

RIBBON HOMOLOGY COBORDISMS

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ABSTRACT. We study 4-dimensional homology cobordisms without 3-handles, showing that they interact nicely with Thurston geometries, character varieties, and instanton and Heegaard Floer homologies. Using these, we derive obstructions to such cobordisms. As one example of these obstructions, we generalize other recent results on the behavior of knot Floer homology under ribbon concordances. Finally, we provide topological applications, including to Dehn surgery problems.

1. INTRODUCTION

The advent of topological quantum field theories (TQFTs) in the past few decades has renewed interest in smooth cobordisms and the associated category. In dimension 4, Seiberg–Witten Floer homology, a gauge-theoretic TQFT, was recently used by Manolescu [Man16] to study the homology cobordism group $\Theta_{\mathbb{Z}}^3$, disproving the Triangulation Conjecture in dimensions $n \geq 5$. Many questions remain: For example, it is unknown whether $\Theta_{\mathbb{Z}}^3$ has any torsion.

In this article, we study 4-dimensional cobordisms from a new perspective, in terms of their directionality. More precisely, we study *ribbon cobordisms*, which are $2n$ -dimensional manifolds that can be built from k -handles with $k \leq n$. They arise in at least two natural ways: as Stein cobordisms between closed, contact manifolds, and as the exterior of (strongly homotopy-) ribbon surfaces, which are cobordisms between link exteriors [Gor81]. Note that every (homology) cobordism can be split into two ribbon (homology) cobordisms. While homology cobordism is an equivalence relation, ribbon homology cobordisms are not symmetric. In fact, as we will see, ribbon homology cobordisms give rise to a preorder on 3-manifolds that seems to agree with orderings by various invariants and Thurston geometries. We conjecture:

Conjecture 1.1 (cf. [Gor81, Conjecture 1.1]). *The preorder on the set of homeomorphism classes of closed, connected, oriented 3-manifolds given by ribbon $\mathbb{Q}$ -homology cobordisms is a partial order.*

Here, an R -homology cobordism between two compact, oriented 3-manifolds Y_1 and Y_2 is an oriented, smooth cobordism $W: Y_1 \rightarrow Y_2$ such that $H_*(W, Y_i; R) = 0$ for $i \in \{1, 2\}$. For example, the exterior of a knot concordance is a $\mathbb{Z}$ -homology cobordism. Our results can be summarized as:

Metatheorem. *Let Y_- and Y_+ be compact, connected, oriented 3-manifolds possibly with boundary, and suppose that there exists a ribbon homology cobordism from Y_- to Y_+ . Then the complexity of Y_- is no greater than that of Y_+ , as measured by each of the following invariants:*

- (A) *The fundamental group;*
- (B) *The G -character variety, for a compact, connected Lie group G , and its Zariski tangent space at a conjugacy class;*
- (C) *Various flavors of instanton and Heegaard Floer homologies.*

These comparisons are sometimes realized by explicit morphisms in the appropriate category.

Note that (A) was proved by Gordon [Gor81] in the case that Y_- and Y_+ have toroidal boundary, and his proof immediately generalizes to the closed case following Geometrization; see Section 1.1 for context. We will provide more precise statements for (B) and (C) in Section 1.2 and Section 1.3.

Our metatheorem has many topological applications, which we will discuss below. But first, (A) and Geometrization together give us the following:

Theorem 1.2. *There is a hierarchy among the Thurston geometries with respect to ribbon $\mathbb{Q}$ -homology cobordisms, given by the diagram*

$$S^3 \rightarrow (S^2 \times \mathbb{R}) \rightarrow \mathbb{R}^3 \rightarrow \text{Nil} \rightarrow \text{Sol} \rightarrow (\mathbb{H}^2 \times \mathbb{R}) \cup \widetilde{\text{SL}(2, \mathbb{R})} \cup \mathbb{H}^3.$$

In other words, suppose that Y_- and Y_+ are compact 3-manifolds with empty or toroidal boundary that admit distinct geometries, and that there exists a ribbon $\mathbb{Q}$ -homology cobordism from Y_- to Y_+ . Then there is a sequence of arrows from the geometry of Y_- to that of Y_+ in the diagram above.

A more refined version of Theorem 1.2 is stated in Theorem 3.4.

Remark 1.3. It is natural to ask how the ribbon $\mathbb{Q}$ -homology cobordism preorder interacts with the JSJ decomposition. Unfortunately, there exist examples where $Y_{\pm}$ is hyperbolic and $Y_{\mp}$ has non-trivial JSJ decomposition; see Remark 8.10 for more details.

Evidence for Conjecture 1.1 is provided by the metatheorem and Theorem 1.2, as well as Corollary 1.16, and Corollary 4.3 below. Conjecture 1.1 is analogous to [Gor81, Conjecture 1.1], which states that the preorder on the set of knots in S^3 by ribbon concordance is a partial order.

Remark 1.4. Another major open problem regarding ribbon concordance is the Slice–Ribbon Conjecture. In a similar spirit, a natural question to ask is whether a $\mathbb{Z}$ -homology sphere bounding a $\mathbb{Z}$ -homology 4-ball always bounds a $\mathbb{Z}$ -homology 4-ball without 3-handles.

We now turn to some new applications. We begin with an application to Seifert fibered homology spheres that illustrates the use of several different tools described above.

Theorem 1.5. *Suppose that Y_- and Y_+ are the Seifert fibered homology spheres $\Sigma(a_1, \dots, a_n)$ and $\Sigma(a'_1, \dots, a'_m)$ respectively, and that there exists a ribbon $\mathbb{Q}$ -homology cobordism from Y_- to Y_+ . Then*

- (1) *The Casson invariants of Y_- and Y_+ satisfy $|\lambda(Y_-)| \leq |\lambda(Y_+)|$;*
- (2) *Either Y_- and Y_+ both bound negative-definite plumbings, or both bound positive-definite plumbings; and*
- (3) *The numbers of exceptional fibers satisfy $n \leq m$.*

The first two items above can be proved with either instanton or Heegaard Floer homology using Metatheorem (C). However, the authors do not know of a Floer-homology proof of (3).

Next, we have applications to ribbon concordance. Recall that a strongly homotopy-ribbon concordance is a knot concordance in $S^3 \times I$ whose exterior is ribbon.¹

Corollary 1.6. *Suppose that K_- and K_+ are Montesinos knots in S^3 of determinant 1, and that the number of rational tangles in K_- with denominator at most 2 is greater than that of K_+ . Then there does not exist a strongly homotopy-ribbon concordance from K_- to K_+ .*

Recall that a knot in S^3 is *small* if there are no closed, non-boundary-parallel, incompressible surfaces in its exterior, and that torus knots are small.

Corollary 1.7. *Suppose that K_- is a composite knot in S^3 , and that K_+ is a small knot in S^3 . Then there does not exist a strongly homotopy-ribbon concordance from K_- to K_+ .*

¹All ribbon concordances are strongly homotopy-ribbon.

We also obtain applications to reducible Dehn surgery problems. The following is a sample theorem; see [Section 9](#) for its proof, as well as another similar result. The same techniques can be used to obtain other results along the same lines, which we do not pursue in this article.

Theorem 1.8. *Suppose that Y is an irreducible $\mathbb{Q}$ -homology sphere, L is a null-homotopic link in Y of ℓ components, and $Y_0(L) \cong N \sharp \ell(S^1 \times S^2)$, where $Y_0(L)$ denotes the result of 0-surgery along each component of L . Then N is orientation-preserving homeomorphic to Y .*

Remark 1.9. Since the first appearance of this article, Hom and the second author [[HL20](#), Corollary 1.2] have used [Theorem 1.8](#) to show that when L is a knot, L must in fact be trivial.

Remark 1.10. The technique used to prove [Theorem 1.8](#) can also be applied to the case where Y is an L -space $\mathbb{Z}$ -homology sphere and L is a non-trivial knot. In this case, one could show that $Y_0(L)$ does not contain an $S^1 \times S^2$ summand, which follows from Ni [[Ni13](#), p. 1144].

Finally, we also obtain an application to the computation of the Furuta–Ohta invariant λ_{FO} for $\mathbb{Z}[\mathbb{Z}]$ -homology $S^1 \times S^3$'s [[FO93](#)].

Corollary 1.11. *Suppose that Y_- and Y_+ are $\mathbb{Z}$ -homology spheres, and that $W: Y_- \rightarrow Y_+$ is a ribbon $\mathbb{Q}$ -homology cobordism, and that $\overline{D(W)}$ is the $\mathbb{Z}[\mathbb{Z}]$ -homology $S^1 \times S^3$ obtained by gluing the ends of $D(W)$ by the identity. Then $\lambda_{\text{FO}}(D(W)) = \lambda(Y_-)$. In particular, $\lambda_{\text{FO}}(D(W))$ agrees with the Rokhlin invariant of $Y_- \bmod 2$.*

Remark 1.12. We believe that the proof of [Corollary 1.11](#) can be adapted to show the analogous statement $\lambda_{\text{SW}}(\overline{D(W)}) = -\lambda(Y_-)$ for the Mrowka–Ruberman–Saveliev invariant λ_{SW} [[MRS11](#)]. This would verify the conjecture that $\lambda_{\text{SW}} = -\lambda_{\text{FO}}$ [[MRS11](#), Conjecture B], for $\mathbb{Z}[\mathbb{Z}]$ -homology $S^1 \times S^3$'s that are of the form $\overline{D(W)}$. See [Remark 4.16](#) for more details.

Remark 1.13. Suppose that Y_+ is a $\mathbb{Z}$ -homology sphere. Then for any $\mathbb{Q}$ -homology sphere Y_- , a ribbon $\mathbb{Q}$ -homology cobordism from Y_- to Y_+ is in fact a ribbon $\mathbb{Z}$ -homology cobordism, and the existence of such a cobordism implies that Y_- is also a $\mathbb{Z}$ -homology sphere. See [Lemma 3.2](#) for the proof. This is relevant, for example, to [Theorem 1.5](#) and [Corollary 1.11](#), as well as [Theorem 4.1](#) and [Theorem 4.8](#) later.

To ease our discussion, we set up some conventions for the article.

Conventions. All 3- and 4-manifolds are assumed to be oriented and smooth, and, except in [Section 7.1](#), they are also assumed to be connected.² Accordingly, we also assume that handle decompositions of cobordisms between non-empty 3-manifolds have no 0- or 4-handles. We say that a handle decomposition is *ribbon* if it has no 3-handles. We always denote the ends of a ribbon homology cobordism by $Y_{\pm}$; for results that hold for more general cobordisms, we typically denote the cobordism by, for example, $W: Y_1 \rightarrow Y_2$. All sutured manifolds are assumed to be balanced. We denote by I the interval $[0, 1]$. Unless otherwise specified, all singular homologies have coefficients in $\mathbb{Z}$, instanton Floer homologies have coefficients in $\mathbb{Q}$, and Heegaard Floer homologies have coefficients in $\mathbb{Z}/2$.

1.1. Context. In the seminal work of Gordon [[Gor81](#)] on ribbon concordance, the key theorem, which is very special to the absence of 3-handles, is the following.

Theorem 1.14 (Gordon [[Gor81](#), Lemma 3.1]). *Let Y_- and Y_+ be compact 3-manifolds possibly with boundary, and suppose that $W: Y_- \rightarrow Y_+$ is a ribbon $\mathbb{Q}$ -homology cobordism. Then*

²Connectedness is often not essential in our statements, but we impose it for ease of exposition.

