}\right)ds + dD_s^\beta\right\}\right].\end{aligned} \tag{3.17}$$

Next, we show that the left side of inequality (3.17) vanishes as $t \to \infty$. Because ϕ_β solves (3.3), by considering the continuous function $h_\beta(x) = \frac{|\phi_\beta(x)|}{x}$ on the interval $[0,\beta]$, in which we define $h_\beta(0) = \lim_{x\to 0^+} \frac{|\phi_\beta(x)|}{x} = |\phi'_\beta(0)|$, we obtain that

$$|\phi_\beta(x)| \le C_\beta x$$

for all $x \in [0,\beta]$, in which C_β is a constant depending on β. In addition, because $x \in [0,\beta]$ and X is controlled by the β-threshold strategy D^β, we must have $X_{t\wedge\tau} \le \beta$ for all $t \ge 0$. This bound on X implies

$$|e^{-\varrho(t\wedge\tau)}\phi_\beta(X_{t\wedge\tau})| \le |\phi_\beta(X_{t\wedge\tau})| \le C_\beta \beta. \tag{3.18}$$

Now, because $X_\tau \mathbb{1}_{\{\tau<\infty\}} = 0$ and $\phi_\beta(0) = 0$,

$$\lim_{t\to\infty} e^{-\varrho(t\wedge\tau)}\phi_\beta(X_{t\wedge\tau})\mathbb{1}_{\{\tau<\infty\}} = 0. \tag{3.19}$$

On the other hand

$$\lim_{t\to\infty} |e^{-\varrho(t\wedge\tau)}\phi_\beta(X_{t\wedge\tau})\mathbb{1}_{\{\tau=\infty\}}| \le \lim_{t\to\infty} e^{-\varrho t} C_\beta \beta \mathbb{1}_{\{\tau=\infty\}} = 0. \tag{3.20}$$

From (3.18), (3.19), and (3.20), and from the dominated convergence theorem, we obtain

$$\lim_{t\to\infty} \mathbb{E}\left[e^{-\varrho(t\wedge\tau)\phi_\beta(X_{t\wedge\tau})}\right] = 0.$$

The last limit together with the monotone convergence theorem and the nonnegativity of the terms within the integral in the right side of (3.17) imply

$$\phi_\beta(x) \le \inf_{\mathbb{Q}\in\mathcal{Q}(x)} J(x, D^\beta, \mathbb{Q}; \kappa) \tag{3.21}$$

for all $\kappa > 0$, and for $\kappa = 0$,

$$\phi_\beta(x) \le J(x, D^\beta; 0). \tag{3.22}$$

Case 2. Consider now the case for which $x \in (\beta, \infty)$. From (3.3),

$$\phi_\beta(x) = (x-\beta) + \phi_\beta(\beta).$$

Because the strategy D^β starts with an instantaneous dividend payment of $x - \beta$, there is an immediate payoff of $(x-\beta)$ and, hence, for any $\mathbb{Q} \in \mathcal{Q}(x)$, we have

$$J(x, D^\beta, \mathbb{Q}; \kappa) = (x-\beta) + J(\beta, D^\beta, \mathbb{Q}; \kappa).$$

From (3.21) and (3.22) applied to $x = \beta$, and the last two equalities, we have the desired bound when $x \in [\beta, \infty)$. ∎

We next show that (HJB(κ)) admits a unique smooth solution for every $\kappa \in [0,\infty)$. However, the nonlinearity in the differential equation prevents us from providing an explicit solution as, for example, provided in Asmussen and Taksar [1, Theorem 3.2] for a model with zero reward function and no uncertainty in the model. Instead, we will use the *shooting method* to prove that there exists a unique smooth solution to (HJB(κ)). Cohen [7] applied this approach to a similar HJB equation, one that arises out of a queuing framework. Specifically, for every $s \geq 1$, consider the Cauchy problem

$$\begin{cases} (\varphi^{(s)})''(x) + H_F(x, \varphi^{(s)}(x), (\varphi^{(s)})'(x)) = 0, & x > 0, \\ \varphi^{(s)}(0) = 0, \quad (\varphi^{(s)})'(0) = s, \end{cases} \tag{3.23}$$

in which

$$H_F(x, y, z) := H(x, y, F(z)). \tag{3.24}$$

Here, F is a $\mathcal{C}^1(\mathbb{R})$ mollifier that satisfies

$$F(z) = z \text{ on } [-\bar{s}, \bar{s}], \quad |F| \leq 2\bar{s}, \quad \text{and} \quad |F'| \leq 1, \tag{3.25}$$

in which $\bar{s} = \frac{2m}{\sigma^2 \kappa}$. Therefore, F is Lipschitz continuous with constant 1. For example, we may choose the following mollifier:

$$F(z) = \begin{cases} -(3/2)\bar{s}, & z < -2\bar{s}, \\ (1/2)\bar{s} + 2z + z^2/(2\bar{s}), & -2\bar{s} \leq z < -\bar{s}, \\ z, & -\bar{s} \leq z \leq \bar{s}, \\ -(1/2)\bar{s} + 2z - z^2/(2\bar{s}), & \bar{s} \leq z < 2\bar{s}, \\ (3/2)\bar{s}, & 2\bar{s} \leq z. \end{cases}$$

When $\kappa > 0$, the function F and its derivative are bounded, and because the function f is Lipschitz, H_F is uniformly Lipschitz. Namely, there is a constant L such that for every $(x, y, z), (x', y', z') \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$, one has

$$|H_F(x, y, z) - H_F(x', y', z')| \leq L(|x - x'| + |y - y'| + |z - z'|). \tag{3.26}$$

From Polyanin and Zaitsev [23, section 0.3.1], the Cauchy problem in (3.23) admits a unique $\mathcal{C}^2(\mathbb{R}_+)$ solution when $\kappa > 0$. For $\kappa = 0$, (3.23) is a nonhomogeneous linear equation, and existence of a unique solution follows from classical results; see, for example, Polyanin and Zaitsev [23]. Define $\beta^{(s)}$ by

$$\beta^{(s)} := \inf\{x > 0 : (\varphi^{(s)})'(x) \leq 1\}. \tag{3.27}$$

The smoothness of $\varphi^{(s)}$ implies that

$$\text{if } \beta^{(s)} < \infty, \text{ then } (\varphi^{(s)})'(\beta^{(s)}) = 1. \tag{3.28}$$

The following lemma provides some continuity properties that serve us in the proof of Proposition 3.10. The continuity results are obtained for the norm

$$\|\phi\|_h := \int_0^\infty e^{-hu^2} \left(1 \wedge \sup_{x \in [0,u]} |\phi(x)|\right) du,$$

which is defined for $\phi \in \mathcal{C}(\mathbb{R}_+)$ and $h > 0$.

Lemma 3.14. *The function* $s \mapsto (\varphi^{(s)}, (\varphi^{(s)})', (\varphi^{(s)})'')$ *on* $[1,\infty)$ *is continuous in the* $\|\cdot\|_h$*-norm topology for any* $h > L/2$*, in which* L *is the Lipschitz constant in* (3.26) *for* H_F*. Moreover, the mapping* $s \mapsto \beta^{(s)}$ *is continuous on the set of* $s \geq 1$ *for which either* (i) $\beta^{(s)} = \infty$ *holds or* (ii) *the following two conditions hold:* $\beta^{(s)} < \infty$ *and* $(\varphi^{(s)})''(\beta^{(s)}) \neq 0$*. If* (ii) *holds, we conclude that the mapping* $s \mapsto (\varphi^{(s)}(\beta^{(s)}), (\varphi^{(s)})'(\beta^{(s)}), (\varphi^{(s)})''(\beta^{(s)}))$ *is also continuous.*

Proof. Step 1*: Continuity of* $s \mapsto (\varphi^{(s)}, (\varphi^{(s)})', (\varphi^{(s)})'')$. Fix $s \geq 1$. For every $\delta_1 \in \mathbb{R}$ and $x \in \mathbb{R}_+$, set

$$\Delta\varphi_{s,\delta_1}(x) := \left|\varphi^{(s+\delta_1)}(x) - \varphi^{(s)}(x)\right|,$$

and define $\Delta\varphi'_{s,\delta_1}(x)$ and $\Delta\varphi''_{s,\delta_1}(x)$ similarly. From the initial value of $\varphi^{(s)}$ in (3.23), we deduce

$$\Delta\varphi_{s,\delta_1}(x) = \left|\int_0^x ((\varphi^{(s+\delta_1)})'(y) - (\varphi^{(s)})'(y))dy\right| \leq \int_0^x \Delta\varphi'_{s,\delta_1}(y)dy. \tag{3.29}$$

Furthermore, we also have from (3.23),

$$(\varphi^{(s)})'(x) = s - \int_0^x H_F(y, \varphi^{(s)}(y), (\varphi^{(s)})'(y))dy,$$
$$(\varphi^{(s+\delta_1)})'(x) = s + \delta_1 - \int_0^x H_F(y, \varphi^{(s+\delta_1)}(y), (\varphi^{(s+\delta_1)})'(y))dy.$$

Then, from (3.26), the following inequality holds, uniformly in x, s, and δ_1:

$$\Delta\varphi'_{s,\delta_1}(x) \leq |\delta_1| + L\int_0^x \left(\Delta\varphi_{s,\delta_1}(y) + \Delta\varphi'_{s,\delta_1}(y)\right) dy. \tag{3.30}$$

