Waveguides with Exceptional Points of Degeneracy of Order 2, 3, 4 and 6 without Loss and Gain

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Abstract—**We show various examples of fully-planar two-way and three-way microstrip coupled waveguides that exhibit second, third, fourth and sixth order exceptional points of degeneracy (EPD). EPDs are shown in reciprocal waveguides that do not have gain or loss. The occurrence of EPDs is observed using three methods: (i) the mathematical description of modes and the similarity of the system matrix to a Jordan block; (ii) the dispersion diagram with prescribed flatness; (iii) the coalescence of eigenvectors that is observed analytically or experimentally . We have defined a coalescence parameter that is used in conjunction with experimental results to measure the separation of the coalescing eigenvectors (polarization states). The presented structures can serve various applications like leaky-wave antennas, distributed amplifiers, oscillators, delay lines, pulse generators, and sensors.**

Index Terms—**Frozen modes, Coupled transmission line, Degeneracy, Band edge, Stationary inflection point, Exceptional point.**

I. Introduction

Exceptional points of degeneracy (EPDs) are special points in a system's parameter space where two or more eigenmodes coalesce in both their eigenvalues and eigenvectors into a single eigenmode. Here we focus on EPDs in waveguides, hence eigenvalues represent modal wavenumbers and eigenvectors represent polarization states. In particular we focus on guiding structures without loss and gain. In the literature, parity-time (PT) symmetric systems have been also studied where the presence of loss and gain along the waveguide is required [1], [2]. Some EPDs have been obtained by using magnetic materials to break reciprocity [3]–[8]. Here we stress that EPDs are obtained in a variety of systems without PT symmetry and without the need to break reciprocity. EPD conditions can be reached in the absence of loss and gain in the waveguide [9]–[11]. The dispersion relation around an EPD at (ω_e, k_e) has the behavior of $(\omega-\omega_e) \propto (k-k_e)^m$ where m is the degeneracy order, that also represents the number of coalescing eigenmodes. Here, ω and k are the angular frequency and the wavenumber (or block wavenumber in case the waveguide is periodic), and the EPD is denoted by the subscript e.

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There has been a lot of interest in concepts related to EPDs in recent years. EPDs occur in systems where the evolution of a system vector, in space (waveguides) or time (resonators), is described by a non-Hermitian matrix, using periodicity [12] or by having losses and gain in the system [13]. Propagation in periodic systems is described by using the concept of system matrix, transforming the state vector from cell to cell.

In this article, we focus on EPDs occurring in multimode waveguides without loss and gain, namely we focus on the regular band edge (RBE) [10] and the degenerate band edge (DBE) that occurs in two-way wavguides, and on the stationary inflection point (SIP) at microwave frequencies, that occur in three-way waveguides. The fourth-order EPDs is called DBE and it occurs in uniform [10] or periodic guiding structures [9], described by two coupled transmission lines (TLs). The SIP is a third-order EPD that occurs in three-way waveguides [11]. Finally, we also discuss the sixth-order EPDs in a triple ladder (or three-way) microwave waveguide realized using three coupled microstrips on a grounded dielectric substrate. Some applications are discussed at the end of the paper.

II. Examples and Discussion

We demonstrate here some examples of the realization of EPDs conditions in microstrip coupled TLs without the presence of gain and loss. The general term of EPD describes a degeneracy represented by the coalescence of eigenvectors. Depending on the kind of systems is considered, the various EPDs in lossless and gainless waveguides are called: RBE, that is an EPD of order 2, the SIP, that is an EPD of order 3, the DBE that is an EPD of order 4, and the 6DBE that is an EPD of order 6. We provide examples for each of these degeneracy conditions.

In all cases, propagation in a multi-way waveguide is described in matrix formalism by

$$\frac{d\mathbf{\Psi}(z)}{dz} = -j\mathbf{M}\mathbf{\Psi}(z), \tag{1}$$

where $\mathbf{\Psi}(z)$ is a state vector and $\mathbf{M}$ is the system matrix [10], [9]. The state vector elements are voltages and currents of the coupled equivalent TLs, or alternatively, they are forward and backward wave amplitudes.