- (1) The map $\iota_*: \pi_1(Y_-) \rightarrow \pi_1(W)$ induced by inclusion is injective; and
- (2) The map $\iota_*: \pi_1(Y_+) \rightarrow \pi_1(W)$ induced by inclusion is surjective.

(While Gordon's original statement is only for exteriors of ribbon concordances, the more general result holds, as explained in this subsection.) Gordon uses the theorem above, combined with various properties of knot groups, to study questions related to ribbon concordance.

Our employment of several different approaches above is motivated by two observations. First, since Gordon's work, there have been many breakthroughs in low-dimensional topology, including the Geometrization Theorem for 3-manifolds, the applications of representation theory and gauge theory, and, relatedly, the advent of Floer theory. Each of these constitutes a new, powerful tool that can be applied in the context of ribbon homology cobordisms, and a major goal of the present article is to systematically carry out these applications. In particular, we will develop obstructions from these theories, which we will then use for topological gain.

Second, while the approaches reflect very different perspectives, there are interesting theoretical connections between them. To illustrate this point, we discuss [Theorem 1.14](#) further. This theorem follows from the deep property of residual finiteness of 3-manifold groups together with the elegant results of Gerstenhaber and Rothaus [\[GR62\]](#) on the representations of finitely presented groups to a compact, connected Lie group G . (The residual finiteness of closed 3-manifold groups has only been known after the proof of the Geometrization Theorem; this new development is the ingredient that extends Gordon's original statement to closed 3-manifolds in [Theorem 1.14](#).) The statement of Gerstenhaber and Rothaus can be reinterpreted as saying that the G -representations of $\pi_1(Y_-)$ extend to those of $\pi_1(W)$, and [Theorem 1.14 \(2\)](#) implies that any non-trivial representation of $\pi_1(W)$ determines a non-trivial representation of $\pi_1(Y_+)$ by pullback. Thus, [Theorem 1.14](#) naturally leads to the study of the character varieties of $Y_{\pm}$. Moving further, focusing on $G = \mathrm{SU}(2)$, we observe that the $\mathrm{SU}(2)$ -representations of $\pi_1(Y_{\pm})$ are related to the instanton Floer homology of $Y_{\pm}$. Like instanton Floer homology, Heegaard Floer homology is defined by considering certain moduli spaces of solutions; however, while they share many formal properties, the exact relationship between these two theories remains somewhat unclear. Finally, we note that the Geometrization Theorem implies that if $Y_{\pm}$ is geometric, its geometry can be determined from $\pi_1(Y_{\pm})$ in many situations.

In fact, apart from theoretical connections, there is considerable interplay among these perspectives even in their *applications*. We direct the interested reader to [Theorem 1.5](#), [Remark 1.17](#), and [Remark 3.6](#) for a few examples.

1.2. Character varieties and ribbon homology cobordisms. As briefly mentioned above, the proof of [Theorem 1.14](#) requires understanding the relationship between G -representations of $\pi_1(Y_{\pm})$ and $\pi_1(W)$. Consequently, given a ribbon $\mathbb{Q}$ -homology cobordism $W: Y_- \rightarrow Y_+$, we will also obtain relations between the character varieties of Y_- and Y_+ . Recall that for a group π and compact, connected Lie group G (e.g. $\mathrm{SU}(2)$), we can define the *representation variety* $\mathcal{R}_G(\pi)$, which is the set of G -representations of π ; we can also quotient by the conjugation action to obtain the *character variety* $\mathcal{X}_G(\pi)$. For a path-connected space X , we will write $\mathcal{R}_G(X)$ for $\mathcal{R}_G(\pi_1(X))$, and $\mathcal{X}_G(X)$ for $\mathcal{X}_G(\pi_1(X))$. As discussed above, we have the following proposition.

Proposition 1.15. *Let Y_- and Y_+ be compact 3-manifolds possibly with boundary, and suppose that $W: Y_- \rightarrow Y_+$ is a ribbon $\mathbb{Q}$ -homology cobordism. Then any $\rho_- \in \mathcal{R}_G(Y_-)$ can be extended to an element $\rho_W \in \mathcal{R}_G(W)$ that pulls back to an element $\rho_+ \in \mathcal{R}_G(Y_+)$, and distinct elements in $\mathcal{R}_G(Y_-)$ corresponds to distinct elements in $\mathcal{R}_G(Y_+)$. The analogous statement for $\mathcal{X}_G$ also holds.*

See [Proposition 2.1](#) for a restatement and proof. Recall that the Chern–Simons functional gives an $\mathbb{R}/\mathbb{Z}$ -valued function on $\mathcal{R}_G(Y)$; the image of this function is a finite subset of $\mathbb{R}/\mathbb{Z}$, which

we call the *G-Chern-Simons invariants* of Y . [Proposition 1.15](#) implies a relation between the G -Chern-Simons invariants of Y_- and Y_+ .

Corollary 1.16. *Let Y_- and Y_+ be closed 3-manifolds, and suppose that there exists a ribbon $\mathbb{Q}$ -homology cobordism from Y_- to Y_+ . Then the set of G -Chern-Simons invariants of Y_- is a subset of that of Y_+ .*

Remark 1.17. Stein manifolds provide a large family of 4-manifolds without 3-handles. It is interesting to compare the discussion above with the work of Baldwin and Sivek [\[BS18\]](#), who use instanton Floer homology to prove that if Y is a $\mathbb{Z}$ -homology sphere that admits a Stein filling with non-trivial homology, then $\pi_1(Y)$ admits an irreducible $\mathrm{SU}(2)$ -representation. In comparison, if $W: Y_- \rightarrow Y_+$ is a Stein $\mathbb{Q}$ -homology cobordism, and $\pi_1(Y_-)$ admits a non-trivial $\mathrm{SU}(2)$ -representation, then it extends to an $\mathrm{SU}(2)$ -representation of $\pi_1(W)$ that pulls back to a non-trivial $\mathrm{SU}(2)$ -representation of $\pi_1(Y_+)$ by [Proposition 1.15](#), which requires no gauge theory.

In fact, with a bit more work, we can compare the local structures of the character varieties. For a path-connected space X and a representation $\rho: \pi_1(X) \rightarrow G$, recall that the *Zariski tangent space* to $\mathcal{X}_G(X)$ at the conjugacy class $[\rho]$ is the first group cohomology of $\pi_1(X)$ with coefficients in the adjoint representation associated to ρ , denoted by $H^1(X; \mathrm{Ad}_\rho)$; see [Section 2.2](#) for more details. Below, we also consider the zeroth group cohomology $H^0(X; \mathrm{Ad}_\rho)$.

Proposition 1.18. *Let Y_- and Y_+ be compact 3-manifolds possibly with boundary, and suppose that $W: Y_- \rightarrow Y_+$ is a ribbon $\mathbb{Q}$ -homology cobordism. Fix $\rho_- \in \mathcal{R}_G(Y_-)$, choose an extension $\rho_W \in \mathcal{R}_G(W)$, and denote by $\rho_+ \in \mathcal{R}_G(Y_+)$ the pullback of ρ_W . Suppose that $\dim_{\mathbb{R}} H^0(Y_-; \mathrm{Ad}_{\rho_-}) = \dim_{\mathbb{R}} H^0(Y_+; \mathrm{Ad}_{\rho_+})$. Then*

$$\dim_{\mathbb{R}} H^1(Y_-; \mathrm{Ad}_{\rho_-}) \leq \dim_{\mathbb{R}} H^1(W; \mathrm{Ad}_{\rho_W}) \leq \dim_{\mathbb{R}} H^1(Y_+; \mathrm{Ad}_{\rho_+}).$$

This seemingly technical result, applied to ribbon $\mathbb{Q}$ -homology cobordisms between Seifert fibered homology spheres, will be our avenue to prove [Theorem 1.5 \(3\)](#).

1.3. Floer homologies and ribbon homology cobordisms. Another way that representations appear in 3- and 4-manifold topology is through instanton Floer homology, where we specialize to $G = \mathrm{SU}(2)$ or $\mathrm{SO}(3)$. Recall that a Floer homology associates a vector space or module to a 3-manifold, and a linear transformation or homomorphism to a cobordism. In the case of instanton Floer homology, the associated group comes roughly from counting $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$ representations of the fundamental group. Below, we state a theorem for the behavior of a *general* Floer homology theory under ribbon homology cobordisms. In [Section 4](#), we give results for most versions of Floer homology with precise conditions on the 3-manifolds and the ribbon homology cobordism.

Theorem 1.19. *Let F be one of the 3-manifold Floer homology theories discussed in [Section 4](#). Let Y_- and Y_+ be compact 3-manifolds, and suppose that $W: Y_- \rightarrow Y_+$ is a ribbon homology cobordism. Then $F(W)$ includes $F(Y_-)$ into $F(Y_+)$ as a summand.³*

Very recently, Zemke and his collaborators [\[Zem19c, MZ21, LZ19\]](#) have shown that ribbon concordances induce injections on knot Heegaard Floer homology and Khovanov homology, and this has led to several other interesting results [\[JMZ20, Sar20\]](#), including an exciting relationship between knot Heegaard Floer homology and the bridge index [\[JMZ20, Corollary 1.9\]](#). In the special case that F is sutured Heegaard Floer homology, and W is the exterior of a strongly-homotopy ribbon concordance, [Theorem 1.19](#) recovers the results of Zemke [\[Zem19c, Theorem 1.1\]](#) and Miller

³For some flavors of Floer homology, we prove the weaker statement that $F(Y_-)$ is isomorphic to a summand of $F(Y_+)$.

and Zemke [MZ21, Theorem 1.2] on knot Heegaard Floer homology. (For a more precise statement, see Corollary 4.13.)

While much of the work involving Floer homologies is inspired by the work of Zemke et al., our proofs use a different argument that holds in a more general context.

Organization. In Section 2, we study the relationship between ribbon homology cobordisms and character varieties, proving Proposition 1.15 and Proposition 1.18. In Section 3, we prove Theorem 1.2, pertaining to Thurston geometries.

Next, in Section 4, we give the precise statements associated with Theorem 1.19, on the behavior of various versions of Floer homology under ribbon homology cobordisms. The following three sections are then devoted to proving these Floer-theoretic results. First, in Section 5, we give the necessary topological background to analyze the double of a ribbon homology cobordism, and give a short application to metrics with positive scalar curvature. In Section 6, after giving an overview of the Chern–Simons functional (proving Corollary 1.16) and instanton Floer homology, we prove Theorem 4.1 to Theorem 4.8 which are instantiations of Theorem 1.19 for instanton Floer homology, as well as Corollary 1.11; we also outline a proof of one of these theorems via character varieties. In Section 7, we set up the necessary tools for Heegaard Floer homology and prove Theorem 4.10 to Theorem 4.15 which are versions of Theorem 1.19 for Heegaard Floer homology.

Combining the results above, in Section 8, we prove some specific obstructions that arise from results discussed so far, including Theorem 1.5, Corollary 1.6, Corollary 1.7, and other statements. Finally, in Section 9, we provide further applications of ribbon homology cobordisms to Dehn surgery problems, proving Theorem 1.8.

We provide a few routes for the reader. The reader solely interested in character varieties, Thurston geometries, or Dehn surgeries can read only Section 2, Section 3, or Section 9, respectively. If the sole interest is in instanton Floer homology, then refer to Section 4, Section 5 and Section 6. For Heegaard Floer homology, see Section 4, Section 5 and Section 7.

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2. THE FUNDAMENTAL GROUP AND CHARACTER VARIETIES

In this section, we study the fundamental groups and character varieties of 3-manifolds related by ribbon cobordisms.

