

Sensing with exceptional points in coupled resonator or waveguide systems

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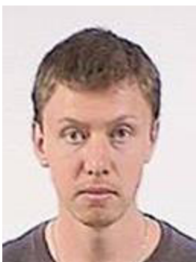
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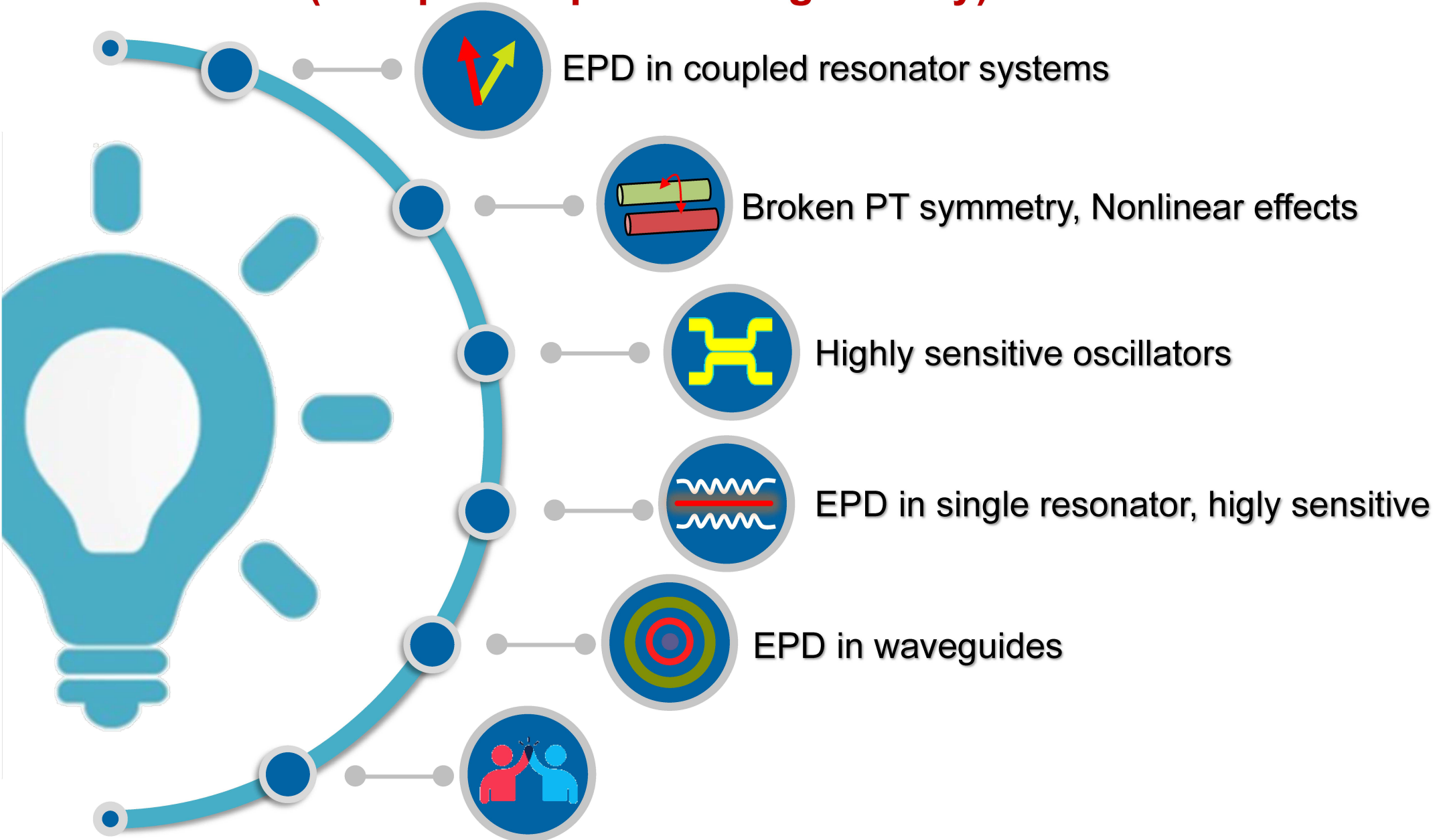


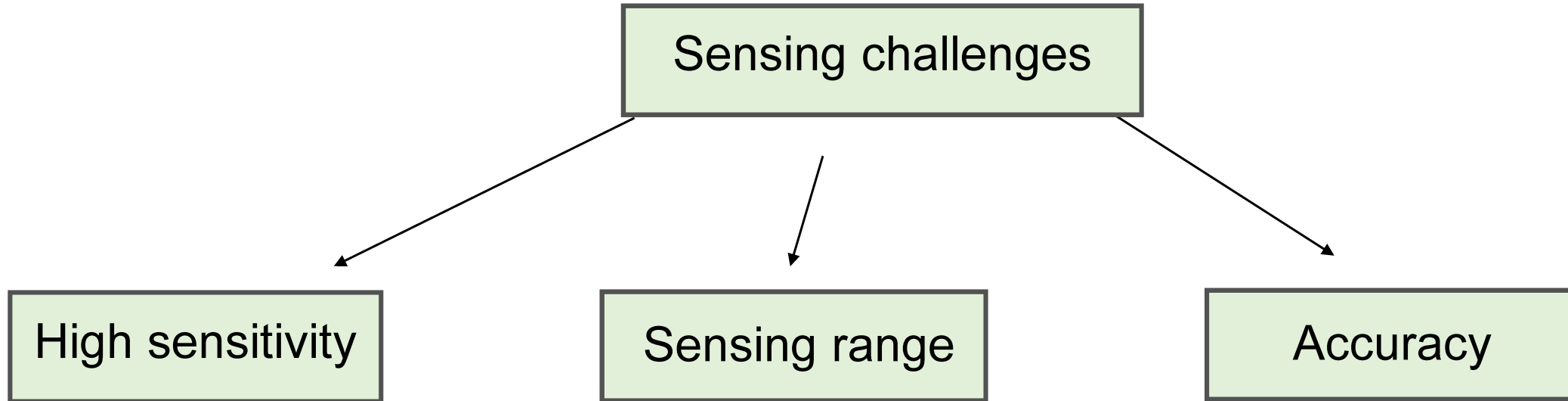
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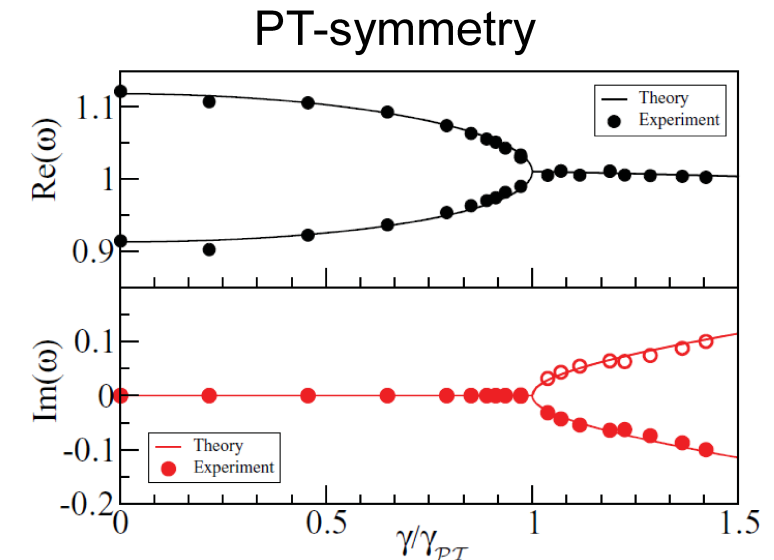
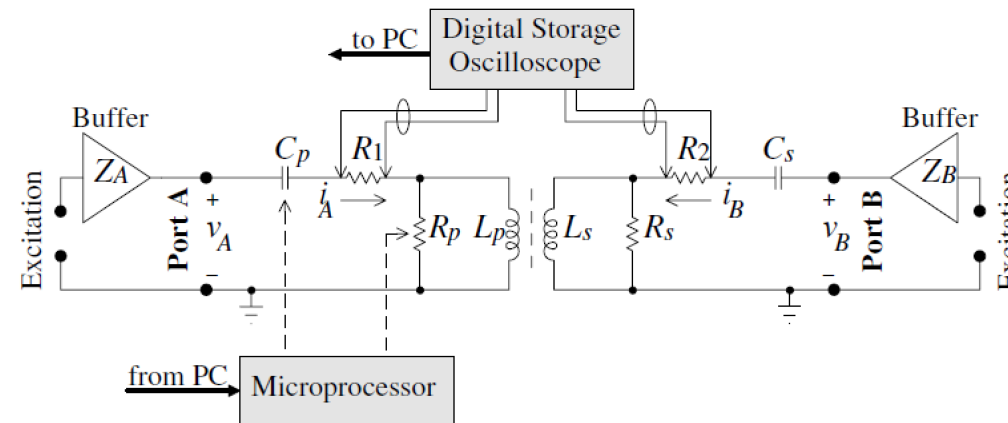
Outline EPD (exceptional point of degeneracy)





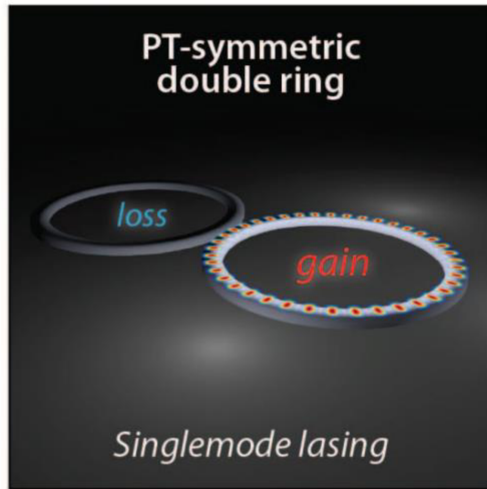
The EPD concept provides some solutions for sensing applications

EPD is obtained in this simple structure:



Stehmann, Heiss, Scholtz, J. Phys. A: Math Gen 37 (2004) 7813–7819, 2004
Schindler, et al., *Physical Review A*, 84.4, 2011

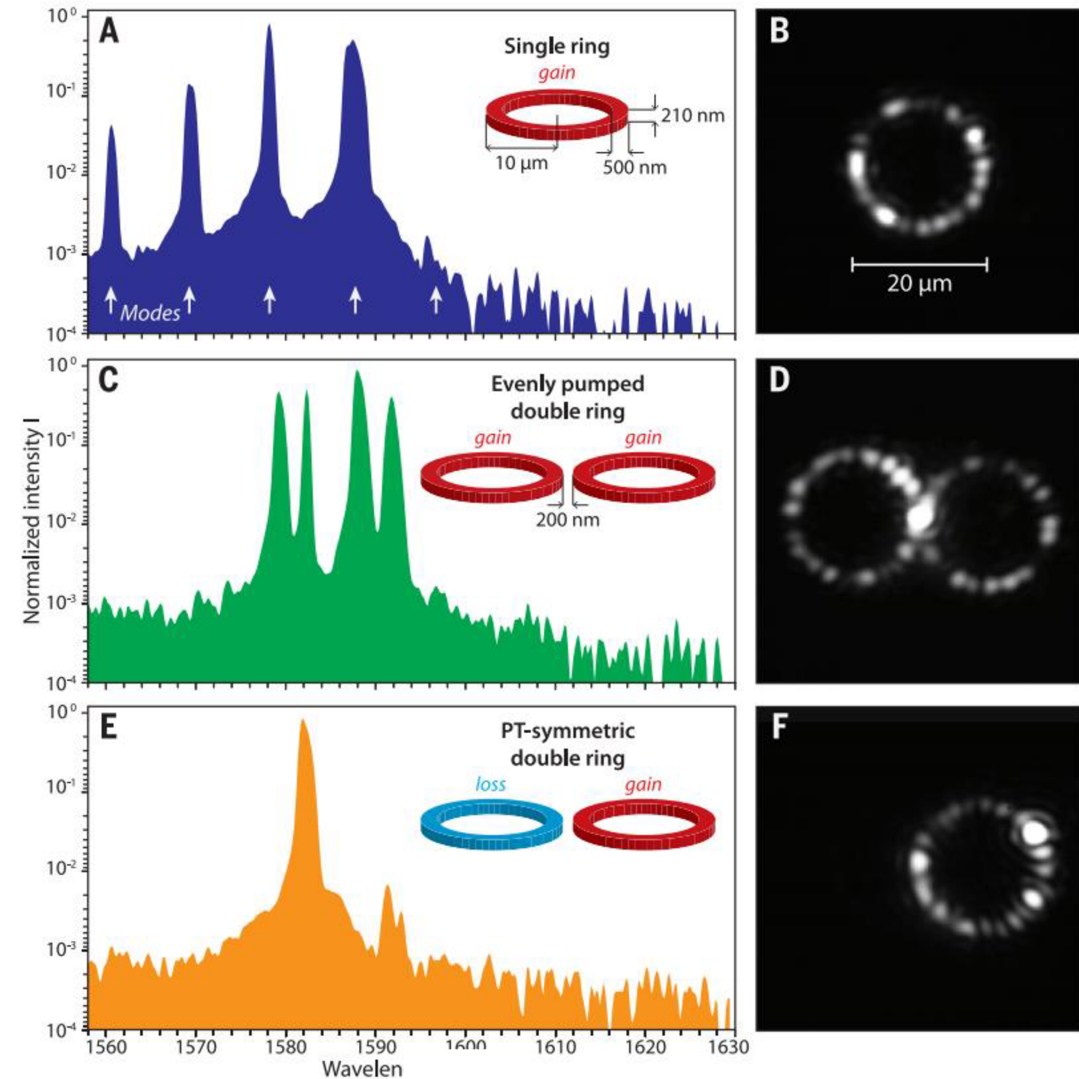
Gain and loss balanced resonators



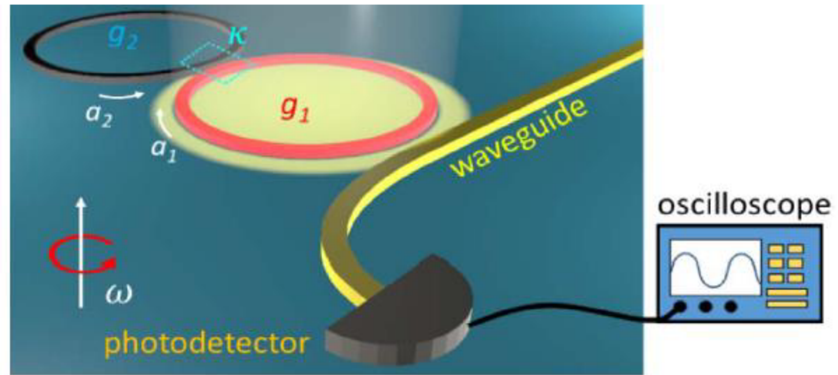
Example based on PT-symmetry

- The emergence of 2nd order EPD in coupled ring resonators with balanced loss and gain
- EPD can be used to enhance some properties of a desired lasing mode

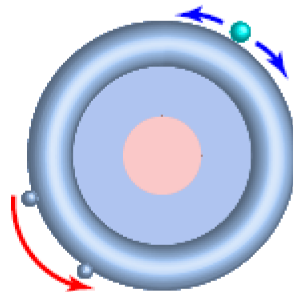
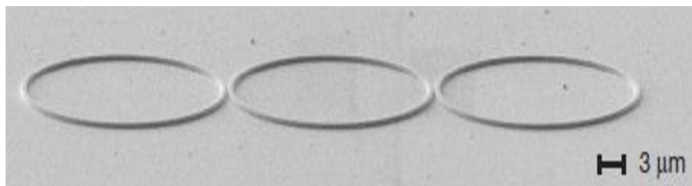
Hodaei, Miri, Heinrich, Christodoulides, Khajavikhan,
Science **346**, no. 6212 (2014)



EPD enhanced sensitivity



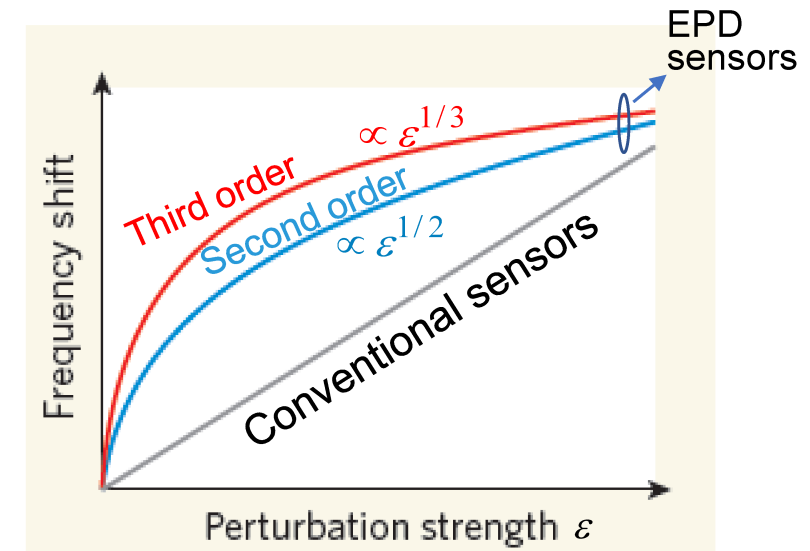
Ren et al, " Ultrasensitive micro-scale parity-time-symmetric ring laser gyroscope," Opt Letts **42**, **8**, (2017)



Wiersig, *Phys. Rev. Lett.* **112**, 203901 (2014)

Hodaei et al. *Nature* **548**, 187–191 (2017)

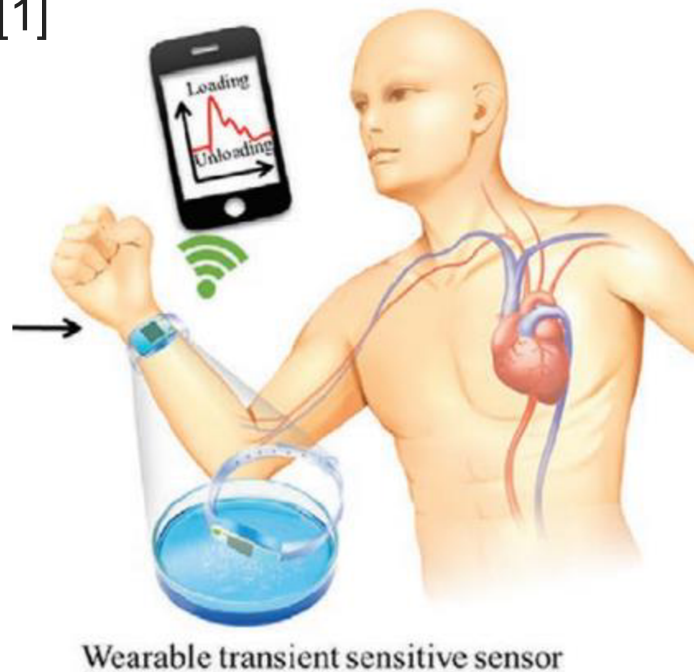
Chen, Özdemir, Zhao, Yang, *Nature* **548**, 192–196 (2017)



Sensitivity demonstrations

- **Sensing** and data acquisition is an essential part of many medical, industrial, and automotive applications
- **Sensing** of local physical, biological, or chemical quantities

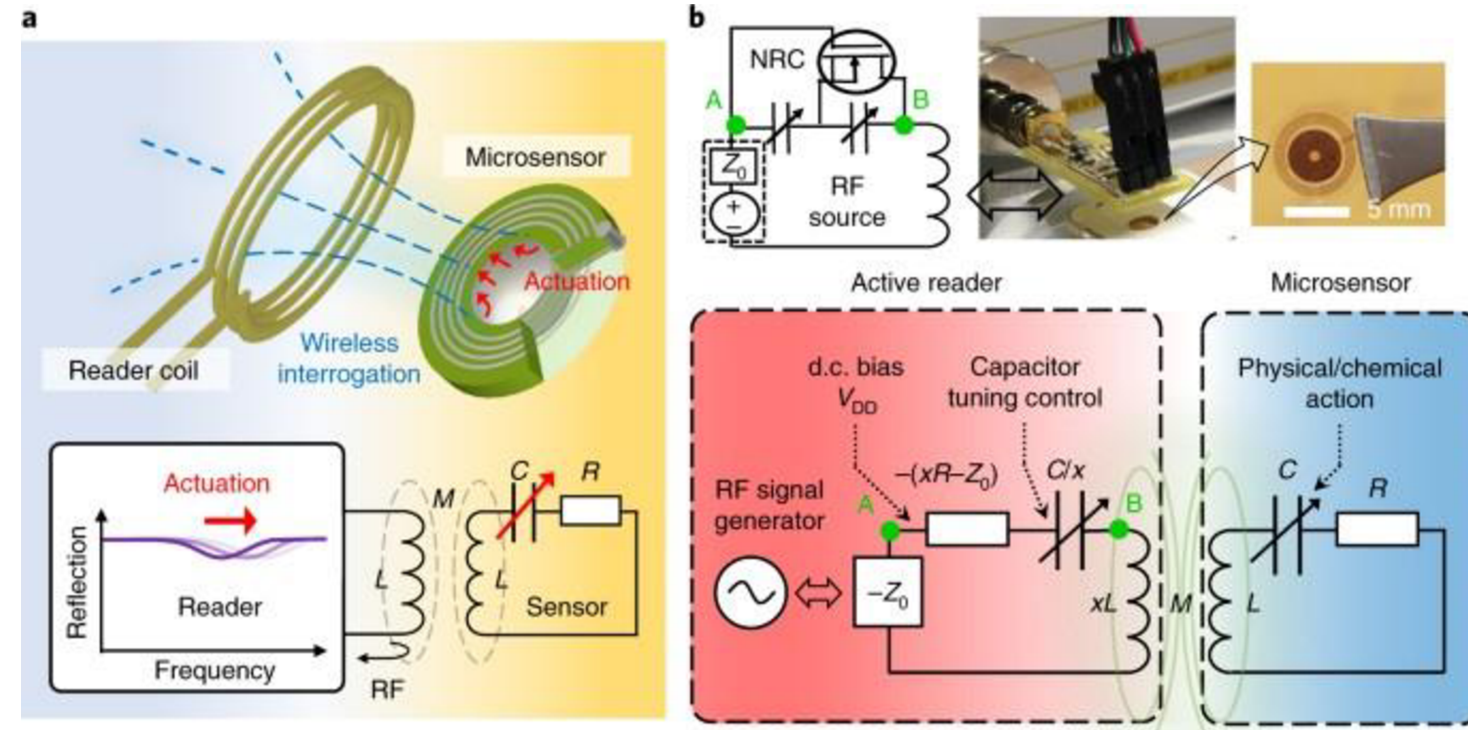
[1]



[1] Yin, Depeng, Shufang, Zheng, Guozhen. *Adv. Funct. Mater.* (2021)

[2] P.-Y. Chen, et al., *Nat. Electron.* (2018)

[2]



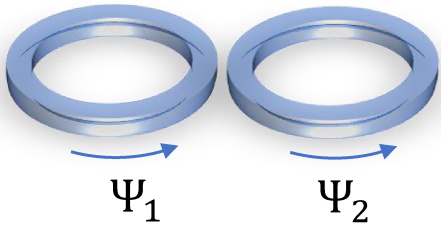
Here the C on the left is finely tuned based on the measuring C on the right (unknown till it is measured). → The current scheme cannot be for sensing unless this is fixed.