By substituting (3.29) into (3.30), we obtain

$$\begin{aligned}\Delta\varphi'_{s,\delta_1}(x) &\leq |\delta_1| + L\int_0^x \left(\int_0^y \Delta\varphi'_{s,\delta_1}(z)dz + \Delta\varphi'_{s,\delta_1}(y)\right) dy \\ &= |\delta_1| + L\int_0^x \left((x-y)\Delta\varphi'_{s,\delta_1}(y) + \Delta\varphi'_{s,\delta_1}(y)\right) dy \\ &\leq |\delta_1| + L(1+x)\int_0^x \Delta\varphi'_{s,\delta_1}(y)dy.\end{aligned}$$

Grönwall's inequality implies (see, for example, Willett [25, Theorem 0])

$$\Delta\varphi'_{s,\delta_1}(x) \leq |\delta_1| \left[1 + Lx(1+x)\exp\left(Lx(1+x/2)\right)\right].$$

Thus,

$$\sup_{x\in[0,u]} \Delta\varphi'_{s,\delta_1}(x) \leq |\delta_1| \left[1 + Lu(1+u)\exp\left(Lu(1+u/2)\right)\right]. \tag{3.31}$$

Now we are ready to bound $\|\Delta\varphi'_{s,\delta_1}\|_h$ from above. Specifically,

$$\begin{aligned}\|\Delta\varphi'_{s,\delta_1}\|_h &= \int_0^\infty e^{-hu^2}\left(1 \wedge \sup_{x\in[0,u]} \Delta\varphi'_{s,\delta_1}(x)\right) du \\ &\leq |\delta_1| \int_0^\infty e^{-hu^2}\left[1 + Lu(1+u)e^{Lu(1+u/2)}\right] du = |\delta_1|\, C_1\end{aligned} \tag{3.32}$$

for some $C_1 > 0$ that depends only on L and h, as long as $h > L/2$. Similarly, from (3.29)

$$\sup_{x\in[0,u]} \Delta\varphi_{s,\delta_1}(x) \le u \sup_{x\in[0,u]} \Delta\varphi'_{s,\delta}(x) \le |\delta_1|\, u\,[1 + Lu(1+u)\exp\left(Lu(1+u/2)\right)],$$

in which the last inequality follows from (3.31). Therefore,

$$\begin{aligned}\|\Delta\varphi_{s,\delta_1}\|_h &= \int_0^\infty e^{-hu^2}\left(1 \wedge \sup_{x\in[0,u]} \Delta\varphi_{s,\delta_1}(x)\right) du \\ &\le |\delta_1| \int_0^\infty u e^{-hu^2}\left[1 + Lu(1+u)e^{Lu(1+u/2)}\right] du = |\delta_1|\, C_2\end{aligned} \tag{3.33}$$

for some $C_2 > 0$ that depends only on L and h, as long as $h > L/2$. A similar result holds for $\|\Delta\varphi''_{s,\delta_1}\|_h$ by the relation $(\varphi^{(s)})''(x) = -H_F(x, \varphi^{(s)}(x), (\varphi^{(s)})'(x))$ and the Lipschitz continuity of H_F stated in (3.26). Indeed, for all $x \ge 0$,

$$\begin{aligned}\Delta\varphi''_{s,\delta_1}(x) &= \left| H_F(x, \varphi^{(s+\delta_1)}(x), (\varphi^{(s+\delta_1)})'(x)) - H_F(x, \varphi^{(s)}(x), (\varphi^{(s)})'(x)) \right| \\ &\le L\left(\Delta\varphi_{s,\delta_1}(x) + \Delta\varphi'_{s,\delta_1}(x)\right),\end{aligned}$$

which implies

$$1 \wedge \sup_{x\in[0,u]} \Delta\varphi''_{s,\delta_1}(x) \le L\left(1 \wedge \sup_{x\in[0,u]} \Delta\varphi_{s,\delta_1}(x) + 1 \wedge \sup_{x\in[0,u]} \Delta\varphi'_{s,\delta_1}(x)\right).$$

By integrating with respect to e^{-hu^2} and by using (3.32) and (3.33), we obtain that there exists $C_3 > 0$ depending only on h and L such that

$$\|\Delta\varphi''_{s,\delta_1}\|_h \le \delta_1 C_3. \tag{3.34}$$

The uniform bounds in δ_1, given in (3.32), (3.33), and (3.34), imply the continuity of the map $s \mapsto (\varphi^{(s)}, (\varphi^{(s)})', (\varphi^{(s)})'')$ in the $\|\cdot\|_h$-norm topology.

Step 2: *Continuity of* $s \mapsto \beta^{(s)}$ *under the conditions mentioned in the lemma.* Fix $s \ge 1$. It is enough to show that if $(\varphi^{(s)})''(\beta^{(s)}) \ne 0$, then

$$\limsup_{\delta\to 0} \beta^{(s+\delta)} \le \beta^{(s)} \le \liminf_{\delta\to 0} \beta^{(s+\delta)}. \tag{3.35}$$

The first inequality is obvious when $\beta^{(s)} = \infty$. If $\beta^{(s)} < \infty$ and $(\varphi^{(s)})''(\beta^{(s)}) \ne 0$, we necessarily have $(\varphi^{(s)})''(\beta^{(s)}) < 0$. Otherwise, if $(\varphi^{(s)})''(\beta^{(s)}) > 0$, then (3.28) implies $(\varphi^{(s)})'(\beta^{(s)} - \nu) < 1$ for sufficiently small $\nu > 0$, a contradiction to the definition of $\beta^{(s)}$.

From $(\varphi^{(s)})''(\beta^{(s)}) < 0$, we deduce that, for sufficiently small $\nu > 0$, $(\varphi^{(s)})'(\beta^{(s)} + \nu) < 1$. From the continuity of $s \mapsto (\varphi^{(s)})'$, we obtain that, for every δ_2 with sufficiently small absolute value, $(\varphi^{(s+\delta_2)})'(\beta^{(s)} + \nu) < 1$. Therefore, $\beta^{(s+\delta_2)} < \beta^{(s)} + \nu$ and $\limsup_{\delta\to 0} \beta^{(s+\delta)} \le \beta^{(s)} + \nu$. Because $\nu > 0$ can be arbitrarily small, we get the first inequality in (3.35).

We now turn to proving the second inequality in (3.35). Let $\gamma_1 > 0$ be arbitrary, and set $\hat\beta^{(s)} := \liminf_{\delta\to 0} \beta^{(s+\delta)}$. If $\hat\beta^{(s)} = \infty$, then the second inequality in (3.35) is immediate, so we consider the case when $\hat{\beta}^{(s)} < \infty$. Let $\{\delta_j\}_{j\in\mathbb{N}}$ be a sequence such that $\lim_{j\to\infty}\delta_j = 0$ and $\lim_{j\to\infty}\beta^{(s+\delta_j)} = \hat{\beta}^{(s)}$, with $\beta^{(s+\delta_j)} < \infty$ for every $j \in \mathbb{N}$. Because $\varphi^{(s)} \in \mathcal{C}^2(\mathbb{R}_+)$, we deduce that, for sufficiently large j,

$$\left|(\varphi^{(s)})'(\beta^{(s+\delta_j)}) - (\varphi^{(s)})'(\hat{\beta}^{(s)})\right| < \gamma_1. \tag{3.36}$$

Also, the continuity of $s \mapsto (\varphi^{(s)})'$ implies that, for sufficiently large j,

$$\left|(\varphi^{(s+\delta_j)})'(\beta^{(s+\delta_j)}) - (\varphi^{(s)})'(\beta^{(s+\delta_j)})\right| < \gamma_1. \tag{3.37}$$

Recall $\beta^{(s+\delta_j)} < \infty$; thus, from (3.28), we know $(\varphi^{(s+\delta_j)})'(\beta^{(s+\delta_j)}) = 1$. From (3.36)–(3.37), we obtain

$$\left|(\varphi^{(s)})'(\hat{\beta}^{(s)}) - 1\right| < 2\gamma_1.$$

Because $\gamma_1 > 0$ can be arbitrarily small, we get $(\varphi^{(s)})'(\hat{\beta}^{(s)}) = 1$, which implies $\beta^{(s)} \le \hat{\beta}^{(s)}$. ■

Remark 3.15. Observe that, in the proof of (3.35), the condition $(\varphi^{(s)})''(\beta^{(s)}) \neq 0$ is only used to show the first inequality. The second inequality holds even when $(\varphi^{(s)})''(\beta^{(s)}) = 0$.

Proof of Proposition 3.10. We break the proof into two parts. In the first part, we show existence of a solution to (HJB(κ)); in the second part, we prove existence of uniform bounds for the derivative of this solution and the corresponding parameter β_κ. Note that once existence of a solution is shown, the verification result provided by Propositions 3.8 and 3.9 proves that this solution equals the value function given by (2.7). Thus, we obtain uniqueness.

Existence. We now construct a $\mathcal{C}^2(\mathbb{R}_+)$ solution of (3.3) for some $\beta_\kappa \in \mathbb{R}_+$ that also solves (HJB(κ)) by considering two cases for the parameters of the problem.

Case 1: $2m \le \sigma^2\kappa$. In this case the choice of $\phi(x) = x$ provides a solution to 3.1. This is readily checked since $H(x,\phi(x),\phi'(x)) = \frac{2}{\sigma^2}((m - \frac{\sigma^2\kappa}{2}) + (f(x) - \varrho x)) \le 0$ by the Lipschitz continuity of f in Assumption 3.6, while $\phi'(x) = 1$ for all $x \in [0,\infty)$. Note that the corresponding $\beta_\kappa = 0$.