2.1. Background. Throughout, we let G denote a compact, connected Lie group. For a group π , let $\mathcal{R}_G(\pi)$ denote the space of G -representations. If X is a connected manifold, we write $\mathcal{R}_G(X)$ for $\mathcal{R}_G(\pi_1(X))$. We write $\mathcal{X}_G(\pi)$ for the set of conjugacy classes of G -representations. We will omit G from the notation when $G = \mathrm{SU}(2)$.

We first prove the following proposition, which is a restatement of [Theorem 1.14](#) and [Proposition 1.15](#). The argument, using work of Gerstenhaber and Rothaus [\[GR62\]](#), repeats that of Gordon [\[Gor81\]](#) and also that of Cornwell, Ng, and Sivek [\[CNS16\]](#).

Proposition 2.1. *Let Y_- and Y_+ be compact 3-manifolds possibly with boundary, and suppose that $W: Y_- \rightarrow Y_+$ is a ribbon $\mathbb{Q}$ -homology cobordism. Then the inclusion $\iota_+: Y_+ \rightarrow W$ induces a surjection $(\iota_+)_*: \pi_1(Y_+) \rightarrow \pi_1(W)$ and an injection $\iota_+^*: \mathcal{R}_G(W) \rightarrow \mathcal{R}_G(Y_+)$, and the inclusion $\iota_-: Y_- \rightarrow W$ induces an injection $(\iota_-)_*: \pi_1(Y_-) \rightarrow \pi_1(W)$ and a surjection $\iota_-^*: \mathcal{R}_G(W) \rightarrow \mathcal{R}_G(Y_-)$.*

Proof. Since W consists entirely of 1- and 2-handles, we may flip W upside down and view it as a cobordism from $-Y_+$ to $-Y_-$. From this perspective, W is obtained by attaching 2- and 3-handles to $-Y_+$. It follows that the inclusion from $-Y_+$ into W induces a surjection from $\pi_1(-Y_+)$ to $\pi_1(W)$.

For $\iota_-: Y_- \rightarrow W$, we will prove the second claim first. Choose a representation $\rho: \pi_1(Y_-) \rightarrow G$. Since W is a $\mathbb{Q}$ -homology cobordism, it admits a handle decomposition with an equal number m of 1- and 2-handles. This allows us to write $\pi_1(W) \cong (\pi_1(Y_-) * \langle b_1, \dots, b_m \rangle) / \langle\langle v_1, \dots, v_m \rangle\rangle$, where the generators b_i are induced by the 1-handles and the relators v_i are induced by the 2-handles. The words v_i induce a map $K: G^m \rightarrow G^m$, and the existence of an extension of ρ to $\pi_1(W)$ is equivalent to solving the equation $K = \vec{e}$. (To handle the elements in $\pi_1(Y_-)$ that appear in v_i , we apply ρ to the element to view it in G .) By quotienting out by $\pi_1(Y_-)$, each element v_i induces a word v'_i in the free group $\langle b_1, \dots, b_m \rangle$. Consider the matrix B whose $(ij)^{\mathrm{th}}$ coordinate is the signed number of times that b_j appears in v'_i . Since $H_1(W, Y_-; \mathbb{Q}) = 0$, we see that $\det(B) \neq 0$. It now follows from [\[GR62, Theorem 1\]](#) that there exists a solution to the equation $K = \vec{e}$.

Now we show that the inclusion map $(\iota_-)_*$ from $\pi_1(Y_-)$ to $\pi_1(W)$ is injective. The residual finiteness property of 3-manifold groups implies that for any non-trivial $x \in \pi_1(Y_-)$, there exists a finite quotient H of $\pi_1(Y_-)$ by a normal subgroup N such that $x \notin N$. We claim that the induced map $(\iota_-)_*: H \rightarrow \pi_1(W) / \langle\langle (\iota_-)_*(N) \rangle\rangle$ is injective; this will imply that $(\iota_-)_*(x)$ is a non-trivial element of $\pi_1(W)$. To prove our claim, note that $\pi_1(W) / \langle\langle (\iota_-)_*(N) \rangle\rangle \cong (H * \langle b_1, \dots, b_m \rangle) / \langle\langle v''_1, \dots, v''_m \rangle\rangle$, where v''_i is obtained from v_i by reducing the elements in $\pi_1(Y_-)$ to H . Now [\[GR62, Theorem 2\]](#) says that there is a finite extension $\tilde{H}$ containing elements $\beta_1, \dots, \beta_m$ such that $v''_i(\beta_1, \dots, \beta_m) = e$ for each $i \in \{1, \dots, m\}$. In other words, there is a homomorphism $\Phi: \pi_1(W) / \langle\langle (\iota_-)_*(N) \rangle\rangle \rightarrow \tilde{H}$ such that $\Phi \circ \overline{(\iota_-)_*}$ is the inclusion of H into $\tilde{H}$. This implies that $\overline{(\iota_-)_*}$ is injective, and our proof is complete. $\square$

Corollary 2.2. *Let Y_- and Y_+ be compact 3-manifolds possibly with boundary, and suppose that $W: Y_- \rightarrow Y_+$ is a ribbon $\mathbb{Q}$ -homology cobordism. If $\mathcal{X}_G(Y_+)$ is finite, then $\mathcal{X}_G(Y_-)$ is finite.*

Proof. This is a direct consequence of [Proposition 2.1](#). $\square$

In the next subsection, we give a more structured comparison of the character varieties with a bit more work.

2.2. Group cohomology computations and Zariski tangent spaces. We briefly review some definitions and constructions in group cohomology; see [\[Bro94\]](#) for more details. Let π be a group and let M be a $\mathbb{Z}[\pi]$ -module. The group cohomology $H^*(\pi; M)$ with coefficients in M is defined by taking a projective $\mathbb{Z}[\pi]$ -resolution $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{Z}$ of $\mathbb{Z}$, where $\mathbb{Z}$ has the $\mathbb{Z}[\pi]$ -module structure

where π acts by the identity. Then $C^*(\pi; M)$ is defined by applying $\text{Hom}_{\mathbb{Z}[\pi]}(-, M)$, and omitting $\mathbb{Z}$, as in

$$0 \rightarrow \text{Hom}_{\mathbb{Z}[\pi]}(P_0, M) \rightarrow \text{Hom}_{\mathbb{Z}[\pi]}(P_1, M) \rightarrow \cdots,$$

and $H^*(\pi; M)$ is the cohomology of this cochain complex. A natural way of constructing a free resolution of $\mathbb{Z}$ is as follows. Consider an aspherical CW complex X with $\pi_1(X) = \pi$, and lift this to a CW structure on the universal cover $\tilde{X}$. Then, the (augmented/reduced) CW chain complex for $\tilde{X}$ naturally inherits a $\mathbb{Z}[\pi]$ -module structure, where π acts by the deck transformation group action, and this is a free $\mathbb{Z}[\pi]$ -resolution of $\mathbb{Z}$. (The lift of an individual cell in X yields a π 's worth of cells upstairs, and these constitute a single copy of $\mathbb{Z}[\pi]$ in the cellular chain complex for the universal cover.) Recall that a presentation $\pi = \langle a_\alpha \mid w_\beta \rangle$ determines a CW structure on X with one 0-cell, one 1-cell e_α^1 for each generator a_α , and one 2-cell e_β^2 for each relator w_β ; then $H^*(\pi; M)$ can be computed from a cochain complex with Abelian groups

$$C^0(\pi; M) = \text{Hom}_{\mathbb{Z}[\pi]}(\mathbb{Z}[\pi], M) \cong M, \quad C^1(\pi; M) \cong \prod_{\alpha} M, \quad C^2(\pi; M) \cong \prod_{\beta} M,$$

and possibly non-trivial higher cochain groups $C^i(\pi; M)$ for $i > 2$ that we will not be concerned with. The $(\alpha\beta)^{\text{th}}$ component of the differential from $C^1(\pi; M)$ to $C^2(\pi; M)$ is non-zero only if a_α appears in w_β . Indeed, in X , if $\overline{e_\alpha^1} \cap \overline{e_\beta^2} = \emptyset$, then the same is true in the universal cover.

Now, given a representation $\rho \in \mathcal{R}_G(\pi)$, we can consider the $\mathbb{Z}[\pi]$ -module Ad_ρ , which is the Lie algebra $\mathfrak{g}$ of G with the $\mathbb{Z}[\pi]$ -action where π acts by the composition of ρ and the adjoint representation. Note that Ad_ρ is in fact an $\mathbb{R}[\pi]$ -module, and so $H^*(\pi; \text{Ad}_\rho)$ is an $\mathbb{R}$ -vector space. Recall also that $H^1(\pi; \text{Ad}_\rho)$ is the *Zariski tangent space* of $\mathcal{X}_G(\pi)$ at $[\rho]$. We are now ready to show that ribbon homology cobordisms induce relations between the Zariski tangent spaces.

Proof of Proposition 1.18. We begin by comparing $\dim_{\mathbb{R}} H^1(W; \text{Ad}_{\rho_W})$ and $\dim_{\mathbb{R}} H^1(Y_+; \text{Ad}_{\rho_+})$. First, we recall the inflation–restriction exact sequence in group cohomology (see, for example, [Wei94, 6.8.3]), which says that, given a normal subgroup N of π and a $\mathbb{Z}[\pi]$ -module M , there exists an injection of $H^1(\pi/N; M^N)$ into $H^1(\pi; M)$, where M^N is the subgroup of elements of M fixed by the action of π restricted to N . It is clear that M^N naturally inherits a $\mathbb{Z}[\pi/N]$ -module structure. (Further, if M actually has an $\mathbb{R}[\pi]$ -module structure, then everything respects the $\mathbb{R}$ -vector space structures.)

In our case, we take $\pi = \pi_1(Y_+)$, take N to be the kernel of the quotient map from $\pi_1(Y_+)$ to $\pi_1(W)$, and take $M = \text{Ad}_{\rho_+}$; then $(\text{Ad}_{\rho_+})^N$ is a $\mathbb{Z}[\pi_1(W)]$ -module. By construction, $N \subset \ker(\rho_+) \subset \ker(\text{Ad} \circ \rho_+)$; thus, N acts by the identity on Ad_{ρ_+} , and so $(\text{Ad}_{\rho_+})^N$ is in fact Ad_{ρ_W} . Therefore, we conclude that $\dim_{\mathbb{R}} H^1(W; \text{Ad}_{\rho_W}) \leq \dim_{\mathbb{R}} H^1(Y_+; \text{Ad}_{\rho_+})$.

Next, we consider the restriction of ρ_W to $\pi_1(Y_-)$. Suppose that $\pi_1(Y_-)$ has a presentation of the form $\pi_1(Y_-) = \langle a_1, \dots, a_g \mid w_1, \dots, w_r \rangle$. (We do not require $Y_{\pm}$ to be closed, and so there may not exist a balanced presentation.) Then $\pi_1(W)$ admits a presentation of the form

$$\pi_1(W) = \langle a_1, \dots, a_g, b_1, \dots, b_m \mid w_1, \dots, w_r, v_1, \dots, v_m \rangle.$$

As discussed above, $H^*(Y_-; \text{Ad}_{\rho_-})$ is the cohomology of a cochain complex of the form

$$(2.3) \quad 0 \rightarrow \mathfrak{g} \xrightarrow{\psi} \bigoplus_{i=1}^g \mathfrak{g} \xrightarrow{\phi} \bigoplus_{j=1}^r \mathfrak{g} \rightarrow \cdots.$$

Thus, $H^0(Y_-; \text{Ad}_{\rho_-}) = \ker(\psi)$, and $\dim_{\mathbb{R}} H^1(Y_-; \text{Ad}_{\rho_-}) = \dim_{\mathbb{R}} \ker(\phi) - \dim_{\mathbb{R}} \text{im}(\psi)$. We consider a similar setup for $\pi_1(W)$, where $C^1(W; \text{Ad}_{\rho_W})$ (resp. $C^2(W; \text{Ad}_{\rho_W})$) has $g + m$ (resp. $r + m$) copies

of $\mathfrak{g}$, and write ψ' and ϕ' for the associated differentials. It is obvious now that the condition $\dim_{\mathbb{R}} H^0(Y_-; \text{Ad}_{\rho_-}) = \dim_{\mathbb{R}} H^0(W; \text{Ad}_{\rho_W})$ implies that $\dim_{\mathbb{R}} \text{im}(\psi) = \dim_{\mathbb{R}} \text{im}(\psi')$.

