Exceptional points of degeneracy with indirect band gap induced by mixing forward and backward propagating waves

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We demonstrate that exceptional points of degeneracy (EPDs) are obtained in two coupled waveguides without resorting to gain and loss. We show the general concept that modes resulting from a proper coupling of forward and backward waves exhibit EPDs of order two and that there the group velocity vanishes. We verify our insight by using coupled-mode theory and also by full wave numerical simulations of light in a dielectric slab coupled to a grating, when one supports a forward wave, whereas the other (the grating) supports a backward wave. We also demonstrate how to realize a photonic indirect band gap in guiding systems supporting a backward and a forward wave, show its relations to the occurrence of EPDs, and offer a design procedure.

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I. INTRODUCTION

An exceptional point of degeneracy (EPD) is a point in the parameter space of a system at which the system's eigenvalues and eigenvectors coalesce [1–4]. The term exceptional point (EP) and the associated perturbation theory were discussed in the well-known Kato's book in 1966 [4]. The phenomenon of degeneracy of both eigenvalues and eigenvectors (polarization states), studied here, is a stronger degeneracy compared to the traditional degeneracy of only two eigenvalues.

Non-Hermitian Hamiltonian can possess entirely real spectra when the system obeys a parity-time ($\mathcal{PT}$) symmetry condition [5]. A system is said to be $\mathcal{PT}$ symmetric if the $\mathcal{PT}$ operator commutes with the Hamiltonian [6,7], where the $\mathcal{PT}$ operator applies a parity reflection and time reversal [5]. When the time-reversal operator is applied to physical systems, energy changes from damping to growing and vice versa [8]. Based on this simple concept, two symmetrical coupled waveguides with balanced gain and loss satisfy $\mathcal{PT}$ symmetry [9–11], where the individual application of each of the space or time reversals would swap the gain and loss; therefore, the simultaneous application of the space and time-reversal operator to the system would end up with the same system. The point separating the complex and real spectra regimes of $\mathcal{PT}$-symmetric Hamiltonians has been called exceptional point (EP) [4], also known as transition point. Here, beside the mathematical aspects, we stress the role of degeneracy, as implied also in [12], and hence include the "D" in the EPD acronym.

In this paper, we present a class of two coupled waveguides where EPDs exist without resorting to the presence of gain and loss. By using coupled-mode theory [13,14], we show that two coupled waveguides, where one waveguide supports forward propagation (i.e., where the phase and power propagate in the same direction) and the other one supports backward propagation (i.e., where the phase and power propagate in the opposite direction), experience a phase transition as in the $\mathcal{PT}$-symmetric case. We show the general conditions for modes resulting from coupling two coupled waves to exhibit an EPD looking at both the degenerate eigenvalues and eigenvectors. We show that the coupling of two waves, that carry power in opposite directions, leads to an EPD and we explain how this results in the vanishing of the group velocity of the degenerate mode. We illustrate the concepts in a simple system made of two coupled waveguides, i.e., a dielectric slab coupled to a grating, when one supports a forward wave and the other (the grating) supports a backward wave. Other general conditions that lead to exceptional degeneracies of two modes in a uniform waveguide were studied in [15] using a transmission line approach. Finally, we relate the occurrence of EPDs to the presence of a photonic indirect band gap.

II. SECOND-ORDER EPD BY MIXING TWO WAVES

We consider two coupled electromagnetic waves as shown in Fig. 1(a). These two waves are described by the complex time-domain notation

$$a(\mathbf{r},t) = A(z) f_a(\boldsymbol{\rho}) e^{i\omega t},$$
$$b(\mathbf{r},t) = B(z) f_b(\boldsymbol{\rho}) e^{i\omega t}, \quad (1)$$