Case 2: $2m > \sigma^2\kappa$. To solve (3.3), we consider the Cauchy problem given in (3.23). Because below we find a solution φ of (3.23) that satisfies $1 \le \varphi' \le \bar{s}$, with $\bar{s} = \frac{2m}{\sigma^2\kappa}$, it also solves the same ordinary differential equation in (3.23) with H replacing H_F.

The outline of the proof is as follows. First, we prove the existence of $s_\kappa \in [1,\bar{s}]$ for which the parameter $\beta^{(s_\kappa)} \in [0,\infty)$ in (3.27) satisfies the following conditions:

$$(\varphi^{(s_\kappa)})'(x) > 1 \quad\text{on}\quad [0,\beta^{(s_\kappa)}), \tag{3.38}$$

$$(\varphi^{(s_\kappa)})'(\beta^{(s_\kappa)}) = 1, \tag{3.39}$$

and

$$(\varphi^{(s_\kappa)})''(\beta^{(s_\kappa)}) = 0 \quad\text{if}\quad \beta^{(s_\kappa)} > 0. \tag{3.40}$$

Note that (3.40) implies

$$H(\beta^{(s_\kappa)}, \varphi^{(s_\kappa)}(\beta^{(s_\kappa)}), (\varphi^{(s_\kappa)})'(\beta^{(s_\kappa)})) = \frac{2}{\sigma^2}\left(m - \frac{\sigma^2\kappa}{2} - \varrho\varphi^{(s_\kappa)}(\beta^{(s_\kappa)}) + f(\beta^{(s_\kappa)})\right) = 0.$$

Then, we define the function φ by

$$\varphi(x) = \begin{cases} \varphi^{(s_\kappa)}(x), & 0 \le x < \beta^{(s_\kappa)}, \\ \varphi^{(s_\kappa)}(\beta^{(s_\kappa)}) + (x - \beta^{(s_\kappa)}), & \beta^{(s_\kappa)} \le x < \infty, \end{cases} \tag{3.41}$$

and note that $\varphi \in \mathcal{C}^2(\mathbb{R}_+)$ solves (HJB(κ)). Indeed, for $0 \le x \le \beta^{(s_\kappa)}$, $\varphi = \varphi^{(s_\kappa)}$ solves the ODE in (3.3) with $\varphi' \ge 1$. Moreover, for $x > \beta^{(s_\kappa)}$, we have

$$\begin{aligned} &\varphi''(x) + H(x, \varphi(x), \varphi'(x)) \\ &= 0 + \frac{2}{\sigma^2}\left(m - \frac{\sigma^2\kappa}{2} - \varrho\varphi(x) + f(x)\right) \\ &= \frac{2}{\sigma^2}\left(m - \frac{\sigma^2\kappa}{2} - \varrho\varphi^{(s_\kappa)}(\beta^{(s_\kappa)}) + f(\beta^{(s_\kappa)})\right) \\ &\quad - \frac{2}{\sigma^2}\left(\varrho(x - \beta^{(s_\kappa)}) - (f(x) - f(\beta^{(s_\kappa)}))\right) \\ &= 0 - \frac{2}{\sigma^2}\left(\varrho(x - \beta^{(s_\kappa)}) - (f(x) - f(\beta^{(s_\kappa)}))\right) \le 0, \end{aligned}$$

in which the last inequality follows by the Lipschitz continuity of f in Assumption 3.6. Finally, the proof of the existence part is complete by setting

$$\beta_\kappa := \beta^{(s_\kappa)}, \qquad \text{and} \qquad g_{\beta_\kappa} := \varphi, \tag{3.42}$$

in which φ is given in (3.41).[2] As a conclusion, we get

$$s_\kappa = (g_{\beta_\kappa})'(0). \tag{3.43}$$

The rest of the proof in this part is, therefore, dedicated to showing the existence of s_κ that satisfies (3.38)–(3.40). We break the remaining part of our proof into two steps: (1) showing the existence of a sufficiently small $s_1 > 1$ for which $(\varphi^{(s_1)})''(\beta^{(s_1)}) < 0$; (2) showing the existence of s_κ such that (3.38)–(3.40) hold.

Step 1. Set $1 \le s \le \bar{s} = \frac{2m}{\sigma^2\kappa}$. For $0 \le x \le \beta^{(s)}$ (which implies $1 \le (\varphi^{(s)})'(x) \le s \le \bar{s}$), we have

$$(\varphi^{(s)})''(x) = \kappa((\varphi^{(s)})'(x))^2 - \frac{2m}{\sigma^2}(\varphi^{(s)})'(x) + \frac{2}{\sigma^2}\left(\varrho\varphi^{(s)}(x) - f(x)\right). \tag{3.44}$$

Then, note that $\bar{s}_1 \in (1, \bar{s})$, in which

$$\bar{s}_1 := \inf\left\{ s > 1 : 8\varrho s(s-1) > \sigma^2\left[s^2\left(\frac{2m}{\sigma^2} - \kappa s\right)^2 \wedge \left(\frac{2m}{\sigma^2} - \kappa\right)^2\right]\right\}. \tag{3.45}$$

[2]Recall that in the proof of Theorem 3.7, we use g_{β_κ} to denote the $\mathcal{C}^2(\mathbb{R}_+)$ solution of (HJB(κ)). In the rest of the proof, we write g_{β_κ} instead of φ to highlight the solution's dependence on κ via β_κ.

Fix an arbitrary $s_1 \in (1, \bar{s}_1)$, and set

$$M := \min\left\{ s_1 \left(\frac{2m}{\sigma^2} - \kappa s_1 \right), \frac{2m}{\sigma^2} - \kappa \right\}, \qquad \text{and} \qquad N := \frac{2(s_1 - 1)}{M}. \tag{3.46}$$

Note that $M > 0$ and $N > 0$. On the interval $[0, N]$, $(\varphi^{(s_1)})' \le s_1$. Indeed, if that is not the case, we have $y_{s_1} = \inf\{x \ge 0 : (\varphi^{(s_1)})'(x) > s_1\} \le N$. By continuity of $(\varphi^{(s_1)})'$ and the definition of y_{s_1}, we have $(\varphi^{(s)})'(y_{s_1}) = s_1$ and $(\varphi^{(s_1)})''(y_{s_1}) > 0$. However, from (3.44), (3.45), and (3.46), we compute

$$(\varphi^{(s_1)})''(y_{s_1}) = \kappa s_1^2 - \frac{2m}{\sigma^2} s_1 + \frac{2}{\sigma^2}(\varrho s_1 y_{s_1} - f(y_{s_1})) \le -M + \frac{2}{\sigma^2} \varrho s_1 y_{s_1} \le -\frac{M}{2} < 0,$$

a contradiction. Consequently, we have $(\varphi^{(s_1)})' \le s_1$ on $[0, N]$. Furthermore, recall that $(\varphi^{(s_1)})'(x) \ge 1$ for $x \in [0, \beta^{(s_1)}]$. These two bounds on $(\varphi^{(s_1)})'$ allow us to bound $(\varphi^{(s_1)})''$ on $[0, N \wedge \beta^{(s_1)}]$. Specifically,

$$(\varphi^{(s_1)})''(x) \le -M + \frac{2}{\sigma^2} \varrho s_1 x \le -\frac{M}{2} \qquad \text{for all } x \in [0, N \wedge \beta^{(s_1)}]. \tag{3.47}$$

By integrating both sides of inequality (3.47), and by using $(\varphi^{(s_1)})'(0) = s_1$, we obtain

$$(\varphi^{(s_1)})'(x) \le s_1 - \frac{M}{2} x \qquad \text{for all } x \in [0, N \wedge \beta^{(s_1)}].$$

If $\beta^{(s_1)} > N$, then the inequality above holds for $x = N$, that is, $(\varphi^{(s_1)})'(N) \le 1$. From the definition of $\beta^{(s_1)}$, we then deduce that $\beta^{(s_1)} \le N$, a contradiction to $\beta^{(s_1)} > N$. This contradiction implies that we must have $\beta^{(s_1)} \le N$, so from (3.47),

$$(\varphi^{(s_1)})''(\beta^{(s_1)}) \le -\frac{M}{2} < 0,$$

as we wished to show.

Step 2. In this step, we show the existence of $s_\kappa \in (1, \bar{s}]$ that satisfies (3.38)–(3.40). We define

$$s_\kappa := \sup\left\{ s \in (s_1, \bar{s}) : \text{for all } s_1 < u < s,\ (\varphi^{(u)})''(\beta^{(u)}) < 0 \ \text{ and } \ \beta^{(u)} < \infty \right\}. \tag{3.48}$$

Observe that s_κ is potentially infinite when $\kappa = 0$. Assume first that $s_\kappa < \infty$. We will show that $\beta^{(s_\kappa)} < \infty$. If not, then we have $\beta^{(s_\kappa)} = \infty$, and Lemma 3.14 implies $\lim_{s \to s_\kappa} \beta^{(s)} = \infty$. Consequently,

$$\varrho \varphi(\beta^{(s)}) - f(\beta^{(s)}) \ge \varrho \beta^{(s)} - f(\beta^{(s)}) \ge \delta \beta^{(s)} \to \infty, \text{ as } s \to s_\kappa,$$

in which $\varrho - \delta$ is the Lipschitz constant for f in Assumption 3.6. By applying this limit to (3.44), we obtain

$$(\varphi^{(s)})''(\beta^{(s)}) = \kappa \left(1 - \frac{2m}{\sigma^2 \kappa} \right) + \frac{2}{\sigma^2} \left(\varrho \varphi^{(s)}(\beta^{(s)}) - f(\beta^{(s)}) \right) \to \infty, \text{ as } s \to s_\kappa,$$

which contradicts $(\varphi^{(s)})''(\beta^{(s)}) \le 0$. Thus, we have $\beta^{(s_\kappa)} < \infty$.