In periodic waveguides, the system matrix is z-dependent and propagation is described by using a transfer matrix formalism

$$\mathbf{\Psi}(z+d) = \underline{\mathbf{T}}_u \mathbf{\Psi}(z), \tag{2}$$

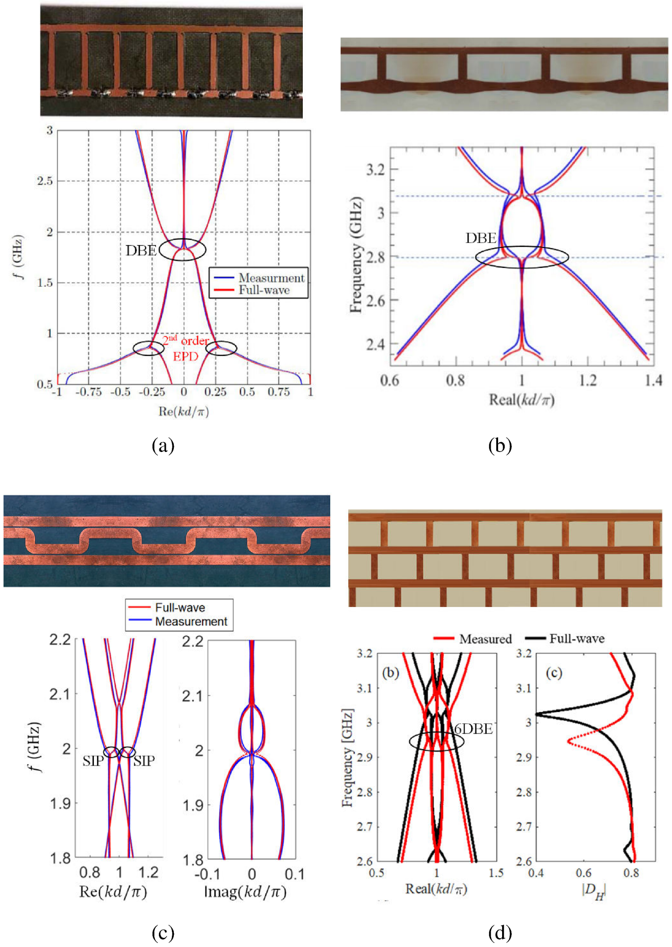

Fig. 1. Four different coupled microstrip waveguide geometries and relative wavenumber dispersion diagrams showing different EPDs: RBE, DBE, SIP and 6DBE. (a) Sub-wavelength unit cell implementation of two uniform TLs supporting forward and backward waves, by loading one line with distributed series capacitors to exhibit second (RBE) and fourth (DBE) order EPDs [10]. (b) Two periodic coupled microstrips that exhibit a DBE at $kd = \pi$ [9]. (c) Three-way periodic microstrip waveguide that exhibits an SIP (a frozen mode regime), i.e., an EPD of order 3 [11]. The dispersion diagram shows the existence of an SIP at frequency $f = 2$ GHz, where three branches of the dispersion diagram, (one real and two complex) coalesce. (d) Periodic three-way waveguide, made of three coupled microstrip lines that exhibits a sixth order DBE. The dispersion diagram shows the existence of 6DBE around 3 GHz. The coalescence parameter $D_H(\omega)$ is a figure of merit that describes the coalescence of eigenvectors; it can be calculated based on numerical or measurement results.

where d is the period and $\underline{\mathbf{T}}_u$ is the transfer matrix of a unit cell of length d. When we look at eigenmodes, the system vector takes the form of $\mathbf{\Psi}(z+d) = \lambda\mathbf{\Psi}(z)$ and the formalism reduces to the eigenvalue problem

$$\underline{\mathbf{T}}_u \mathbf{\Psi} = \lambda \mathbf{\Psi}, \tag{3}$$

where the eigenvalues $\lambda_n \equiv e^{-jk_n d}$, with $n = 1, 2, \ldots, N$, are obtained by solving the dispersion characteristic equation $D(k, \omega) \equiv \det[\underline{\mathbf{T}}_u - \lambda\underline{\mathbf{1}}]$=0, with $\underline{\mathbf{1}}$ being the $N \times N$ identity matrix and N is the size of transfer matrix $\underline{\mathbf{T}}_u$. Due to reciprocity, the determinant of $\underline{\mathbf{T}}_u$ is always equal to unity [14] which implies that the eigenvalues solutions of the characteristic equation must appear in reciprocal pairs, i.e., the six k-solutions must come in positive-negative pairs.