EPD in PT-symmetric systems (Eigenvalue is ω)

state vector: $\Psi(t) = \begin{bmatrix} \Psi_1(t) \\ \vdots \\ \Psi_N(t) \end{bmatrix}$

Temporal evolution: $\frac{d}{dt}\Psi(t) = \underline{\underline{\mathbf{M}}}\Psi(t)$

time-invariant
system matrix



Eigenvalues (resonant frequencies)

Solutions like $\Psi(t) \propto e^{-i\omega t}$

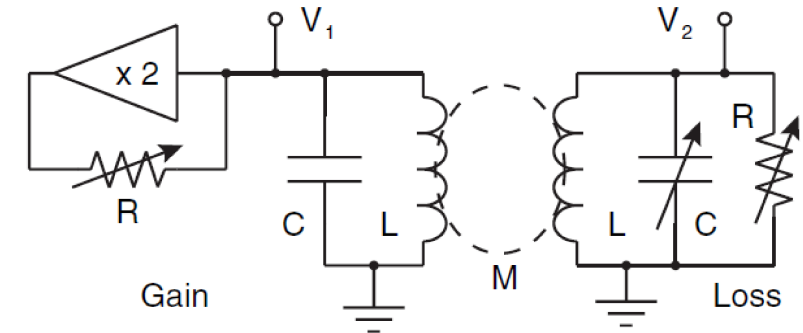


$$-i\omega\Psi = \underline{\underline{\mathbf{M}}}\Psi$$

EPD means degeneracy!

At EPD, the Puiseux series expansion describes the perturbation of resonance frequencies

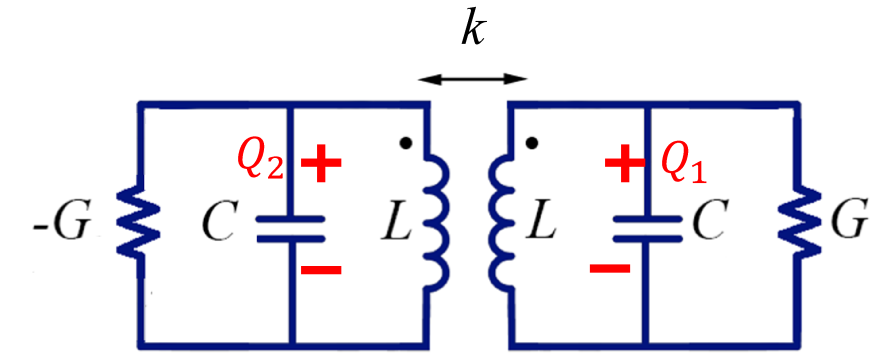
$$\omega_n \approx \omega_e + (-1)^n \alpha_1 \sqrt{\delta}$$



Stehmann, Heiss, Scholtz, *J. Phys A*, 37, 7813 (2004)

Schindler, et al., *Phys. Rev. B*, 84, 040101 (2010)

EPD in two mutually coupled resonators



- State vector $\Psi(t) \equiv [Q_1, Q_2, \dot{Q}_1, \dot{Q}_2]^T$

- System dynamics: first-order differential equation $\frac{d}{dt}\Psi(t) = \underline{\mathbf{M}}\Psi(t)$

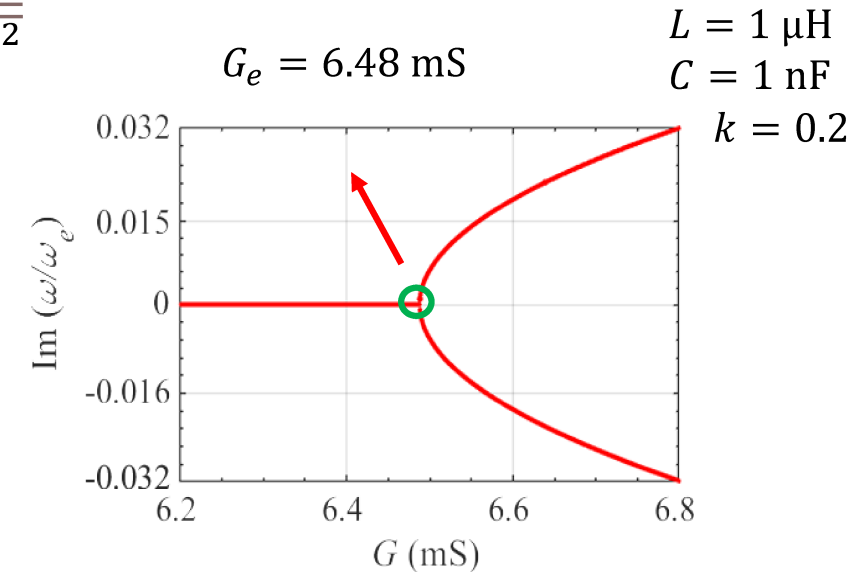
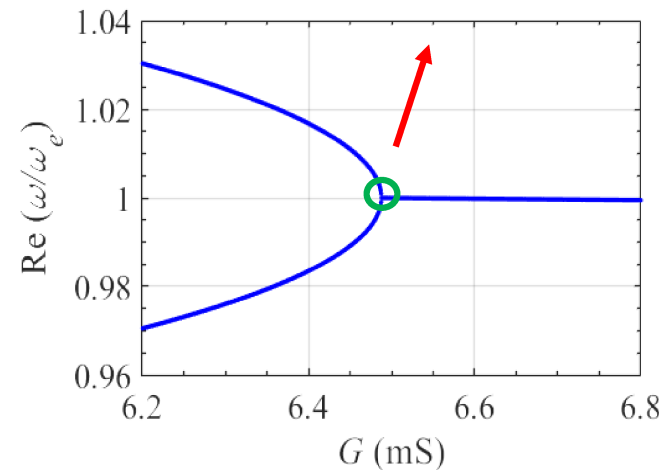
Eigenfrequencies found by solving

$$\det(\underline{\mathbf{M}} + i\omega\underline{\mathbf{I}}) = 0$$

$$\underline{\mathbf{M}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & k & \frac{G_1}{C_1} & 0 \\ \frac{k}{LC_1(1-k^2)} & \frac{-1}{LC_2(1-k^2)} & 0 & -\frac{G_2}{C_2} \end{bmatrix}$$

$$\begin{cases} \omega_{1,3} = \pm\omega_0 \sqrt{\frac{1}{1-k^2} - \frac{\gamma^2}{2} - \sqrt{b}} \\ \omega_{2,4} = \pm\omega_0 \sqrt{\frac{1}{1-k^2} - \frac{\gamma^2}{2} + \sqrt{b}} \end{cases}$$

2nd order EPD at $\omega_e = \frac{\omega_0}{\sqrt[4]{1-k^2}}$



Schindler, et al., *Physical Review A*, 84.4, 2011

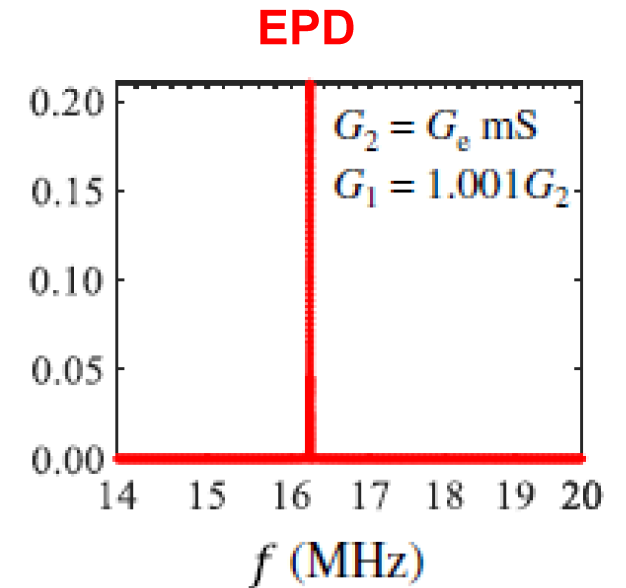
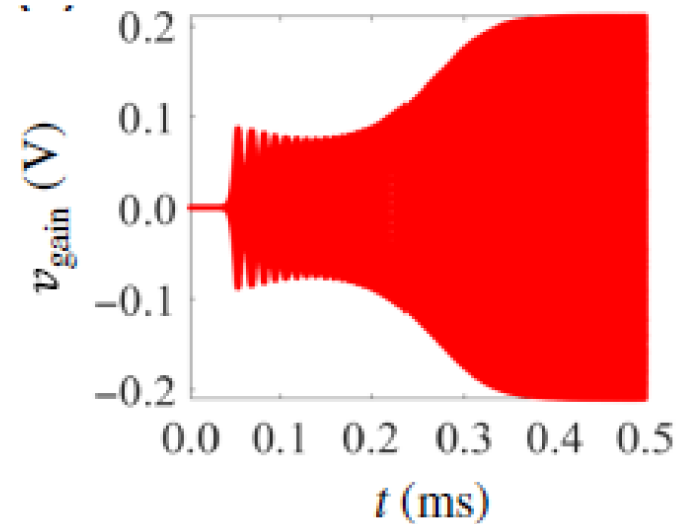
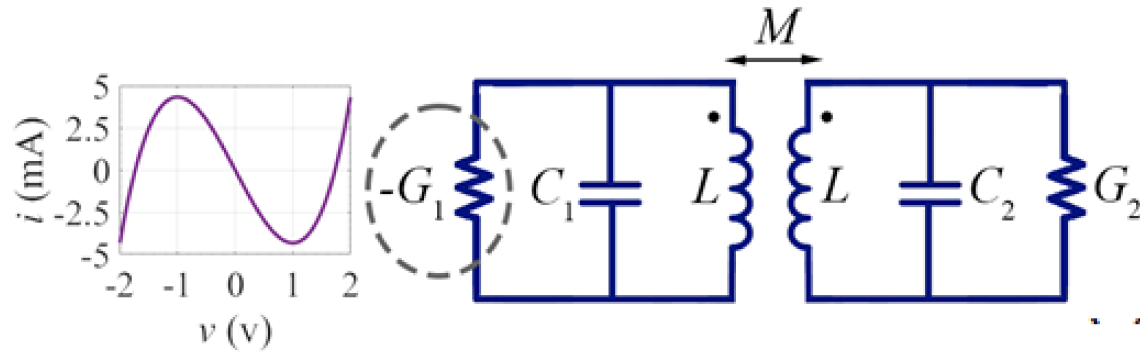
Nikzamid, Rouhi, Figotin, Capolino, *EPJ Appl. Metamat.* 2022

Nikzamid, Capolino, *Phys. Rev. Applied* 18, 054059, 2022

When measuring C changes, and the system is not PT symmetric anymore → It becomes unstable. This is fixed using the method in the slides: we make an oscillator.

Nonlinearity in two mutually coupled resonators

Using a nonlinear gain and making the system unstable → **oscillator**

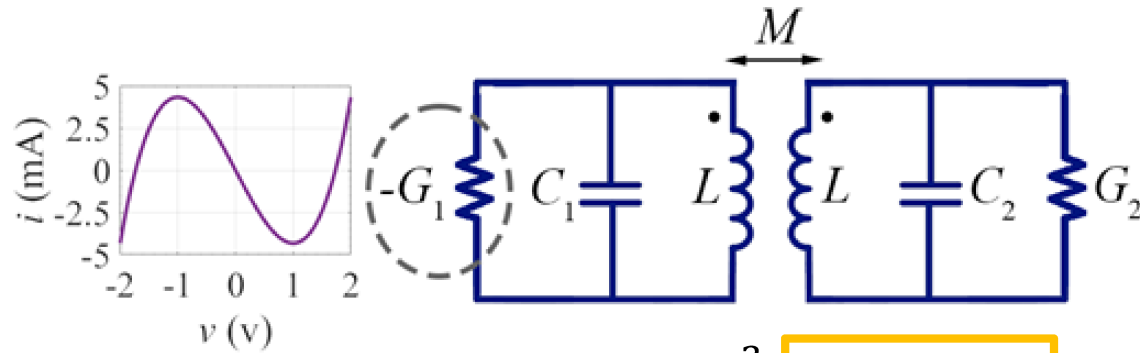


Nikzamir, Capolino, *Phys. Rev. Applied*
18, 054059, 2022

The oscillation frequency is at the EPD!!!

Nonlinearity in two mutually coupled resonators

Using a nonlinear gain and making the system unstable

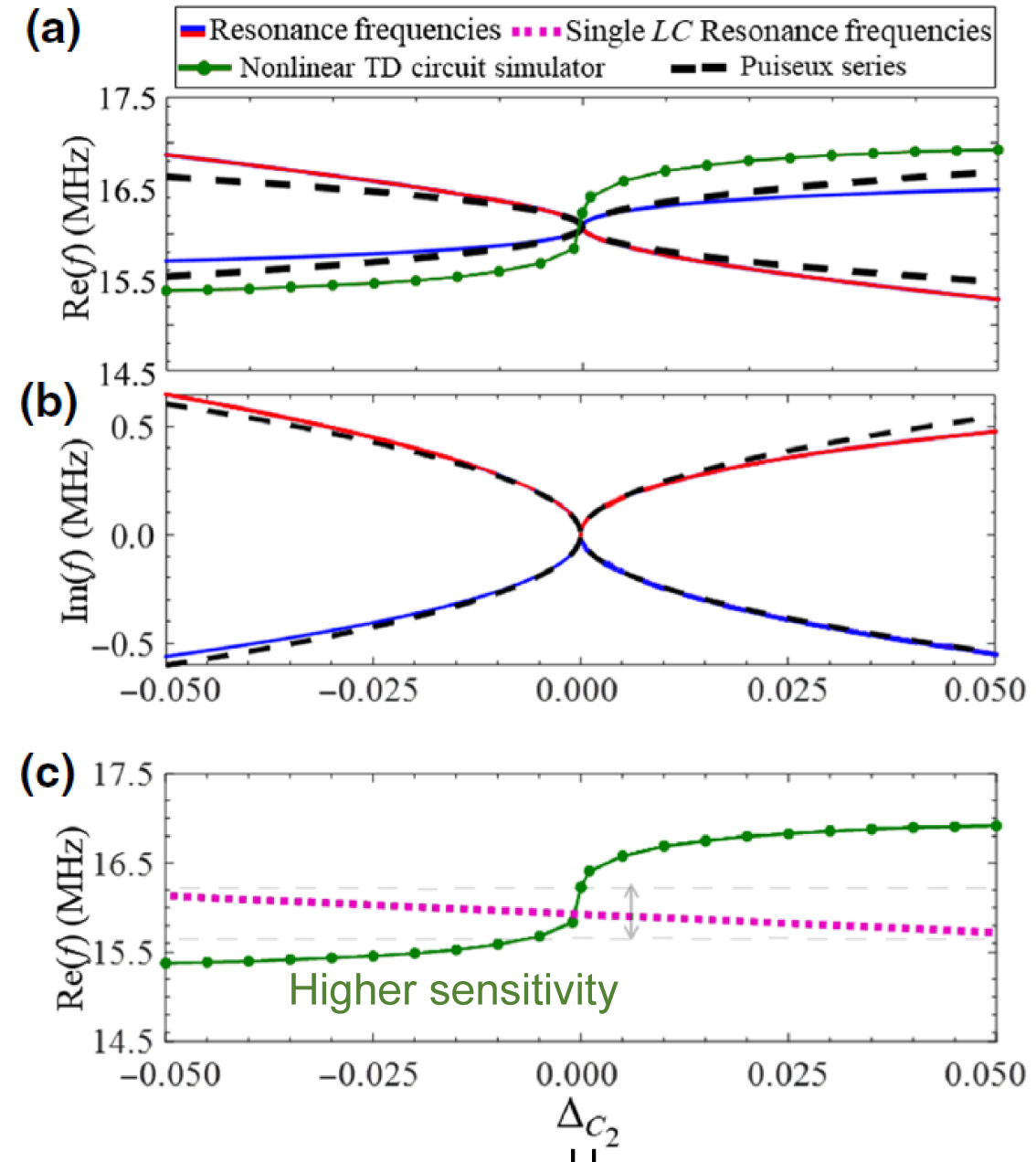


$$i = -G_1 v + \alpha v^3$$

Oscillator

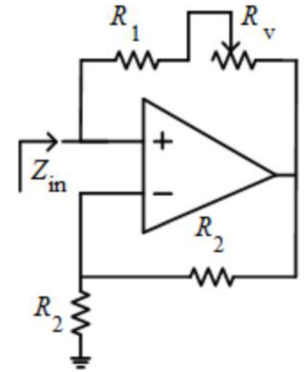
Perturbing by $\Delta C_2 = (C_2 - C_{2,e})/C_{2,e}$

- $\det(\underline{\mathbf{M}} - j\omega \underline{\mathbf{I}}) = 0$
- $\omega_p(\Delta C_2) \approx \omega_e + \alpha_1(-1)^p \sqrt{\Delta C_2}$
- The oscillation frequency variation follows the square root sensitivity due to the perturbation



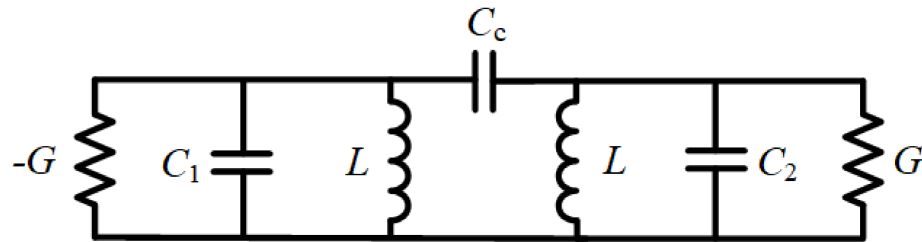
Experimental demonstration – excellent agreement

- How to realize the nonlinear gain $-G$

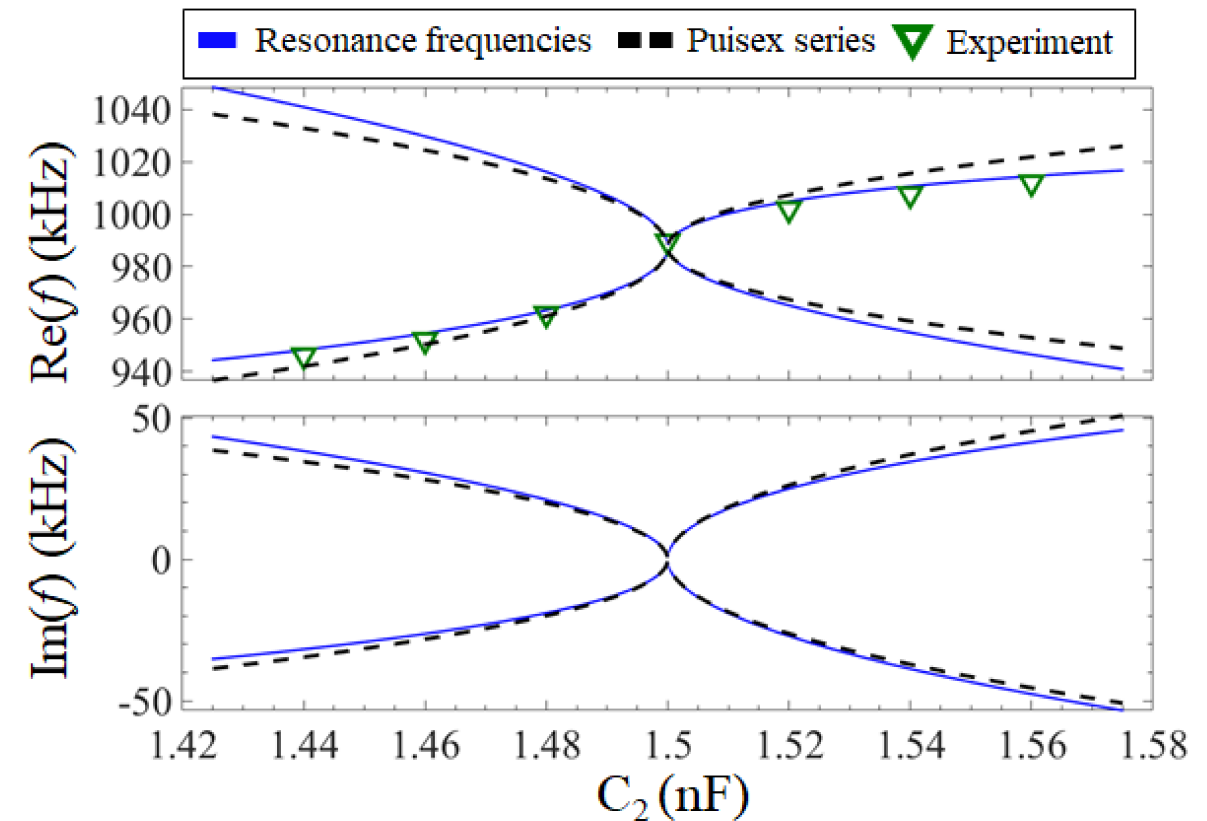
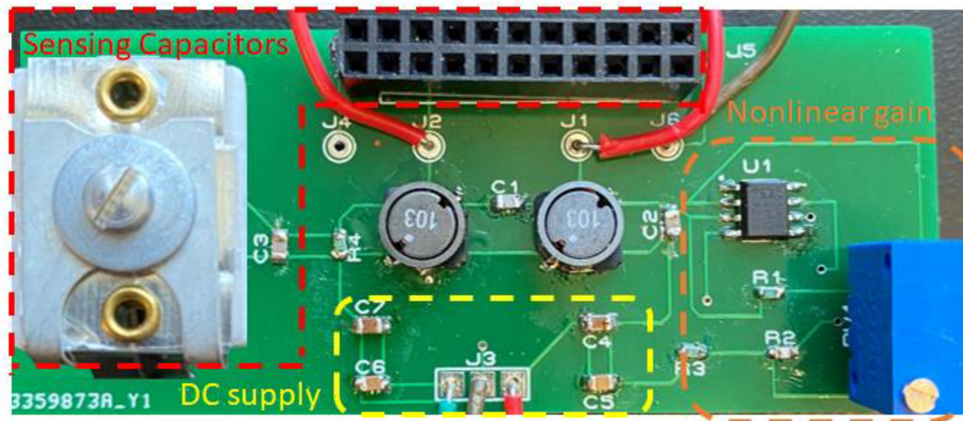