When $\kappa = 0$, we now show that it is not possible for s_κ in (3.48) to be infinite. If $s_\kappa = \infty$, we must have $\beta^{(s)} < \infty$ for all $s > s_1$. By following the same logic as in the first paragraph of Step 2, we deduce $\lim_{s\to\infty} \beta^{(s)} < \infty$, which implies that $\beta^{(s)}$ is uniformly bounded as $s \to \infty$. This uniform bound implies that $f(\beta^{(s)})$ is also uniformly bounded. By substituting $x = \beta^{(s)}$ in (3.44) and using $(\varphi^{(s)})''(\beta^{(s)}) \le 0$ and $(\varphi^{(s)})'(\beta^{(s)}) = 1$, we obtain that $\varphi^{(s)}(\beta^{(s)})$ is also uniformly bounded. Assumption 3.6 and (3.44) with $\kappa = 0$ imply, for $0 \le x \le \beta^{(s)}$,

$$\begin{aligned}(\varphi^{(s)})''(x) &= -\frac{2m}{\sigma^2}(\varphi^{(s)})'(x) + \frac{2}{\sigma^2}(\varrho\varphi^{(s)}(x) - f(x))\\ &\ge -\frac{2m}{\sigma^2}(\varphi^{(s)})'(x) + \frac{2}{\sigma^2}\delta x.\end{aligned} \tag{3.49}$$

By integrating both sides of the inequality from $x = 0$ to $x = \beta^{(s)}$, we get

$$(\varphi^{(s)})'(\beta^{(s)}) \ge s - \frac{2m}{\sigma^2}\varphi^{(s)}(\beta^{(s)}) + \frac{\delta(\beta^{(s)})^2}{\sigma^2} \to \infty, \text{ as } s \to \infty,$$

which contradicts $(\varphi^{(s)})''(\beta^{(s)}) \le 0$. Thus, when $\kappa = 0$, then s_κ in (3.48) is finite.

Observe that, because $\beta^{(s_\kappa)} < \infty$, (3.28) implies $(\varphi^{(s_\kappa)})'(\beta^{(s_\kappa)}) = 1$. Next, we show that the smooth pasting boundary condition (3.40) is satisfied for the above choice of s_κ, that is, $(\varphi^{(s_\kappa)})''(\beta^{(s_\kappa)}) = 0$. If not, then $(\varphi^{(s_\kappa)})''(\beta^{(s_\kappa)}) \ne 0$, and by Lemma 3.14, we have continuity of the mapping $s \mapsto (\beta^{(s)}, (\varphi^{(s)})''(\beta^{(s)}))$ at $s = s_\kappa$. In particular, if $(\varphi^{(s_\kappa)})''(\beta^{(s_\kappa)}) > 0$, there exists $\nu > 0$ sufficiently small such that $\beta^{(s_\kappa-\nu)} < \infty$ and $(\varphi^{(s_\kappa-\nu)})''(\beta^{(s_\kappa-\nu)}) > 0$, contradicting the definition of s_κ in (3.48). On the other hand if $(\varphi^{(s_\kappa)})''(\beta^{(s_\kappa)}) < 0$, there exists $\nu_0 > 0$ sufficiently small such that $\beta^{(s_\kappa+\nu)} < \infty$ and $(\varphi^{(s_\kappa+\nu)})''(\beta^{(s_\kappa+\nu)}) < 0$ for all $\nu \in [0, \nu_0)$, again contradicting the definition of s_κ.

Boundedness of β_κ *and* $(g_{\beta_\kappa})'$. We now show that the solution to (HJB(κ)) has bounded derivatives uniformly in κ. To that aim, we provide some additional insight about the solution g_{β_κ} and $\varphi^{(s_\kappa)}$, in which s_κ is given in (3.48); note that s_κ depends on κ. We have shown that for each $\kappa \ge 0$, $\beta_\kappa = \beta^{(s_\kappa)} < \infty$. Consequently, because $(\varphi^{(s_\kappa)})'(\beta^{(s_\kappa)}) = 1$ and $(\varphi^{(s_\kappa)})''(\beta^{(s_\kappa)}) = 0$, Assumption 3.6 and (3.44) imply

$$\begin{aligned}0 &= \kappa - \frac{2m}{\sigma^2} + \frac{2}{\sigma^2}(\varrho\varphi^{(s_\kappa)}(\beta^{(s_\kappa)}) - f(\beta^{(s_\kappa)}))\\ &\ge \kappa - \frac{2m}{\sigma^2} + \frac{2}{\sigma^2}\delta\beta^{(s_\kappa)}.\end{aligned} \tag{3.50}$$

Thus, $\beta_\kappa = \beta^{(s_\kappa)} \le \frac{2m-\sigma^2\kappa}{2\delta} \le \frac{m}{\delta}$, uniformly in $\kappa \ge 0$.

Therefore, for any $\kappa \ge 0$, we can restrict our analyses of $\varphi^{(s_\kappa)}$ or related ODEs to the interval $[0, m/\delta]$ as long as the coefficients are continuous, for which existence and uniqueness of solution follows from classical results. Consider the case $2m > \sigma^2\kappa$ because the other cases are trivial.

We first find a uniform-in-κ bound for s_κ. Observe that, from (3.49),

$$(\varphi^{(s_\kappa)})''(x) \ge -\frac{2m}{\sigma^2}(\varphi^{(s_\kappa)})'(x)$$

for $x \in [0, \beta_\kappa]$. Because $(\varphi^{(s_\kappa)})'(x) \geq 1$ for $x \in [0, \beta_\kappa]$, we have

$$\frac{(\varphi^{(s_\kappa)})''(x)}{(\varphi^{(s_\kappa)})'(x)} \geq -\frac{2m}{\sigma^2}.$$

By integrating this inequality from 0 to $x \in [0, \beta_\kappa] \subset [0, m/\delta]$, and by exponentiating the result, we obtain

$$(\varphi^{(s_\kappa)})'(x) \geq s_\kappa \exp(-2mx/\sigma^2).$$

If $s_\kappa > \exp(2m^2/(\sigma^2\delta))$, then $(\varphi^{(s_\kappa)})'$ is never equal to 1 on $[0, \beta_\kappa = \beta^{(s_\kappa)}]$, which contradicts the fact that $(\varphi^{(s_\kappa)})'(\beta^{(s_\kappa)}) = 1$. Consequently, we have a uniform bound

$$s_\kappa \leq \exp(2m^2/(\sigma^2\delta)) < 1 + \exp(2m^2/(\sigma^2\delta)) =: \tilde{s} \tag{3.51}$$

for all $\kappa \geq 0$.

Next, observe that, because $\kappa < \frac{2m}{\sigma^2}$, then from (3.44), we have

$$(\varphi^{(s)})''(x) < \frac{2m}{\sigma^2}((\varphi^{(s)})'(x))^2 + \frac{2\varrho}{\sigma^2}\varphi^{(s)}(x)$$

for all $x \in [0, \beta^{(s)}]$ and for all $1 \leq s < \tilde{s}$. We claim that for any $s < \tilde{s}$, $(\varphi^{(s)})'(x) \leq (\Psi^{(\tilde{s})})'(x)$, in which $\Psi^{(\tilde{s})} \in \mathcal{C}^2(\mathbb{R}_+)$ uniquely solves

$$(\Psi^{(\tilde{s})})''(x) = \frac{2m}{\sigma^2}((\Psi^{(\tilde{s})})'(x))^2 + \frac{2\varrho}{\sigma^2}\Psi^{(\tilde{s})}, \qquad \Psi^{(\tilde{s})}(0) = 0, \qquad (\Psi^{(\tilde{s})})'(0) = \tilde{s}.$$