From (3) also the eigenvectors can be obtained, and this is useful to test their coalescence. Indeed, during the past years, we have found it very convenient to introduce a coalescence parameter, that is a figure of merit to observe the degree of coalescence of the system's eigenvectors. As an example, the coalescence parameters $D_H(\omega)$ for the 6DBE is defined as

$$D_H = \frac{1}{15} \sum_{p=1,\, n=1 (p \neq n)}^{6} \left|\sin\left(\theta_{np}\right)\right|, \; \cos\theta_{np} = \frac{\left|\langle \mathbf{\Psi}_n, \mathbf{\Psi}_p \rangle\right|}{\|\mathbf{\Psi}_n\| \, \|\mathbf{\Psi}_p\|}, \tag{4}$$

where θ_{np} represents the angle between two eigenvectors $\mathbf{\Psi}_n, \mathbf{\Psi}_p$ and it is defined via the inner product $\langle \mathbf{\Psi}_n, \mathbf{\Psi}_p \rangle$, $|\,|$ represents the absolute value and $\|\,\|$ represents the norm of a complex vector. The exact occurrence of an EPD implies that $D_H(\omega) = 0$, however numerical tolerances, fabrication tolerances and losses prevent a simulated or experimentally tested system to be exactly at an ideal EPD. However, it is still important that m eigenvectors are close to each other for an EPD of order m, hence we need to observe a minimum in $D_H(\omega_e)$ if we want to retain some properties related to the EPD.

The coalescence parameter can be estimated by calculating the eigenvectors based on measurements or simulations. Indeed, the eigenvectors are calculated using a MATLAB routine based on the system or transfer matrix obtained from a multi port simulation of a unit cell or by a multiport measurement of the scattering parameters. Some examples are provided in this paper.

We theoretically and experimentally demonstrated in [10] that a second order EPD and a fourth order (DBE, at $k = 0$) are supported in *uniform* coupled TLs when there is proper coupling between forward and backward propagating modes. We provided a possible microstrip implementation of a uniform coupled TLs exhibiting such EPDs using periodic series capacitors with very sub-wavelength unit-cell length, shown in Fig. 1(a). We show in Fig. 1(a) the dispersion relation calculated based on scattering (S)-parameters obtained from measurement and from full-wave simulations where a RBE is obtained around 0.86 GHz while a DBE is obtained at $k = 0$ (1/m) around 1.85 GHz. In [9], two periodic microstrip coupled TLs were proposed to exhibit a fourth order EPD (DBE) at $kd = \pi$ as shown in Fig. 1(b). The DBE has been also demonstrated experimentally in a circular metallic waveguide with periodic inclusions (geometry not shown here) for possible applications in electron beam devices [15].

In [11], the frozen mode was shown in a reciprocal three-way waveguide supporting three modes in each direction. The three-way waveguide comprises two uniform TLs that are periodically coupled through a third serpentine-shaped TL as shown in Fig. 1(c). We illustrated the occurrence of the SIP by observing the dispersion diagram based on full-wave simulations and measurements as shown in Fig. 1(c). An EPD of order six (6DBE) has been obtained in a three-way periodic

CTL shown in Fig. 1(d).

Applications of the DBE and the RBE have been proposed, to realize low noise, robust, and low threshold oscillators [16], [17]. The concept has been also used at optical frequencies where low threshold, single-frequency of oscillation, lasers have been proposed using the DBE [18] and the 6DBE [11] concepts. Note that any amount of losses or gain disrupts the occurrence of these kinds of EPDs in lossless and gainless structures. Therefore, oscillators based on the RBE, DBE and 6DBE retain the properties associated to these degeneracies only when small gain per unit length is introduced. Despite the gain per unit length shall not be high, oscillators based on these concepts may exhibit other important properties like robust single frequency of oscillation, low phase noise and stability of the oscillation frequency to variation of loads [16], [17]. The concept of DBE oscillator has been proposed also in the area of electron-beam devices [19]. The SIP has been proposed to realize electron beam-based amplifiers with high gain-bandwidth product [20], delay lines [21], and one wave lasers [22]. Finally, EPDs in waveguides can be used to conceive leaky wave antennas that are very sensitive to tuning or environment perturbations for sensing applications [13]. Since the discovery of such EPDs is recent, we foresee that other possible applications will be proposed in the near future.

III. Conclusion

We have shown various examples of fully-planar microstrip geometries that exhibit different orders of EPDs without gain and loss in the system. The occurrence of the EPDs is demonstrated using (i) analytical formulations; (ii) full-wave simulations, and (iii) scattering parameters measurements. The presented structures can be useful in various applications like leaky-wave antennas, oscillators, distributed amplifiers, delay lines, pulse generators, and sensors. In summary, to realize EPDs of order 2 and 4 in waveguides, we need at least two coupled modes (e.g., a two-way waveguide), whereas to realize EPDs of order 3 and 6, we need at least three coupled modes (e.g., a three-way waveguide). These orders of EPDs have been demonstrated theoretically and experimentally in planar coupled waveguides implemented using microstrip technology. Computational tools have been developed in Refs. [9], [10], and [13] to design and characterize such EPDs.

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