where $A(z)$ and $B(z)$ are the complex amplitudes of waves along the z direction, $\mathbf{r} = \boldsymbol{\rho} + z\hat{\mathbf{z}}$, and $\boldsymbol{\rho}$ is the transverse coordinate. $f_a(\boldsymbol{\rho})$ and $f_b(\boldsymbol{\rho})$ are the normalized modal field profile in the transverse direction for each mode. When the two waves are uncoupled, i.e., when their waveguides are far from each other, the evolution of the amplitudes along the z direction is simply described by $dA(z)/dz = -i\beta'_a A(z)$ and $dB(z)/dz = -i\beta'_b B(z)$, where β'_a and β'_b are the uncoupled propagation constants of each wave (that is also an eigenmode of the structure since there is no coupling). The solutions are $A(z) = A_0 \exp(-i\beta'_a z)$ and $B(z) = B_0 \exp(-i\beta'_a z)$. The power

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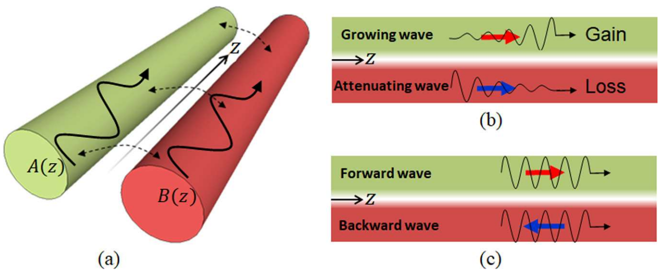

FIG. 1. (a) Coupling between two electromagnetic waves whose complex amplitudes are A and B. Conditions that lead to EPDs are obtained by introducing proper coupling between (b) two waveguides with $\mathcal{PT}$ symmetry where the two media have gain and loss supporting exponentially growing and attenuating waves and (c) two waveguides with forward and backward propagating waves, without resorting to $\mathcal{PT}$ symmetry (i.e., in this case the waveguides do not have gain and loss). The waves with black arrows represent the directions of phase propagation. The blue and red arrows represent the directions of power flow.

carried by each wave in the positive z direction is given by $p_a(z) = \pm|A(z)|^2$ and $p_b(z) = \pm|B(z)|^2$, where the sign depends on the type of wave. The sign is positive when the wave is forward, i.e., when the power is carried in the same direction of wave propagation (i.e., when the phase and group velocity have the same directions). The sign is negative when the wave is backward, i.e., when the phase propagates along the positive z direction and the power flows in the negative z direction (i.e., when the phase and group velocity have opposite directions).

When coupling is introduced to those two waves, the system eigenmode is found by solving the spatial-evolution equation that, based on coupled-mode theory [13,14], is given by

$$\frac{d}{dz}\begin{pmatrix} A(z) \\ B(z) \end{pmatrix} = -i\begin{pmatrix} \beta_a & \kappa_{ab} \\ \kappa_{ba} & \beta_b \end{pmatrix}\begin{pmatrix} A(z) \\ B(z) \end{pmatrix}, \tag{2}$$

where β_a and β_b are "perturbed" propagation constants for the coupled system and κ_{ab} and κ_{ba} are the coupling coefficients between the two modes. The relation between κ_{ab} and κ_{ba} is determined by applying the power conservation principle. The total power carried in the coupled structure is $p_t(z) = |A(z)|^2 \pm |B(z)|^2$, assuming wave A is a forward wave and wave B is either forward or backward when taking the $+$ or the $-$ sign, respectively. Thus there are two possible scenarios: (i) "codirectional coupling" when both waves carry power in the same direction and (ii) "contradirectional coupling" when the two waves carry power in the opposite direction [13].

When the system does not have gain and loss, conservation of energy states that $dp_t(z)/dz = 0$, and by using (2), one finds that the constraint $\mathrm{Re}[AB^*(\kappa_{ba} \mp \kappa_{ab}^*)] = 0$ should be satisfied. Therefore, we have $\kappa_{ab} = \kappa_{ba}^*$ in the case of codirectional coupling where the two waves are forward and $\kappa_{ab} = -\kappa_{ba}^*$ in the case of contradirectional coupling where one wave is forward and the other one is backward [13].