If not, let $s < \tilde{s}$ and $\hat{x}_s = \hat{x} = \inf\{x > 0 : (\varphi^{(s)})'(x) > (\Psi^{(\tilde{s})})'(x)\}$, which implies $(\varphi^{(s)})'(x) \leq (\Psi^{(\tilde{s})})'(x)$ and, thus, $\varphi^{(s)}(x) \leq \Psi^{(\tilde{s})}(x)$ for all $x \in [0, \hat{x}]$. Note that $\hat{x} > 0$ because $(\varphi^{(s)})'(0) = s < \tilde{s} = (\Psi^{(\tilde{s})})'(0)$. We then obtain

$$\begin{aligned}
(\varphi^{(s)})'(\hat{x}) - s = \int_0^{\hat{x}} (\varphi^{(s)})''(u)du &< \int_0^{\hat{x}} \left(\frac{2m}{\sigma^2}((\varphi^{(s)})'(u))^2 + \frac{2}{\sigma^2}\varrho\varphi^{(s)}(u)\right) du \\
&\leq \int_0^{\hat{x}} \left(\frac{2m}{\sigma^2}((\Psi^{(\tilde{s})})'(u))^2 + \frac{2}{\sigma^2}\varrho\Psi^{(\tilde{s})}(u)\right) du \\
&= \int_0^{\hat{x}} (\Psi^{(\tilde{s})})''(u)du = (\Psi^{(\tilde{s})})'(\hat{x}) - \tilde{s},
\end{aligned}$$

which implies

$$(\varphi^{(s)})'(\hat{x}) < (\Psi^{(\tilde{s})})'(\hat{x}) - (\tilde{s} - s) < (\Psi^{(\tilde{s})})'(\hat{x}),$$

thereby contradicting the definition of $\hat{x}$. Thus, $(\varphi^{(s)})'(x) \leq (\Psi^{(\tilde{s})})'(x)$ for all $x \in [0, \beta^{(s)}]$ and all $s < \tilde{s}$. Because $\sup_{x \in [0, m/\delta]} |(\Psi^{(\tilde{s})})'(x)|$ is finite, we have a bound for $(\varphi^{(s_\kappa)})'$ on $[0, \beta_\kappa]$, uniformly in κ. We thus deduce that the solution g_{β_κ} of (HJB(κ)), for any $\kappa \geq 0$, satisfies

$$(g_{\beta_\kappa})'(x) \leq \bar{c}, \qquad x \geq 0, \tag{3.52}$$

in which the constant $\bar{c}$ is independent of κ. ■

Remark 3.16. Because we have proved the existence of an upper bound for $(g_{\beta_\kappa})'$, uniform in κ, we can replace $\bar{s} = 2m/(\sigma^2\kappa)$ in the definition of H_F in (3.24) with the constant $\bar{c}$.

Proof of Proposition 3.11. Recall the definition of β_κ from Theorem 3.7. Define

$$\hat{\beta}_\kappa := \sup\left\{x > 0 : \text{for all } y \le x,\ V''(y;\kappa) + H(y, V(y;\kappa), V'(y;\kappa)) = 0\right\}. \tag{3.53}$$

From Proposition 3.10 along with the relation (3.3) it follows that $\hat{\beta}_\kappa \ge \beta_\kappa$. Because on the interval $[\beta_\kappa, \hat{\beta}_\kappa]$, both of the conditions

$$V''(x;\kappa) + H(x, V(y;\kappa), V'(x;\kappa)) = 0 \qquad \text{and} \qquad V'(x;\kappa) = 1 \tag{3.54}$$

hold, it follows that V solves (HJB(κ)) for any $\beta \in [\beta_\kappa, \hat{\beta}_\kappa]$. Propositions 3.8 and 3.9 imply that every such β-threshold strategy is optimal. Thus, the nonuniqueness of the optimal threshold strategy is equivalent to the existence of nondegenerate interval $[\beta_\kappa, \hat{\beta}_\kappa]$, on which the equations in (3.54) hold. By combining them, we get $V(x;\kappa) = (m - \sigma^2\kappa/2 + f(x))/\varrho$. By using again $V'(x;\kappa) = 1$, we deduce that $f' \equiv \varrho$ on $[\beta_\kappa, \hat{\beta}_\kappa]$, which contradicts Assumption 3.6, namely, that f is Lipschitz continuous with the Lipschitz constant strictly smaller than ϱ. We have thereby shown uniqueness of the optimal threshold strategy. ■

Proof of Proposition 3.12. From Propositions 3.8–3.10, we know that the value function $V(\cdot;\kappa)$ given by (2.7) satisfies the conditions in (3.3) for $\beta = \beta_\kappa$. Thus, $V''(x;\kappa) = 0$ for $x \ge \beta_\kappa$, while for $x < \beta_\kappa$, we have

$$V''(x;\kappa) = \kappa(V'(x;\kappa))^2 - \frac{2m}{\sigma^2}V'(x;\kappa) + \frac{2}{\sigma^2}\left(\varrho V(x;\kappa) - f(x)\right). \tag{3.55}$$

To derive a contradiction, assume there exists some $x_0 < \beta_\kappa$ such that $V''(x_0;\kappa) > 0$. Thus, $V'(x_1;\kappa) > V'(x_0;\kappa) > 1$ for some $x_1 > x_0$, sufficiently close to x_0. However, because $\beta_\kappa < \infty$ and $V'(\beta_\kappa;\kappa) = 1$, in order for V' to penetrate the $y = V'(x_0;\kappa)$ barrier, there must exist $z \in (x_1, \beta_\kappa)$ such that $V'(z;\kappa) = V'(x_0;\kappa)$ and $V''(z;\kappa) \le 0$. From Assumption 3.6, note that $f(z) \le f(x_0) + (\varrho - \delta)(z - x_0)$. Furthermore, since $V' \ge 1$, we have $V(z;\kappa) \ge V(x_0;\kappa) + (z - x_0)$. Thus, from (3.55), we get

$$\begin{aligned} V''(z;\kappa) &= \kappa(V'(z;\kappa))^2 - \frac{2m}{\sigma^2}V'(z;\kappa) + \frac{2}{\sigma^2}\left(\varrho V(z;\kappa) - f(z)\right) \\ &> \kappa(V'(x_0;\kappa))^2 - \frac{2m}{\sigma^2}V'(x_0;\kappa) + \frac{2}{\sigma^2}\left(\varrho V(x_0;\kappa) - f(x_0)\right) = V''(x_0;\kappa) > 0, \end{aligned}$$

which contradicts $V''(z;\kappa) \le 0$. Hence, $V''(x;\kappa) \le 0$ for all $x \in [0,\infty)$. ■

4. Optimal strategy and dependence on the ambiguity parameter.

In this section, we study the dependence of the value function V and optimal threshold β_κ on the ambiguity parameter κ. We show continuity of V and β_κ on κ, and we show that, as $\kappa \to 0^+$, our model converges to the risk-neutral model. For the latter, recall the definition of $V(\cdot;0)$ given in (2.10).

Theorem 4.1. *The mapping $[0,\infty) \ni \kappa \mapsto V(x;\kappa)$ is decreasing and continuous, uniformly in $x \in \mathbb{R}_+$. Moreover, there is a constant $C > 0$ such that for every $\kappa \in (0,\infty)$, $\sup_{x\in[0,\infty)} |V(x;\kappa) - V(x;0)| \le C \cdot \kappa$. Also, $\lim_{\kappa\to\infty} V(x;\kappa) = x$ for all $x \in \mathbb{R}_+$.*

Proof. We start by showing the monotonicity and continuity of the mapping $(0,\infty)\ni\kappa\mapsto V(\cdot;\kappa)$. The proof for $\kappa=0$ is given separately. Fix $0<\kappa_1<\kappa_2$. Let $\mathbb{Q}^i$ and $\xi_i=\{\xi_{i,t}\}_{t\geq 0}$ denote $\mathbb{Q}^{V(\cdot;\kappa_i)}$ and $\xi^{V(\cdot;\kappa_i)}=\{\xi_t^{V(\cdot;\kappa_i)}\}_{t\geq 0}$, respectively, for $i=1,2$. Then, for every $x>0$, from (3.5), we have

$$V(x;\kappa_1)=\sup_{D\in\mathcal{A}(x)} J(x,D,\mathbb{Q}^1;\kappa_1)$$
$$=\sup_{D\in\mathcal{A}(x)}\left[J(x,D,\mathbb{Q}^1;\kappa_2)+\frac{1}{2}\left(\frac{1}{\kappa_1}-\frac{1}{\kappa_2}\right)\int_0^\tau e^{-\varrho t}\mathbb{E}^{\mathbb{Q}^1}[(\xi_{1,t})^2]dt\right].$$

Because $\kappa_2>\kappa_1>0$ and $\xi_{1,t}=-\kappa_1\sigma V'(X_t;\kappa_1)\leq-\kappa_1\sigma<0$, we deduce

$$V(x;\kappa_1)>\sup_{D\in\mathcal{A}(x)} J(x,D,\mathbb{Q}^1;\kappa_2)\geq\sup_{D\in\mathcal{A}(x)}\inf_{\mathbb{Q}\in\mathcal{Q}(x)} J(x,D,\mathbb{Q};\kappa_2)=V(x;\kappa_2). \tag{4.1}$$

We have thus shown $\kappa\mapsto V(\cdot;\kappa)$ is strictly decreasing on the interval $(0,\infty)$. We argue monotonicity at $\kappa=0$ as follows: for $x>0$ and $\kappa_1>0$,

$$\begin{aligned} V(x;0)&=\sup_{D\in\mathcal{A}(x)} J(x,D,\mathbb{P};\kappa_1)\\ &\geq\sup_{D\in\mathcal{A}(x)}\inf_{\mathbb{Q}\in\hat{\mathcal{Q}}(x)} J(x,D,\mathbb{Q};\kappa_1)\\ &=V(x;\kappa_1). \end{aligned} \tag{4.2}$$

Also, for $0<\kappa_1<\kappa_2$,

$$V(x;\kappa_1)=\inf_{\mathbb{Q}\in\mathcal{Q}(x} J(x,D^1,\mathbb{Q};\kappa_1)\leq J(x,D^1,\mathbb{Q}^2;\kappa_1)\leq\sup_{D\in\mathcal{A}(x)} J(x,D,\mathbb{Q}^2;\kappa_1)$$
$$=\sup_{D\in\mathcal{A}(x)}\left[J(x,D,\mathbb{Q}^2;\kappa_2)+\frac{1}{2}\left(\frac{1}{\kappa_1}-\frac{1}{\kappa_2}\right)\int_0^\tau e^{-\varrho t}\mathbb{E}^{\mathbb{Q}^2}[(\xi_{2,t})^2]dt\right].$$