We now aim to compare $H^1(Y_-; \text{Ad}_{\rho_-})$ and $H^1(W; \text{Ad}_{\rho_W})$. Note that we have an $\mathbb{R}$ -vector space decomposition $C^i(W; \text{Ad}_{\rho_W}) = C^i(Y_-; \text{Ad}_{\rho_-}) \oplus \mathfrak{g}^m$ for $i \in \{1, 2\}$. Since the relators $w_1, \dots, w_r$ do not interact with the m additional generators in $\pi_1(W)$, we have a block decomposition

$$\phi' = \begin{pmatrix} \phi & 0 \\ \eta & \gamma \end{pmatrix}.$$

Writing $\dim_{\mathbb{R}} \mathfrak{g} = d$, we note that η is a $(dm \times dg)$ -matrix and γ is a $(dm \times dm)$ -matrix. We deduce

$$\begin{aligned} \dim_{\mathbb{R}} H^1(Y_-; \text{Ad}_{\rho_-}) &= \dim_{\mathbb{R}} \ker(\phi) - \dim_{\mathbb{R}} \text{im}(\psi) \\ &= dg - \dim_{\mathbb{R}} \text{im}(\phi) - \dim_{\mathbb{R}} \text{im}(\psi) \\ &= dg + dm - (\dim_{\mathbb{R}} \text{im}(\phi) + dm) - \dim_{\mathbb{R}} \text{im}(\psi') \\ &\leq d(g + m) - \dim_{\mathbb{R}} \text{im}(\phi') - \dim_{\mathbb{R}} \text{im}(\psi') \\ &= \dim_{\mathbb{R}} \ker(\phi') - \dim_{\mathbb{R}} \text{im}(\psi') \\ &= \dim_{\mathbb{R}} H^1(W; \text{Ad}_{\rho_W}), \end{aligned}$$

which completes the proof. $\square$

3. THURSTON GEOMETRIES

In this section, we study the relationship between ribbon $\mathbb{Q}$ -homology cobordisms between compact 3-manifolds and the Thurston geometries that these manifolds admit.

We first prove a homology version of [Theorem 1.14](#).

Lemma 3.1. *Let Y_- and Y_+ be compact 3-manifolds possibly with boundary, and suppose that $W: Y_- \rightarrow Y_+$ is a ribbon $\mathbb{Q}$ -homology cobordism. Then*

- (1) *The inclusion of Y_- into W induces an injection on H_1 ; and*
- (2) *The inclusion of Y_+ into W induces a surjection on H_1 .*

Proof. For (1), view W as constructed by attaching 1- and 2-handles to Y_- ; the fact that W is a $\mathbb{Q}$ -homology cobordism implies that the attaching circles of the 2-handles are linearly independent in $H_1(Y_- \# m(S^1 \times S^2))/H_1(Y_-)$, implying that $H_2(W, Y_-) = 0$. The statement now follows from the long exact sequence associated to the pair (W, Y_-) .

The statement (2) follows from Abelianizing the statement of [Theorem 1.14](#) (2). $\square$

This has the following consequence, explaining [Remark 1.13](#):

Lemma 3.2. *Suppose that Y_- and Y_+ are $\mathbb{Q}$ -homology spheres such that $H_1(Y_-)$ and $H_1(Y_+)$ are isomorphic. Then any ribbon $\mathbb{Q}$ -homology cobordism from Y_- to Y_+ is in fact a ribbon $\mathbb{Z}$ -homology cobordism. In particular, in view of [Lemma 3.1](#), the same conclusion holds in the case that Y_+ is a $\mathbb{Z}$ -homology sphere.*

Proof. First note that for any ribbon $\mathbb{Q}$ -homology cobordism, we have $H_2(W, Y_-) = H_3(W, Y_-) = 0$, by considering the attachment of 1- and 2-handles to Y_- to form W . Analogously, we also have $H_1(W, Y_+) = H_3(W, Y_+) = 0$. Thus, in general, the only possibly nonzero relative homology groups are $H_1(W, Y_-) \cong H_2(W, Y_+)$, which are torsion. (These are isomorphic via $H^2(W, Y_-)$.)

When $H_1(Y_-) \cong H_1(Y_+)$, [Lemma 3.1](#) implies that $H_1(Y_-)$, $H_1(W)$, and $H_1(Y_+)$ all have the same finite cardinality, and so the injection of $H_1(Y_-)$ into $H_1(W)$ must be an isomorphism; thus, $H_1(W, Y_-) = 0$. $\square$

The authors thank Cameron Gordon for pointing out that [Lemma 3.2](#) is false when $b_1(Y_-) > 0$; this case was mistakenly included in a previous version.

We now turn to the key lemma that relates $\pi_1(Y_-)$ and $\pi_1(Y_+)$ under a ribbon $\mathbb{Q}$ -homology cobordism.

Lemma 3.3. *Let $\mathcal{P}$ be one of the following properties of groups:*

- (1) *Finite;*
- (2) *Cyclic;*
- (3) *Abelian;*
- (4) *Nilpotent;*
- (5) *Solvable; or*
- (6) *Virtually $\mathcal{P}'$, where $\mathcal{P}'$ is one of the properties above.*

Let Y_- and Y_+ be compact 3-manifolds. Suppose that $\pi_1(Y_+)$ has property $\mathcal{P}$, while $\pi_1(Y_-)$ does not. Then there does not exist a ribbon $\mathbb{Q}$ -homology cobordism from Y_- to Y_+ .

Proof. Suppose that there exists a ribbon $\mathbb{Q}$ -homology cobordism $W: Y_- \rightarrow Y_+$. By [Theorem 1.14](#), $\pi_1(Y_-)$ is a subgroup of $\pi_1(W)$, which is a quotient of $\pi_1(Y_+)$. For (1) to (5), the lemma is now evident. For (6), a simple algebraic argument shows that if $\mathcal{P}'$ is a property inherited by subgroups (resp. quotients), then so is the property “virtually $\mathcal{P}'$ ”. $\square$

Let Y be a compact 3-manifold with empty or toroidal boundary. These are the only cases that we will be interested in. Then according to [\[AFW15, Theorem 1.11.1\]](#), Y belongs to one of the classes in [Figure 1](#) (if Y is closed) or [Figure 2](#) (if Y has toroidal boundary). Indeed, if Y is spherical or has a finite solvable cover $\tilde{Y}$ that is a torus bundle, then Y is obviously closed. In the latter case, by [\[AFW15, Theorem 1.10.1\]](#), $\tilde{Y}$ admits either a Euclidean, Nil-, or Sol-geometry; by [\[AFW15, Theorem 1.9.3\]](#), Y is itself geometric, and, according to [\[AFW15, Table 1.1\]](#), also admits one of these geometries. That the last rows of [Figure 1](#) and [Figure 2](#) encompass all remaining cases is a consequence of the Geometric Decomposition Theorem; see [\[AFW15, Theorem 1.9.1\]](#). Note that five out of seven $(S^2 \times \mathbb{R})$ -manifolds [\[Sco83, p. 457\]](#) either have S^2 as a boundary component or are not orientable; the other two are $S^1 \times S^2$ and $\mathbb{RP}^3 \# \mathbb{RP}^3$. Also, if Y is geometric and has toroidal boundary, and is not homeomorphic to $K \tilde{\times} I$, $S^1 \times D^2$, or $T^2 \times I$, then it must have $(\mathbb{H}^2 \times \mathbb{R})$ -, $\widetilde{\text{SL}(2, \mathbb{R})}$ -, or hyperbolic geometry.

Theorem 3.4. *Suppose that Y_- and Y_+ are compact 3-manifolds with empty or toroidal boundary that belong to distinct classes in [Figure 1](#) or [Figure 2](#), such that there does not exist a sequence of arrows from the class of Y_- to the class of Y_+ . Then there does not exist a ribbon $\mathbb{Q}$ -homology cobordism from Y_- to Y_+ .*

Proof. We begin by inspecting [Figure 1](#), which consists of two columns corresponding to whether $\pi_1(Y)$ is finite; we call them the *finite column* and the *infinite column* respectively. Focusing on each of these columns separately, successive application of [Lemma 3.3](#) shows that there are no arrows that point up. Of course, one must check that the manifolds in each class indeed have fundamental groups that are characterized by the property on the left. For the finite column, Y is a lens space if and only if $\pi_1(Y)$ is cyclic, and $\pi_1(Y)$ is solvable unless it is the direct sum of a cyclic group and the binary icosahedral group P_{120} (in which case Y is known as a *type-I manifold*); the only spherical 3-manifold with fundamental group P_{120} is the Poincaré homology sphere $\Sigma(2, 3, 5)$. See [\[AFW15, Section 1.7\]](#) for a discussion. For the infinite column, the classification by π_1 follows from [\[AFW15, Table 1.1 and Table 1.2\]](#); the fact there are no arrows between $\mathbb{RP}^3 \# \mathbb{RP}^3$ and $S^1 \times S^2$ reflects the fact that their $\mathbb{Q}$ -homologies have different ranks.

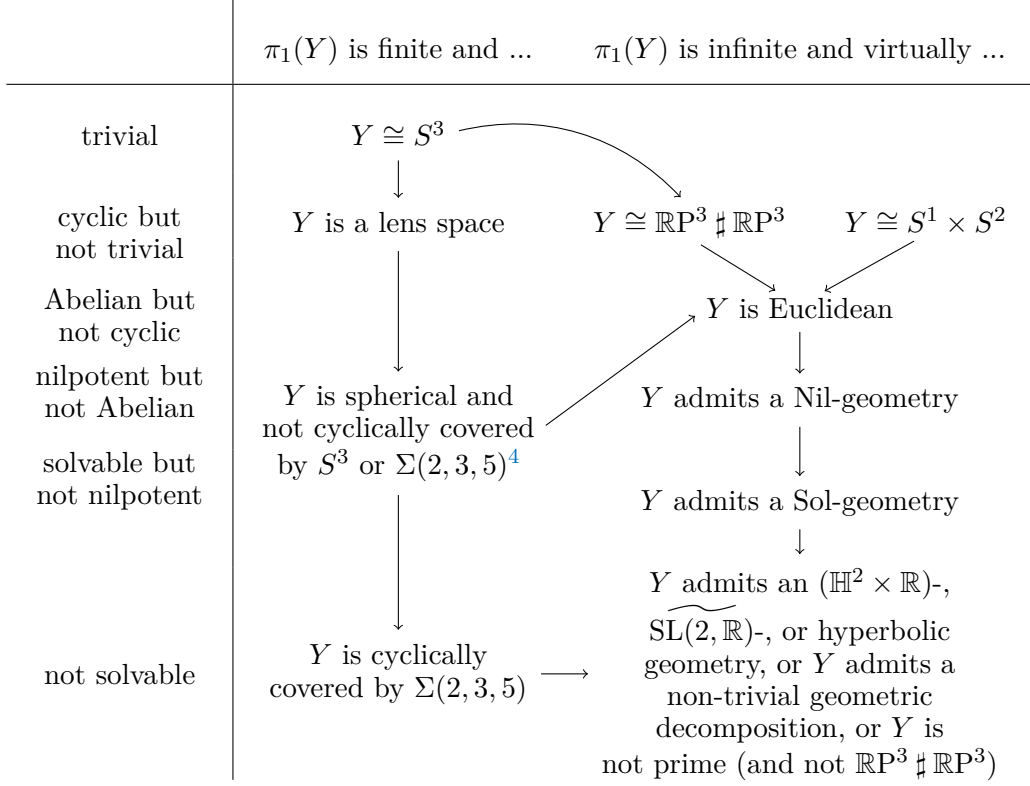


FIGURE 1. Hierarchy of ribbon $\mathbb{Q}$ -homology cobordisms of closed 3-manifolds. For 3-manifolds with infinite π_1 , the adverb “virtually” applies to all adjectives in the leftmost column. (For example, the fundamental group of a Euclidean manifold is virtually Abelian but not virtually cyclic.)