The mixing of the two waves constitutes what is called the guiding system's eigenmode (some call it "supermode"), which is a weighted sum of the individual guided waves. The eigenmode propagation constant is determined by solving the characteristic equation of the coupled system in (2) assuming the wave amplitudes to be in the form of $[A(z), B(z)]^T \propto e^{-ikz}$, which yields $k^2 - k(\beta_a + \beta_b) + (\beta_a\beta_b - \kappa_{ab}\kappa_{ba}) = 0$. The characteristic equation has two solutions that are given by

$$k_n = \frac{\beta_a + \beta_b}{2} + (-1)^n\sqrt{\left(\frac{\beta_a - \beta_b}{2}\right)^2 - (-1)^p\kappa^2}, \tag{3}$$

where $\kappa = |\kappa_{ab}|$ and the indices $n = 1, 2$ denote the two modes of the coupled system. Furthermore, $p = 1$ and $p = 2$ represent the case of codirectional and contradirectional coupling, respectively. An EPD occurs when two eigenmodes coalesce, i.e., $k_1 = k_2 = k_e$, with $k_e = (\beta_a + \beta_b)/2$. This EPD occurs when $\beta_a - \beta_b = 2\kappa\sqrt{(-1)^p}$. Alternatively, the condition is satisfied by setting $\kappa = \kappa_e$, where $\kappa_e = (\beta_a - \beta_b)/(2\sqrt{(-1)^p})$. At an EPD, the eigenvectors must coalesce and in this simple system their coalescence follows from the coalescence of the eigenvalues. Indeed, the two eigenvectors are $[A_n, \quad B_n]^T = [1, \quad (k_n - \beta_a)/\kappa_{ab}]^T$ and it is easy to see that they coalesce when $k_1 = k_2$.

The group velocity of the eigenmode with wave number k_n is determined as (assuming k_n to be purely real)

$$v_{g,n} = \frac{1}{d_\omega k_n} = \frac{k_n - k_e}{2k_n d_\omega(\beta_a + \beta_b) + 2d_\omega[\beta_a\beta_b + (-1)^p\kappa^2]}, \tag{4}$$

where $d_\omega \equiv d/d\omega$ denotes the derivative with respect to angular frequency ω. It is clear from the expression that $v_{g,1,2} = 0$ when $k_1 = k_2 = k_e$, i.e., exactly at the EPD. Next, we also show what happens near the EPD.

A. Codirectional coupling ($\kappa_{ab} = \kappa_{ba}^*$)

For the case of codirectional coupling, $p = 1$, the EPD condition is simplified to $\beta_a - \beta_b = \pm 2i\kappa$. The EPD condition puts a constraint that the difference between the propagation constants of the uncoupled waves has to be purely imaginary in order to exhibit an EPD. Thus we conclude that the EPD can never be obtained for any value of the coupling parameter κ in the case of a lossless and gainless system. If we resort to a $\mathcal{PT}$ symmetry, as in Fig. 1(b), where the system has balanced gain and loss, we have $\beta_a = \beta_0 + i\alpha$ and $\beta_b = \beta_0 - i\alpha$, and an EPD is obtained when $\alpha = \kappa$ [9–11,16,17] and the degenerate wave number is $k_e = \beta_0$.

For this case, the two propagation constants of the coupled system in the vicinity of the EPD are $k_{1,2} = \beta_0 \pm \sqrt{\kappa^2 - \alpha^2}$ and their derivatives are $d_\omega k_{1,2} = d_\omega\beta_0 \pm (2\kappa d_\omega\kappa - 2\alpha d_\omega\alpha)/(k_2 - k_1)$. This is also illustrated by determining the eigenvector of the degenerate eigenmode from (2) (for the codirectional coupling case where $p = 1$) as

$$\begin{pmatrix} A_e(z) \\ B_e(z) \end{pmatrix} = \begin{pmatrix} 1 \\ -i\, e^{-i\arg(\kappa_{ab})} \end{pmatrix} e^{-ik_e z}, \tag{5}$$

and one finds that the total power carried by the degenerate eigenmode $p_t(z) = |A_e(z)|^2 - |B_e(z)|^2 = 0$ vanishes, in agreement with the vanishing of the group velocity. In the vicinity of an EPD we have $k_1 \approx k_2$ and by neglecting the $d_\omega\beta_0$, usually smaller than the other term, the group velocities of the two modes are $v_{g,1,2} \approx \pm(k_2 - k_1)/(2\kappa d_\omega\kappa - 2\alpha d_\omega\alpha)$ when $\kappa > \kappa_e$, i.e., where the two eigenmodes are propagating