Note that by using (3.52), $\xi_{2,t}=-\kappa_2\sigma V'(X_t;\kappa_2)\geq-\kappa_2\sigma\bar{c}$. This bound implies $(\xi_{2,t})^2\leq\kappa_2^2\sigma^2\bar{c}^2$ and, consequently,

$$\begin{aligned} V(x;\kappa_1)&\leq\sup_{D\in\mathcal{A}(x)}\left[J(x,D,\mathbb{Q}^2;\kappa_2)+\frac{1}{2}\left(\frac{1}{\kappa_1}-\frac{1}{\kappa_2}\right)\kappa_2^2\sigma^2\bar{c}^2\frac{1-e^{-\varrho\tau}}{\varrho}\right]\\ &=V(x;\kappa_2)+(\kappa_2-\kappa_1)\frac{\kappa_2}{\kappa_1}\frac{\sigma^2\bar{c}^2}{2\varrho}. \end{aligned} \tag{4.3}$$

By combining (4.1) and (4.3), we obtain

$$V(x;\kappa_2)<V(x;\kappa_1)\leq V(x;\kappa_2)+\frac{\kappa_2\sigma^2\bar{c}^2}{2\kappa_1\varrho}(\kappa_2-\kappa_1),$$

from which follows continuity of $\kappa\mapsto V(\cdot;\kappa)$ on the interval $(0,\infty)$, uniformly in x.

We next prove continuity of $\kappa\mapsto V(\cdot;\kappa)$ at $\kappa=0$. Note that in the arguments above, we cannot relax the inequality $\kappa_1>0$ to $\kappa_1\geq 0$ because we divide by κ_1. Therefore, we use a different proof to show continuity at $\kappa=0$. Let β_0 denote the optimal threshold for the risk-neutral problem. Let $\kappa > 0$, and let $\mathbb{Q}^\kappa$ and ξ^κ denote, respectively, $\mathbb{Q}^{V(\cdot;\kappa)}$ and $\xi^{V(\cdot;\kappa)}$. Observe that, from (4.2),

$$V(x;0) \geq V(x;\kappa) = \sup_{D\in\mathcal{A}(x)} J(x,D,\mathbb{Q}^\kappa;\kappa) \geq J(x,D^{\beta_0},\mathbb{Q}^\kappa;\kappa). \tag{4.4}$$

From (2.9) and (3.4),

$$\begin{aligned} J(x,D^{\beta_0},\mathbb{Q}^\kappa;\kappa) = \mathbb{E}^{\mathbb{Q}^\kappa}&\left[\int_0^\tau e^{-\varrho t}\left\{f(X_t^{\beta_0})dt + dD_t^{\beta_0}\right\}\right] \\ &+ \frac{\kappa\sigma^2}{2}\mathbb{E}^{\mathbb{Q}^\kappa}\left[\int_0^\tau e^{-\varrho t}(V'(X_t;\kappa))^2 dt\right]. \end{aligned} \tag{4.5}$$

Because $1 \leq V' \leq \bar{c}$, the second term in the right side of (4.5) vanishes as $\kappa \to 0^+$. Thus, owing to (4.4) and (4.5), and by recalling the payoff function for the risk-neutral problem, to show continuity of $\kappa \mapsto V(\cdot;\kappa)$ at $\kappa = 0$, it is enough to show

$$\lim_{\kappa\to 0^+} \mathbb{E}^{\mathbb{Q}^\kappa}\left[\int_0^\tau e^{-\varrho t}\left\{f(X_t^{\beta_0})dt + dD_t^{\beta_0}\right\}\right] = \mathbb{E}^{\mathbb{P}}\left[\int_0^\tau e^{-\varrho t}\left\{f(X_t^{\beta_0})dt + dD_t^{\beta_0}\right\}\right]. \tag{4.6}$$

To that end, we construct two coupled processes, one each for $\kappa = 0$ and $\kappa > 0$, on the probability space $(\Omega,\mathcal{F},\mathbb{F},\check{\mathbb{P}})$ that supports a one-dimensional standard Brownian motion B. By using Definition 3.2, we construct the processes as follows: for $t \geq 0$,

$$\begin{aligned} (X_t^\kappa, D_t^\kappa) &:= \Gamma_{\beta_0}\left(x + m\,\cdot - \int_0^\cdot \kappa\sigma^2 V'(X_s^\kappa;\kappa)ds + \sigma B_\cdot\right)_t, \\ (X_t^0, D_t^0) &:= \Gamma_{\beta_0}(x + m\,\cdot + \sigma B_\cdot)_t. \end{aligned} \tag{4.7}$$

Please refer to [20] or [24] for the existence and uniqueness of a solution of (4.7). Note that, by (3.4), (X^0, D^0) (resp., $(X^\kappa, D^{\beta_\kappa})$) has the same distribution under the measure $\check{\mathbb{P}}$ as $(X^{\beta_0}, D^{\beta_0})$ under the measure $\mathbb{P}$ (resp., $\mathbb{Q}^\kappa$). Hence, (4.6) is equivalent to

$$\lim_{\kappa\to 0^+} \mathbb{E}^{\check{\mathbb{P}}}\left[\int_0^\tau e^{-\varrho t}\left\{f(X_t^\kappa)dt + dD_t^\kappa\right\}\right] = \mathbb{E}^{\check{\mathbb{P}}}\left[\int_0^\tau e^{-\varrho t}\left\{f(X_t^0)dt + dD_t^0\right\}\right]. \tag{4.8}$$

Now

$$\begin{aligned} &\left|\mathbb{E}^{\check{\mathbb{P}}}\left[\int_0^\tau e^{-\varrho t}\left\{f(X_t^\kappa)dt + dD_t^\kappa\right\}\right] - \mathbb{E}^{\check{\mathbb{P}}}\left[\int_0^\tau e^{-\varrho t}\left\{f(X_t^0)dt + dD_t^0\right\}\right]\right| \\ &\quad = \left|\mathbb{E}^{\check{\mathbb{P}}}\left[\int_0^\tau e^{-\varrho t}(f(X_t^\kappa) - f(X_t^0))dt + \int_0^\tau e^{-\varrho t}(dD_t^\kappa - dD_t^0)\right]\right| \\ &\quad \leq \mathbb{E}^{\check{\mathbb{P}}}\left[\int_0^\tau e^{-\varrho t}\left|f(X_t^\kappa) - f(X_t^0)\right|dt + \int_0^\tau \varrho e^{-\varrho t}\left|D_t^\kappa - D_t^0\right|dt + e^{-\varrho\tau}\left|D_\tau^\kappa - D_\tau^0\right|\right]. \end{aligned}$$

From the Lipschitz continuity of f in Assumption 3.6 and from Lemma 3.1, we obtain

$$\begin{aligned}
&\left|\mathbb{E}^{\check{\mathbb{P}}}\left[\int_0^\tau e^{-\varrho t}\left\{f(X_t^\kappa)dt + dD_t^\kappa\right\}\right] - \mathbb{E}^{\check{\mathbb{P}}}\left[\int_0^\tau e^{-\varrho t}\left\{f(X_t^0)dt + dD_t^0\right\}\right]\right| \\
&\quad \le \mathbb{E}^{\check{\mathbb{P}}}\left[\int_0^\tau \varrho e^{-\varrho t}\left(\left|X_t^\kappa - X_t^0\right| + \left|D_t^\kappa - D_t^0\right|\right)dt + e^{-\varrho\tau}\left|D_\tau^\kappa - D_\tau^0\right|\right] \\
&\quad \le \mathbb{E}^{\check{\mathbb{P}}}\left[\int_0^\tau \varrho e^{-\varrho t} c_S \sup_{s\in[0,t]}\left|\kappa\sigma^2\int_0^s V'(X_u^\kappa;\kappa)du\right| dt \right.\\
&\qquad \left. + e^{-\varrho\tau}\sup_{s\in[0,\tau]}\left|\kappa\sigma^2\int_0^s V'(X_u^\kappa;\kappa)du\right|\right] \\
&\quad \le \mathbb{E}^{\check{\mathbb{P}}}\left[\kappa\sigma^2 c_S\bar{c}\int_0^\tau \varrho t e^{-\varrho t}dt + \tau e^{-\varrho\tau}\kappa\sigma^2\bar{c}\right] \le \frac{\kappa\bar{c}\sigma^2}{\varrho}(c_S + e^{-1}) \to 0,
\end{aligned}$$

as $\kappa \to 0^+$, as required.