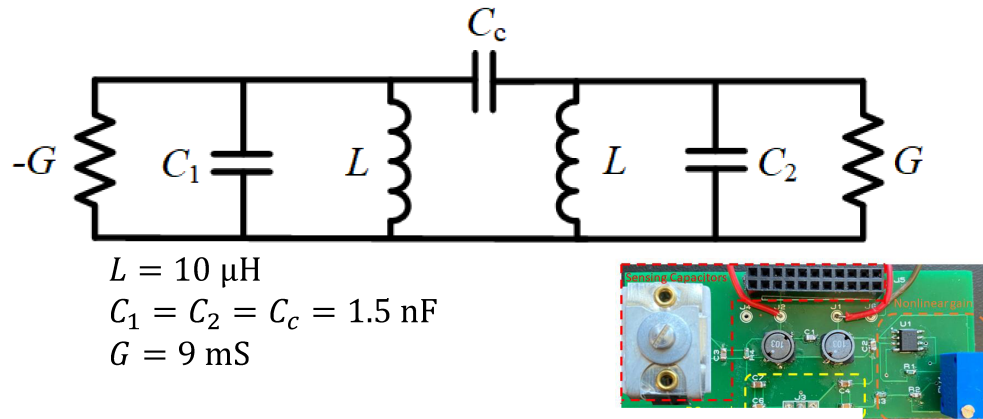
Square root sensitivity to the capacitance change C_2

$$\omega_p(\Delta C_2) \approx \omega_e + \alpha_1(-1)^p \sqrt{\Delta C_2}$$



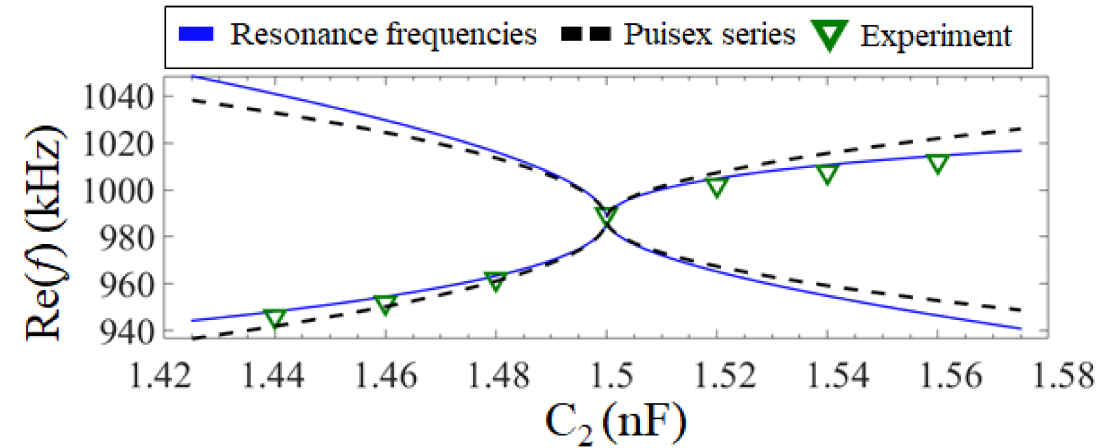
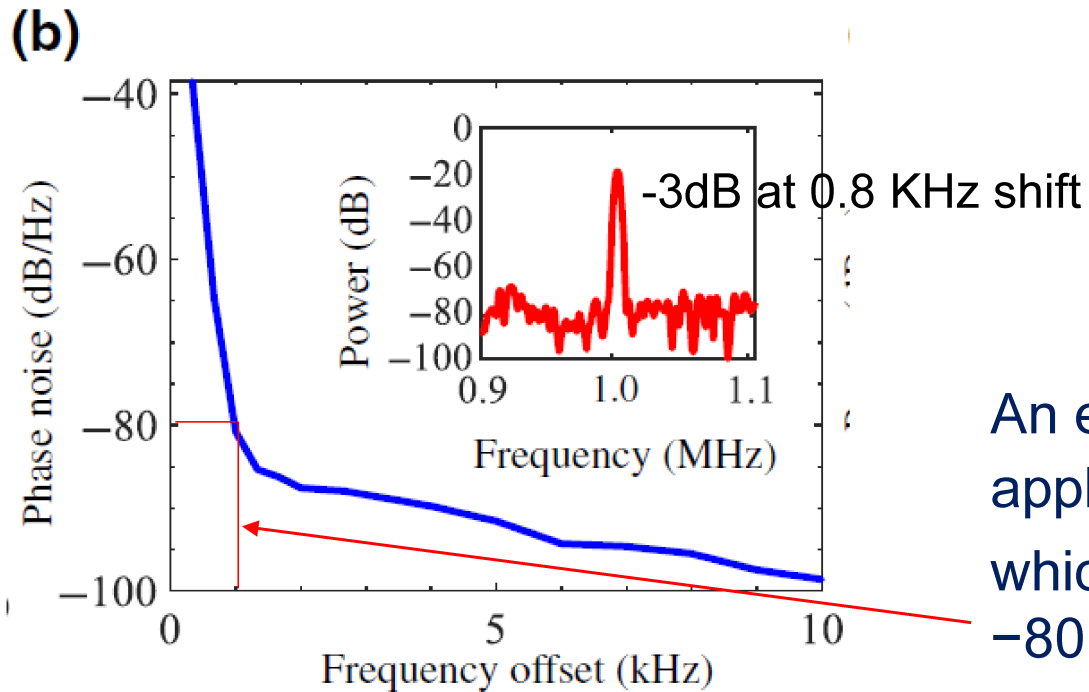
$$\begin{aligned} L &= 10 \mu\text{H} \\ C_1 &= C_2 = C_c = 1.5 \text{ nF} \\ G &= 9 \text{ mS} \end{aligned}$$





Square root sensitivity to the capacitance change C_2

$$\omega_p(\Delta C_2) \approx \omega_e + \alpha_1(-1)^p \sqrt{\Delta C_2}$$

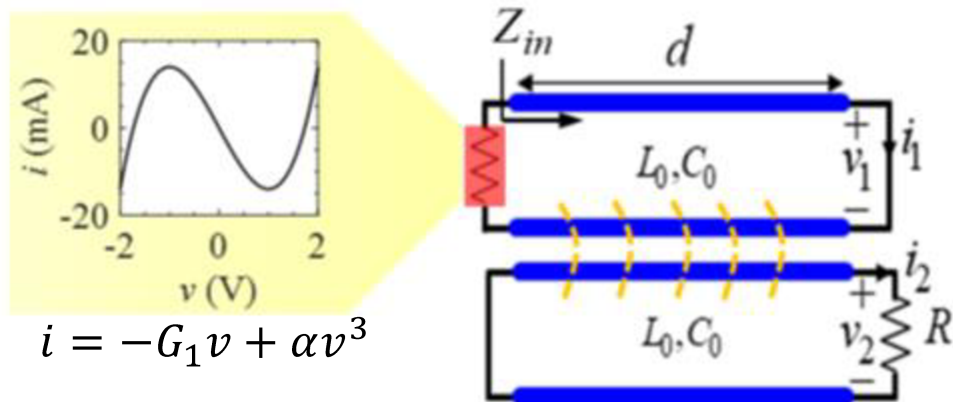


An experimental relative perturbation $\Delta C_2 = 0.013$ (i.e., 1.3%) applied to C_2 leads to a frequency shift $\Delta f = 12.4 \text{ kHz}$, which is much larger than the 1 kHz offset associated to -80.8 dB/Hz noise.

Very low noise!

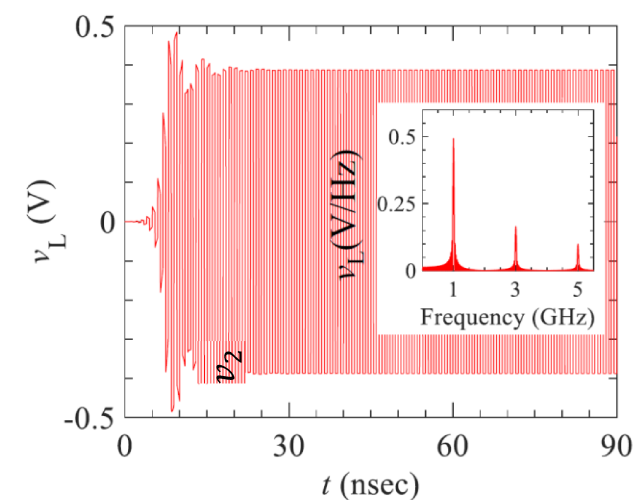
Nonlinearity in two mutually coupled resonators

of Irvine

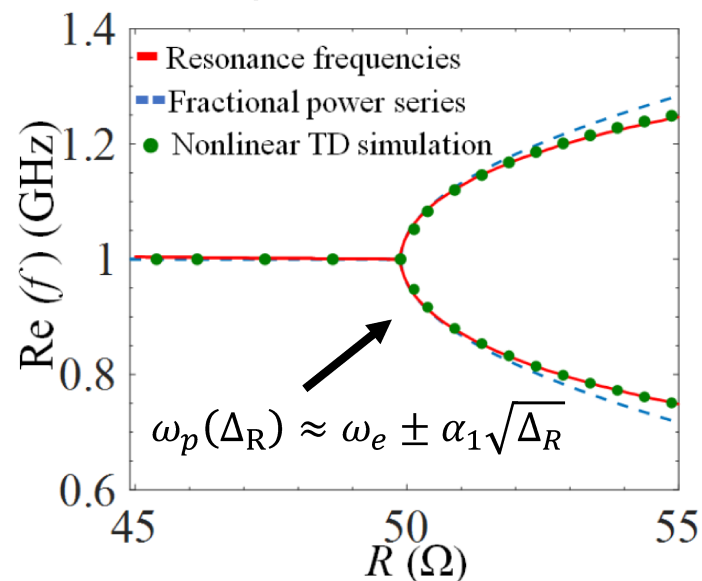


Voltage across the load

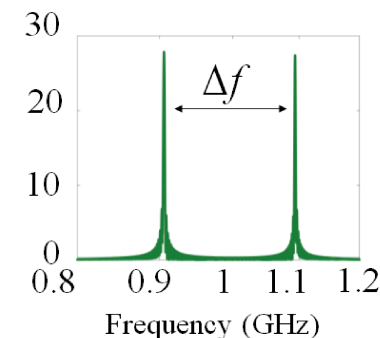
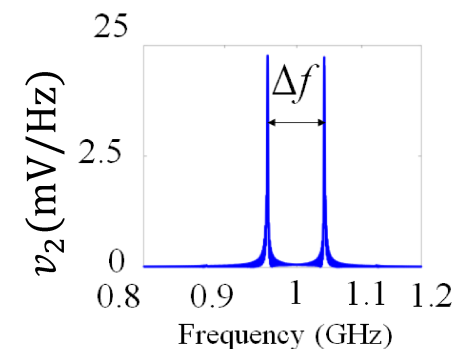
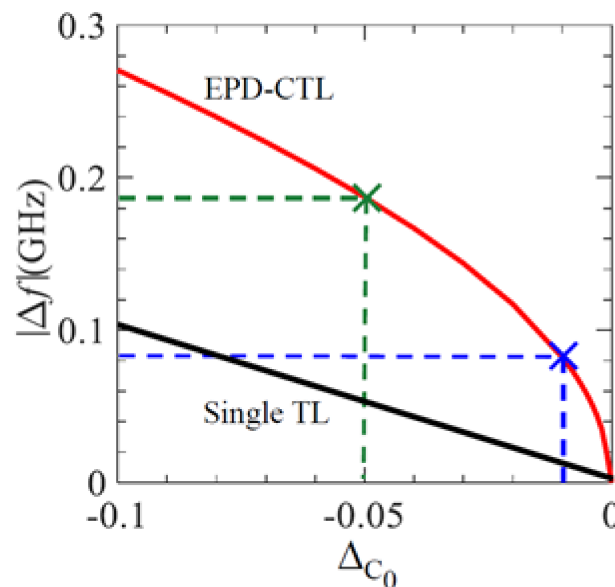
$v_2(V)$



Perturbing load resistance R



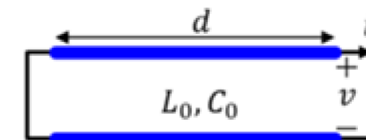
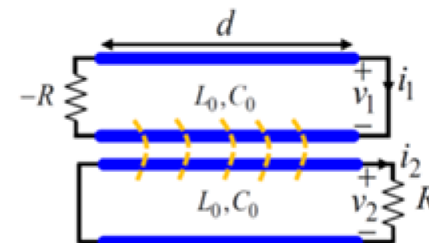
Perturbing per-unit-length capacitance C_0



$$\Delta\omega_p(\Delta_C) \approx \alpha_1 \sqrt{\Delta_C}$$

vs

$$\Delta\omega_p(\Delta_C) \approx \beta \Delta_C$$

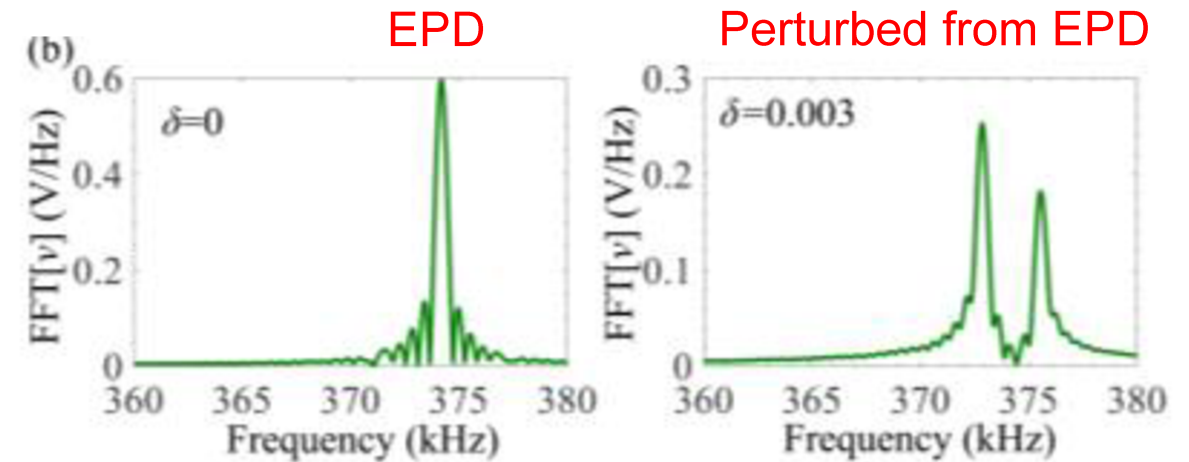
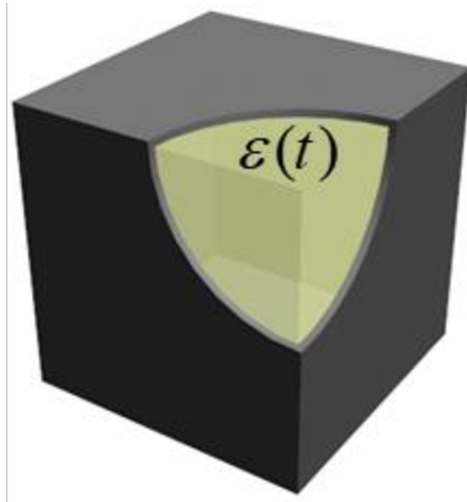


Can you have an EPD in a single resonator?

(Yes!)

Solution: Time-varying system

EPD is obtained in this simple structure:



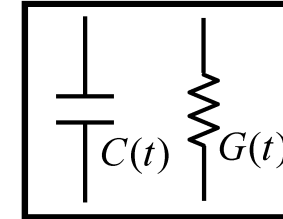
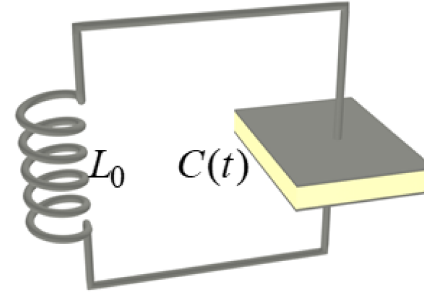
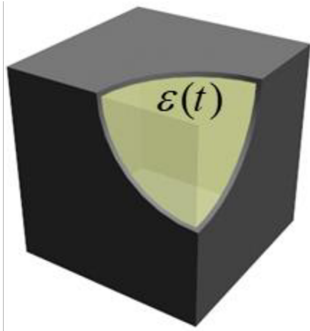
Kazemi, Nada, Mealy, Abdelshafy, Capolino, *Phys. Rev. Applied*, 11, 014007 (2019)

Kazemi, Nada, Nikzamir, Maddaleno, Capolino. *J. Appl. Phys.* 131, 144502 (2022)

Nikzamir, Rouhi, Figotin, Capolino. *EPJ Appl. Metamat.* (2022)

Linear time-varying (LTV) single resonator

One of the parameters is changing with time i.e., the capacitance or dielectric permittivity



Temporal evolution of the state vector:

$$\frac{d}{dt} \Psi(t) = \underline{\mathbf{M}}(t) \Psi(t)$$

Periodic in time (time-variant) system matrix

$$\Psi(t) = \begin{bmatrix} \Psi_1(t) \\ \vdots \\ \Psi_N(t) \end{bmatrix} \text{ state vector}$$

State vector evolution from t to $t + T_m$:

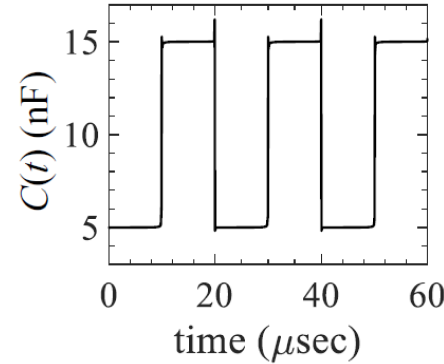
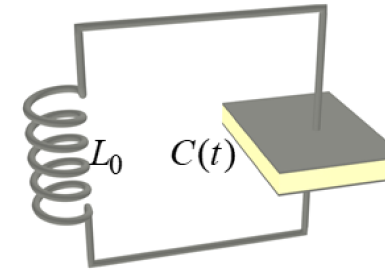
$$\Psi(t + T_m) = \underline{\Phi}(t + T_m, t) \Psi(t)$$

State transition matrix

T_m : period of time variation

General condition on second order EPD

We assume the system matrix $\underline{\mathbf{M}}(t)$ is piecewise constant



$$\underline{\Phi} = e^{\underline{\mathbf{M}}_1 T_1} e^{\underline{\mathbf{M}}_2 T_2}$$

$$T_m = T_1 + T_2$$

$$\Psi(t + T_m) = \underline{\Phi}(t + T_m, t) \Psi(t)$$

State vector satisfies the Floquet theorem

$$\Psi(t + T_m) = e^{-i\omega T_m} \Psi(t)$$

For a general second order system, it is necessary and sufficient to have degenerate eigenvalues if

$$\lambda = \text{trace}(\underline{\Phi}) / 2 \pm \sqrt{\frac{\text{trace}(\underline{\Phi})^2}{4} - \det(\underline{\Phi})}$$

EPD condition

$$\text{trace}(\underline{\Phi}) / 2 = \pm \sqrt{\det(\underline{\Phi})}$$

Eigenvalue problem

$$\underline{\Phi} \Psi(t) = \lambda \Psi(t)$$

$$\lambda_n = e^{-i\omega_n T_m}$$

$$n = 1, 2$$

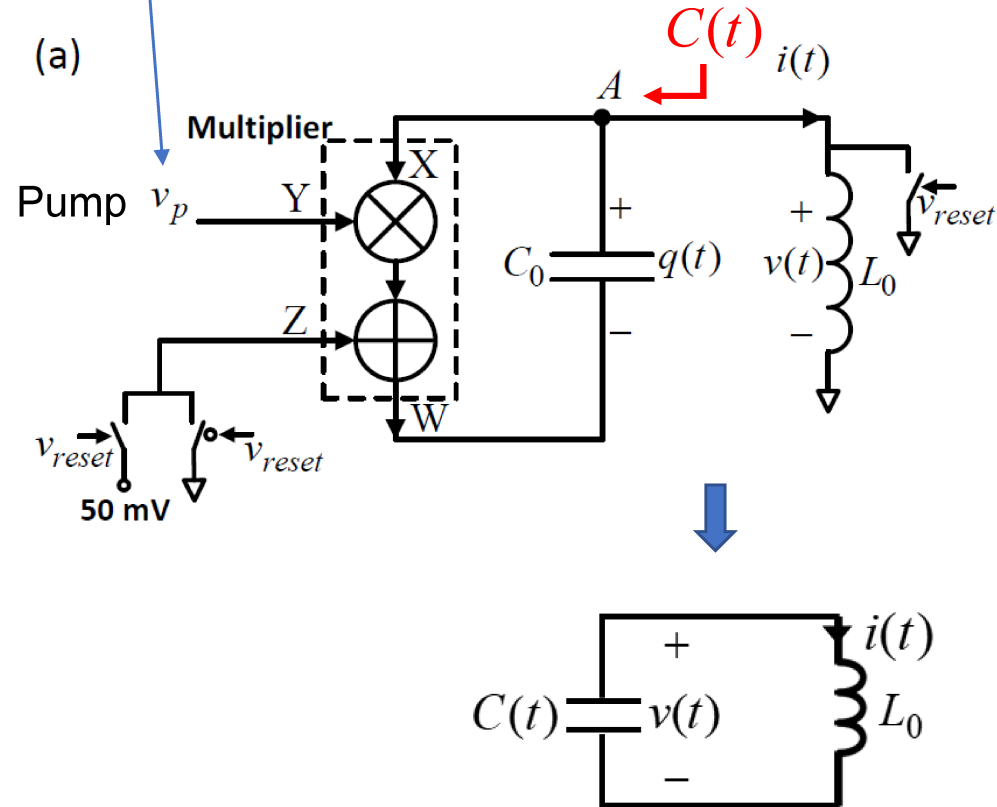
Resonance frequencies:

$$\omega_n + p\omega_m$$

$$p = 0, \pm 1, \pm 2, \dots$$

Experimental demonstration

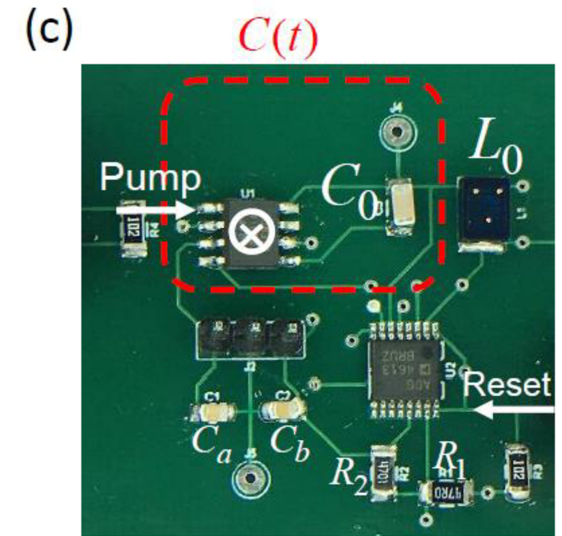
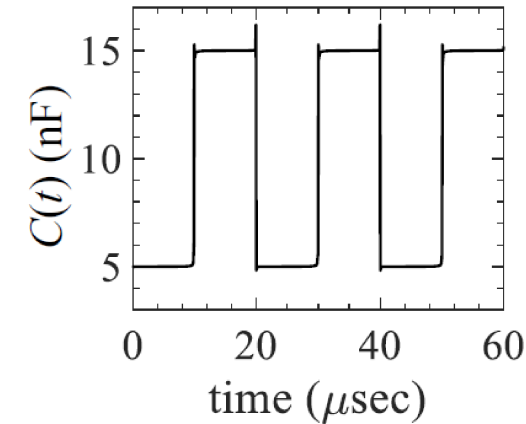
The time-periodic capacitance is achieved using a modulation pump = $v_p(t)$



$$I_c = j\omega C_0 (1 - v_p(t)/V_0) V_c$$

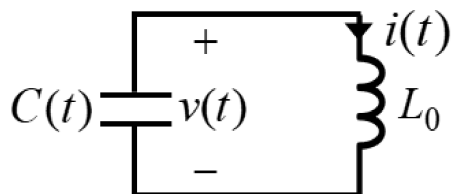
$C(t)$

The admittance seen on the right, changes with $v_p(t)$

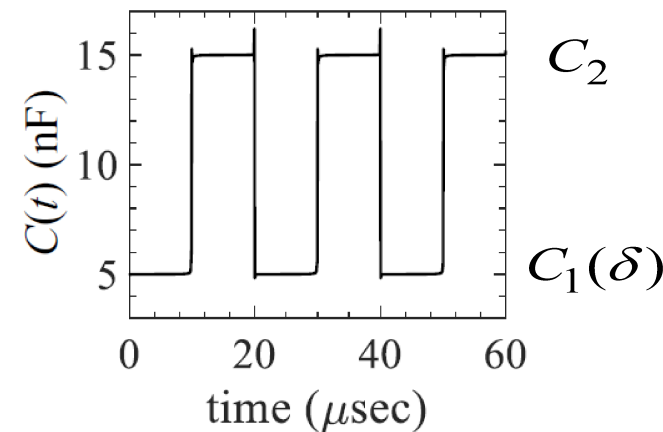


Kazemi, Nada, Nikzamir, Maddaleno, Capolino, "Experimental demonstration of exceptional points of degeneracy in linear time periodic systems and exceptional sensitivity," *Journal of Applied Physics*, 131, 144502, 2022