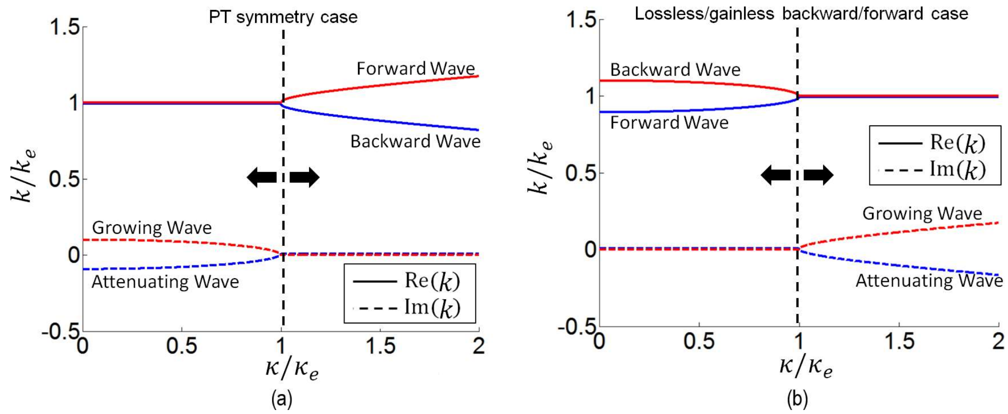

FIG. 2. Two wave numbers of the guiding system versus the coupling parameter κ, showing the existence of an EPD. Two cases are examined: (a) coalescence of modes in $\mathcal{PT}$-symmetrical waveguides and (b) coalescence of modes obtained by coupling a forward wave (phase and group velocities have the same direction) and a backward wave (phase and group velocities have opposite directions). Both cases exhibit an EPD, represented by the bifurcation point.

with purely real wave numbers. Therefore, we conclude that near an EPD in a $\mathcal{PT}$-symmetric guiding system, the two modes of the coupled system are phase synchronized ($k_1 \approx k_2$), i.e., with almost identical phase velocity, but have opposite group velocity ($v_{g,1} \approx -v_{g,2}$), which eventually results in having a wave in the guiding structure with vanishing group velocity when the system is exactly at the EPD.

At the EPD, the system matrix is not diagonalizable but rather similar to a 2×2 Jordan matrix. The fields in the two-waveguide system are represented using the degenerate and generalized eigenvectors as

$$\begin{pmatrix} A(z) \\ B(z) \end{pmatrix} = \begin{pmatrix} 1 \\ -i\, e^{-i\arg(\kappa_{ab})} \end{pmatrix} (u_1 - izu_2) e^{-ik_e z} + \begin{pmatrix} 0 \\ e^{-i\arg(\kappa_{ab})}/\kappa_e \end{pmatrix} u_2 e^{-ik_e z}, \quad (6)$$

where u_1 and u_2 are proper coefficients that depend on the system excitation and boundary conditions.

As an example, for a system with $\mathcal{PT}$ symmetry where the uncoupled waveguides have, respectively, a growing wave with $\beta_a = 100 + 10i$ (1/m) and an attenuating wave with $\beta_b = 100 - 10i$ (1/m), an EPD is obtained when the coupling parameter is $\kappa = 10$ 1/m as shown in Fig. 2(a).

