*14th International Congress on Artificial Materials for Novel Wave Phenomena – Metamaterials 2020*
New York, USA, Sept. 28th – Oct. 3rd, 2020

# Sensing Via Exceptional Points in Space and Time Periodic Systems and in PT-Symmetric Systems

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***Abstract* – We review and explore sensor applications based on electromagnetic systems operating near an exceptional point of degeneracy (EPD). The EPD is defined as the point at which the system eigenmodes coalesce in both their eigenvalues and eigenvectors varying a system parameter. Sensors based on EPDs show sensitivity proportional to $\delta^{1/m}$, where $\delta$ is a perturbation of a system parameter and $m$ is the order of the EPD. EPDs manifest in PT-symmetric systems or periodic systems that can be periodic in either time or space. We review all the methods to obtain EPD based sensors, and we focus on two classes of ultra-sensitive EPD systems: i) periodic linear time-varying single oscillators, and ii) optical gyroscopes based on a modified coupled resonators optical waveguide (CROW) exhibiting 4th order EPD.**

## I. INTRODUCTION

Sensing is a crucial part of various medical, industrial, and automotive applications that require sensing of local physical or biological quantities [1]–[4]. Hence, low-profile, low-cost, highly-sensitive electromagnetic (EM) sensing systems are desired to achieve continuous and accurate measurement for various applications. We exploit the concept of exceptional point of degeneracy (EPD) to enhance the sensitivity of EM systems, at which the observables are no longer linearly proportional to a system perturbation but rather have an $m$th root dependence with $m$ being the order of the EPD [3], [5]. In this paper we review all EPD-based techniques to achieve high sensitivity, and we present a highly sensitive time varying LC oscillator and a sensitive gyroscope as applications where the EPD is employed to enhance sensitivity.

## II. SYSTEMS WITH EXCEPTIONAL POINTS

We discuss two categories of systems that exhibit EPDs. Such categories are classified based on how EPDs are induced. The first category includes PT-symmetric systems where we introduce gain into the naturally lossy structures so that exceptional points of non-Hermitian degeneracies are manifested [2], [6]. The second category includes periodic EM structures where the periodicity is either in space, by the designing the waveguides to be periodic, or in time, through utilizing a periodic time varying parameter in a resonating system. Both categories of EPD systems are based on the same mathematical concept: having a system eigenvalue with geometrical multiplicity less than its algebraic multiplicity, i.e., the system matrix describing modes evolution is similar to a Jordan block matrix or a matrix containing Jordan blocks, and at least two eigenvectors coalesce.

An interesting property of EPDs is their ultra-sensitivity to small perturbations that make them promising for sensor applications [1], [7]. To explore such a property, we consider the perturbed eigenvalues and dispersion relation of the periodic circuit around the EPD condition. A tiny physical perturbation $\epsilon$ of a system parameter (frequency, refractive index, losses, geometrical dimensions, etc) perturbs the system eigenvalues. The perturbed eigenvalues $\zeta_q$ of the system are obtained using the Puiseux series expansion [8] around the ideal degenerate eigenvalue $\zeta_e$, which is approximated to the first order as

$$\zeta_q(\epsilon) \approx \zeta_e + \alpha_1 e^{i2q\pi/m} \epsilon^{1/m} \tag{1}$$

for $q = 1, 2, 3, ..., m$, with $m$ the order of the EPD. The Puiseux series coefficient $\alpha_1$ is calculated using the formula provided in [8], and it depends on both the perturbed system parameter and the flatness of the EPD dispersion curve. This equation shows that the perturbation of the eigenvalues is proportional to $\epsilon^{1/m}$ which means the observables in an EPD based sensor show much higher sensitivity to small values of the perturbation $\epsilon$ than the linear proportionality in conventional sensors. Also, we can see from (1) that the higher the order of the EPD, the better the sensitivity to small perturbations. Examples of EPD based sensor utilizing PT-symmetric systems can be found in [1], [2], systems utilizing periodic linear time varying resonator in [3], [9], whereas systems utilizing spatially-periodic waveguide can be found in [5].