We now turn to proving the limit $\lim_{\kappa\to\infty} V(x;\kappa) = x$. To show this limit, given $\kappa > 0$, consider the probability measure $\mathbb{Q}^\kappa$ that is associated with the strategy $\xi_t^\kappa \equiv -\kappa^{1/4}$ for $t \ge 0$. Under ξ^κ, for any $D \in \mathcal{A}(x)$, X follows the dynamics

$$X_t = x + (m - \sigma\kappa^{3/4})t + \sigma B_t^{\mathbb{Q}^\kappa} - D_t =: Z_t - D_t \tag{4.9}$$

for $t \ge 0$, in which $B^{\mathbb{Q}^\kappa}$ is the $\mathbb{F}$-standard Brownian motion under $\mathbb{Q}^\kappa$ given by $B_t^{\mathbb{Q}^\kappa} = B_t + \kappa^{1/4}t$. The payoff function is given by

$$\begin{aligned}
J(x, D, \mathbb{Q}^\kappa;\kappa) &= \mathbb{E}^{\mathbb{Q}^\kappa}\left[\int_0^\tau e^{-\varrho t}\left(f(X_t)dt + dD_t + \frac{1}{2\kappa}(\xi_t^\kappa)^2 dt\right)\right] \\
&\le \mathbb{E}^{\mathbb{Q}^\kappa}\left[\int_0^\tau e^{-\varrho t}\left(f(X_t)dt + dD_t\right)\right] + \frac{1}{2\varrho\sqrt{\kappa}} \\
&\le \mathbb{E}^{\mathbb{Q}^\kappa}\left[\int_0^\tau e^{-\varrho t}\left(f(Z_t) + \varrho Z_t\right)dt + e^{-\varrho\tau}D_\tau\right] + \frac{1}{2\varrho\sqrt{\kappa}},
\end{aligned} \tag{4.10}$$

in which the second inequality follows from integration by parts and $Z_t \ge \min(X_t, D_t)$ for all $t \ge 0$. Furthermore, one can check that $\tau^* \to 0^+$ $\mathbb{Q}^\kappa$-a.s. as $\kappa \to \infty$, in which $\tau^* := \inf\{t > 0 : Z_t = 0\} \ge \tau$. Consequently, $\tau \to 0$ $\mathbb{Q}^\kappa$-a.s. as $\kappa \to \infty$. Because $X_\tau = 0$, by right continuity of D, we also have $D_\tau \to x$ $\mathbb{Q}^\kappa$-a.s. as $\kappa \to \infty$. By using these limits, we see that the expectation in (4.10) converges to x as $\kappa \to \infty$. We thus have for any $D \in \mathcal{A}(x)$,

$$\lim_{\kappa\to\infty} J(x, D, \mathbb{Q}^\kappa;\kappa) = x,$$

and our result follows. ■

In the following theorem, we show that the minimizer's optimal control is continuous with respect to κ, which will follow readily from (3.4) and from showing that the map $\kappa \mapsto V'(\cdot;\kappa)$ is continuous with respect to κ. We also show continuity of the threshold β_κ with respect to κ; note that we do not have an explicit expression for β_κ.

Theorem 4.2. *The mapping* $[0,\infty) \ni \kappa \mapsto V'(x;\kappa)$ *is continuous, uniformly in* $x \in \mathbb{R}_+$. *In addition, for any given* $\kappa \geq 0$,

$$\beta_\kappa = \lim_{\delta \to 0} \beta_{\kappa+\delta}. \tag{4.11}$$

Proof. For ease of reading, we present the proof in four steps.

Step 1: *Continuity of* $\kappa \mapsto V'(0;\kappa)$. Observe that because $V(0;\kappa)=0$ and $\kappa \mapsto V(x;\kappa)$ is decreasing (Theorem 4.1), we deduce that if $\kappa_1 < \kappa_2$, then

$$V'(0;\kappa_1) \geq V'(0;\kappa_2). \tag{4.12}$$

Suppose $\kappa \mapsto V'(0;\kappa)$ is not continuous. Then, there exists $0 \leq \tilde{\kappa} < \frac{2m}{\sigma^2}$ (recall that if $\kappa \geq \frac{2m}{\sigma^2}$, then $\beta_\kappa = 0$ with $V' \equiv 1$ on $\mathbb{R}_+$) such that

$$a_1 := \liminf_{\varepsilon \to 0^+} V'(0;\tilde{\kappa}-\varepsilon) > \limsup_{\varepsilon \to 0^+} V'(0;\tilde{\kappa}+\varepsilon) =: a_2 \geq 1,$$

since $V'(0;\kappa) \geq 1$. It follows that, for all $0 < \varepsilon < \tilde{\kappa} \wedge (\frac{2m}{\sigma^2} - \tilde{\kappa})$,

$$V'(0;\tilde{\kappa}-\varepsilon) > V'(0;\tilde{\kappa}+\varepsilon) + \frac{a_1 - a_2}{2}.$$

Because $V'(\cdot;\kappa) \leq \bar{c}$ (recall 3.52) and $\beta_\kappa \leq \frac{m}{\delta}$ are uniformly bounded in κ, from (HJB(κ)), (3.2), and (3.3), we get that $V''(\cdot;\kappa)$ is also uniformly bounded in κ. From $V''(\cdot;\kappa)$'s uniform bound and the fact that $V(\cdot;\kappa) \in \mathcal{C}^2(\mathbb{R}_+)$, we now obtain that there exists $\bar{\delta} > 0$, independent of κ, such that for all $0 < \varepsilon < \tilde{\kappa} \wedge (\frac{2m}{\sigma^2} - \tilde{\kappa})$,

$$V'(x;\tilde{\kappa}-\varepsilon) > V'(x;\tilde{\kappa}+\varepsilon) + \frac{a_1 - a_2}{2} \qquad \text{for all } x \in [0,\bar{\delta}].$$

By integrating in $x = 0$ to $\bar{\delta}$, we obtain

$$V(\bar{\delta};\tilde{\kappa}-\varepsilon) > V(\bar{\delta};\tilde{\kappa}+\varepsilon) + \frac{a_1 - a_2}{2}\,\bar{\delta}$$

for all $0 < \varepsilon < \tilde{\kappa} \wedge (\frac{2m}{\sigma^2} - \tilde{\kappa})$. Finally, by taking limit $\varepsilon \to 0^+$, the continuity of $\kappa \mapsto V(\cdot;\kappa)$ implies

$$V(\bar{\delta};\tilde{\kappa}) \geq V(\bar{\delta};\tilde{\kappa}) + \frac{a_1 - a_2}{2}\,\bar{\delta},$$

contradicting the continuity of $\kappa \mapsto V(x;\kappa)$. Thus, we have shown $\kappa \mapsto V'(0;\kappa)$ is continuous.

Step 2: *Continuity of* $\kappa \mapsto (l^{(\kappa)}(\cdot), (l^{(\kappa)})'(\cdot))$. Recall the definition of $\varphi^{(s)}$ given in (3.23), and recall that $\beta_\kappa \leq \frac{m}{\delta}$. Let $l^{(\kappa)}(x)$ denote $\varphi^{(V'(0;\kappa))}(x)$ for $x \in [0, \frac{m}{\delta}]$. More explicitly,

$$\begin{cases} (l^{(\kappa)})''(x) + H_F(x, l^{(\kappa)}(x), (l^{(\kappa)})'(x)) = 0, & x \in [0, \frac{m}{\delta}], \\ (l^{(\kappa)})(0) = 0, \quad (l^{(\kappa)})'(0) = V'(0;\kappa), \end{cases} \tag{4.13}$$

with $\bar{s} = 2m/(\sigma^2\kappa)$ replaced by the constant $\bar{c}$ in F, in conjunction with Remark 3.16. For any $\kappa \geq 0$, the function $V(\cdot;\kappa)$ satisfies (3.3) and, therefore, also (4.13) on $[0,\beta_\kappa]$. Uniqueness of the solution implies that

$$V(x;\kappa) = l^{(\kappa)}(x), \quad V'(x;\kappa) = (l^{(\kappa)})'(x), \qquad x \in [0, \beta_\kappa]. \tag{4.14}$$

We now show that the mapping $[0,\infty) \ni \kappa \mapsto (l^{(\kappa)}(x), (l^{(\kappa)})'(x))$ is continuous, uniformly in $x \in [0, \frac{m}{\delta}]$. Fix $\kappa_1, \kappa_2 \in \mathbb{R}_+$. For simplicity of notation, let f_i denote $l^{(\kappa_i)}$ for $i = 1, 2$; then,

$$\begin{aligned} f_1'(x) &= V'(0;\kappa_1) - \int_0^x H_F^{\kappa_1}(y, f_1(y), f_1'(y))dy, \\ f_2'(x) &= V'(0;\kappa_2) - \int_0^x H_F^{\kappa_2}(y, f_2(y), f_2'(y))dy \\ &= V'(0;\kappa_2) - \int_0^x H_F^{\kappa_1}(y, f_2(y), f_2'(y))dy + \int_0^x (\kappa_2 - \kappa_1)F^2(f_2'(y))dy, \end{aligned} \tag{4.15}$$

in which the superscript κ_i in $H_F^{\kappa_i}$ emphasizes H_F's dependence on κ_i for $i = 1, 2$. Also, from (3.25), we know $|F(f_2'(y))| \le 2\bar{c}$. From (3.26) and the expressions in (4.15), it follows that there exists a constant $\tilde{L} > 0$, independent of κ_1 and κ_2, such that

$$\begin{aligned} |f_1'(x) - f_2'(x)| \le \tilde{L} \left(\int_0^x \Big[|f_1(y) - f_2(y)| + |f_1'(y) - f_2'(y)| \Big] dy + |\kappa_1 - \kappa_2| \right) \\ + |V'(0;\kappa_1) - V'(0;\kappa_2)|. \end{aligned} \tag{4.16}$$

Also, because $f_i(0) = 0$ for $i = 1, 2$, we have

$$f_1(x) - f_2(x) = \int_0^x (f_1'(y) - f_2'(y))dy,$$

which implies

$$|f_1(x) - f_2(x)| \le \int_0^x |f_1'(y) - f_2'(y)| dy. \tag{4.17}$$

From inequalities (4.16) and (4.17) and from Grönwall's inequality, we deduce there is a constant $C > 0$, independent of κ_1 and κ_2, such that

$$\sup_{x \in [0, \frac{m}{\delta}]} \big(|f_1(x) - f_2(x)| + |f_1'(x) - f_2'(x)| \big) \le C \big(|V'(0;\kappa_1) - V'(0;\kappa_2)| + |\kappa_1 - \kappa_2| \big).$$

Recalling from Step 1 that $\kappa \mapsto V'(\cdot;\kappa)$ is continuous, we obtain

$$\lim_{|\kappa_1 - \kappa_2| \to 0+} \sup_{x \in [0, \frac{m}{\delta}]} \big(|f_1(x) - f_2(x)| + |f_1'(x) - f_2'(x)| \big) = 0. \tag{4.18}$$

We have thus shown that the mapping $[0,\infty) \ni \kappa \mapsto (l^{(\kappa)}(x), (l^{(\kappa)})'(x))$ is continuous, uniformly in $x \in [0, \frac{m}{\delta}]$.