B. Contradirectional coupling ($\kappa_{ab} = -\kappa_{ba}^*$)

We consider coupling between a forward wave with wave number $\beta_a > 0$ and $v_{g,a} > 0$ ($v_{g,a} = d_\omega \beta_a$) and a backward wave with wave number $\beta_b > 0$ and $v_{g,b} < 0$ ($v_{g,b} = d_\omega \beta_b$). The two propagating waves carry power in opposite directions and therefore they exhibit contradirectional coupling; we use $p = 2$ in Eq. (3). The EPD condition ($k_1 = k_2$) for this case is simplified to $\beta_a - \beta_b = \pm 2\kappa$, which means that the difference between the propagation constants should be purely real to have an EPD, which is possible for a lossless and gainless system. Therefore, the EPD condition for this case is satisfied through the proper design of the coupling parameters, i.e., when $\kappa = |\beta_a - \beta_b|/2$ (we recall that κ was defined as purely real positive). This means that there are two possible EPD conditions, $\beta_a - \beta_b = 2\kappa$ and $\beta_a - \beta_b = -2\kappa$, that may both occur when varying frequency. At those two frequencies one has $\beta_a > \beta_b$ and $\beta_b > \beta_a$, respectively. A more detailed discussion is provided later on when discussing the indirect band gap.

In the *vicinity* of an EPD, the two propagation constants of the coupled system are given by Eq. (3) and the derivatives of the two wave numbers with respect to the angular frequency are

$$d_\omega k_{1,2} = \frac{1}{2} d_\omega (\beta_a + \beta_b) \pm \frac{1}{2} \frac{(\beta_a - \beta_b) d_\omega (\beta_a - \beta_b) - 2\kappa\, d_\omega \kappa}{(k_2 - k_1)}. \quad (7)$$

When $\kappa < \kappa_e$, i.e., where the two eigenmodes are propagating with purely real wave numbers, in the vicinity of an EPD we have $k_1 \approx k_2$ and, by neglecting the term $d_\omega(\beta_a + \beta_b)$ with respect to the second one, the group velocities of the two modes are

$$v_{g,1,2} \approx \pm \frac{k_2 - k_1}{\frac{1}{2}(\beta_a - \beta_b)\left(v_{g,a}^{-1} - v_{g,b}^{-1}\right) - 2\kappa\, d_\omega \kappa}. \quad (8)$$

Therefore, we conclude that near an EPD, the coupled forward and backward waves are synchronized in phase, i.e., $k_1 \approx k_2$, but have nearly opposite group velocity ($v_{g,1} \approx -v_{g,2}$), which eventually results in having a wave in the guiding structure with vanishing group velocity when the system is exactly at an EPD.

This is also illustrated by determining the eigenvector of the degenerate eigenmode from (2) (for a contradirectional coupling case where $p = 2$) as

$$\begin{pmatrix} A_e(z) \\ B_e(z) \end{pmatrix} = \begin{pmatrix} 1 \\ -e^{-i\arg(\kappa_{ab})} \end{pmatrix} e^{-ik_e z}, \quad (9)$$

and one finds that the total power carried by the degenerate eigenmode $p_t(z) = |A_e(z)|^2 - |B_e(z)|^2 = 0$ vanishes, in agreement with the vanishing of the group velocity. At the EPD, the system matrix is not diagonalizable but rather similar to a 2×2 Jordan matrix. The fields in the two-waveguide system are represented using the degenerate and generalized eigenvectors as

$$\begin{pmatrix} A(z) \\ B(z) \end{pmatrix} = \begin{pmatrix} 1 \\ -e^{-i\arg(\kappa_{ab})} \end{pmatrix}(u_1 - izu_2)e^{-ik_e z} + \begin{pmatrix} 0 \\ e^{-i\arg(\kappa_{ab})}/\kappa_e \end{pmatrix}u_2 e^{-ik_e z}, \quad (10)$$

where u_1 and u_2 are proper coefficients that depend on the system excitation and boundary conditions.

The two waveguides with contradirectional coupling are schematically shown in Fig. 1(c) and the dispersion diagram is in Fig. 2 where we see a forward wave with $\beta_a = 110$ (1/m) and backward wave with $\beta_b = 90$ (1/m), and the EPD is obtained at $k = k_e = 100$ (1/m), where $\kappa_e = 10$ (1/m) as shown in Fig. 2(b).