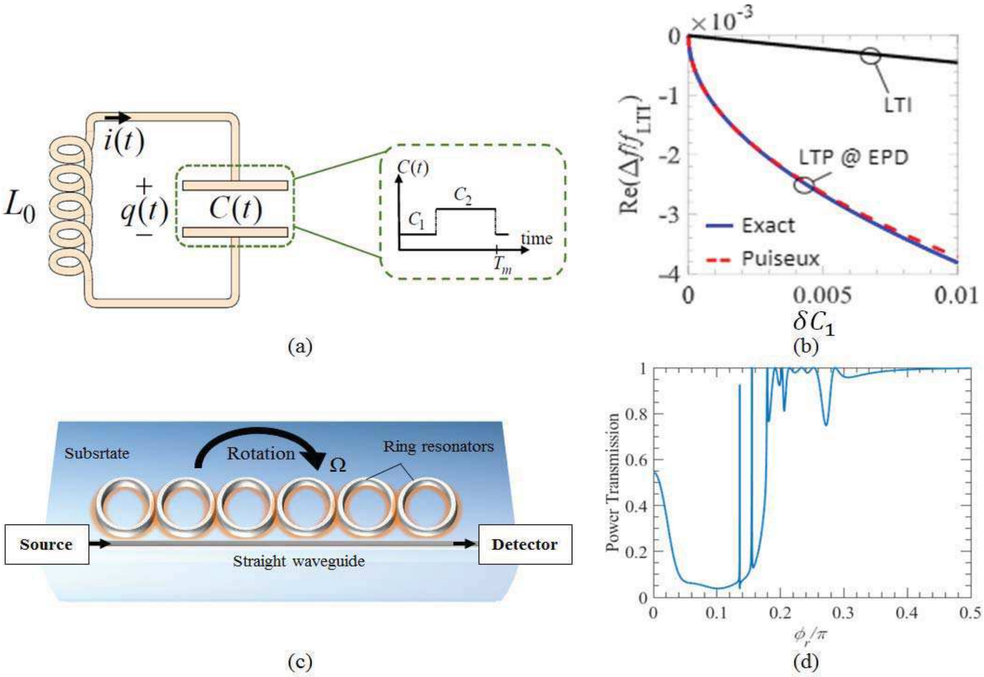


Fig. 1. (a) Sensor circuit consisting of a periodic time-varying LC resonator operating at an EPD. (b) Relative change in the resonance frequency of the EPD sensor as a function of the relative change in the capacitance $\delta C_1$. (c) Optical gyroscope that utilizes a CROW structure operating near a 4th order EPD. (d) Power transmission function varying the phase shift induced by the rotation of the structure.

## III. Examples of Systems Utilizing The Ultra Sensitivity of EPDs

### *A. Sensor Utilizing Periodic Linear Time-Varying Systems*

We illustrate a simple resonating system comprised of an inductor $L$ and a linear time-varying (LTV) capacitor $C(t)$ that exhibits an EPD of second order by using a periodic modulation of the time varying capacitor, without the need of incorporating loss or gain, as shown in Fig. 1(a). The theory of such a simple resonator was first presented in [3], and it is experimentally verified in [9]. Designing the system to resonate at an EPD provides the advantage that any small perturbation to the inductor or the LTV capacitor leads to a huge shift in the resonance frequency which boosts the sensitivity of such a system. In Fig. 1(b) we show the change in the resonance frequency $\Delta f$ due to a small perturbation of the LTV capacitor lower value $\delta C_1$. In this figure we compare the exact calculated $\Delta f$ and the approximate one obtained using (1) with $m = 2$, which are in a very good agreement. We also show that the EPD-induced shift is much larger than $\Delta f$ of a conventional linear time invariant (LTI) sensor. The superiority of the EPD based sensors over the conventional sensor is clear for such a small perturbation.