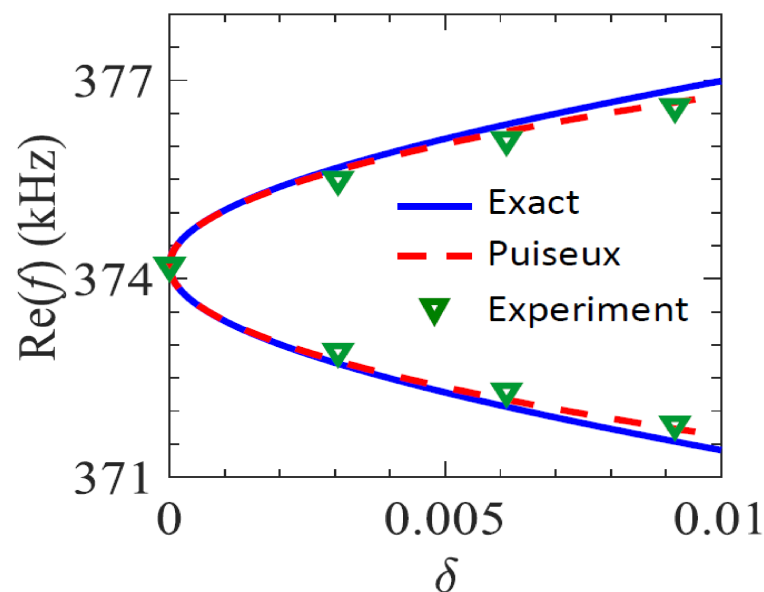
Perturbation analysis for sensing applications



EPD at $f_e = 374 \text{ kHz}$ (6th harmonic)
with modulation $f_m = 62.8 \text{ kHz}$



Perturbation of eigenfrequencies (i.e., resonance frequencies) for LTV-LC resonator when $C_1(\delta) = (1 + \delta)C_1$



Puisseux series expansion describes the perturbation of resonance frequencies:

$$f_n(\delta) \approx f_e + (-1)^n \frac{f_m}{2\pi} i \alpha_1 \sqrt{\delta}$$

↑ unperturbed eigenvalue at EPD
↑ Capacitance perturbation

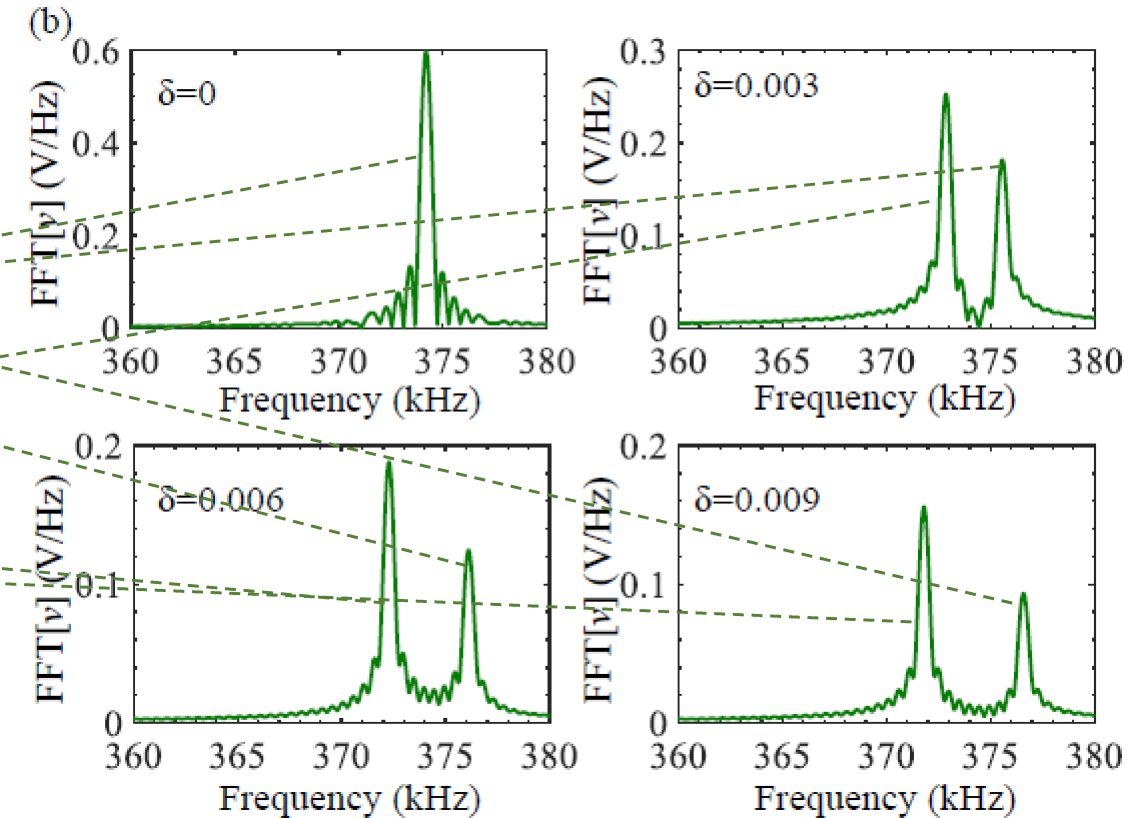
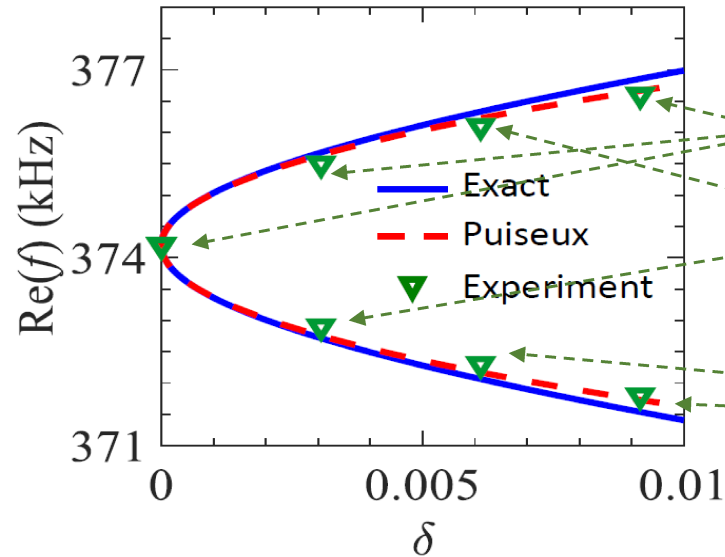
Perturbed frequencies are real valued

Kazemi, Nada, Nikzamir, Maddaleno, Capolino, "Experimental demonstration of exceptional points of degeneracy in linear time periodic systems and exceptional sensitivity," *Journal of Applied Physics*, 131, 144502, 2022

Perturbation of resonance frequencies for LTV-LC resonator when $C_1(\delta) = (1 + \delta)C_1$

$$f_n(\delta) \approx f_e + (-1)^n \frac{f_m}{2\pi} i \alpha_1 \sqrt{\delta}$$

unperturbed eigenvalue at EPD
↓

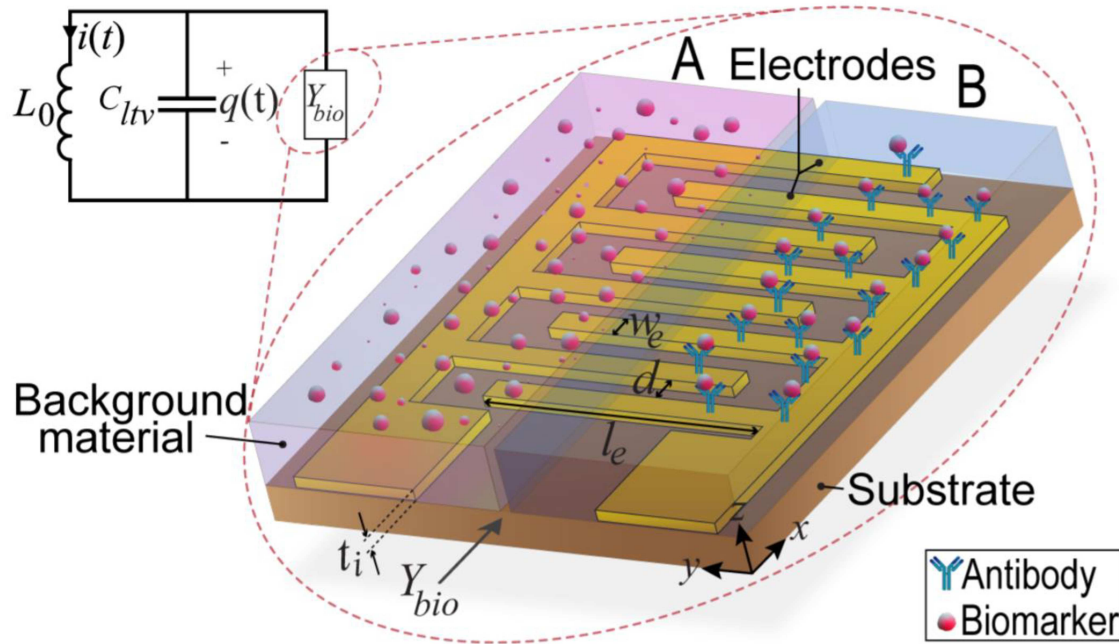


Kazemi, Nada, Maddaleno, Nikzamir, Capolino, "Experimental demonstration of exceptional points of degeneracy in linear time periodic systems and exceptional sensitivity," arXiv:1908.08516, 2019

Very distinct peaks! We can measure the distance

Application: sensor

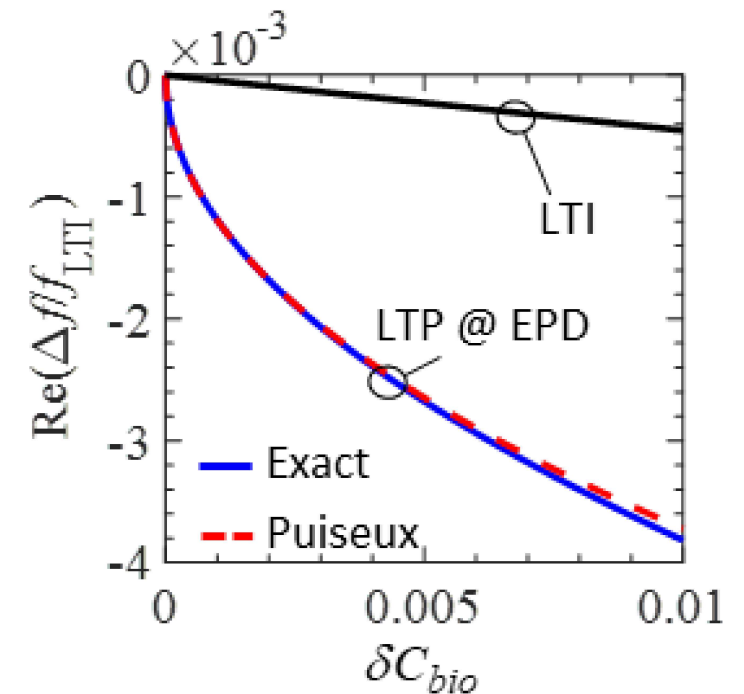
High sensitivity of resonances at an EPD



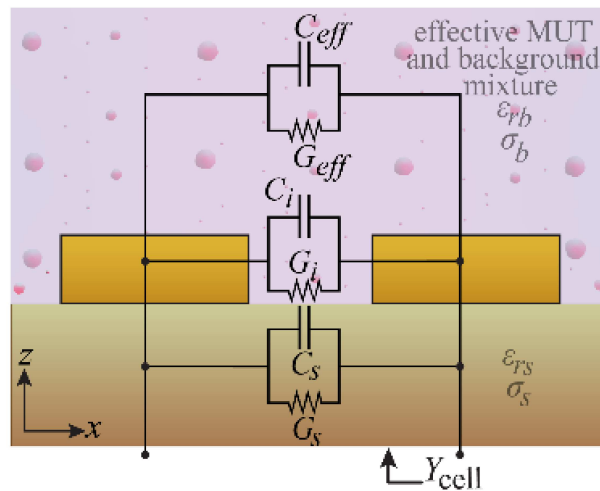
Puiseux:

$$f_p(\delta) \approx f_e \pm j \frac{f_m}{2\pi} (-1)^p \alpha_1 \sqrt{\delta}$$

$$\Delta f = f_p(\delta) - f_e$$



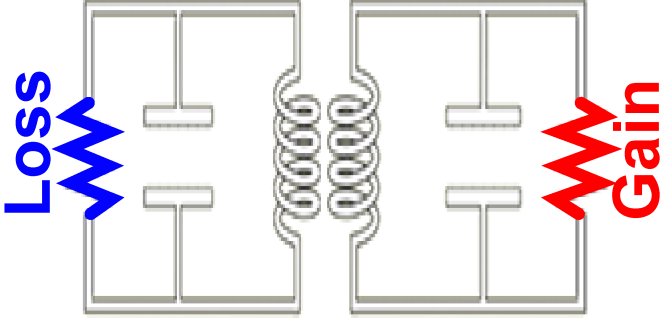
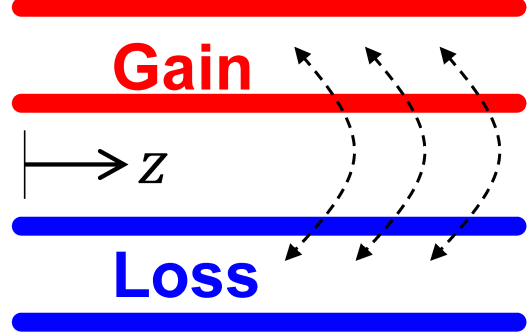
$\delta C_{bio} \ll 1 \implies f_p$ changes dramatically



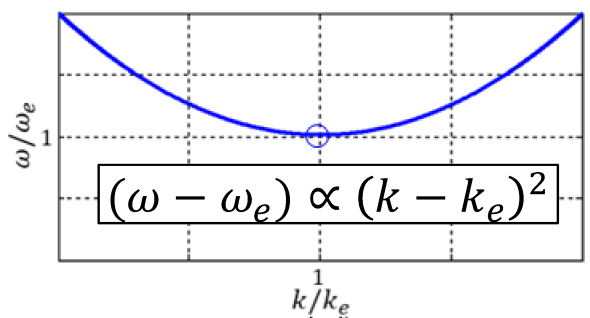
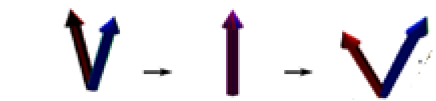
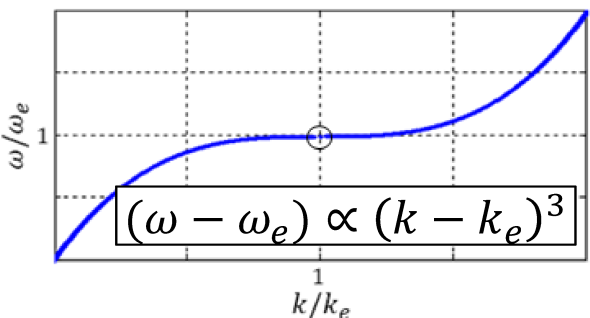
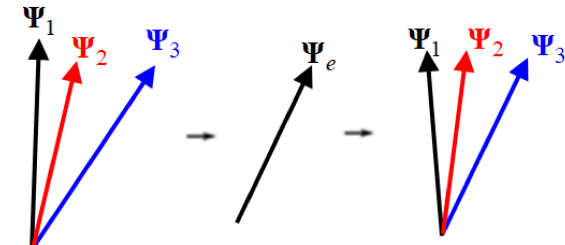
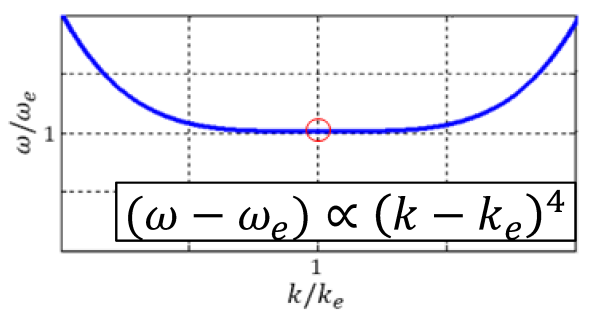
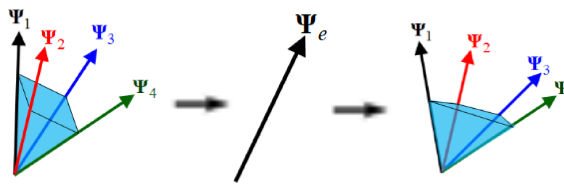
LTP sensor shows much higher sensitivity compared to a *linear-time-invariant* (LTI) LC-based sensor

Kazemi et al., "Ultra-Sensitive Radio Frequency Biosensor...",
IEEE Sensors Journ., 21 (6), 2021

EPD is found also in two coupled waveguides

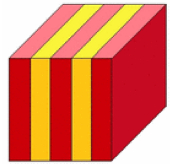
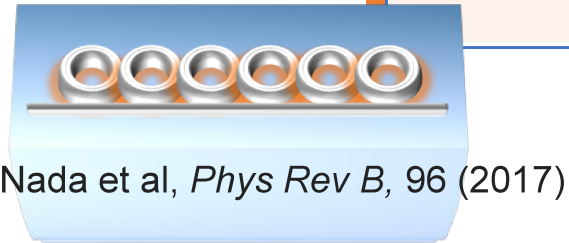
	Temporal EPD	Spatial EPD
Structure	 2 coupled resonators Schindler, Joseph, et al, <i>Phys. Review A</i>, vol. 84. no. 4 (2011)	 2 coupled waveguides Klaiman, et al, <i>Physical Rev Lett</i>, 101, 8, 2008. Othman, Capolino, <i>IEEE Trans. Antennas Propagat.</i>, 65, 10, 2017
State vector	$\Psi(t) \propto e^{-i\omega t}$	$\Psi(z, t) \propto e^{-i\omega t} e^{ikz}$
Degeneracy	$\omega_1 = \omega_2$ <i>Coalescence in resonance frequency</i>	$k_1 = k_2 \quad @ \quad \omega = \omega_e$ <i>Coalescence in wavenumber</i>

Dispersion diagram with EPD from linear algebra perspective

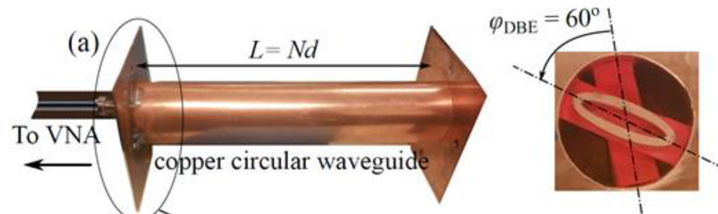
EPD order	Purely-real wavenumber	Eigenvector evolution	System matrix @ EPD	Eigenmode distribution
2 (RBE)		 Varying system parameter	$\mathbf{M} \sim \begin{bmatrix} k_e & 1 \\ 0 & k_e \end{bmatrix}$ 2x2 Jordan block	$\Psi(z) \propto z e^{ik_e z}$
3 (SIP)		 Varying system parameter	$\mathbf{M} \sim \begin{bmatrix} k_e & 1 & 0 \\ 0 & k_e & 1 \\ 0 & 0 & k_e \end{bmatrix}$ 3x3 Jordan block	$\Psi(z) \propto z^2 e^{ik_e z}$
4 (DBE)		 Varying system parameter	$\mathbf{M} \sim \begin{bmatrix} k_e & 1 & 0 & 0 \\ 0 & k_e & 1 & 0 \\ 0 & 0 & k_e & 1 \\ 0 & 0 & 0 & k_e \end{bmatrix}$ 4x4 Jordan block	$\Psi(z) \propto z^3 e^{ik_e z}$

EPD in Waveguides

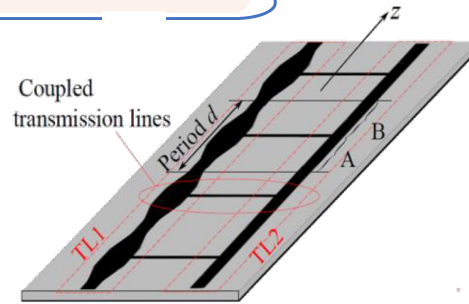
Lossless



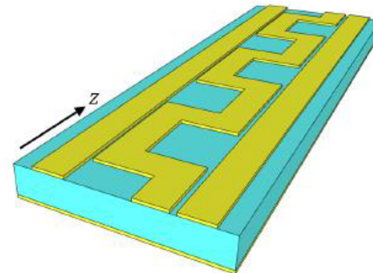
Figotin, Vitebskiy, *Phys. Rev. E* **68**(3), (2003)



