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# Two resonators with negative and positive reactive components to achieve an exceptional point of degeneracy

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***Abstract* – We present a scheme supporting an exceptional point of degeneracy (EPD) using connected Foster and non-Foster resonators. One resonator contains positive components, whereas the second resonator contains negative components. We show a second-order EPD where two eigenvalues and eigenvectors coalesce. This circuit can be used to make ultra-sensitive sensors.**

## I. Introduction

An exceptional point of degeneracy (EPD) is a point in the system parameter space at which two or more eigenvalues and their corresponding eigenvectors coalesce [1], [2]. Traditionally, EPDs in coupled resonators have been obtained in PT-symmetric circuits [3]–[5]. Here we show a new way to get EPD using connected Foster and non-Foster resonators. It is possible to define a non-Foster network as a network that has one or more non-Foster parts. Non-Foster parts are subnetworks or elements whose admittance or impedance is imaginary at all frequencies and whose derivative of reactance or susceptance is zero or negative. Therefore, networks containing negative inductance or capacitance are inherently non-Foster.

The system supporting a second-order EPD shows a square-root sensitivity of the degenerate eigenfrequency $\omega_e$ to a relative perturbation $\Delta$, leading to $(\omega - \omega_e) \propto \pm\sqrt{\Delta}$. Coupled resonator systems have shown EPDs when they are connected via inductive coupling [4] or through a gyrator [6]–[9].

Here, we show how to obtain an EPD using two directly connected resonators. We demonstrate that (i) EPDs are obtained in resonator-based circuits without any inductive, capacitive, or gyrator coupling and (ii) the physics behind using components with negative values to achieve an EPD. We also study the enhanced sensitivity to perturbations when operating near an EPD. Using time-domain simulations, we show an example of the mentioned circuit operating at an EPD and demonstrate linear growth in the voltage signal. Moreover, we investigate the sensitivity of the circuit's eigenfrequencies to component variations. We show that the Puiseux fractional power series expansion can approximate the eigenfrequency diagram bifurcation (square root behavior) near the EPD [1].

## II. EPD in Two Resonators with Negative and Positive Reactive Components

Referring to the circuit in Fig. 1(a), we obtain the circuit equations using Kirchhoff voltage and current laws. It is convenient to define the state vector as $\mathbf{\Psi} \equiv [Q_1, Q_2, \dot{Q}_1, \dot{Q}_2]^{\mathrm{T}}$, which have a combination of stored charge $Q_n = C_n V_n$ and its time derivative (current) on each capacitor, and T denotes the transpose operation. We write down the circuit equations by making use of Liouvillian formalism to describe the dynamics of the circuit illustrated in Fig. 1(a) as

$$\mathrm{d}\mathbf{\Psi}/\mathrm{d}t = \underline{\mathbf{M}}\mathbf{\Psi}, \quad \underline{\mathbf{M}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(\omega_{01}^2 + \omega_x^2) & \omega_{02}^2 & 0 & 0 \\ \omega_x^2 & -\omega_{02}^2 & 0 & 0 \end{bmatrix}. \tag{1}$$

where $\underline{\mathbf{M}}$ is the circuit matrix, $\omega_{01} = 1/\sqrt{C_1 L_1}$, $\omega_{02} = 1/\sqrt{C_2 L_2}$ are resonance angular frequencies of the two isolated (i.e., uncoupled) parallel and series resonators, and $\omega_x = 1/\sqrt{C_1 L_2}$. Assuming signals of the form $Q_n \propto \exp(j\omega t)$, we write the eigenvalue problem associated with the circuit equations. The circuit's eigenfrequencies are calculated by

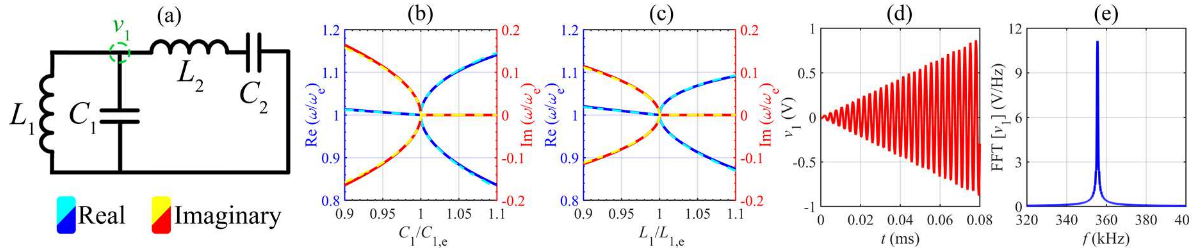


Fig. 1. (a) The schematic illustration of the proposed circuit with negative components in the series resonators. In this circuit, two LC resonators are connected directly. Variation of the real and imaginary parts of the eigenfrequencies to a (b) capacitance and (c) inductance perturbation. Solid lines: solution of the eigenvalue problem of Eq. (1) (blue for real and red for imaginary); dashed lines: Puiseux series approximation truncated to its second term (cyan for real and yellow for imaginary). Voltage under the EPD condition in the (d) time-domain and (e) frequency-domain calculated by using the ADS circuit simulator. The frequency-domain result is calculated by applying an FFT with $10^6$ samples in the time window of 0 to 2 ms.