*B. Optical Gyroscope Based on periodic Coupled Resonator optical waveguide (CROW)*

We present an example where an EPD of 4th order induced in spatially periodic coupled resonator optical waveguide is adopted to enhance the sensitivity of optical gyroscopes. The existence of high order EPDs in photonic structures based on periodic CROW was introduced in [10], where the conventional CROW is modified by side-coupling the chain of ring resonators to a straight waveguide to enable modes mixing. Such EPDs involve the coalescence of four Floquet-Bloch eigenmodes, without the presence of gain and loss. Fig. 1(a) shows a possible utilization of the EPD modified CROW operating near a 4th order EPD as a highly-sensitive gyroscope, where we use a source to feed the straight waveguide and a detector to sense the output power from the other end. In Fig. 1(c), we plot the power transmission versus the phase shift introduced in rings due to rotation. The figure shows very sharp transitions in the calculated power transmission due to very small change in the introduced rotation phase shift. Various parameters must be carefully optimized, and various designs could be used to maximize the sensitivity of the CROW gyroscope.

## V. Conclusion

We explore all various methods to obtain sensors based on electromagnetic systems operating near an EPD. We analyze two distinct categories of such systems: PT-symmetric systems and periodic systems. Periodic systems can be designed to be either periodic waveguides (that are periodic in space) or time-modulated resonators with a periodic time varying system parameter. We show two examples of EPD based systems that are ultra-sensitive to small perturbations: i) a periodic linear time varying simple LC oscillator operating near a 2nd order EPD, and ii) an EPD-based gyroscope utilizing rotating CROW operating near a 4th order EPD.

## Acknowledgement

The authors acknowledge support from the USA National Science Foundation, award NSF ECCS-171197.

## References

[1] J. Wiersig, "Enhancing the Sensitivity of Frequency and Energy Splitting Detection by Using Exceptional Points: Application to Microcavity Sensors for Single-Particle Detection," *Phys. Rev. Lett.*, vol. 112, no. 20, p. 203901, May 2014, doi: 10.1103/PhysRevLett.112.203901.

[2] H. Hodaei *et al.*, "Enhanced sensitivity at higher-order exceptional points," *Nature*, vol. 548, no. 7666, pp. 187–191, Aug. 2017, doi: 10.1038/nature23280.

[3] H. Kazemi, M. Y. Nada, T. Mealy, A. F. Abdelshafy, and F. Capolino, "Exceptional Points of Degeneracy Induced by Linear Time-Periodic Variation," *Phys. Rev. Appl.*, vol. 11, no. 1, p. 014007, Jan. 2019, doi: 10.1103/PhysRevApplied.11.014007.

[4] H. Kazemi *et al.*, "Ultra-Sensitive Radio Frequency Biosensor at an Exceptional Point of Degeneracy induced by time modulation," *arXiv preprint arXiv:1909.03344 Phys.*, Sep. 2019.

[5] M. Y. Nada and F. Capolino, "Exceptional Point of Sixth Order Degeneracy in a Modified Coupled Resonators Optical Waveguide," *arXiv preprint arXiv:1909.03114 Phys.*, Apr. 2020.

[6] C. E. Rüter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, "Observation of parity–time symmetry in optics," *Nat. Phys.*, vol. 6, no. 3, pp. 192–195, 2010.

[7] W. Chen, S. Kaya Özdemir, G. Zhao, J. Wiersig, and L. Yang, "Exceptional points enhance sensing in an optical microcavity," *Nature*, vol. 548, no. 7666, pp. 192–196, Aug. 2017, doi: 10.1038/nature23281.

[8] A. Welters, "On explicit recursive formulas in the spectral perturbation analysis of a Jordan block," *SIAM J. Matrix Anal. Appl.*, vol. 32, no. 1, pp. 1–22, 2011.

[9] H. Kazemi, M. Y. Nada, F. Maddaleno, and F. Capolino, "Experimental Demonstration of Exceptional Points of Degeneracy in Linear Time Periodic Systems and Exceptional Sensitivity," *arXiv preprint arXiv:1908.08516 Phys.*, Oct. 2019.

[10] M. Y. Nada, M. A. K. Othman, and F. Capolino, "Theory of coupled resonator optical waveguides exhibiting high-order exceptional points of degeneracy," *Phys. Rev. B*, vol. 96, no. 18, p. 184304, Nov. 2017, doi: 10.1103/PhysRevB.96.184304.