Othman et al, *IEEE MTT*, 65, 6, 2017

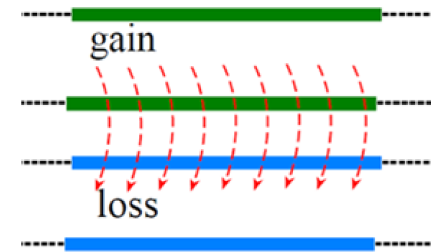


Abdelshafy et al, *IEEE TAP*, 67, no. 9, 2019

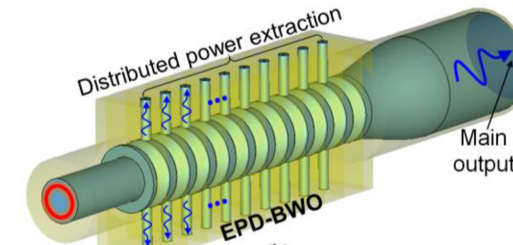
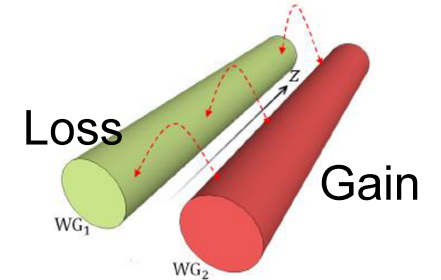


Nada, Mealy, Capolino, *IEEE MWCL*, 2020

With gain/loss



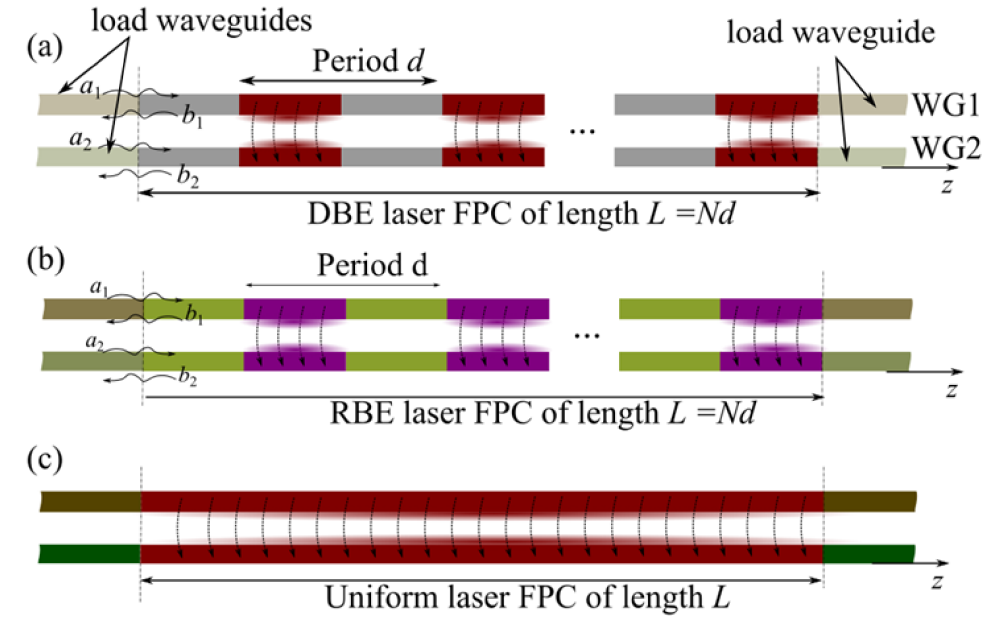
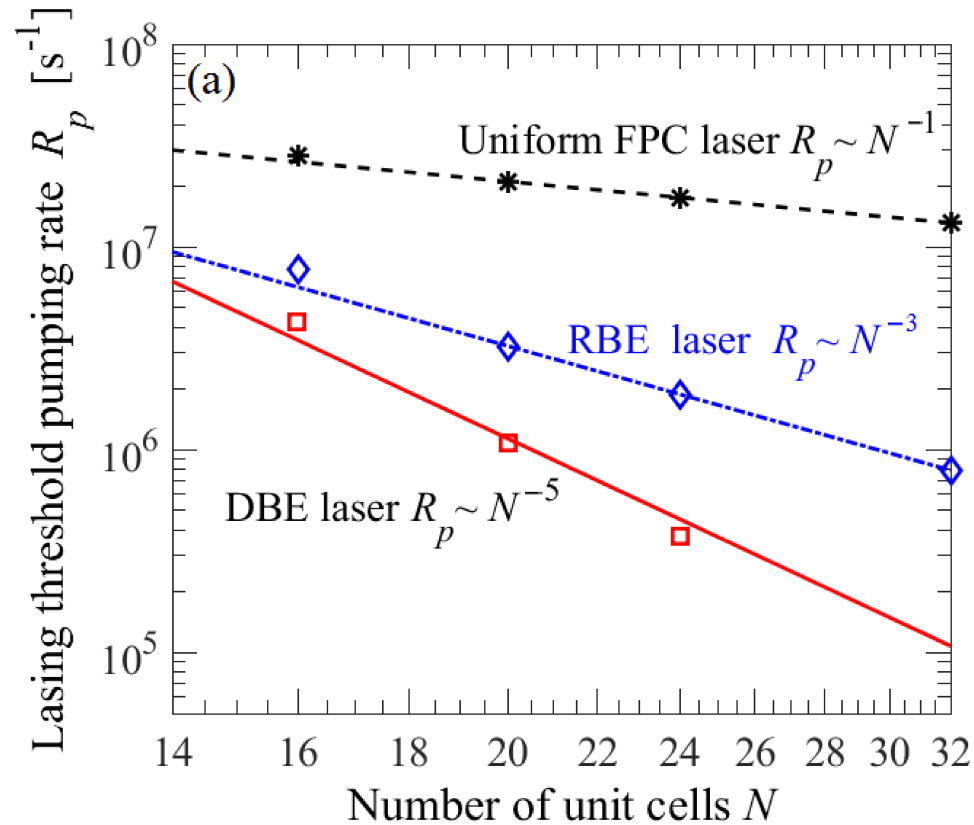
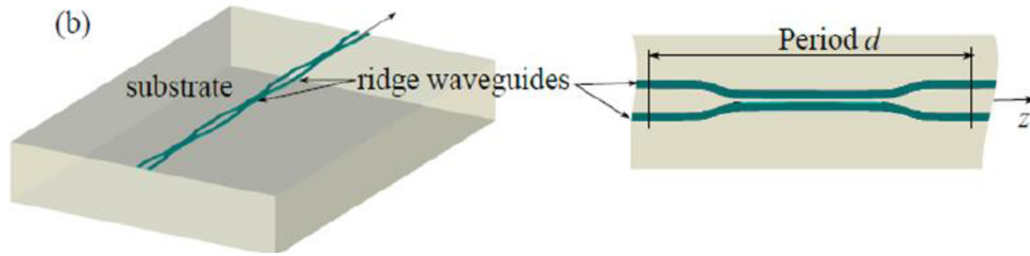
Othman, Capolino, *IEEE TAP*, 65, 10, 2017



Mealy, Abdleshafy, Capolino, *PR Appl*, 2021

Today we focus on EPD in lossless/gainless waveguides!

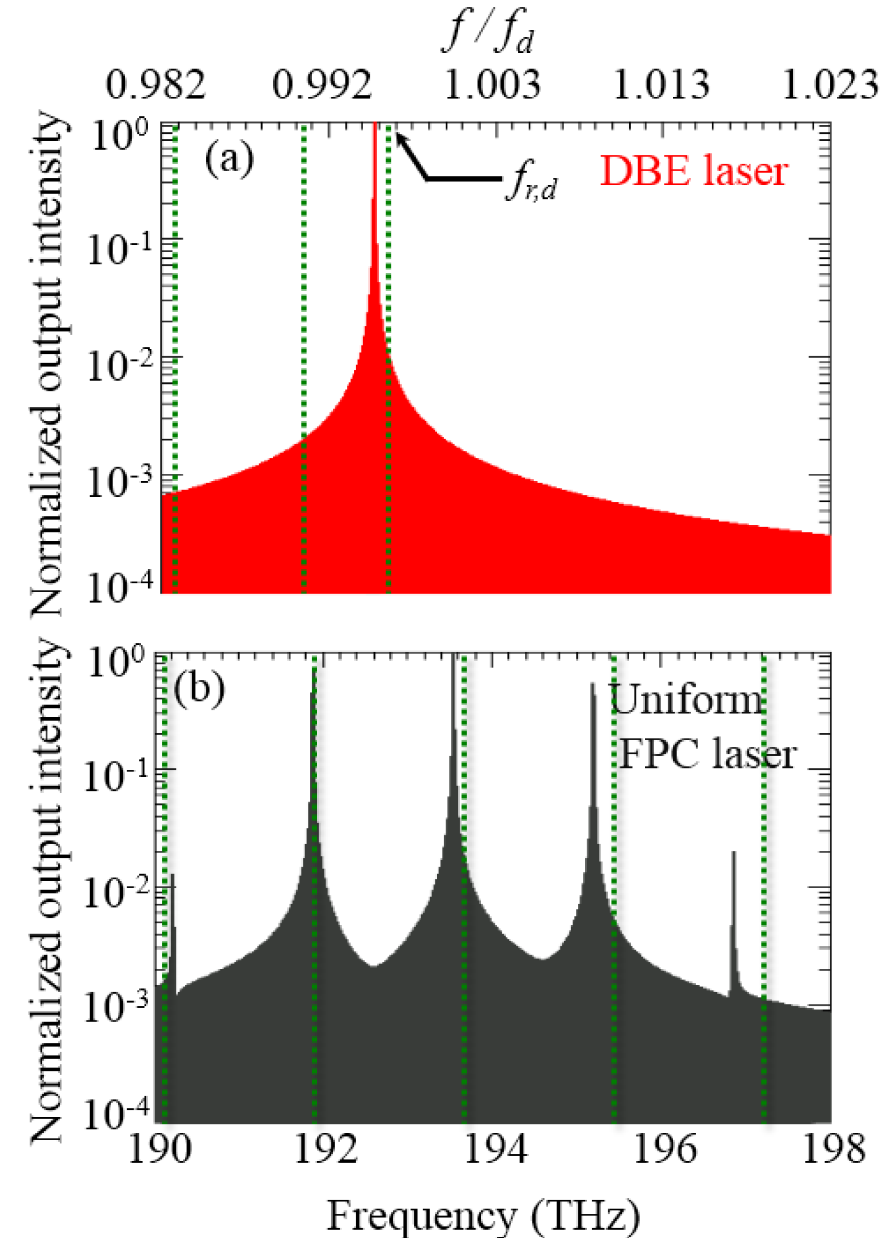
Applications: DBE laser



- DBE laser threshold, calculated using Time-domain coupled mode theory
- DBE laser has the lowest threshold
- Scales as N^{-5}

Applications: DBE laser, mode selectivity

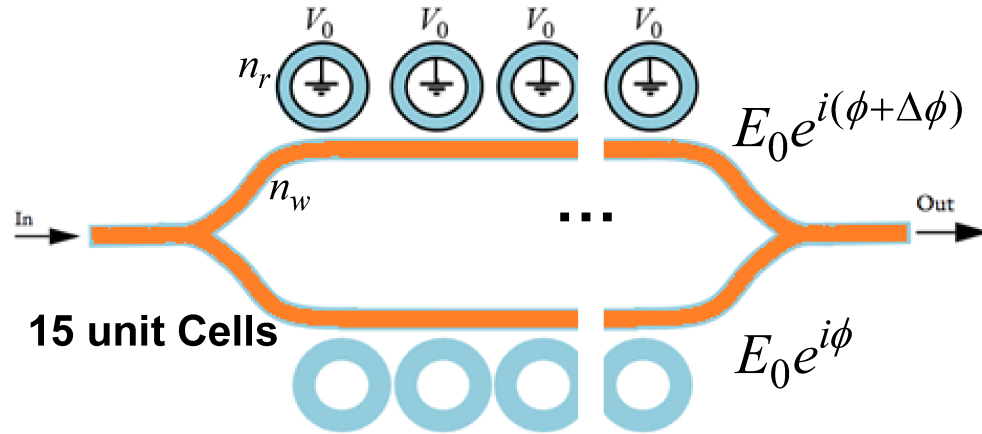
- Comparison between **DBE laser** and **uniform FPC laser**
 - Same Q-factor
 - Both pumped at rate 10^6 s^{-1}
- The **DBE laser** has a **single oscillation mode** (single frequency). **Robust!**
- **Uniform optical cavities** have multimode operation (spectral separation of the FPC resonance frequencies of $\sim 1.6 \text{ THz}$.)



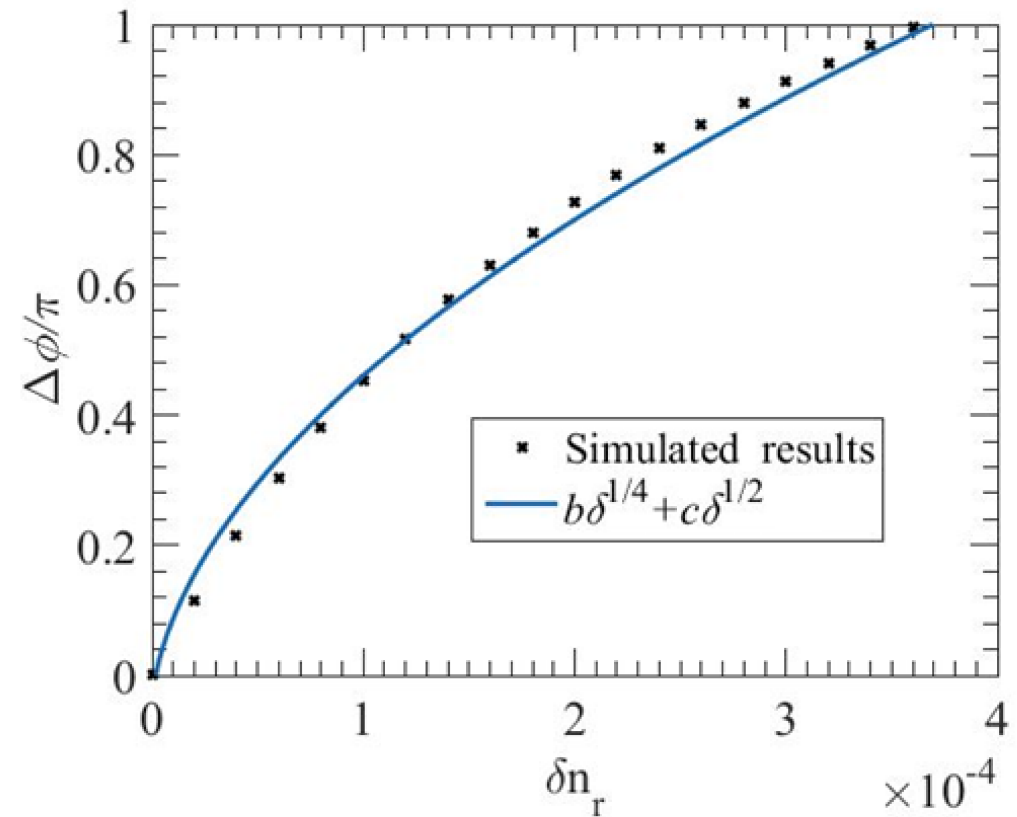
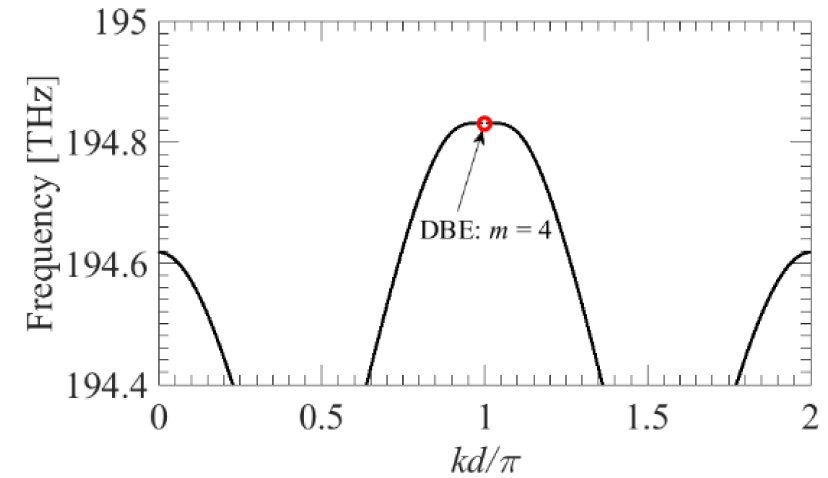
Veysi, Othman, Figotin, Capolino, "DBE Laser,"
Phys. Rev. B, vol. 97, no. 19 (2018)

Not good for sensing: No frequency shift with perturbations. The eigenvalue is k !!!

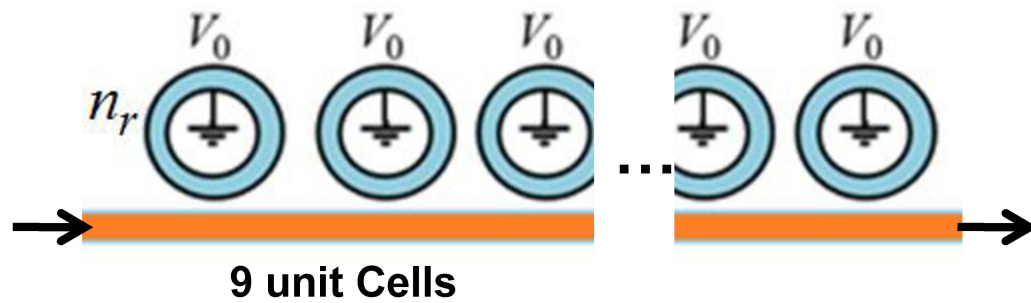
DBE interferometer based on sensitive $\phi(k)$



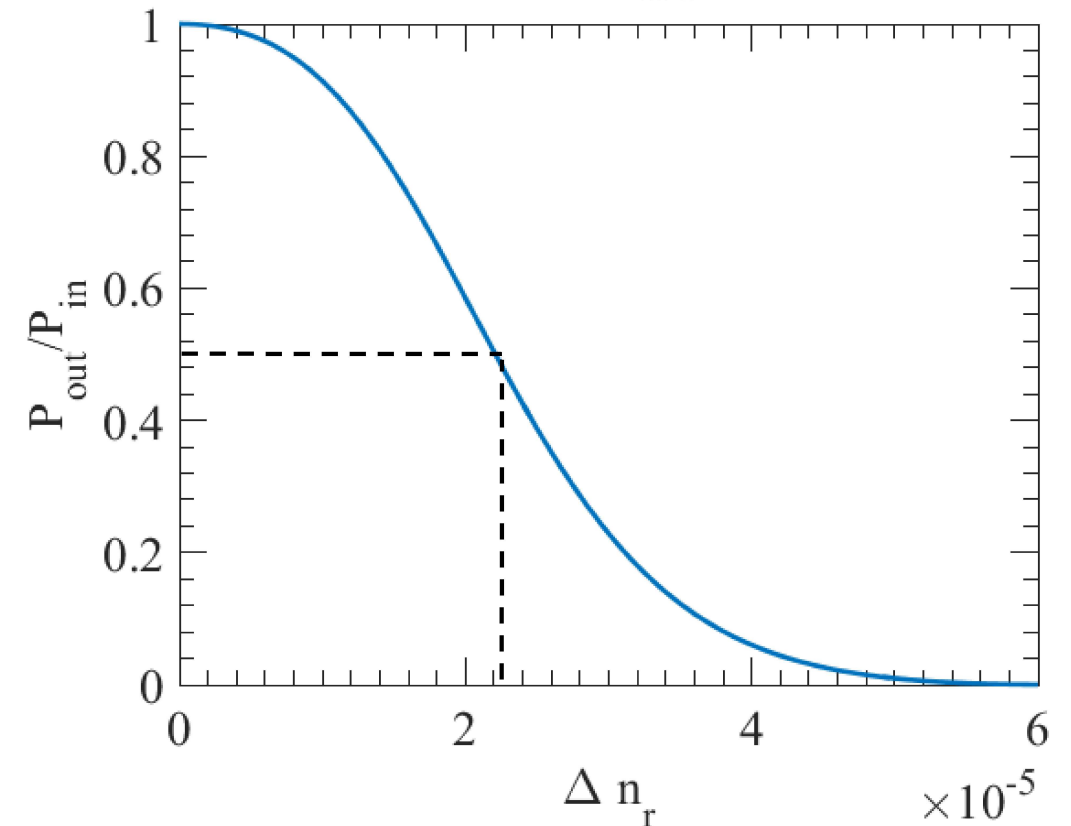
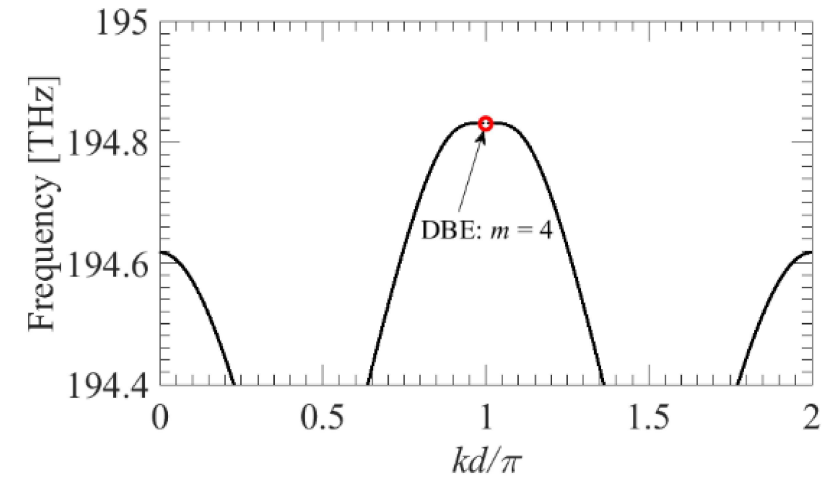
The phase difference is very sensitive to the change in the refractive index: $\Delta\phi \propto \sqrt[4]{\delta n_r}$



DBE Optical Switch based on abrupt mismatch

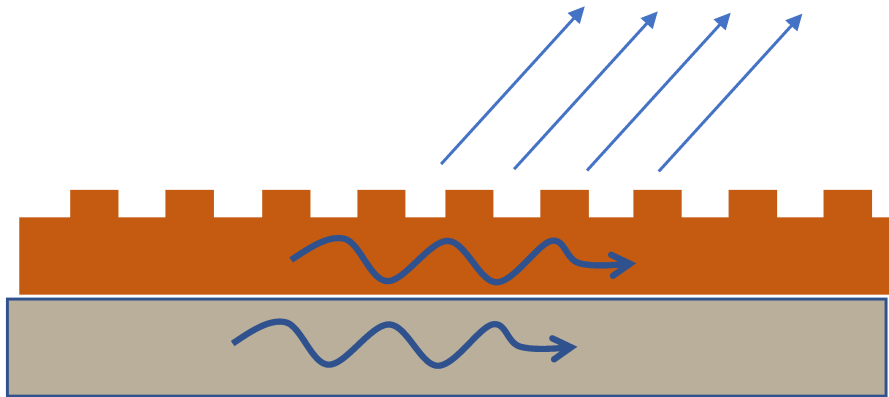


- Electro-optic material in the coupled ring resonators
- Switching happens with very small change in n_r which means very small switching threshold voltage

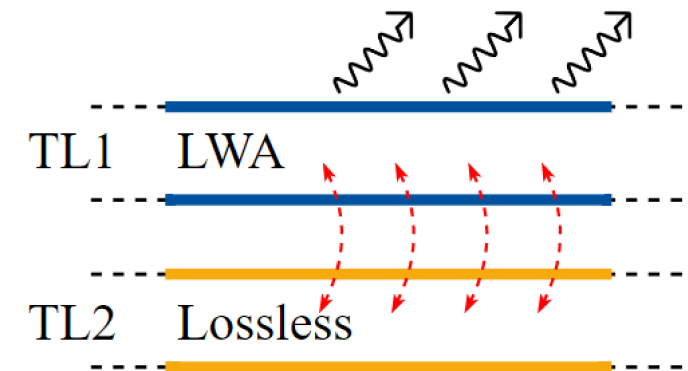


Spectrometer concept: Very sensitive scattering angle

- The degenerate eigenvalue is k , so the beam of a radiating structure (pointing angle or beamwidth) would be very sensitive to frequency changes or other perturbations
- How do we make an EPD including radiation and no gain?