The contradirectional coupling case can be realized in two possible scenarios: (i) two modes exist in two separate waveguides where the first waveguide supports a forward wave and the second waveguide supports a backward wave and the coupling is introduced by bringing them near each other and (ii) two modes exist in the same waveguide having periodicity where the one wave (e.g., the forward) has the fundamental Floquet harmonic equal to $\beta_1 = \beta_0$ and the other wave (e.g., the backward) has its first harmonic Floquet harmonic equal to $\beta_2 = -\beta_0 + 2\pi/d$, where d is the waveguide period, and the EPD is only possible at the band edge $\beta_0 = \pi/d$. An example belonging to the first scenario, where the EPD is found in two coupled dielectric slab waveguides, is shown later on. The second scenario instead exists in conventional periodic waveguides and it is not further considered in this paper. We present an example of a guiding system that supports two waves carrying power in opposite directions and we show that it exhibits two EPDs. Consider the guiding system made of a Si substrate (supporting the forward wave) coupled to a Si grating waveguide as shown in Fig. 3(a), with dimensions $w = p = h =$ 70 nm and $d = 140$ nm. Silicon is modeled with a refractive index $n_{\text{Si}} = 3.45$. We first show in Fig. 3(b) the dispersion of the two wave numbers β'_a and β'_b of the two uncoupled waveguides ($s \to \infty$) as red dashed (uniform waveguide) and blue dashed (grating waveguide). The figure shows that the structures support a forward wave where the group velocity is positive $v'_{g,a} = 1/(d_\omega \beta'_a) > 0$ and a backward wave where the group velocity is negative $v'_{g,b} = 1/(d_\omega \beta'_b) < 0$. In the same Fig. 3(b), we show the dispersion of the two wave numbers k_1 and k_2 of the coupled guiding system, i.e., when the two waveguides are close to each other with a gap of $s = 70$ nm. The dispersion shows the existence of two EPDs associated to a wave number (momentum) displacement. The dispersion diagrams we show are for modes with electric field polarized in the y direction. The dispersion diagrams have been found by using the finite element method-based eigenmode solver implemented in CST Studio Suite, by numerically simulating only one unit cell of the structure. The proposed condition allows one to locate the EPDs at band edges that are not necessarily at the center or at the edge of a Brillioum zone, without using loss and gain. It also shows the capability to engineer an indirect band gap in these simple structures.