$$\omega_{1,4} = \pm\sqrt{a+b}, \qquad \omega_{2,3} = \pm\sqrt{a-b}, \tag{2}$$

$$a = \frac{1}{2}(\omega_{01}^2 + \omega_{02}^2 + \omega_x^2), \tag{3}$$

$$b^2 = a^2 - \omega_{01}^2\omega_{02}^2, \tag{4}$$

A necessary condition to achieve an EPD is that $b = 0$. If this condition is met, we can obtain degenerate real or imaginary EPD frequencies. Here, we consider the case with real EPD frequencies achieved when $\omega_{01}^2$ and $\omega_{02}^2$ are positive, both $C_2$ and $L_2$ are negative, and $a > 0$. The following values are taken into account for the components in the circuit that achieve an EPD: $C_{1,e} = 40$ nF, $L_{1,e} = 10$ μH, $C_{2,e} = -10$ nF, and $L_{2,e} = -10$ μH; then, we calculate $\omega_e = 2.236 \times 10^6$ rad/s. We can use an operational amplifier in the inverter circuit to realize the elements with negative values. In Figs. 1(b) and (c), the complex eigenfrequencies are shown by varying the positive capacitance and inductance. In these plots, only the frequencies with positive real parts are shown. Following the typical bifurcation diagram of an EPD, the eigenfrequencies change from purely real to complex values. The eigenvalues at EPDs are extremely sensitive to changes in the circuit's parameters. The degeneracy of the eigenvalues boosts the sensitivity of a circuit's eigenfrequencies to a specific change in an LC component value in the resonators.

A perturbed circuit matrix results from a perturbation in component value. Hence, the two degenerate eigenvalues occurring at the EPD change due to the slight perturbation in component value, resulting in two distinct eigenfrequencies $\omega_p$, with $p = 1,2$. The corresponding eigenfrequencies are approximated using the Puiseux fractional power series expansion truncated at the second order. A single convergent Puiseux series represents the two perturbed eigenvalues near the EPD. The explicit recursive formulas given in [10] are used to derive the coefficients of the Puiseux series. An approximation of $\omega_p$ around a second-order EPD is given by

$$\omega_p(\Delta) \approx \omega_e + (-1)^p \alpha_1 \sqrt{\Delta} + \alpha_2 \Delta \tag{5}$$

where $\omega_p$ are the two perturbed eigenfrequencies near the EPD, and $\Delta$ indicates relative circuit perturbation, i.e., $\Delta = (C_1 - C_{1,e})/C_{1,e}$ when perturbing the capacitor, and $\Delta = (L_1 - L_{1,e})/L_{1,e}$ when perturbing the inductor. Dashed lines in Figs.1(b) and (c) show the perturbed eigenvalues calculated by a convergent Puiseux series.

The time-domain simulation signal obtained using the Keysight Advanced Design System (ADS) circuit simulator is illustrated in Figs. 1(d), which shows the voltage $v_1(t)$ in the connecting node between the two resonators. We have put 1 mV as an initial voltage on $C_1$. According to Fig. 1(d), the signal's envelope grows linearly with increasing time. This important aspect is peculiar to the second-order EPD, and it is the result of coalescing eigenvalues and eigenvectors that also corresponds to a double pole in the circuit. The system matrix at the EPD is similar to a matrix comprising a non-trivial Jordan block. We take a fast Fourier transform (FFT) of the voltage $v_1(t)$ to show the frequency spectrum, and the calculated spectrum is shown in Fig. 1(e). The observed oscillation frequency is $f_o = 355.9$ kHz, which is in good agreement with the theoretical value $\omega_e/(2\pi)$ calculated above.

## VI. Conclusion

We have shown that connecting two resonators where one includes two reactive components with negative values leads to an EPD. A perturbation of the inductance or capacitance leads to two real-valued frequency shifts from the EPD. Then, the perturbation in the system can be calculated by measuring the changes in two resonance frequencies. Furthermore, we discussed how the eigenfrequencies are extremely sensitive to a system perturbation, and this new scheme of operation leads to improving the sensitivity of sensors.

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