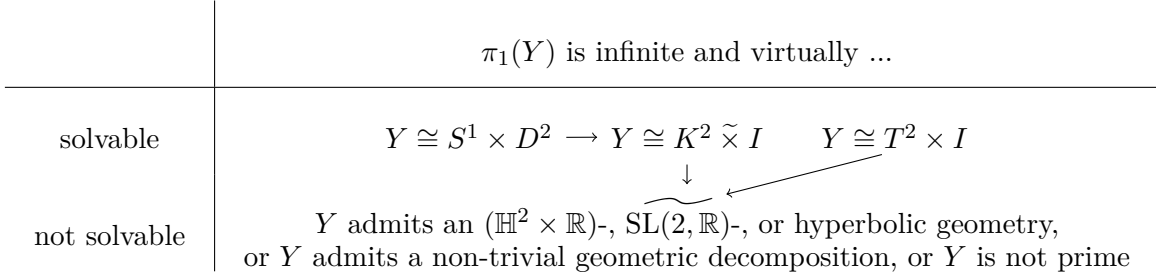


FIGURE 2. Hierarchy of ribbon $\mathbb{Q}$ -homology cobordisms of compact 3-manifolds with toroidal boundary.

We now move on to arrows between the two columns. First, there are clearly no arrows from the infinite to the finite column. Also, the ranks of the $\mathbb{Q}$ -homologies obstruct any arrow from the finite column to $S^1 \times S^2$. The only remaining obstructions are as follows. There are no arrows

- (1) From lens spaces to $\mathbb{RP}^3 \# \mathbb{RP}^3$. Indeed, [Lemma 3.1](#) implies that $H_1(Y_-)$ is a subgroup of a quotient of $H_1(\mathbb{RP}^3 \# \mathbb{RP}^3)$, and thus can only be the trivial group, $\mathbb{Z}/2$, or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. Since Y_- is a lens space, it has non-trivial cyclic H_1 ; thus, $H_1(Y_-) \cong \mathbb{Z}/2$. Suppose there exists a ribbon $\mathbb{Q}$ -homology cobordism from Y_- to $\mathbb{RP}^3 \# \mathbb{RP}^3$; then $(-Y_-) \# \mathbb{RP}^3 \# \mathbb{RP}^3$ bounds a

$\mathbb{Q}$ -homology ball. This implies that $|H_1(-Y_- \# \mathbb{RP}^3 \# \mathbb{RP}^3)| = 8$ is a perfect square, which is a contradiction.

- (2) From spherical manifolds that are not cyclically covered by S^3 to $\mathbb{RP}^3 \# \mathbb{RP}^3$.⁵ Here, Y_- is a non-lens space spherical manifold. Recall that such a manifold has π_1 isomorphic to a central extension of a polyhedral group, which in particular is a non-cyclic group, with elements of order 4, that does not embed into a dihedral group; see [AFW15, Section 1.7] and [Orl72, Section 6.2]. Suppose there exists a ribbon $\mathbb{Q}$ -homology cobordism from Y_- to $\mathbb{RP}^3 \# \mathbb{RP}^3$; then Theorem 1.14 implies that $\pi_1(Y_-)$ is a subgroup of a quotient of $\pi_1(\mathbb{RP}^3 \# \mathbb{RP}^3) \cong \mathbb{Z}/2 * \mathbb{Z}/2$. However, it is an elementary exercise to see that each quotient of $\mathbb{Z}/2 * \mathbb{Z}/2$ is either a cyclic group, a dihedral group, or itself (which does not contain elements of order 4). In any case, $\pi_1(Y_-)$ cannot be a subgroup of a quotient of $\mathbb{Z}/2 * \mathbb{Z}/2$, which is a contradiction.
- (3) From type-I manifolds to any manifold with solvable π_1 ; for manifolds in the infinite column, these are exactly the ones with virtually solvable π_1 (see [AFW15, Theorem 1.11.1]), i.e. all classes except the one in the last row.

For Figure 2, it suffices to observe that, in the first row, the $\mathbb{Q}$ -homology of $T^2 \times I$ differs from those of $S^1 \times D^2$ and $K^2 \tilde{\times} I$, and Lemma 3.1 shows that there is no ribbon $\mathbb{Q}$ -homology cobordism from $K^2 \tilde{\times} I$ to $S^1 \times D^2$. $\square$

Remark 3.5. It is easy to construct a ribbon $\mathbb{Q}$ -homology cobordism from $S^1 \times D^2$ to $K^2 \tilde{\times} I$.

Remark 3.6. Boyer, Gordon, and Watson [BGW13, Theorem 2] show that all $\mathbb{Q}$ -homology spheres with Sol-geometry are L -spaces. By Corollary 8.11, there do not exist ribbon $\mathbb{Z}/2$ -homology cobordisms from any $\mathbb{Q}$ -homology sphere that is not an L -space to a manifold that admits a Sol-geometry. Observe that this is consistent with Figure 1, since $\mathbb{Q}$ -homology spheres with spherical, $(S^2 \times \mathbb{R})$ -, Euclidean, and Nil-geometry are also L -spaces [BGW13, Proposition 5].

4. STATEMENTS OF RESULTS ON FLOER HOMOLOGIES

In the next few sections of this article, we will prove a number of results of the following flavor: If $W: Y_- \rightarrow Y_+$ is a ribbon homology cobordism, then $F(Y_-)$ is a summand of $F(Y_+)$, where F is a version of Floer homology (e.g. sutured instanton Floer homology, involutive Heegaard Floer homology, etc.). In the theorems below, we give the precise statements, which have varying technical hypotheses and conclusions. However, the rough idea is the same throughout and indeed quite simple, which is to show that the double $D(W)$ of W induces an isomorphism on Floer homology. All cobordism maps and isomorphisms can easily be checked to be graded; we leave this task to the reader, although we do use this fact in Theorem 1.5 and Corollary 8.12 below.

We begin with results for instanton Floer homology. We start with Floer's original homology I for $\mathbb{Z}$ -homology spheres [Flo88].

Theorem 4.1. *Let Y_- and Y_+ be $\mathbb{Z}$ -homology spheres, and suppose that $W: Y_- \rightarrow Y_+$ is a ribbon $\mathbb{Q}$ -homology cobordism. Then the cobordism map $I(D(W)): I(Y_-) \rightarrow I(Y_+)$ is the identity map up to a sign, and $I(W)$ includes $I(Y_-)$ into $I(Y_+)$ as a summand.*

⁴For this class, $\pi_1(Y)$ is solvable but not Abelian; it is nilpotent if and only if it is a direct sum of a cyclic group and the generalized quaternion group Q_{2^n} ; all such manifolds Y are prism (i.e. type-D) manifolds. One could accordingly stratify the class into two classes with an arrow between them.

⁵An alternative proof can be given here as follows. First deduce that $H_1(Y_-) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ with an argument involving perfect squares, which implies that the ribbon $\mathbb{Q}$ -homology cobordism is a $\mathbb{Z}$ -homology cobordism. Then observe that by [Doi15, Example 15], the set of d -invariants of Y_- does not match that of $\mathbb{RP}^3 \# \mathbb{RP}^3$, a contradiction.

Next, we have an analogous statement for the framed instanton Floer homology $I^\sharp$ [KM11b].

Theorem 4.2. *Let Y_- and Y_+ be closed 3-manifolds, and suppose that $W: Y_- \rightarrow Y_+$ is a ribbon $\mathbb{Q}$ -homology cobordism. Then the cobordism map $I^\sharp(D(W)): I^\sharp(Y_-) \rightarrow I^\sharp(Y_+)$ satisfies*

$$I^\sharp(D(W)) = |H_1(W, Y_-)| \cdot \mathbb{I}_{I^\sharp(Y_-)}$$

up to a sign, and $I^\sharp(W)$ includes $I^\sharp(Y_-)$ into $I^\sharp(Y_+)$ as a summand.

[Theorem 4.2](#) implies the following corollary, which may also be proved using [Theorem 4.10](#) below for ribbon $\mathbb{Z}/2$ -homology cobordisms.

Corollary 4.3. *Let Y_- and Y_+ be closed 3-manifolds, and suppose that there exists a ribbon $\mathbb{Q}$ -homology cobordism from Y_- to Y_+ . Then the unit Thurston norm ball of Y_- includes that of Y_+ .*

Proof. This follows from the fact that $I^\sharp$ detects the Thurston norm [KM10], together with the fact that a ribbon $\mathbb{Q}$ -homology cobordism induces a concrete identification between $H_2(Y_-; \mathbb{Q})$ and $H_2(Y_+; \mathbb{Q})$. $\square$

The following is an analogue for the sutured instanton Floer homology SHI [KM10]. Here and below, by a cobordism between sutured manifolds, we mean a cobordism obtained by attaching interior handles to a product cobordism; this means that the 3-manifolds have isomorphic sutured boundaries. This definition is narrower than the one used by Juhász [Juh16]. See [Definition 6.10](#) for a precise definition.

Theorem 4.4. *Let (M_-, η_-) and (M_+, η_+) be sutured manifolds, and suppose that $N: (M_-, \eta_-) \rightarrow (M_+, \eta_+)$ is a ribbon $\mathbb{Q}$ -homology cobordism. Then the cobordism map $\text{SHI}(D(N)): \text{SHI}(M_-, \eta_-) \rightarrow \text{SHI}(M_+, \eta_+)$ satisfies*

$$\text{SHI}(D(N)) = |H_1(N, M_-)| \cdot \mathbb{I}_{\text{SHI}(M_-, \eta_-)}$$

up to a sign, and $\text{SHI}(N)$ includes $\text{SHI}(M_-, \eta_-)$ into $\text{SHI}(M_+, \eta_+)$ as a summand.

Recall that for a knot K in a closed 3-manifold Y , the sutured instanton Floer homology of the exterior of K is also denoted by $\text{KHI}(Y, K)$ [KM10]. By the isomorphism between KHI and the reduced singular knot instanton Floer homology $I^\sharp$ [KM11a], [Theorem 4.4](#) implies the following result.