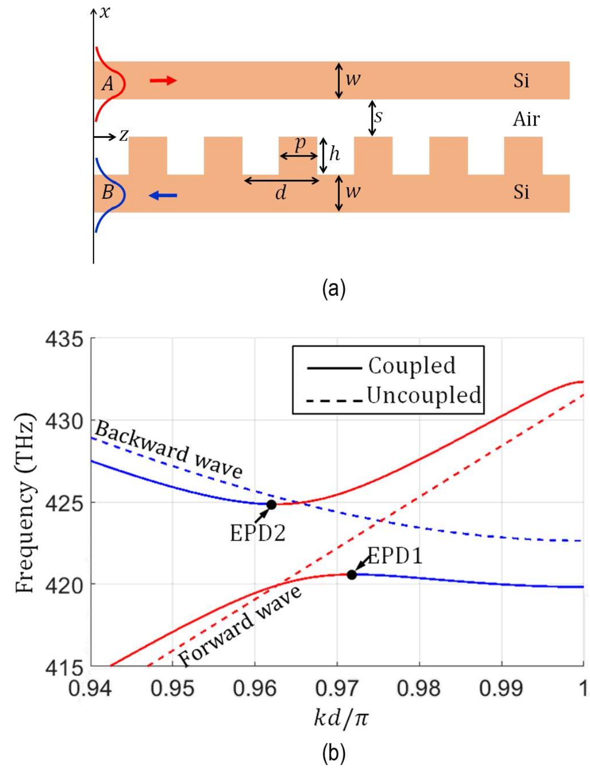

FIG. 3. Example of modal EPD of order 2 between a forward wave (phase and group velocities have the same direction) and backward wave (phase and group velocities have opposite directions). (a) Two coupled Si layers where the top one supports a forward wave (in red) while the bottom one is periodically corrugated to support a backward wave (positive phase velocity and negative group velocity, in blue). The red and blue arrows represent the direction of power flow. (b) Dispersion relation showing the propagating eigenmodes when the two waveguides are uncoupled (dashed) and when they are coupled (solid). The dispersion of modes in the coupled waveguides show the existence of two EPDs. The blue and red colors of the curves are related to the power flow directions. Note also that because two EPDs are found, an indirect band gap is present between the upper and lower branches that can be designed *ad hoc*. In the shown case we have $\Delta k_e \equiv k_{e2} - k_{e1} < 0$.

III. INDIRECT BAND GAP IN THE CONTRADIRECTIONAL CASE

In the contradirectional case where coupling occurs between a forward and backward wave, as in Fig. 1(c), an indirect band gap is possible and we show here how it is formed. Since in this case one wave is forward and one is backward, the uncoupled propagation constants β'_a and β'_b have opposite slopes as schematically shown in Fig. 4 [see also dashed blue and red curves in Fig. 3(b)] and an analogous trend is expected for the parameters β_a and β_b of the coupled system. By looking at the dispersion diagram shown in Fig. 4, the red-dashed curve is the forward wave with wave number $\beta'_a(\omega)$ and the blue-dashed curve is the backward wave with wave number $\beta'_b(\omega)$. Assuming that the coupling is not so strong, one may assume that $\beta_a(\omega) \approx \beta'_a(\omega)$ and $\beta_b(\omega) \approx \beta'_b(\omega)$, at least in trend and hence in slope. The forward wave has positive slope, $v_{g,a} = d\omega/d\beta_a > 0$, whereas the backward wave has negative slope $v_{g,b} = d\omega/d\beta_b < 0$, and the

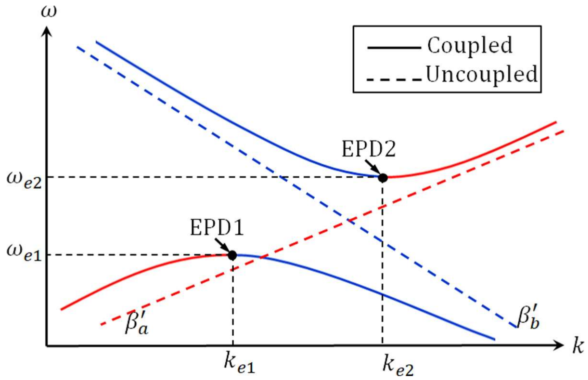

FIG. 4. Schematic of a dispersion diagram showing the indirect band gap that results from two EPDs based on contradirectional coupling. The red-dashed line represents the wave number of the forward wave β_a', whereas the blue-dashed line the one of the backward wave β_b', when the waveguides are uncoupled. The coupling yields the two curves with two EPDs that are labeled as EPD1 and EPD2. The indirect band gap width is $\Delta\omega_{IB}$. In the shown case $\Delta k_e \equiv k_{e2} - k_{e1} > 0$.

dispersion curves for β_a and β_b versus frequency should intersect at some frequency ($\beta_a = \beta_b$) because one wave number is increasing with frequency whereas the other is decreasing with frequency; an example with a grating is illustrated in Fig. 3(b), while a schematic is in Fig. 4. Let us approximate the dispersion curves locally, in the frequency range of interest, as straight lines, i.e., $\beta_a(\omega) \approx a + v_{g,a}^{-1}\omega$ and $\beta_b(\omega) \approx b + v_{g,b}^{-1}\omega$, where now $v_{g,a}$ and $v_{g,b}$ are assumed to have the local fixed value. Furthermore, assuming, for simplicity, that κ is constant within the frequency range of interest, one finds that EPDs occur at two angular frequencies ω_{e1} and ω_{e2} such that $\beta_a(\omega_{e1}) - \beta_b(\omega_{e1}) = -2\kappa$ and $\beta_a(\omega_{e2}) - \beta_b(\omega_{e2}) = 2\kappa$, where $\omega_{e1} < \omega_{e2}$. Subtracting the previous two conditions leads to the indirect band gap determination

$$\Delta\omega_{IB} \equiv \omega_{e2} - \omega_{e1} \approx \frac{4\kappa}{v_{g,a}^{-1} - v_{g,b}^{-1}}. \tag{11}$$