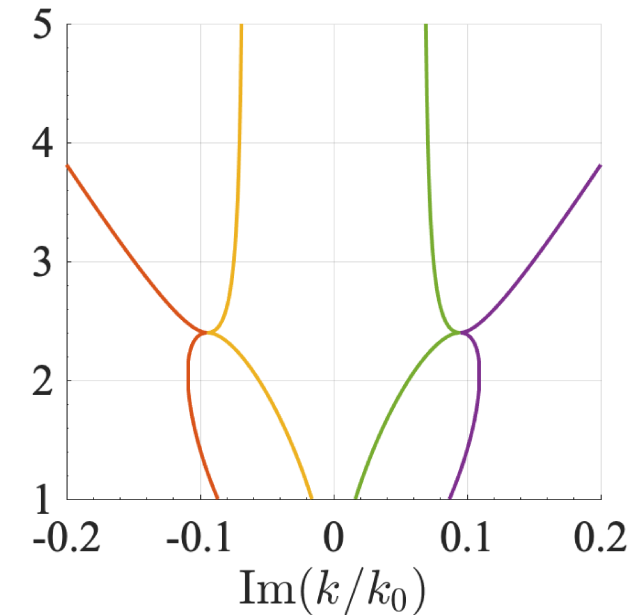
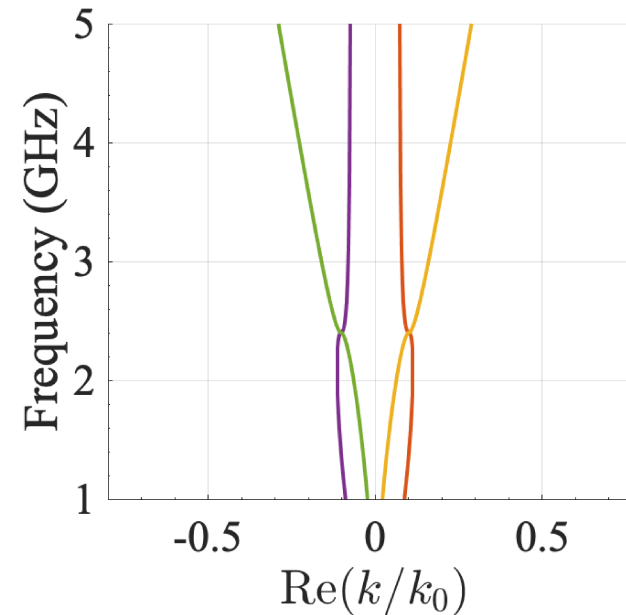
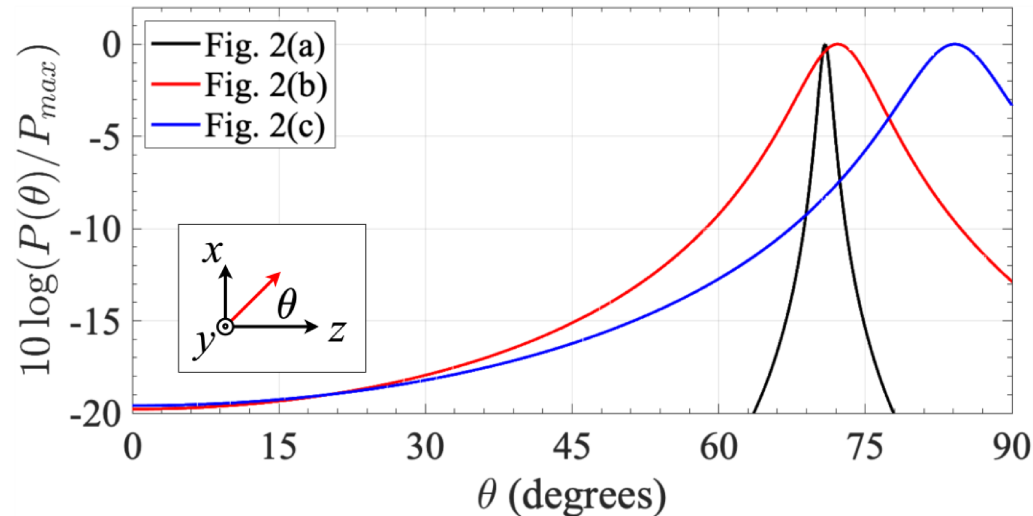
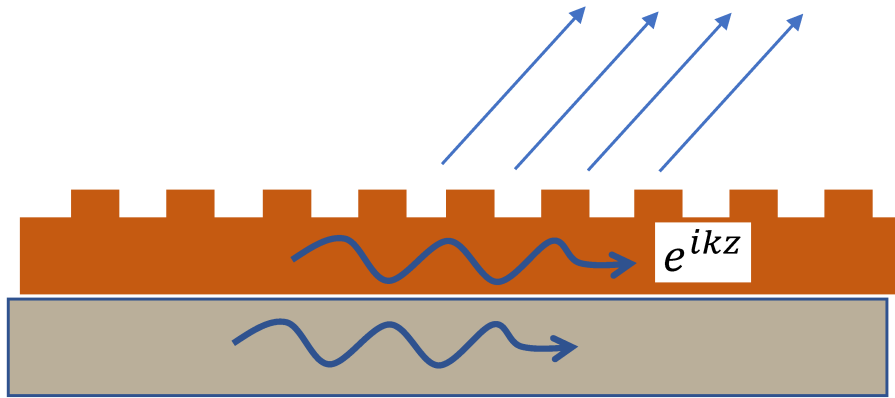
We add a second coupled waveguide to make an EPD at a given k (not the edge of a BZ)



We model it as two coupled TLs

Spectrometer concept

- It is possible to find EPDs in these kind of structures!
- The degenerate eigenvalue is in k , so the beam of a radiating structure (pointing angle and beamwidth) would be very sensitive to frequency changes or other perturbations



Nathaniel Furman

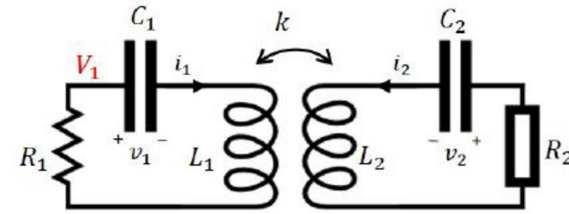
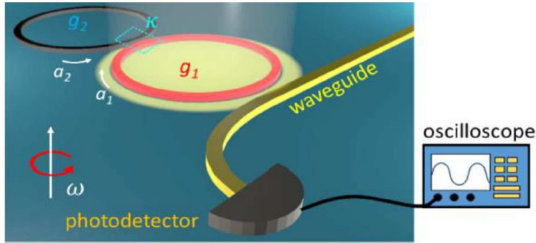


Albert Herrero

Conclusion – EPD for sensing

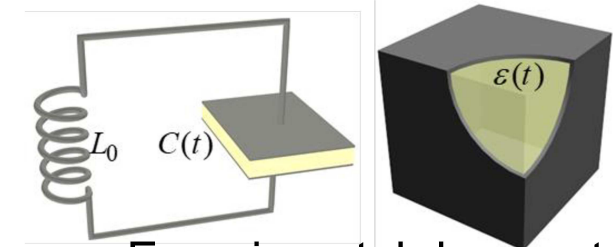
- Two new ways to get EPDs, with experimental demonstrations, solving important problems

1) EPD in (slightly) broken PT symmetry systems (eigenvalue is ω) with *nonlinearity*



$$f_p(\delta) \approx f_e \pm j \frac{f_m}{2\pi} (-1)^p \alpha_1 \sqrt{\delta}$$

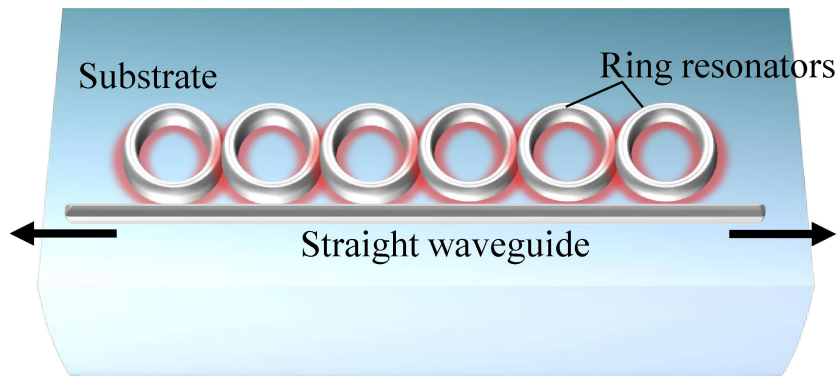
2) EPD in *time varying* systems (eigenvalue is ω)



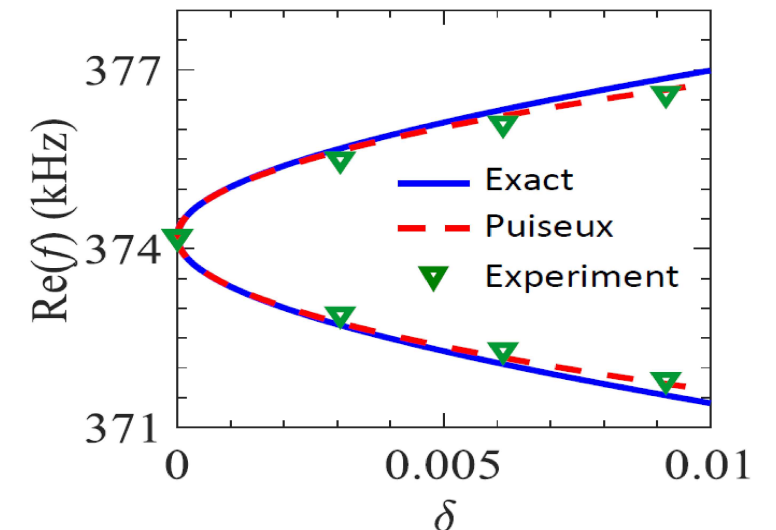
- Experimental demonstration
- High sensitivity to perturbations
- Low noise

- EPDs in gainless waveguides can be used to sense using the strong k -sensitivity to perturbations

3) EPD in *Waveguides* (eigenvalue is k)



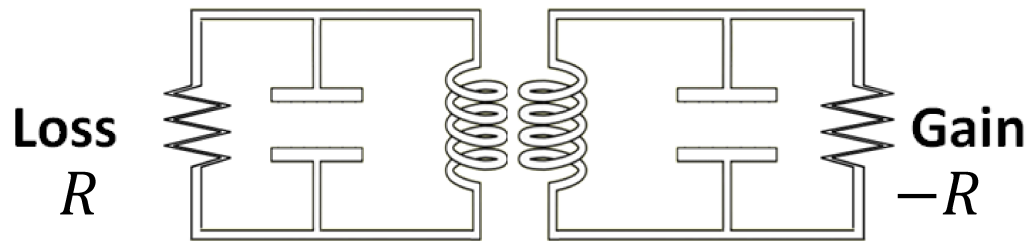
$$k_p(\delta) \approx k_e + (-1)^p \alpha_1 \sqrt{\Delta} \quad (p = 1, 2)$$



Thank You

Auxiliary Slides

EPD in 2 coupled resonators



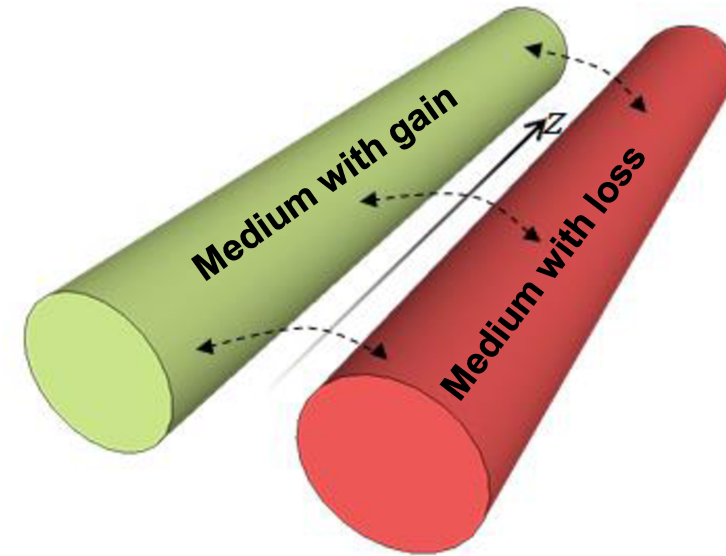
Resonance: $v \propto e^{+j\omega t} \rightarrow$ Solutions: $\pm\omega_1, \pm\omega_2$

EPD occurs when: $\omega_1 = \omega_2$

Coalescence in resonance frequency

Schindler, et al., *Physical Review A* 84.4 (2011)

EPD in 2 coupled waveguides



Wave function: $E \propto e^{-jkz} e^{j\omega t} \rightarrow$ Solutions: $\pm k_1, \pm k_2$

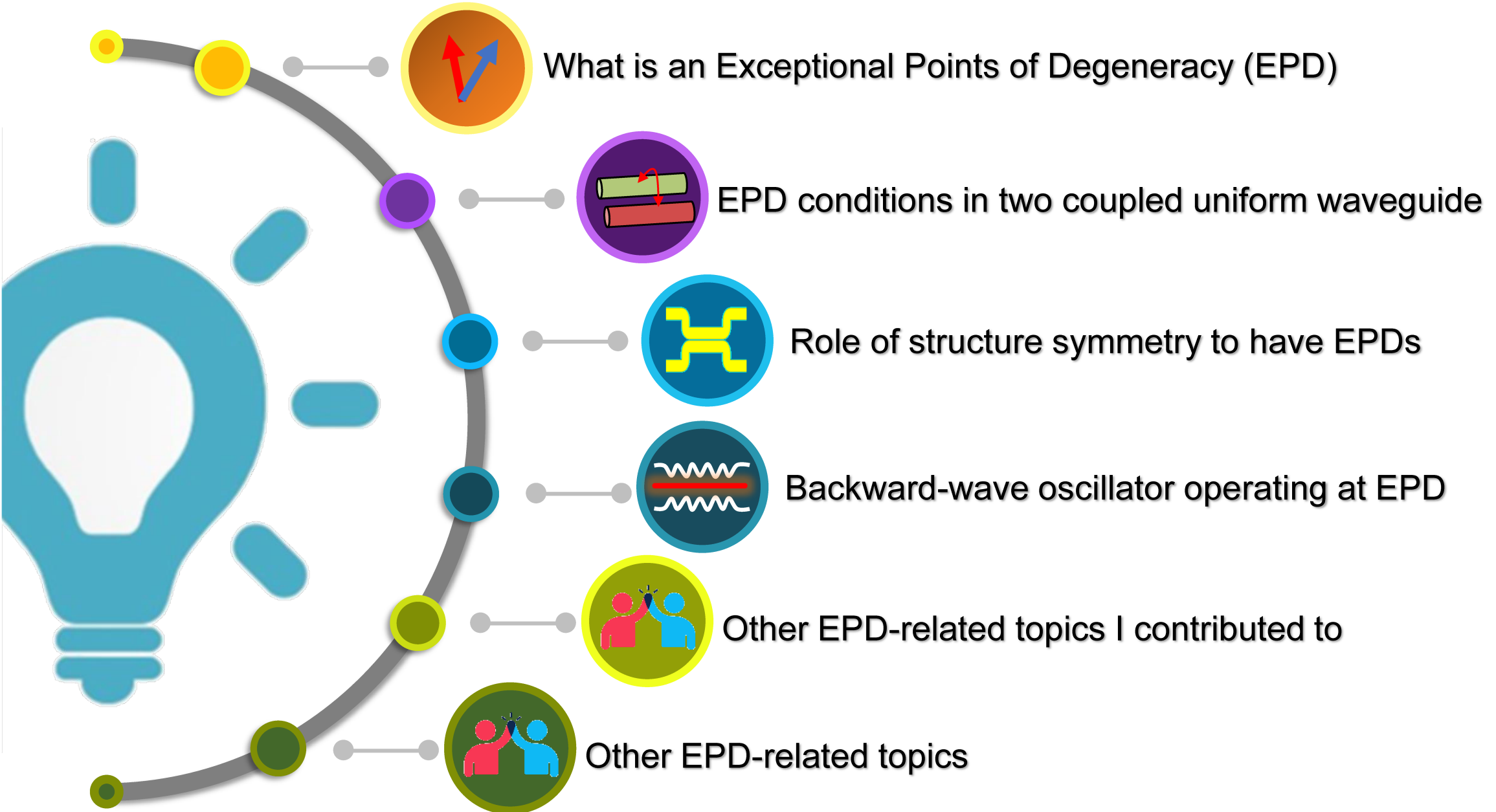
EPD occurs when: $k_1 = k_2$

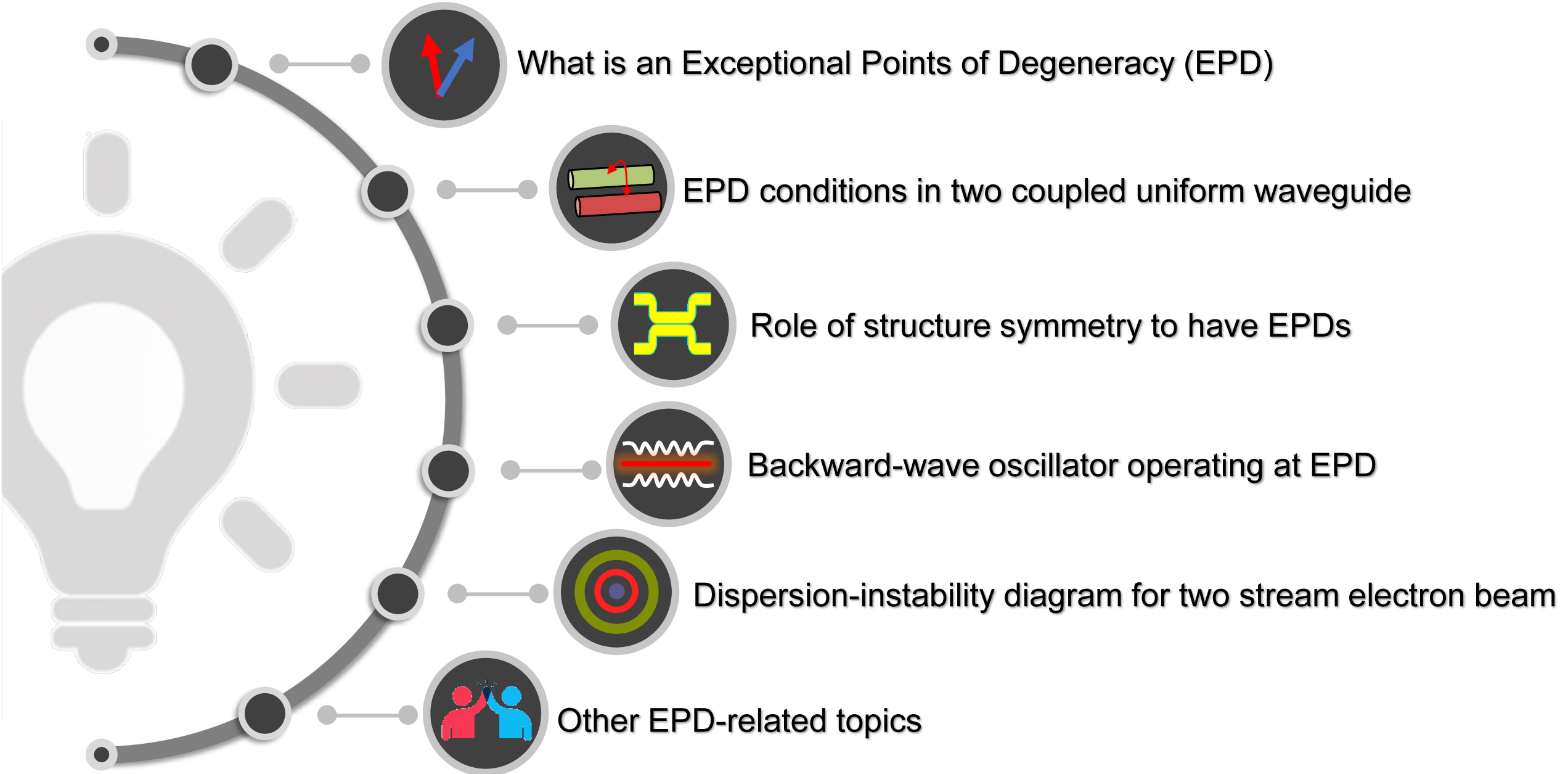
Coalescence in wavenumber

Rüter, et al, *Nature physics*, vol. 6, no. 3, 2010

Othman, Capolino, *IEEE TAP*, vol. 65, no. 10, 2017

**Analogous to EPD that occurs for eigenmodes that represent resonance modes,
EPD occur for eigenmodes that represent EM modes in waveguides**