Step 3: *Continuity of* $\kappa \mapsto \beta_\kappa$. Noting the definition of β_κ in Theorem 3.7 and the definition of $\beta^{(s)}$ in (3.27), and recalling that $l^{(\kappa)} = \varphi^{(V'(0;\kappa))}$, it follows from (4.14) that for any $\kappa \in [0,\infty)$, $\beta_\kappa = \beta^{(V'(0;\kappa))}$. From the continuity of $\kappa \mapsto V'(0;\kappa)$ obtained in Step 1 and from the second inequality in (3.35), we get

$$\beta_\kappa \le \liminf_{\varepsilon\to 0} \beta_{\kappa+\varepsilon}. \tag{4.19}$$

From Theorem 4.1 and the above considerations, we have

$$\lim_{\varepsilon\to 0} V(\cdot;\kappa+\varepsilon) = V(\cdot;\kappa) \quad \text{and} \quad \lim_{\varepsilon\to 0} l^{(\kappa+\varepsilon)}(\cdot) = l^{(\kappa)}(\cdot)$$

uniformly on $[0,\frac{m}{\delta}]$. In addition, (4.14) implies

$$V(x;\kappa+\varepsilon) = l^{(\kappa+\varepsilon)}(x), \qquad x\in[0,\beta_{\kappa+\varepsilon}).$$

By taking $\varepsilon\to 0$, we get $V(x;\kappa) = l^{(\kappa)}(x)$ for $x\in[0,\limsup_{\varepsilon\to 0}\beta_{\kappa+\varepsilon})$. By continuity

$$V(\cdot;\kappa) = l^{(\kappa)} \quad \text{for } x\in[0,\limsup_{\varepsilon\to 0}\beta_{\kappa+\varepsilon}].$$

Note from Proposition 3.11 that $\beta_\kappa = \hat\beta_\kappa$, in which $\hat\beta_\kappa$ is given by (3.53). Consequently,

$$\limsup_{\varepsilon\to 0} \beta_{\kappa+\varepsilon} \le \beta_\kappa. \tag{4.20}$$

From (4.19) and (4.20), we obtain that $\lim_{\varepsilon\to 0}\beta_{\kappa+\varepsilon}$ exists and equals β_κ. Therefore, we have shown the continuity of $\kappa\mapsto\beta_\kappa$.

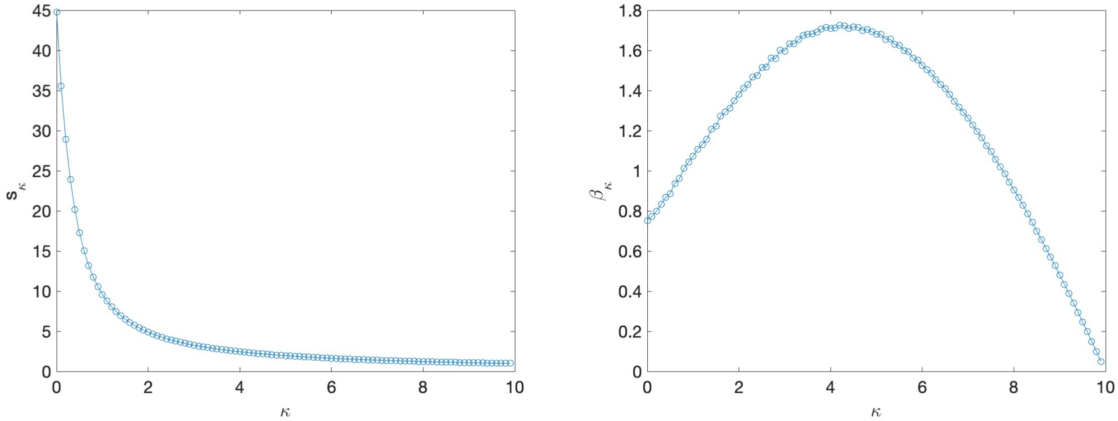
Step 4: *Continuity of* $\kappa\mapsto V'(\cdot;\kappa)$. From (4.14), we have $V'(x;\kappa+\varepsilon) = (l^{(\kappa+\varepsilon)})'(x)$ for $x\in[0,\beta_{\kappa+\varepsilon}]$. It follows from the continuity of $\kappa\mapsto(l^{(\kappa)})'(\cdot)$ obtained in Step 2 that

$$\lim_{\varepsilon\to 0} V'(x;\kappa+\varepsilon) = \lim_{\varepsilon\to 0}(l^{(\kappa+\varepsilon)})'(x) = (l^{(\kappa)})'(x) = V'(x;\kappa) \quad \text{for } x\in[0,\liminf_{\varepsilon\to 0}\beta_{\kappa+\varepsilon}]. \tag{4.21}$$

Recall $V'(\cdot;\kappa)\equiv 1$ on $[\beta_\kappa,\infty)$. From the continuity of $\kappa\mapsto\beta_\kappa$, we now conclude, from (4.21), that $\kappa\mapsto V'(x;\kappa)$ is continuous, uniformly in $x\in\mathbb{R}_+$. ∎

5. Numerical experiments. In this section, we present a few simulations of the value function $V(\cdot;\kappa)$ for different choices of κ under the simplified setting of $f\equiv 0$. From Theorem 3.7, this entails solving numerically (HJB(κ)) or the free-boundary problem (3.3). To this end, we use, as in our proofs in this paper, the shooting method to solve (HJB(κ)). Namely, we consider the Cauchy problem (3.23) and the corresponding threshold β_κ given by (3.27) and (3.42) for the optimal s_κ defined in (3.48). In Figure 5.1, we present the dependance of s_κ and β_κ on κ under the nontrivial setting $\kappa < \frac{2m}{\sigma^2}$ (cf. Case 3 in the existence proof of Proposition 3.10). Observe s_κ is monotonically decreasing with respect to κ, as already shown in (4.12), while β_κ does not share this monotonicity for our choice of parameters. We expected β_κ to be a decreasing function of κ (because β_κ equals zero for $\kappa\ge 2m/\sigma^2$), so we were surprised to see that β_κ is not necessarily decreasing with κ when κ is small. We have no explanation for this result. Recall that Theorem 3.7 shows β_κ equals zero for $\kappa\ge 2m/\sigma^2$, which is indeed reflected in Figure 5.1(b).

Finally Figure 5.2 contains simulations of the value function $V(\cdot;\kappa)$ for varying choices of κ. For each curve, the onset of linearity at the threshold β_κ is indicated by a circle. Note that as κ gets larger, $V(\cdot;\kappa)$ approaches the identity as proved in Theorem 3.7.



(a) s_κ versus κ. (b) β_κ versus κ.

Figure 5.1. *Simulations for $m=5$, $\sigma=1$, $\rho=1$, and $f\equiv 0$.*

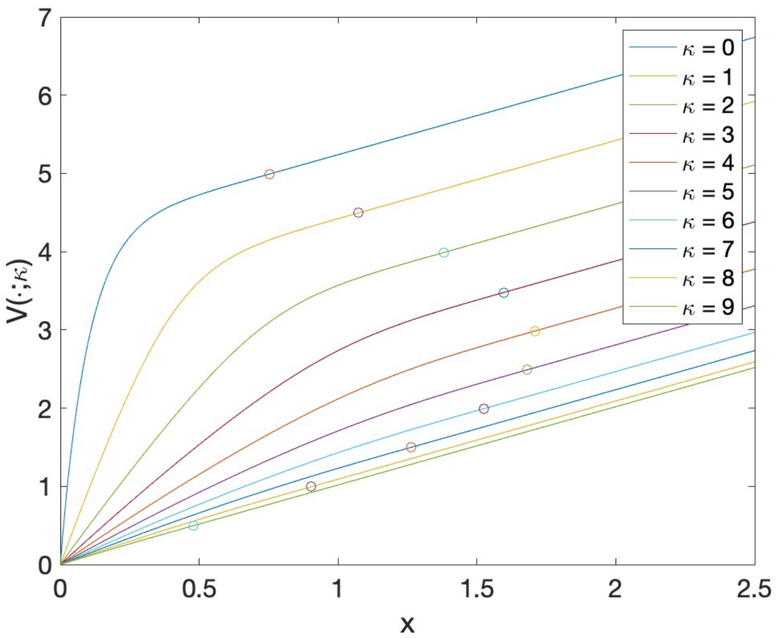


Figure 5.2. $V(\cdot;\kappa)$ *for $m=5$, $\sigma=1$, $\rho=1$, and $f\equiv 0$.*

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