Note that $v_{g,a}^{-1} - v_{g,b}^{-1} > 0$; hence $\Delta\omega_{IB} > 0$. The band-gap width can be controlled by the slope of the two parameters β_a and β_b; indeed, when $v_{g,a}^{-1} \approx -v_{g,b}^{-1}$, the denominator of (11) is small and the band gap is very wide; vice versa, the band gap is narrow when $v_{g,a}^{-1}$ is very different from $-v_{g,b}^{-1}$. If we consider the dispersion of the coupling term κ, a more complicated picture may arise that could be determined by the reasoning just provided.

The degenerate wave numbers at the two EPDs are $k_{e,1} = [\beta_a(\omega_{e1}) + \beta_b(\omega_{e1})]/2$ and $k_{e,2} = [\beta_a(\omega_{e2}) + \beta_b(\omega_{e2})]/2$. Using the linear approximation formulas for the wave numbers $\beta_a(\omega)$ and $\beta_a(\omega)$, one finds that $k_{e1} \approx [a + b + (v_{g,a}^{-1} + v_{g,b}^{-1})\omega_{e1}]/2$ and $k_{e2} \approx [a + b + (v_{g,a}^{-1} + v_{g,b}^{-1})\omega_{e2}]/2$. The difference between the two degenerate wave numbers is

$$\Delta k_e \equiv k_{e2} - k_{e1} \approx \frac{v_{g,a}^{-1} + v_{g,b}^{-1}}{2}\Delta\omega_{IB}. \tag{12}$$

Therefore, it is necessary that $v_{g,a}^{-1} + v_{g,b}^{-1} \neq 0$ in order to have an indirect band gap. When $|v_{g,a}^{-1}| > |v_{g,b}^{-1}|$ we get $k_{e2} > k_{e1}$; hence $\Delta k_e > 0$, i.e., the EPD that occurs at the smaller frequency ω_{e1} occurs also at the smaller degenerate wave number k_{e1} and this condition is depicted in Fig. 4. When $|v_{g,a}^{-1}| < |v_{g,b}^{-1}|$, we get $k_{e1} > k_{e2}$; hence $\Delta k_e < 0$, i.e., the EPD that occurs at the smaller frequency ω_{e1} occurs at larger degenerate wave number k_{e1} and this condition is depicted in Fig. 3(b). Indeed, by looking at Fig. 3(b), one finds by naked eye that $|v_{g,a}| > |v_{g,b}|$ (absolute change with frequency of the dashed red curve is higher than the one of the dashed blue one); therefore, $|v_{g,a}^{-1}| < |v_{g,b}^{-1}|$, resulting in $k_{e1} > k_{e2}$ according to (12), and the EPD at lower frequency, around 420 THz, occurs at a higher degenerate wave number of $k_{e1} = 0.972\,\pi/d$.

IV. CONCLUSION

We have demonstrated that EPDs are not only obtained in $\mathcal{PT}$-symmetric waveguides but they are also obtained in two lossless and gainless waveguides when they support forward and backward waves that are properly coupled. We have shown a simple system that supports this condition made of a grating (supporting a backward guided mode) coupled to a dielectric layer (supporting the forward mode). We have elaborated that the scheme discussed here exhibits a photonic indirect band gap that can be controlled by changing the group velocities of the forward and backward modes along with the coupling coefficient. Two conditions may occur: the band gap is associated either to a positive or negative momentum difference between the two energy levels. The finding in this paper can be useful to design systems with EPDs whose use is of growing importance for enhancing light-matter interactions and nonlinear photonic phenomena.

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