EPD conditions in two uniform waveguides

Based on coupled mode theory

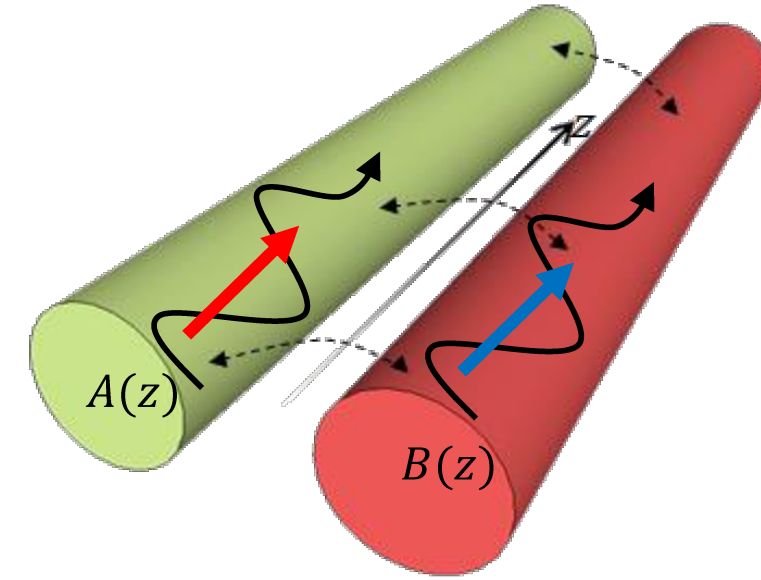
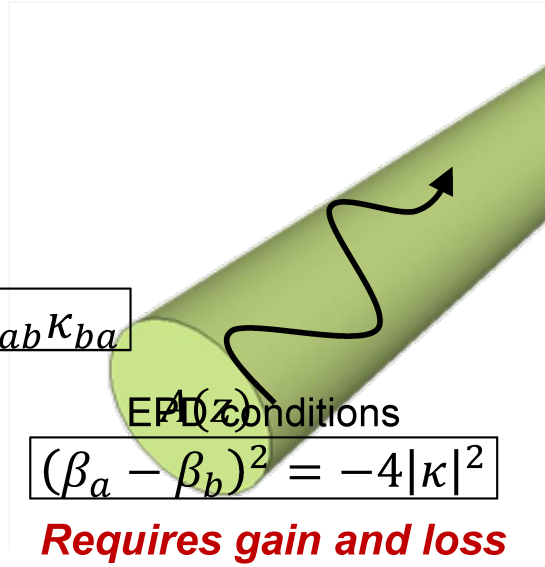
- Coupled mode theory equations

$$\frac{d}{dz} \begin{bmatrix} A(z) \\ B(z) \end{bmatrix} = -j \begin{bmatrix} \beta_a & \kappa_{ab} \\ \kappa_{ba} & \beta_b \end{bmatrix} \begin{bmatrix} A(z) \\ B(z) \end{bmatrix}$$

- EPD condition $k_1 = k_2 \rightarrow$ $(\beta_a - \beta_b)^2 = -4\kappa_{ab}\kappa_{ba}$

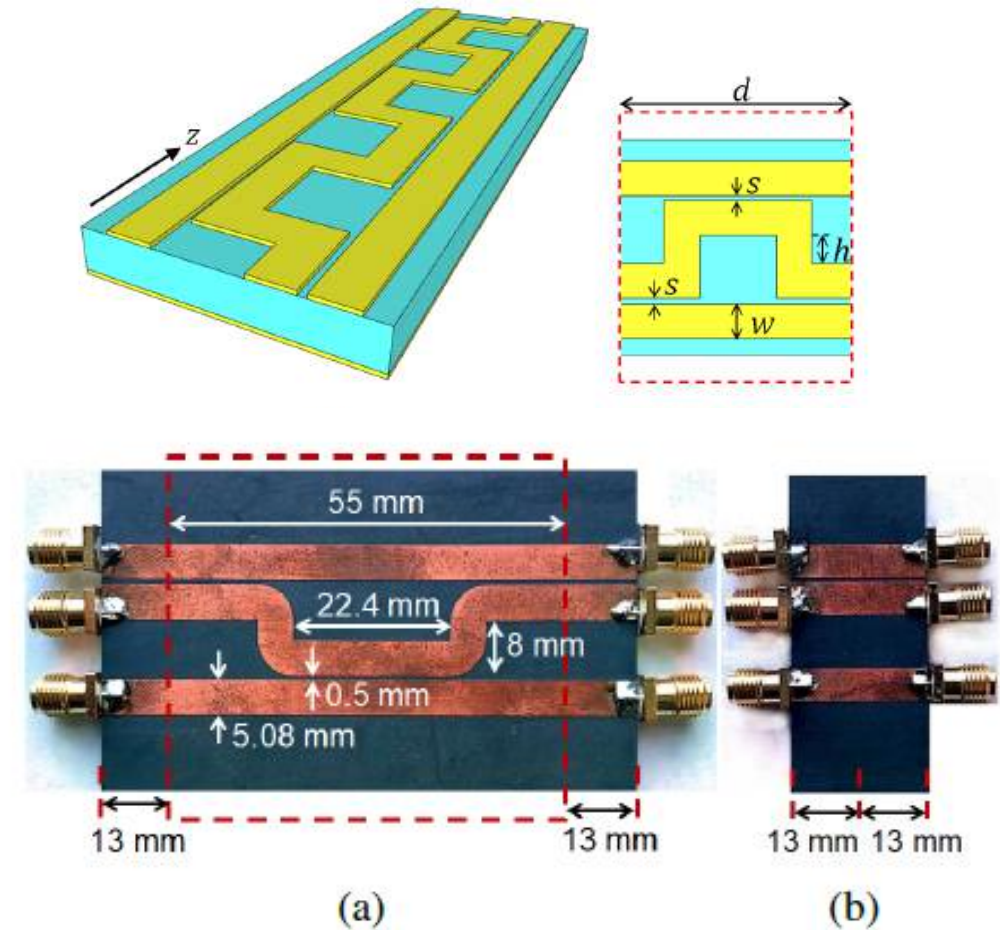
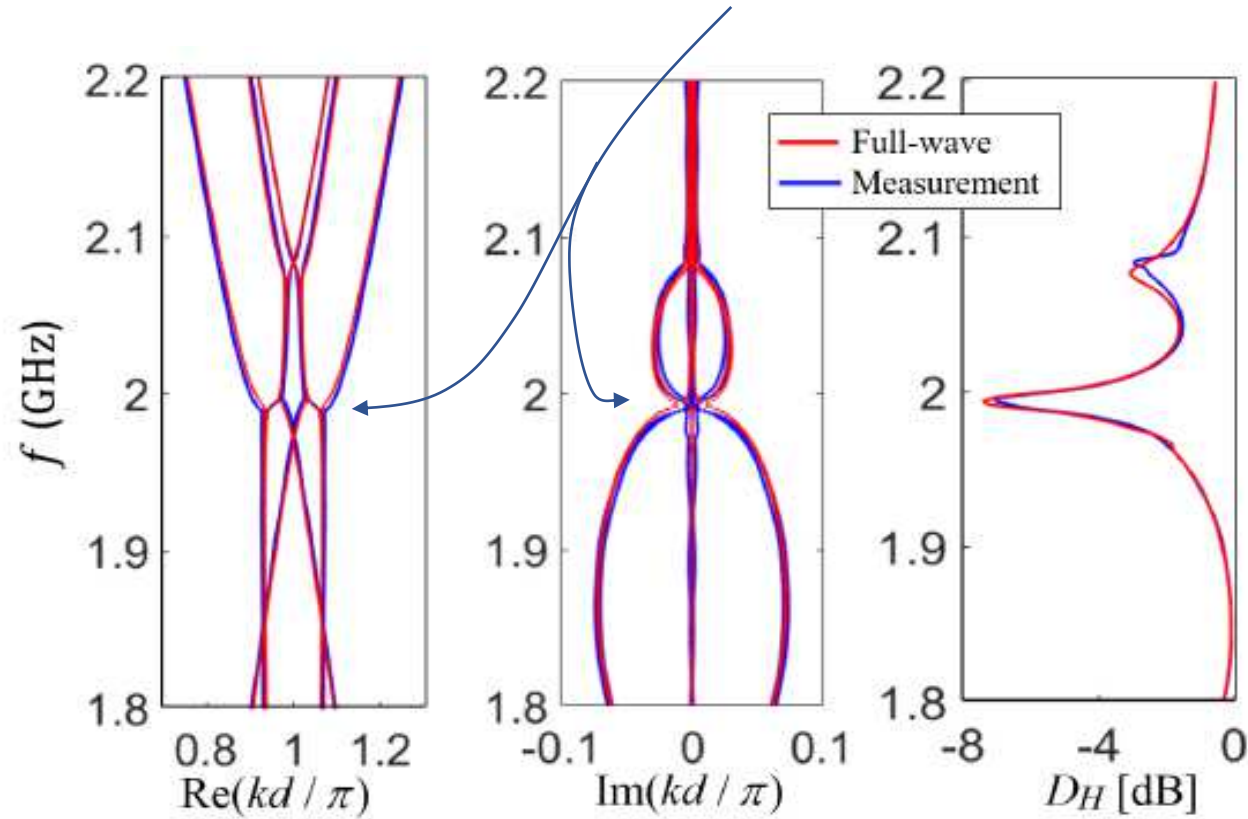
- Possible Scenarios

- Codirectional coupling ($\kappa_{ab} = \kappa_{ba}^*$):



Experimental Demonstrations

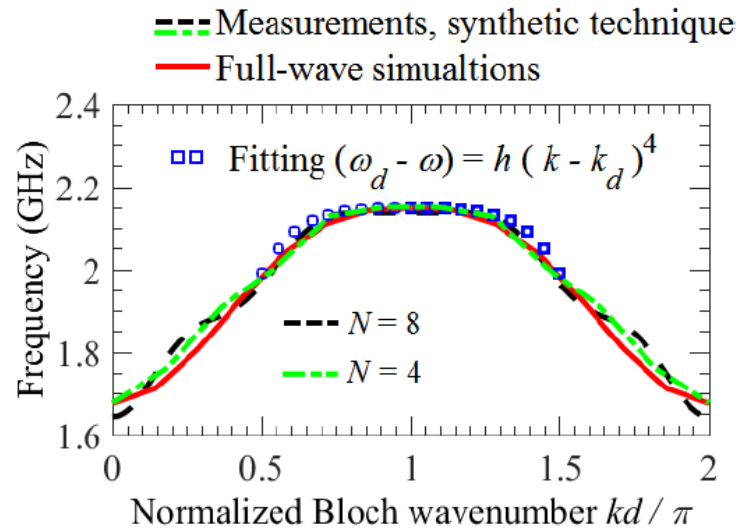
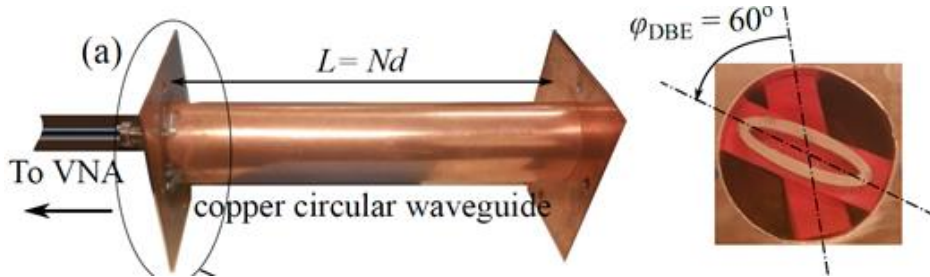
Exper Demonstr.- Stationary inflection point (SIP), 3rd order EPD



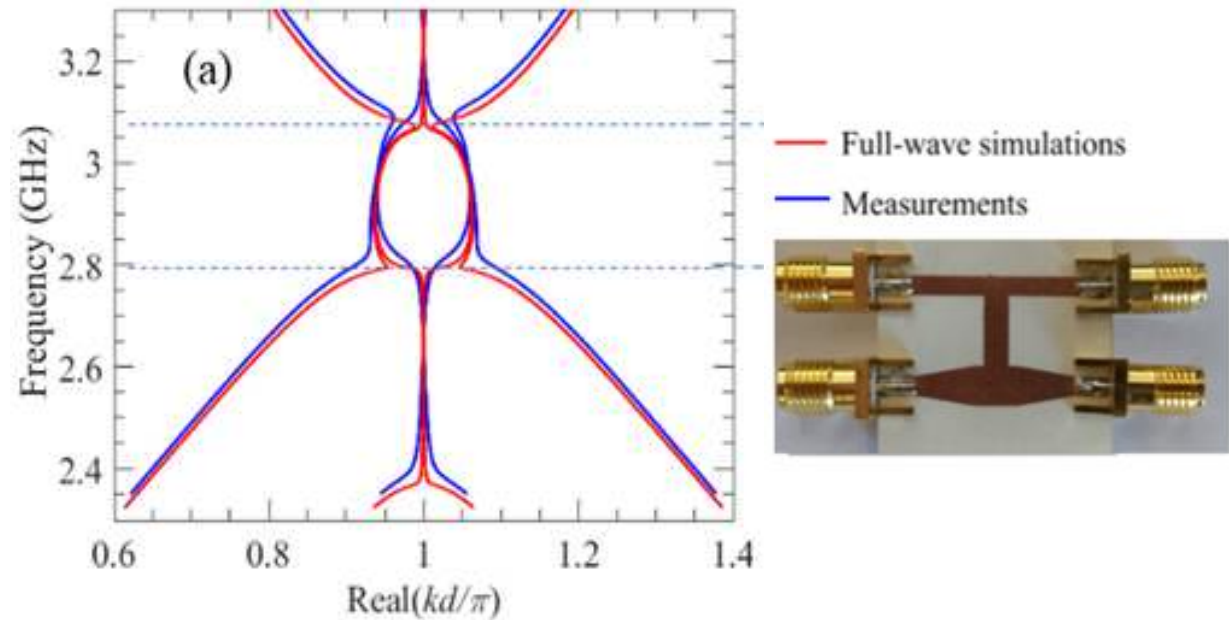
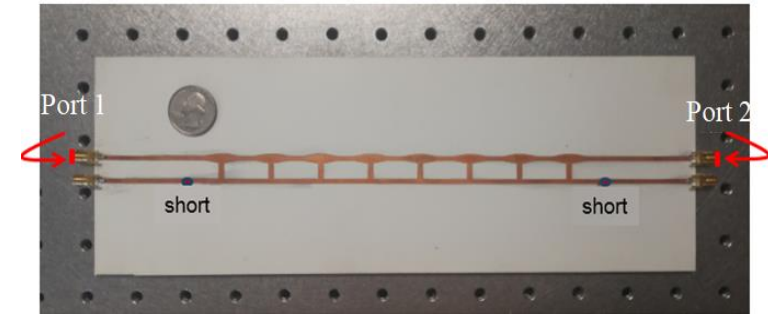
“Coalescing” parameter:
$$D_H = \frac{1}{3} \sum_{\substack{m=1, n=2 \\ n>m}}^3 |\sin(\theta_{mn})|$$

Nada, Mealy, Capolino, IEEE MWCL, 2021

Experimental demonstrations: DBE (no loss, no gain), EPD of order 4

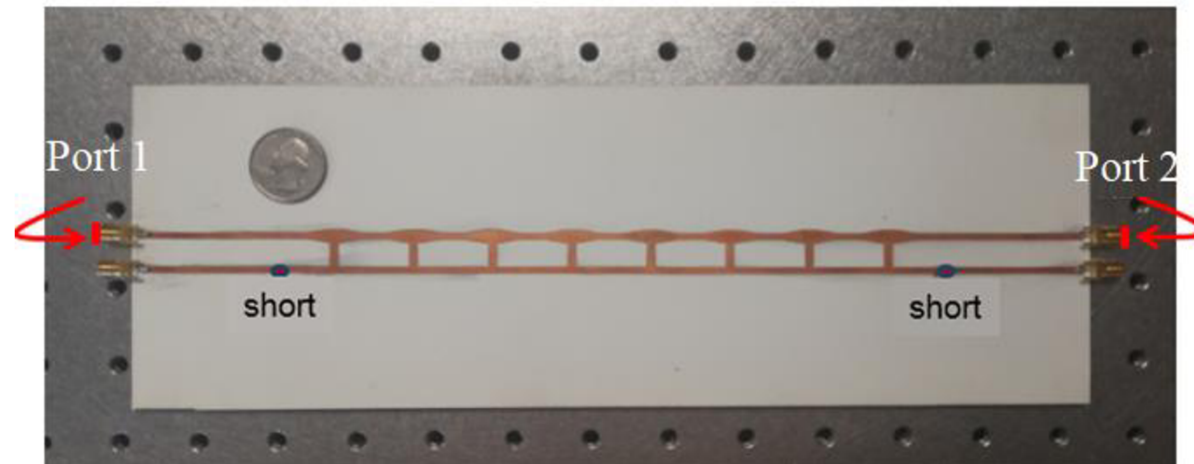


Othman, Pan, Atmatzakis, Christodoulou, Capolino, *IEEE Trans. Microw. Theory Techn.*, vol. 65, no. 6, 2017

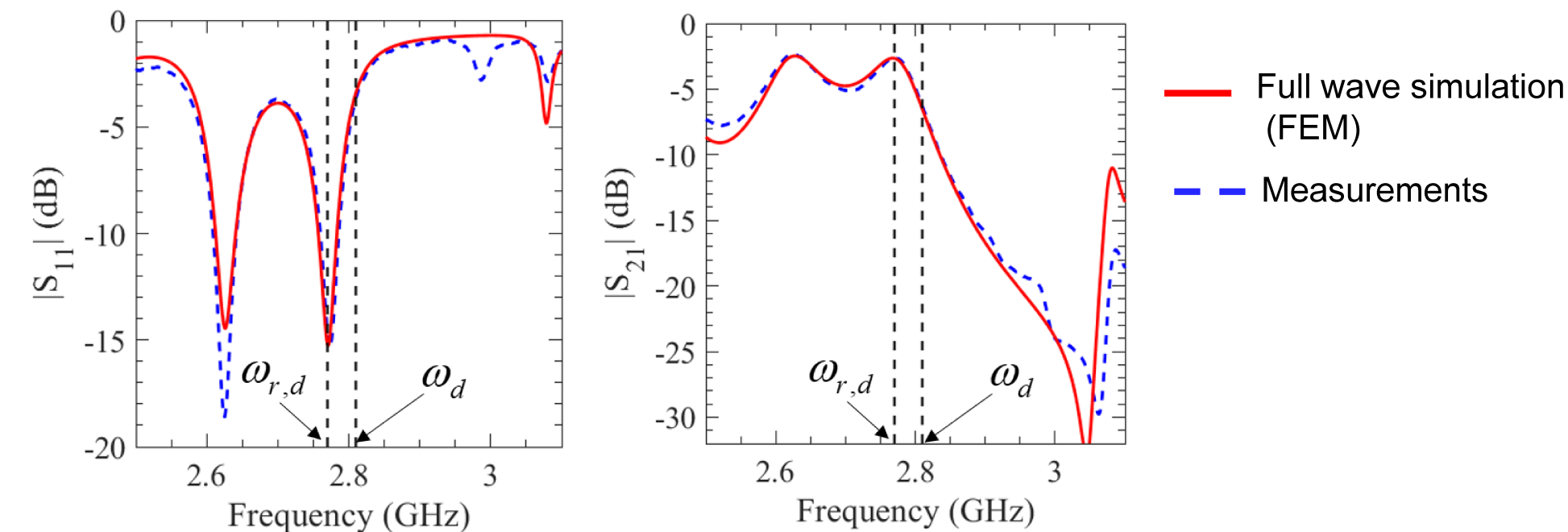


Abdelshafy, et al., *IEEE Trans. Antenna Propag.*, vol. 67, no. 9, 2019

- Finite structure:



- Simulations and measurements of the scattering parameters S_{11} and S_{21} for an 8-unit-cell array



Resonance DBE frequency

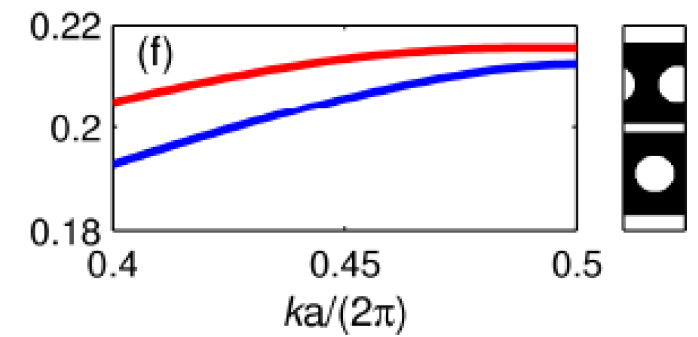
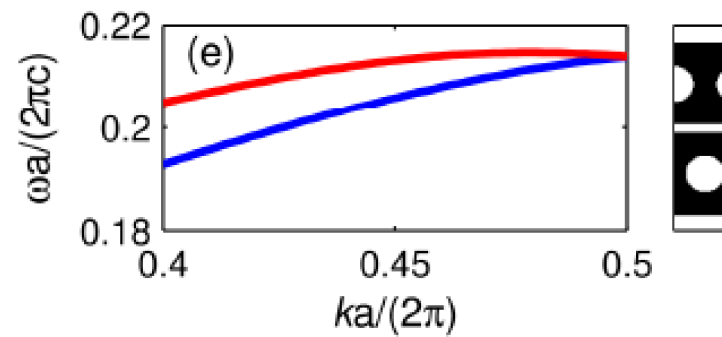
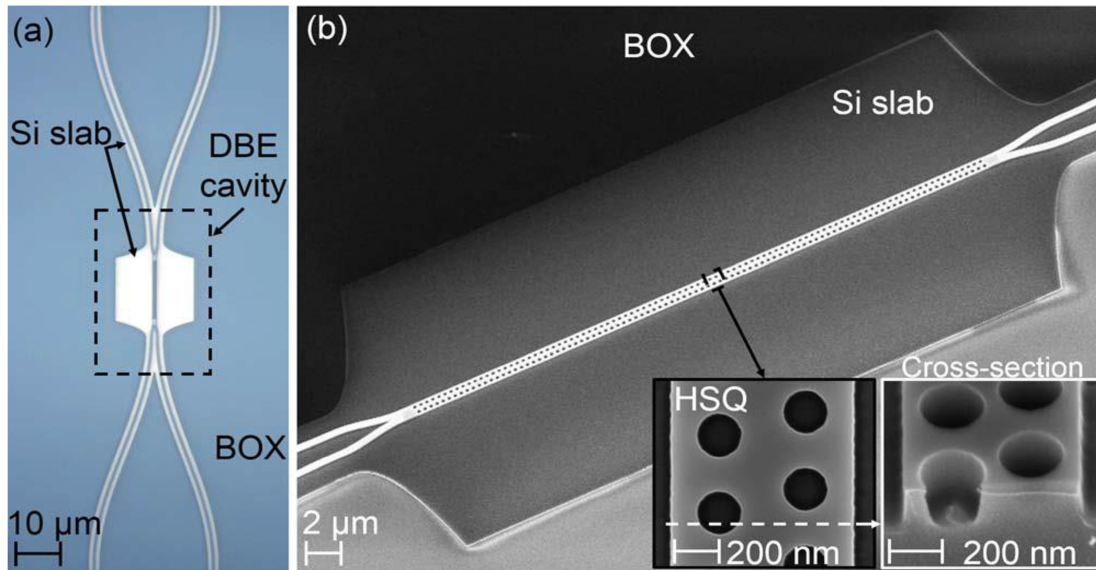
$$(\omega_{r,d} - \omega_d) \propto \left(\frac{\pi}{Nd} \right)^4$$

Figotin, Vitebskiy, "Slow-wave resonance in periodic stacks of anisotropic layers", PRA 2007