Corollary 4.5. *Let Y_- and Y_+ be closed 3-manifolds, and let K_- and K_+ be knots in Y_- and Y_+ respectively. Suppose that there exists a concordance $C: K_- \rightarrow K_+$ in a cobordism $W: Y_- \rightarrow Y_+$, such that the exterior of C is a ribbon $\mathbb{Q}$ -homology cobordism. Then the cobordism map $I^\sharp(D(W), D(C)): I^\sharp(Y_-, K_-) \rightarrow I^\sharp(Y_+, K_+)$ satisfies*

$$I^\sharp(D(W), D(C)) = |H_1(W, Y_-)| \cdot \mathbb{I}_{I^\sharp(Y_-, K_-)}$$

up to a sign, and $I^\sharp(W, C)$ includes $I^\sharp(Y_-, K_-)$ into $I^\sharp(Y_+, K_+)$ as a summand.

Remark 4.6. Sherry Gong has informed the authors of a direct proof of a version of [Corollary 4.5](#) with coefficients in $\mathbb{Z}$ for concordances in $Y_- \times I$, without appealing to the isomorphism between KHI and $I^\sharp$.⁶ Kang [Kan22] has very recently provided a general proof of [Corollary 4.5](#) for conic strong Khovanov–Floer theories for concordances in $S^3 \times I$, which may be used to recover the version of [Corollary 4.5](#) for ribbon concordances in $S^3 \times I$.

⁶Since the first appearance of this article, a version of Gong’s argument has appeared in the work of Kronheimer and Mrowka [KM21, Theorem 7.4].

Remark 4.7. One can easily see that the double cover of $S^3 \times I$ branched along the concordance is a ribbon $\mathbb{Z}/2$ -homology cobordism, and so [Theorem 4.2](#) applies to show an inclusion of $I^\sharp(\Sigma_2(K_-))$ into $I^\sharp(\Sigma_2(K_+))$, for knots K_- and K_+ in S^3 . A similar statement holds for surgeries along $K_\pm$. We omit these statements for brevity.

We also provide a version for equivariant instanton Floer homologies [\[Don02, Dae20\]](#). Denote by I° any of the equivariant instanton Floer homologies $\check{I}$, $\hat{I}$, and $\bar{I}$.⁷ (We adopt the notation in [\[Dae20\]](#) for these homologies.)

Theorem 4.8. *Let Y_- and Y_+ be $\mathbb{Z}$ -homology spheres, and suppose that $W: Y_- \rightarrow Y_+$ is a ribbon $\mathbb{Q}$ -homology cobordism. Then the cobordism map $I^\circ(W)$ includes $I^\circ(Y_-)$ into $I^\circ(Y_+)$ as a summand.*

Remark 4.9. Equivariant instanton Floer homologies can be extended to $\mathbb{Q}$ -homology spheres (with certain auxiliary data) [\[Mil19, AB96\]](#). We expect (but do not prove) that [Theorem 4.8](#) holds also for these extensions.

We now turn to Heegaard Floer homology [\[OSz04c\]](#). Denote by HF° any of the Heegaard Floer homologies $\widehat{\text{HF}}$, HF^+ , HF^- , and HF^∞ , and by F_W° the corresponding cobordism map.

Theorem 4.10. *Let Y_- and Y_+ be closed 3-manifolds, and suppose that $W: Y_- \rightarrow Y_+$ is a ribbon $\mathbb{Z}/2$ -homology cobordism. Then the cobordism map F_W° includes $\text{HF}^\circ(Y_-)$ into $\text{HF}^\circ(Y_+)$ as a summand. In fact, $\widehat{F}_{D(W)}: \widehat{\text{HF}}(Y_-) \rightarrow \widehat{\text{HF}}(Y_+)$ is the identity map.*

Remark 4.11. We also provide a Spin^c -refinement of [Theorem 4.10](#); see [Theorem 7.9](#) for the precise statement.

As in instanton Floer theory, there is also a version for the sutured Heegaard Floer homology SFH [\[Juh06\]](#). We expect that the stated isomorphism below coincides with the cobordism map defined by Juhász [\[Juh16\]](#), although we do not prove it.

Theorem 4.12. *Let (M_-, η_-) and (M_+, η_+) be sutured manifolds, and suppose that there exists a ribbon $\mathbb{Z}/2$ -homology cobordism from (M_-, η_-) to (M_+, η_+) . Then $\text{SFH}(M_-, \eta_-)$ is isomorphic to a summand of $\text{SFH}(M_+, \eta_+)$.*

As in [Corollary 4.5](#), by the isomorphism [\[Juh06, Proposition 9.2\]](#) between the knot Heegaard Floer homology $\widehat{\text{HFK}}$ [\[OSz04a, Ras03\]](#) of a null-homologous knot and SFH of its exterior, [Theorem 4.12](#) immediately implies the following statement for such concordances. This recovers a version of the results in [\[Zem19c\]](#) and [\[MZ21\]](#) when the concordance is in $S^3 \times I$;⁸ again, we do not prove that the stated isomorphism coincides with the knot cobordism map.

Corollary 4.13 (cf. [\[Zem19c, Theorem 1.1\]](#) and [\[MZ21, Theorem 1.2\]](#)). *Let Y_- and Y_+ be closed 3-manifolds, and let K_- and K_+ be null-homologous knots in Y_- and Y_+ respectively. Suppose that there exists a concordance from K_- to K_+ in a cobordism from Y_- to Y_+ , whose exterior is a ribbon $\mathbb{Z}/2$ -homology cobordism. Then $\widehat{\text{HFK}}(Y_-, K_-)$ is isomorphic to a summand of $\widehat{\text{HFK}}(Y_+, K_+)$. $\square$*

Remark 4.14. [Corollary 4.13](#) has been used to obtain a genus bound on knots related by ribbon concordance [\[Zem19c, Theorem 1.5\]](#) analogous to [Corollary 4.3](#), and on band connected sums of knots [\[Zem19c, Theorem 1.6\]](#); [Corollary 4.5](#) provides an alternative proof of these results using knot instanton Floer homology. It also recovers the well-known theorem that, if a ribbon concordance exists in $S^3 \times I$ from K_- to K_+ , where K_- and K_+ have the same genus, then the fiberedness of K_+ implies that of K_- .

⁷The homologies $\check{I}$, $\hat{I}$, and $\bar{I}$ may be viewed as analogues of HF^+ , HF^- , and HF^∞ respectively.

⁸Note that the exterior of a concordance in $S^3 \times I$ is a $\mathbb{Z}$ -homology cobordism.

As explained in [Remark 4.7](#), one could also use [Theorem 4.10](#) to obtain analogous statements for HF° of certain cyclic covers of S^3 branched along $K_\pm$, and for surgeries along $K_\pm$.

We also give an extension for the involutive Heegaard Floer homology $\widehat{\mathrm{HFI}}$ [[HM17](#)].

Theorem 4.15. *Let Y_- and Y_+ be closed 3-manifolds, and suppose that there exists a ribbon $\mathbb{Z}$ -homology cobordism from Y_- to Y_+ . Then $\widehat{\mathrm{HFI}}(Y_-)$ is isomorphic to a summand of $\widehat{\mathrm{HFI}}(Y_+)$.*

The rough strategy for proving all of the theorems above is fairly straightforward. First, a topological argument ([Proposition 5.1](#) below) shows that $D(W)$ is given by surgery along a collection of m loops in $(Y_- \times I) \sharp m(S^1 \times S^3)$. By using surgery formulas, it can be shown that the induced map for $D(W)$ is the same as that for the 4-manifold obtained by surgering along the m cores of the $S^1 \times S^3$ summands, which is just $Y_- \times I$. Of course, this induces the identity map.

We will also outline an alternative proof of [Theorem 4.1](#) in [Section 6.7](#) that passes more directly through the fundamental group and [Theorem 1.14](#).

Remark 4.16. We expect the analogue of [Theorem 4.10](#) to hold also for the monopole Floer homology groups $\widehat{\mathrm{HM}}$, $\overline{\mathrm{HM}}$, and $\widehat{\mathrm{HM}}$ [[KM07](#)]. Note that by the isomorphisms between Heegaard and monopole Floer homologies [[KLT](#), [CGH11](#), [Tau10](#)], we already know that $\mathrm{HM}^\circ(Y_-)$ is isomorphic to a summand of $\mathrm{HM}^\circ(Y_+)$. In order to prove that the isomorphism coincides with the cobordism map, one could, for example, prove a surgery formula analogous to [Proposition 7.2](#) for monopole Floer homology. Although we expect that this surgery formula holds for monopole Floer homology (especially because an analogous result holds for a variation of Bauer–Furuta invariants [[KLS20](#), Example 1.4]), we do not give a proof of this result for brevity.

We also expect an analogue of [Corollary 1.11](#) to hold for the Mrowka–Ruberman–Saveliev invariant λ_{SW} [[MRS11](#)]. Using the splitting theorem [[LRS18](#)], we have

$$\lambda_{\mathrm{SW}}(\overline{D(W)}) = -\mathrm{Lef}(\mathrm{HM}^{\mathrm{red}}(D(W))) - h(Y_-),$$

where h is the monopole Frøyshov invariant. Since the Casson invariant of Y_- can alternatively be computed as $\chi(\mathrm{HM}^{\mathrm{red}}(Y_-)) + h(Y_-)$, we would obtain that $\lambda_{\mathrm{SW}}(\overline{D(W)}) = -\lambda(Y_-)$. In particular, we have $\lambda_{\mathrm{SW}}(\overline{D(W)}) = -\lambda_{\mathrm{FO}}(\overline{D(W)})$. This would verify [[MRS11](#), Conjecture B] for the 4-manifolds with the $\mathbb{Z}[\mathbb{Z}]$ -homology of $S^1 \times S^3$ that have the form $\overline{D(W)}$.

5. TOPOLOGY OF THE DOUBLE OF A RIBBON COBORDISM

Recall that the *double* $D(W)$ of a cobordism $W: Y_1 \rightarrow Y_2$ is formed by gluing W and $-W$ along Y_2 . In analogy with the arguments used in ribbon concordance, our strategy to prove [Theorem 4.1](#), [Theorem 4.2](#), and [Theorem 4.10](#) will be to prove the cobordism map on Floer homology induced by $D(W)$ is an isomorphism, when W is ribbon. First, we need a topological description of $D(W)$. In what follows, we will use $\mathbb{F}$ to denote any field. Note that a ribbon $\mathbb{F}$ -homology cobordism has the same number of 1- and 2-handles.

Proposition 5.1. *Let Y_- and Y_+ be compact 3-manifolds, and suppose that $W: Y_- \rightarrow Y_+$ is a ribbon cobordism, where the number of 1-handles is m , and that of 2-handles is ℓ . Then $D(W)$ can be described by surgery on $X \cong (Y_- \times I) \sharp m(S^1 \times S^3)$ along ℓ disjoint simple closed curves $\gamma_1, \dots, \gamma_\ell$.*

Suppose that, in addition, W is also an $\mathbb{F}$ -homology cobordism, and denote by $\alpha_i \in H_1(X)$ the homology class of the core of the i^{th} $S^1 \times S^3$ summand. Then, writing

$$[\gamma_j] = \sigma_j + \sum_{i=1}^m c_{ij} \alpha_i, \quad \sigma_j \in H_1(Y_-),$$

we have that the matrix $(c_{ij}) \otimes_{\mathbb{Z}} \mathbb{F}$ is invertible over $\mathbb{F}$, and $|\det(c_{ij})| = |H_1(W, Y_-)|$; in particular,

$$[\gamma_1] \wedge \cdots \wedge [\gamma_\ell] = \det(c_{ij}) \cdot \alpha_1 \wedge \cdots \wedge \alpha_\ell \in (\Lambda^*(H_1(X)/\text{Tors})/\langle H_1(Y_-)/\text{Tors} \rangle) \otimes_{\mathbb{Z}} \mathbb{F},$$

where $\langle H_1(Y_-)/\text{Tors} \rangle$ is the ideal generated by $H_1(Y_-)/\text{Tors}$, is an equality of non-zero elements.