DBE in optical periodic coupled mode structures

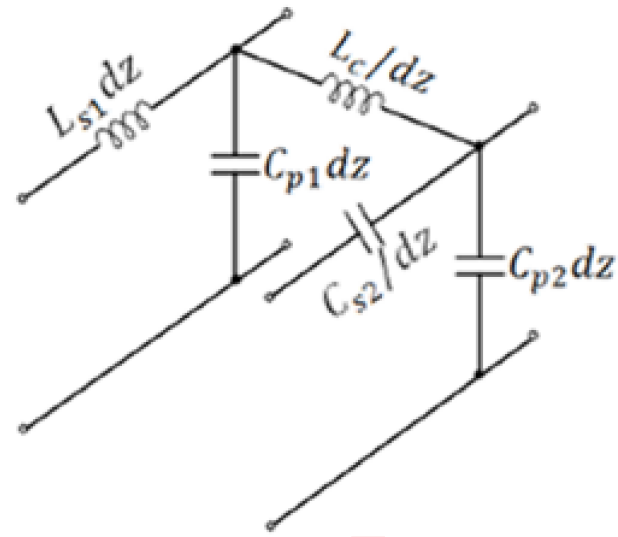
Photonic crystal waveguides in optics

Silicon based waveguides

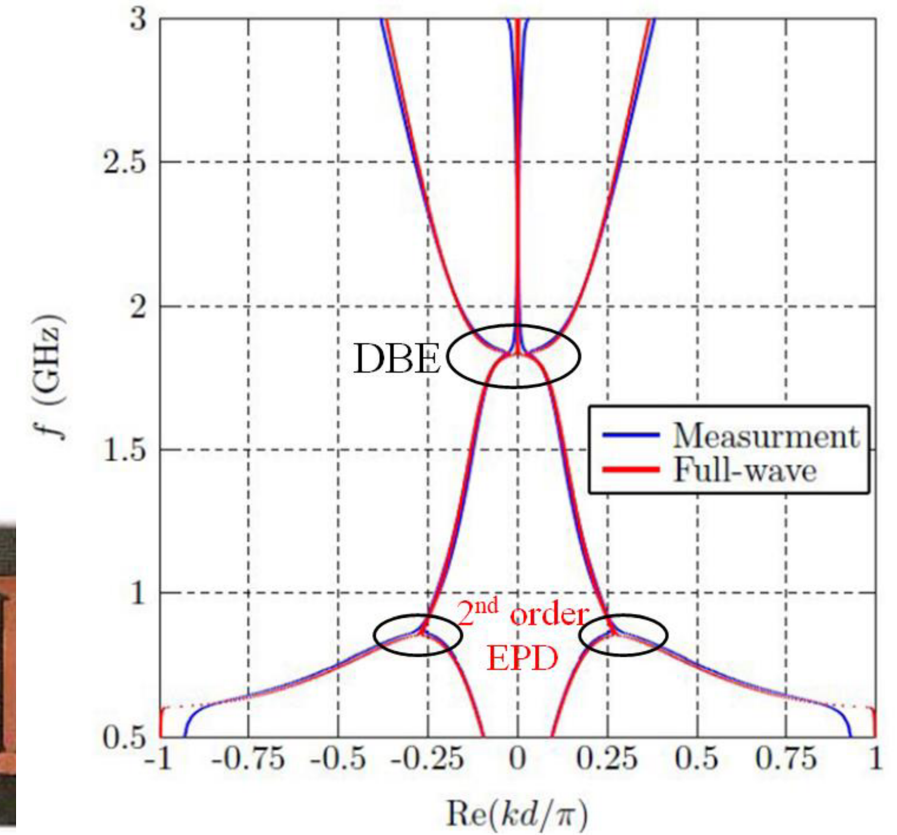
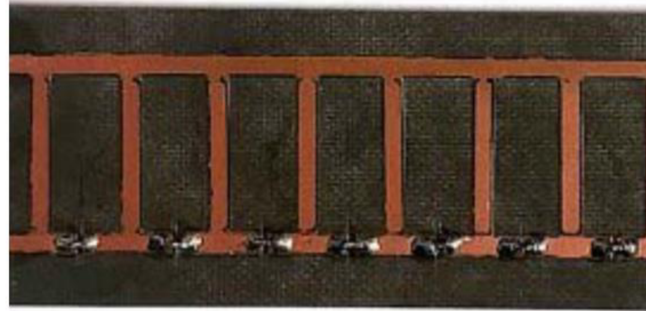
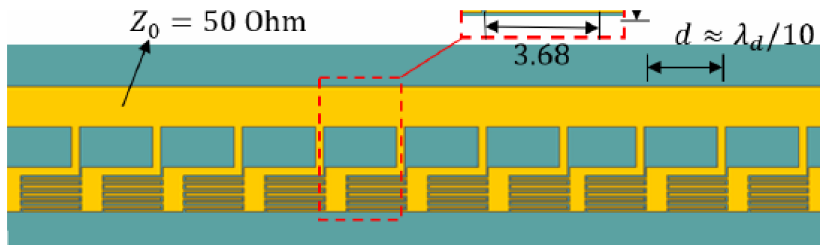


Wood, Burr, Reano, *Opt. Lett.* 40,11, 2015

DBE (EPD of order 4)



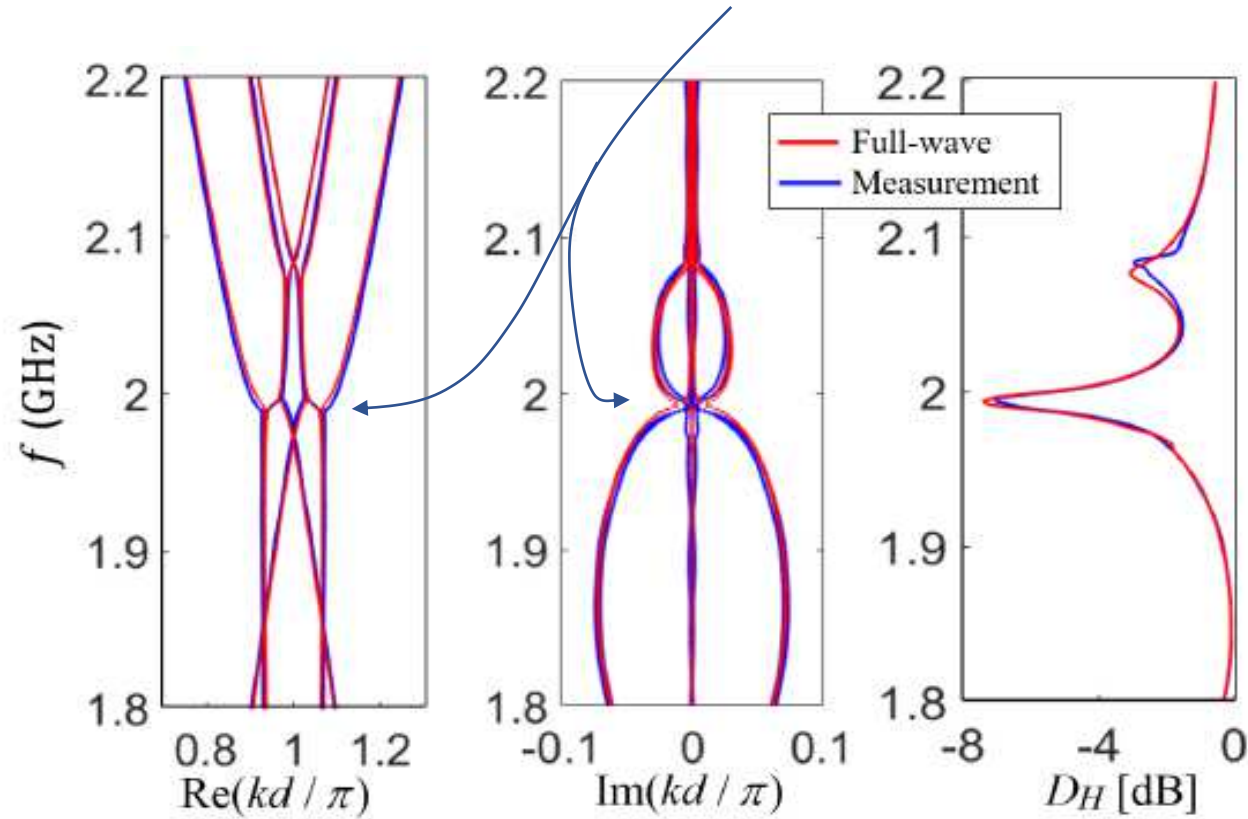
Microstrip implementation



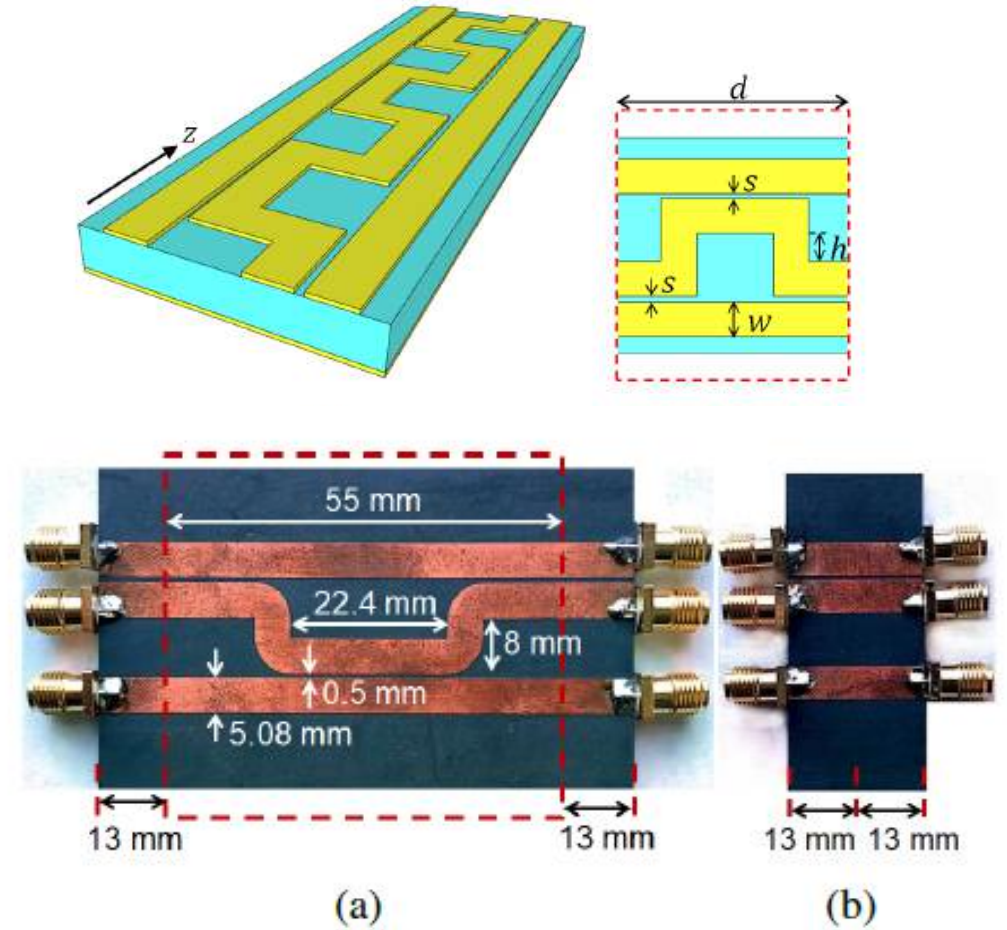
4th order DBE($k=0$) and 2nd order RBE are obtained in uniform coupled transmission lines where one line supports forward propagating modes and the other supports evanescent modes

Tarek Mealy and Filippo Capolino, "General conditions to realize exceptional points of degeneracy in two uniform coupled transmission lines," *IEEE Transactions on Microwave Theory and Techniques*, vol. 68, no. 8, pp. 3342–3354, 2020

Stationary inflection point (SIP), 3rd order EPD



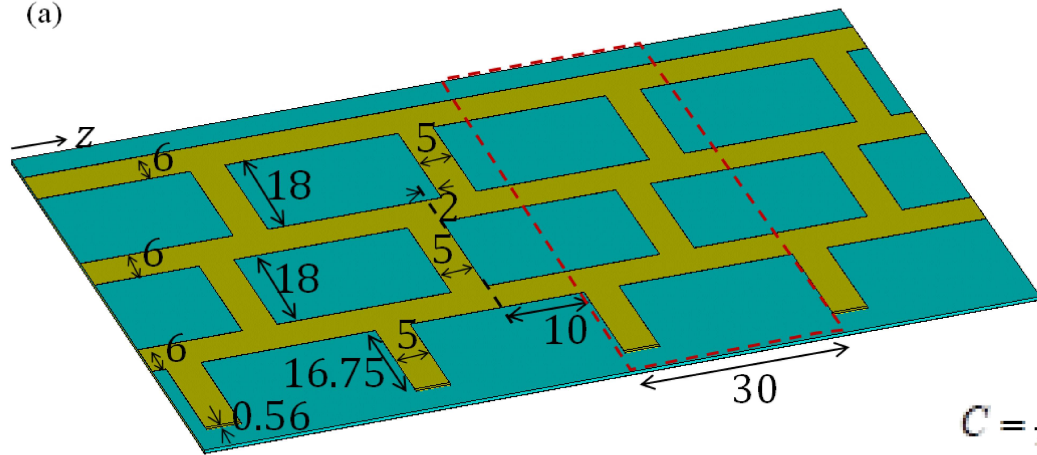
“Coalescing” parameter:
$$D_H = \frac{1}{3} \sum_{\substack{m=1, n=2 \\ n>m}}^3 |\sin(\theta_{mn})|$$



Nada, Mealy, Capolino, IEEE MWCL, 2021

6DBE- 6th order EPD

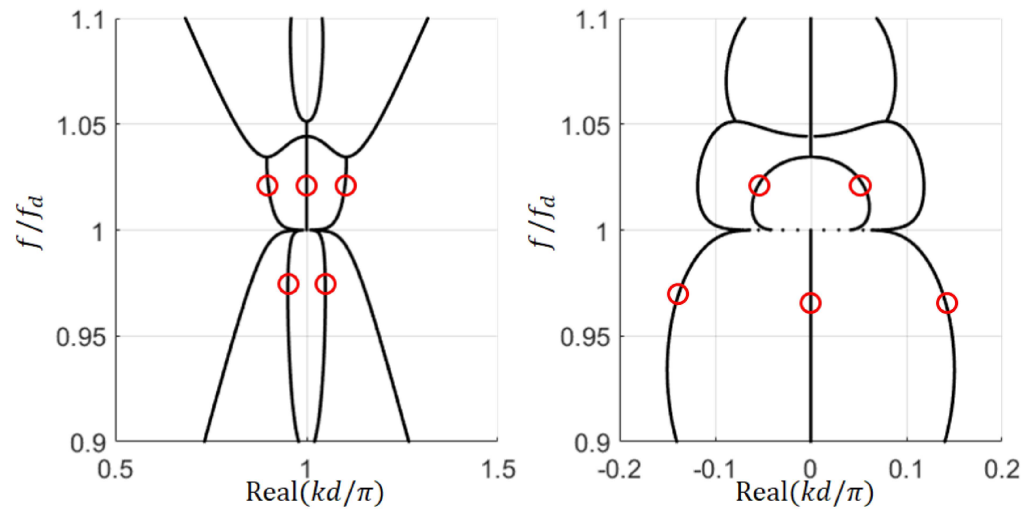
(a)



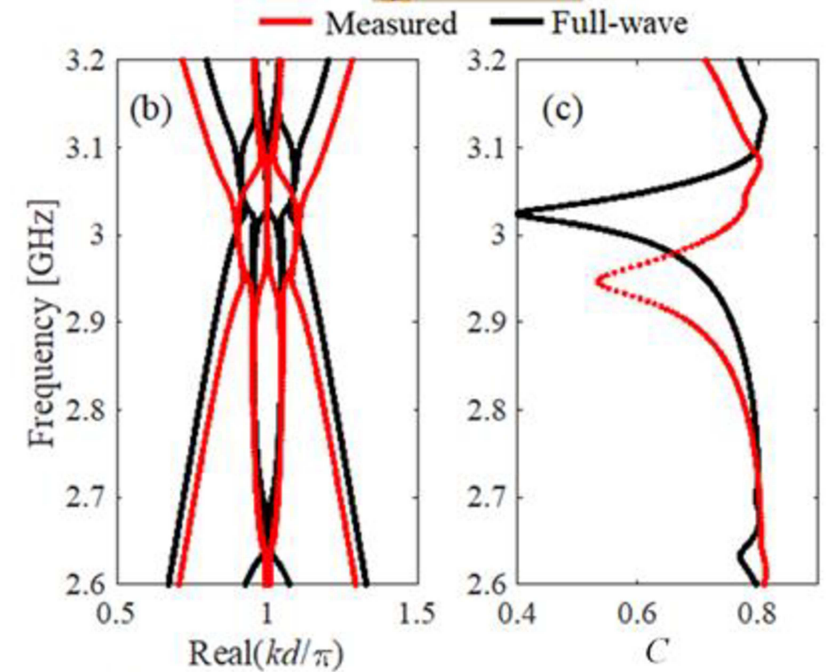
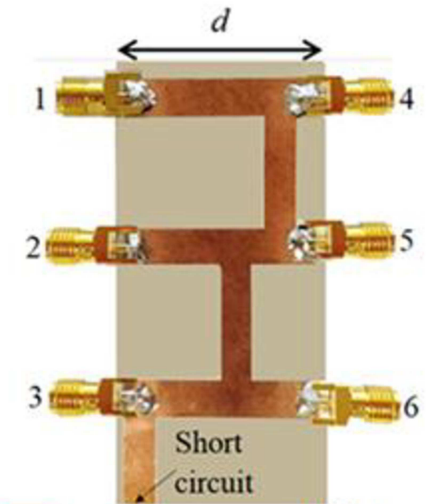
“Coalescence”
parameter:

$$C = \frac{1}{15} \sum_{\substack{m=1, N-1 \\ n=1}}^6 |\sin(\theta_{mn})|, \quad \cos(\theta_{mn}) = \frac{(\Psi_m, \Psi_n)}{\|\Psi_m\| \|\Psi_n\|}$$

(b)

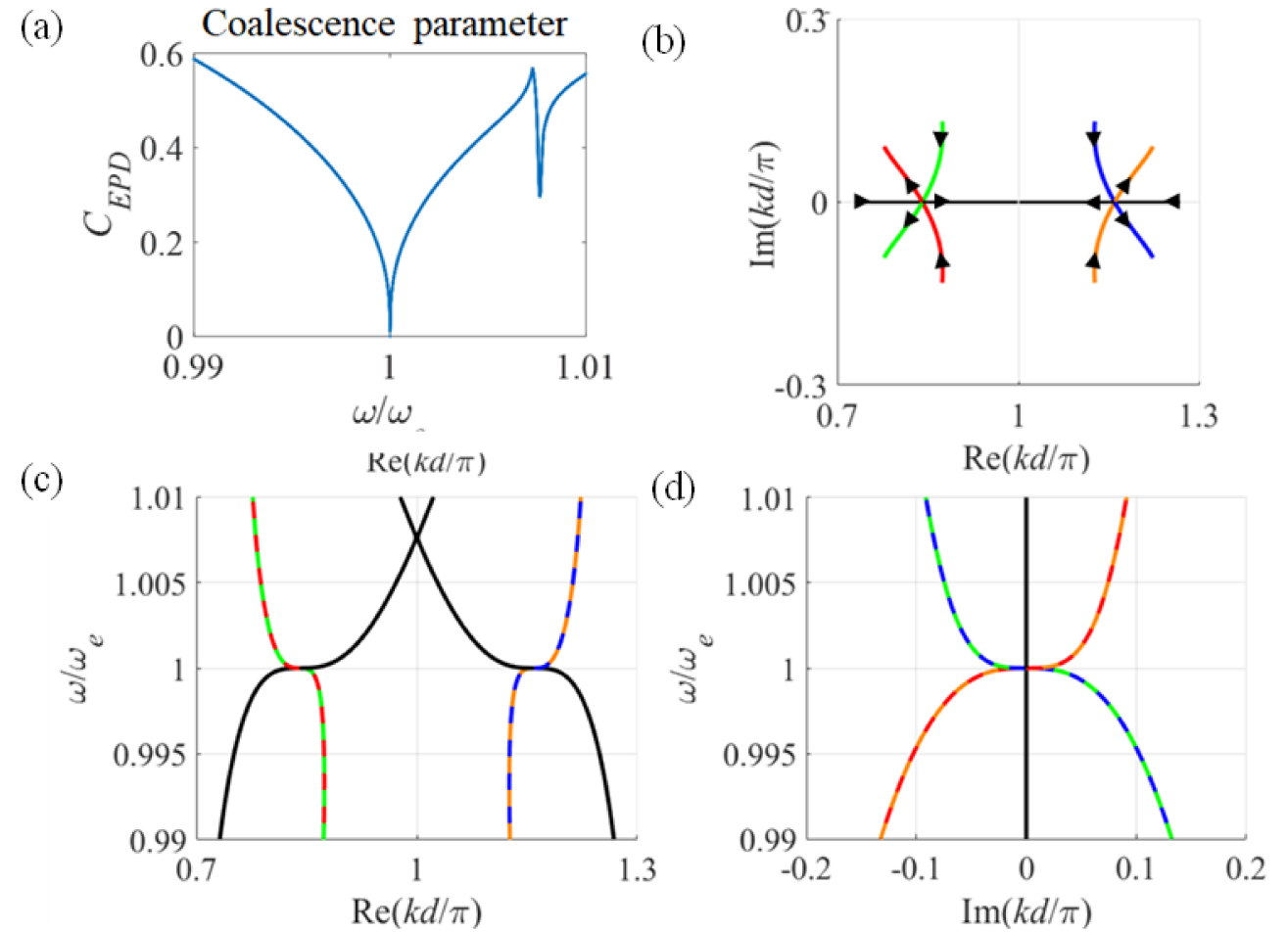
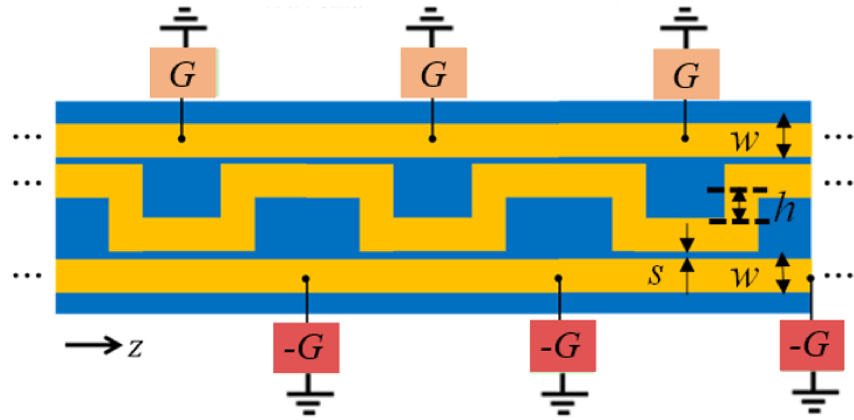


(a)



Yazdi, Oshmarin, Mealy, Almutawa, Nikzamid, Capolino,
arXiv:1906.03331, 2019

EPD in GT symmetric waveguide



EPD is found in Glide-time symmetric waveguide