Of course, working with elements in $\Lambda^*(H_1(X)/\text{Tors})/\langle H_1(Y_-)/\text{Tors} \rangle$ is the same as first projecting $H_1(X)$ to the submodule corresponding to the $S^1 \times S^3$ summands and then working in the exterior algebra there. See Figure 3 for a schematic diagram when $m = 1$.

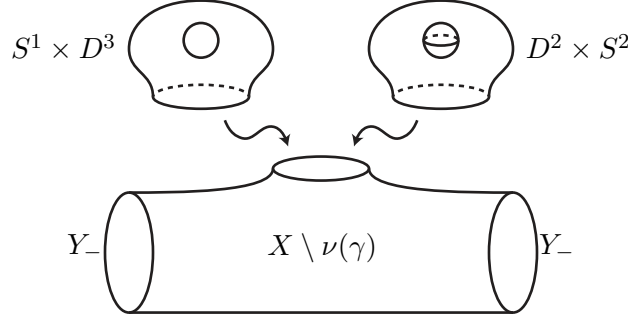


FIGURE 3. An illustration of Proposition 5.1 in the case of $m = \ell = 1$. Here, $\nu(\gamma)$ denotes a neighborhood of γ . Reattaching the $S^1 \times D^3$ would yield $X \cong (Y_- \times I) \# (S^1 \times S^3)$, while we may obtain $D(W)$ by switching it for the $D^2 \times S^2$.

Before we prove Proposition 5.1, we first establish an elementary fact.

Lemma 5.2. *Let M_1 and M_2 be $(n-1)$ -manifolds, and suppose that $N: M_1 \rightarrow M_2$ is a cobordism associated to attaching an n -dimensional k -handle h . Then the double $D(N)$ can be described by surgery on $M_1 \times [-1, 1]$ along some $S^{k-1} \subset M_1 \times \{0\}$ given by the attaching sphere of h .*

Proof. Write $D(N) = (M_1 \times [-1, 0]) \cup h \cup h' \cup (M_1 \times [0, 1])$, where h' is the dual handle of h . The cocore of h and the core of h' together form an S^{n-k} with trivial normal bundle, which may be identified with $h \cup h'$. (The case where $n = 4$ and $k = 2$ is described, for example, in [GS99, Example 4.6.3].) Note that h meets the lower $M_1 \times [-1, 0]$, and h' meets the upper $M_1 \times [0, 1]$, at the same attaching region $S^{k-1} \times D^{n-k} \subset M_1 \times \{0\}$, with the same framing. Thus, removing $h \cup h'$ from $D(N)$ would result in $(M_1 \times [-1, 1]) \setminus (S^{k-1} \times D^{n-k} \times (-\epsilon, \epsilon))$. In other words, $D(N)$ may be formed by removing $S^{k-1} \times D^{n-k} \times (-\epsilon, \epsilon) \cong S^{k-1} \times D^{n-k+1}$ from $M_1 \times [-1, 1]$ and replacing it with $h \cup h' \cong D^k \times S^{n-k}$, which is the definition of surgery. $\square$

In the case where $n = 4$ and $k = 2$, the handles h and h' above can be described by a Kirby diagram consisting of a loop γ with some (possibly non-zero) framing and the linking circle of γ with zero framing; the fact that this corresponds to surgery is well known to experts; see, for example, [Akb99, p. 500].

Proof of Proposition 5.1. First, decompose W into a cobordism W_1 from Y_- to $\tilde{Y} \cong Y_- \# m(S^1 \times S^2)$ and a cobordism W_2 from $\tilde{Y}$ to Y_+ , corresponding to the attachment of 1- and 2-handles respectively. Below, we will compare $D(W) = W_1 \cup W_2 \cup (-W_2) \cup (-W_1)$ with $W_1 \cup (-W_1)$.

Applying Lemma 5.2 to each of the 2-handles in W_2 , we see that $W_2 \cup (-W_2)$ can be described by surgery on $\tilde{Y} \times [-1, 1]$ along some $\gamma_1, \dots, \gamma_\ell$, where the γ_i 's are given by the attaching circles of the 2-handles. (Perform isotopies and handleslides first, if necessary, to ensure that the attaching regions of the 2-handles lie in $\tilde{Y}$ and are disjoint.)

Note that $W_1 \cup (-W_1) \cong W_1 \cup (\tilde{Y} \times [-1, 1]) \cup (-W_1)$ is diffeomorphic to $X \cong (Y_- \times I) \sharp m(S^1 \times S^3)$. Thus, we see that $D(W) = W_1 \cup W_2 \cup (-W_2) \cup (-W_1)$ can be described by surgery on X along $\gamma_1, \dots, \gamma_\ell$.

Finally, suppose W is a ribbon $\mathbb{F}$ -homology cobordism; then $m = \ell$. Present the differential $\partial_2: C_2(Y_-) \rightarrow C_1(Y_-)$ by a matrix A ; then in the corresponding cellular chain complex of W , the presentation matrix Q of the differential $\partial_2: C_2(W) \rightarrow C_1(W)$ is of the form

$$Q = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where C is an $(m \times m)$ -matrix representing the attachment of the 2-handles in W_2 . As the γ_i 's are given by the attaching circles of these 2-handles, we see that C_{ij} is given by the algebraic intersection number of γ_j with $\{p\} \times S^3$ in the i^{th} $S^1 \times S^3$ summand, and so $C_{ij} = c_{ij}$. Since W is an $\mathbb{F}$ -homology cobordism, we have $H_1(W, Y_-; \mathbb{F}) = 0$, implying that $C \otimes_{\mathbb{Z}} \mathbb{F}: \mathbb{F}^m \rightarrow \mathbb{F}^m$ is surjective, and hence invertible. It is now clear that $|\det(C_{ij})| = |H_1(W, Y_-)|$, and the equality in $(\Lambda^*(H_1(X)/\text{Tors})/\langle H_1(Y_-)/\text{Tors} \rangle) \otimes_{\mathbb{Z}} \mathbb{F}$ is obvious. $\square$

While it will not be used later in the paper, we conclude this section with the following geometric result, which may be of independent interest.

Proposition 5.3. *Suppose that W is a compact 4-manifold with connected boundary and a ribbon handle decomposition. Then W admits a metric with positive scalar curvature.*

Proof. By Proposition 5.1, $D(W)$ is obtained by surgery on a collection of ℓ loops in $\sharp m(S^1 \times S^3)$. First, it is well known that $S^1 \times S^3$ has a p.s.c. metric. By the work of Gromov and Lawson [GL80, Theorem A], $\sharp m(S^1 \times S^3)$ admits a p.s.c. metric. Next, surgery on loops is a codimension-3 surgery, and so we may again apply the result of Gromov and Lawson to see that $D(W)$ admits a p.s.c. metric. Since W is a codimension-0 submanifold of $D(W)$, it inherits a p.s.c. metric as well. $\square$

6. INSTANTON FLOER HOMOLOGY

6.1. The Chern–Simons functional. Let G be a compact, connected, simply connected, simple Lie group, and let P be a principal G -bundle on Y . Any such bundle can be trivialized, and we fix one such trivialization. Denote by $\text{ad}(P)$ the adjoint bundle associated to P ; this vector bundle is induced by the adjoint action of G on its Lie algebra $\mathfrak{g}$. The space of connections $\mathcal{A}(P)$ on P is an affine space modeled on $\Omega^1(Y; \text{ad}(P))$, with a distinguished element Θ , which is the trivial connection (associated to the trivialization we chose). Given a connection $B \in \mathcal{A}(P)$, let A be the connection on the bundle $P \times \mathbb{R}$ over $Y \times \mathbb{R}$ that is equal to the pull-back of B on $P \times (-\infty, -1]$ and the pull-back of Θ on $P \times [1, \infty)$. The *Chern–Simons functional* of B is defined by the Chern–Weil integral

$$(6.1) \quad \widetilde{\text{CS}}(B) = -\frac{1}{32\pi^2 \check{h}} \int_{Y \times \mathbb{R}} \text{tr}(\text{ad}(F(A)) \wedge \text{ad}(F(A))),$$

where $F(A)$ is the $\text{ad}(P)$ -valued curvature 2-form, and $\text{ad}(F(A))$ is the corresponding induced element of $\text{End}(\text{ad}(P))$. The constant $\check{h}$ is the *dual Coxeter number*, which depends on G ; it is equal to N when $G = \text{SU}(N)$.

Let $\mathcal{G}_G$ be the space of smooth maps from Y to G . This space can be identified with the group of automorphisms of P , known as the *gauge group*; in particular, any $g \in \mathcal{G}_G$ acts on $\mathcal{A}(P)$ by mapping a connection A to its pull-back $g^*(A)$. The integral in (6.1) is not necessarily invariant with respect to this $\mathcal{G}_G$ -action; however, it always changes by multiples of a fixed constant, and the normalization in (6.1) is chosen such that the change in $\widetilde{\text{CS}}$ is always an integer. In particular, if we denote by

$\mathcal{B}(P)$ the quotient of $\mathcal{A}(P)$ by this $\mathcal{G}_G$ -action, then (6.1) induces a map $\text{CS} : \mathcal{B}(P) \rightarrow \mathbb{R}/\mathbb{Z}$. An important feature of CS is that it is a topological function, in that its definition does not require a metric on Y .

It is not hard to see from the definition that a connection B is a critical point of $\widetilde{\text{CS}}$ if and only if B has vanishing curvature, i.e. if B is flat. Given a flat connection, one may take its holonomy along closed loops in Y to obtain a homomorphism $\rho : \pi_1(Y) \rightarrow G$, i.e. an element of the representation variety $\mathcal{R}_G(Y)$. This is not necessarily a one-to-one correspondence, but if we quotient the space of flat connections by the gauge group action and quotient $\mathcal{R}_G(Y)$ by conjugation, we do get an identification of the isomorphism classes of flat connections with the character variety $\mathcal{X}_G(Y)$. In other words, $\mathcal{X}_G(Y)$ is the set of critical points of CS. Further, the set of critical values of the Chern–Simons functional CS is a finite set, which is a topological invariant of Y .

In the definition of the Chern–Simons functional, the assumptions on the Lie group G are not essential. Indeed, we may take G to be a compact, connected, simple Lie group that is possibly not simply connected, with universal cover $\widetilde{G}$. An important example to keep in mind is when $G = \text{SO}(3)$ and $\widetilde{G} = \text{SU}(2)$. Instead of a trivial principal bundle, we consider a possibly non-trivial principal G -bundle on Y . We may still form the space of connections $\mathcal{A}(P)$ as before, and we may form the configuration space $\mathcal{B}(P)$ by quotienting $\mathcal{A}(P)$ by the $\mathcal{G}_{\widetilde{G}}$ -action (rather than the $\mathcal{G}_G$ -action). There is no longer a distinguished element $\Theta \in \mathcal{A}(P)$. Instead, we arbitrarily choose a connection $B_0 \in \mathcal{A}(P)$, which plays the role of Θ in the definitions of $\widetilde{\text{CS}}$; this determines an $\mathbb{R}/\mathbb{Z}$ -valued functional CS on $\mathcal{B}(P)$ that is well defined up to addition by a constant (representing the indeterminacy of the choice of B_0). The critical points of CS are isomorphism classes of flat connections on P . Moreover, the set of (relative) values of the Chern–Simons functional at this set of critical points is a topological invariant of the pair (Y, P) .

Proof of Corollary 1.16. Let $W : Y_- \rightarrow Y_+$ be a ribbon $\mathbb{Q}$ -homology cobordism. Let α_- be a flat connection on Y_- , whose holonomy gives an element $\rho_- \in \mathcal{R}_G(Y_-)$. By Proposition 1.15, we may extend ρ_- to an element $\rho_W \in \mathcal{R}_G(W)$, which pulls back to an element $\rho_+ \in \mathcal{R}_G(Y_+)$. We may then choose a corresponding flat connection α_+ on Y_+ . By Auckly [Auc94], the Chern–Simons invariants of α_- and α_+ agree. $\square$

6.2. An overview of instanton Floer theory. In this section, we review the two main versions of instanton Floer homology and develop some properties of the associated cobordism maps. (Other versions will be discussed later in this section.) Throughout, we work only with coefficients in $\mathbb{Q}$. We begin with Floer’s original version of instanton Floer homology [Flo88], which associates to any $\mathbb{Z}$ -homology sphere Y a $\mathbb{Z}/8$ -graded vector space $I(Y)$. To a $\mathbb{Q}$ -homology cobordism $W : Y_1 \rightarrow Y_2$ of $\mathbb{Z}$ -homology spheres, the theory associates a homomorphism $I(W) : I(Y_1) \rightarrow I(Y_2)$ of vector spaces [Don02].⁹

The vector space $I(Y)$ is the homology of a chain complex $(C(Y), d)$. The chain complex $C(Y)$ is defined roughly as the Morse homology of the Chern–Simons functional CS with the Lie group $G = \text{SU}(2)$ and the trivial bundle on Y . Recall from Section 6.1 that the critical set of CS is exactly the space of isomorphism classes of flat connections; in this setup, all non-trivial flat connections are irreducible. Here, a connection is *irreducible* if its isotropy group is $\{\pm 1\}$; when the connection is flat, this is equivalent to the condition that the associated representation is irreducible.

In order to achieve Morse–Smale transversality, one perturbs the Chern–Simons functional. The critical set of the perturbed Chern–Simons functional still contains the trivial connection; the other critical points are no longer necessarily flat, but the perturbation can be chosen to be small,

⁹The homomorphism $I(W)$ is also defined for more general cobordisms W ; see [Don02] for details. We focus on $\mathbb{Q}$ -homology cobordisms here for ease of exposition, as this specialization suffices for our purposes.

which guarantees that the non-trivial critical points are still (isomorphism classes of) irreducible connections.¹⁰ We denote the set of all non-trivial critical points by $\mathfrak{S}(Y)$.¹¹ Then $C(Y)$ is the $\mathbb{Q}$ -vector space generated by the elements of $\mathfrak{S}(Y)$, equipped with the differential d , where the coefficients $\langle d(\alpha), \beta \rangle$ are given by the signed count of index-1 gradient flow lines of the perturbation of CS that are asymptotic to α and β . A useful observation, which is also essential in the development of the analytical aspects of the theory, is that the gradient flow lines of (a perturbation of) CS may be viewed as the solutions of (a corresponding perturbation of) the ASD (anti-self-dual) equation for the trivial $SU(2)$ -bundle on $Y \times \mathbb{R}$.

The cobordism map $I(W): I(Y_1) \rightarrow I(Y_2)$ is also defined with the aid of the ASD equation. We first attach cylindrical ends to W and fix a Riemannian metric on this new manifold, which we also denote by W by abuse of notation. For any pair $(\alpha_1, \alpha_2) \in \mathfrak{S}(Y_1) \times \mathfrak{S}(Y_2)$, we may form a moduli space $\mathcal{M}(W; \alpha_1, \alpha_2)$ of connections that satisfy a perturbed ASD equation for the trivial $SU(2)$ -bundle on W and that are asymptotic to α_1 and α_2 on the ends. Here, the perturbation of the ASD equation is chosen such that it is compatible with the perturbations of the Chern–Simons functionals of Y_1 and Y_2 , and guarantees that each connected component of $\mathcal{M}(W; \alpha_1, \alpha_2)$ is a smooth manifold, of possibly different dimensions. We write $\mathcal{M}(W; \alpha_1, \alpha_2)_d$ for the union of the d -dimensional connected components of $\mathcal{M}(W; \alpha_1, \alpha_2)$. The value of $d \bmod 8$ is determined by α_1 and α_2 . We then define a chain map $C(W): C(Y_1) \rightarrow C(Y_2)$ by

$$(6.2) \quad C(W)(\alpha_1) = \sum_{\alpha_2 \in \mathfrak{S}(Y_2)} \# \mathcal{M}(W; \alpha_1, \alpha_2)_0 \cdot \alpha_2 \in C(Y_2).$$

Here, $\# \mathcal{M}(W; \alpha_1, \alpha_2)_0$ is the signed count of the elements of $\mathcal{M}(W; \alpha_1, \alpha_2)_0$. The homomorphism $I(W): I(Y_1) \rightarrow I(Y_2)$ is the map induced by $C(W)$ at the level of homology. It turns out that this map depends only on W and is independent of the choice of Riemannian metric on W and perturbation of the ASD equation.

A variation of instanton Floer homology is obtained by replacing the trivial $SU(2)$ -bundles with non-trivial $SO(3)$ -bundles. Fix a closed 3-manifold Y . The isomorphism class of an $SO(3)$ -bundle P on Y is determined by its second Stiefel–Whitney class $w = w_2(P) \in H^2(Y; \mathbb{Z}/2)$. As described in Section 6.1, we may define a Chern–Simons functional CS_w on the configuration space $\mathcal{B}(P)$ of connections on P up to gauge group action. We say that (Y, w) is an *admissible pair* if the pairing of w with $H_2(Y)$ is not trivial. This condition guarantees that the set of critical points of CS_w , or equivalently, the set of flat connections on P , consists only of irreducible elements of $\mathcal{B}(P)$. This assumption considerably simplifies the analytical aspects of gauge theory and allows us to define an instanton Floer homology $I(Y, w)$ for an admissible pair, analogous to instanton Floer homology of a $\mathbb{Z}$ -homology sphere. As in the previous case, we apply a small perturbation to CS_w to obtain a Morse–Smale functional with the critical set $\mathfrak{S}(Y, w)$. Again, the critical points of the perturbed functional are no longer necessarily flat, but they remain irreducible. We define $C(Y, w)$ to be the $\mathbb{Q}$ -vector space generated by $\mathfrak{S}(Y, w)$, equipped with a differential d defined using gradient flow lines of the perturbed Chern–Simons functional.

Instanton Floer homology of admissible pairs is also functorial with respect to cobordisms. Let (Y_1, w_1) and (Y_2, w_2) be admissible pairs, let $W: Y_1 \rightarrow Y_2$ be an arbitrary cobordism (i.e. not necessarily a $\mathbb{Q}$ -homology cobordism), and let $c \in H^2(W; \mathbb{Z}/2)$ be a cohomology class whose

¹⁰For simplicity, it is customary to blur the line between connections and isomorphism classes of connections (i.e. connections up to the gauge group action). From now on, we will often follow this custom; for example, by an irreducible element of $\mathfrak{S}(Y)$, we will mean an isomorphism class of irreducible connections.

¹¹Although it is not reflected in the notation, the set $\mathfrak{S}(Y)$ depends on the choice of perturbation of the Chern–Simons functional.

restriction to Y_i is equal to w_i . The cohomology class c determines an $\mathrm{SO}(3)$ -bundle on W , and solutions to a perturbed ASD equation for connections on this bundle that are asymptotic to $\alpha_1 \in \mathfrak{S}(Y_1, w_1)$ and $\alpha_2 \in \mathfrak{S}(Y_2, w_2)$ give rise to the moduli space $\mathcal{M}(W, c; \alpha_1, \alpha_2)$. As in the previous case, the perturbation of the ASD equation is chosen such that it is compatible with the perturbations of the Chern–Simons functionals of (Y_1, w_1) and (Y_2, w_2) and that each component of $\mathcal{M}(W, c; \alpha_1, \alpha_2)$ is a smooth manifold. As in (6.2), these moduli spaces can be used to define a homomorphism $I(W, c): I(Y_1, w_1) \rightarrow I(Y_2, w_2)$. In general, this map is defined only up to a sign; this sign can be determined if we fix a homology orientation on W , which is an orientation of $\Lambda^{\mathrm{top}} H^1(W; \mathbb{Q}) \otimes \Lambda^{\mathrm{top}} H^+(W; \mathbb{Q}) \otimes \Lambda^{\mathrm{top}} H^1(Y_2; \mathbb{Q})$. Here $H^+(W; \mathbb{Q})$ is the subspace of $H^2(W; \mathbb{Q})$ represented by L^2 self-dual harmonic 2-forms on W . (See, for example, [KM11b] for more details on how to use homology orientations to remove the sign ambiguity of $I(W, c)$.) In particular, for a $\mathbb{Q}$ -homology cobordism W , there is a canonical choice of homology orientation.

There are more general cobordism maps defined for instanton Floer homology of admissible pairs. Let $\mathbb{A}(W)$ be the $\mathbb{Z}$ -graded algebra $\mathrm{Sym}^*(H_2(W; \mathbb{Q}) \oplus H_0(W; \mathbb{Q})) \otimes \Lambda^*(H_1(W; \mathbb{Q}))$, where the elements in $H_i(W; \mathbb{Q})$ have degree $4 - i$. For any $z \in \mathbb{A}(W)$ with degree i , a standard construction gives rise to a cohomology class $\mu(z)$ of degree i in $\mathcal{M}(W, c; \alpha_1, \alpha_2)_d$, represented by a linear combination of submanifolds $V(W, c; \alpha_1, \alpha_2; z)_{d-i}$ of codimension i ; see, for example, [DK90, Chapter 5]. Then the homomorphism $C(W, c; z): C(Y_1, w_1) \rightarrow C(Y_2, w_2)$ defined by

$$(6.3) \quad C(W, c; z)(\alpha_1) = \sum_{\alpha_2 \in \mathfrak{S}(Y_2, w_2)} \#V(W, c; \alpha_1, \alpha_2; z)_0 \cdot \alpha_2 \in C(Y_2, w_2)$$

is a chain map, and the induced homomorphism $I(W, c; z)$ at the level of homology is independent of the choice of metric, perturbation, and the representative submanifold $V(W, c; \alpha_1, \alpha_2; z)_0$. The homomorphism $I(W, c; z)$ depends linearly on z , and is again defined up to a sign that can be fixed using a homology orientation on W . It is also functorial: Let (Y_1, w_1) , (Y_2, w_2) , and (Y_3, w_3) be admissible pairs, $W: Y_1 \rightarrow Y_2$ and $W': Y_2 \rightarrow Y_3$ be cobordisms equipped with homology orientations, and $c_\circ$ be an element of $H^2(W' \circ W; \mathbb{Z}/2)$ whose restrictions to W and W' are denoted by c and c' respectively, and fix $z \in \mathbb{A}(W)$ and $z' \in \mathbb{A}(W')$; then $I(W' \circ W, c_\circ; z \cdot z')$, defined using the composed homology orientation, is equal to $I(W', c'; z') \circ I(W, c; z)$.

It is natural to ask whether for a cobordism W between $\mathbb{Z}$ -homology spheres, the definition of the cobordism map $I(W)$ can also be extended to a homomorphism $I(W; z)$ for $z \in \mathbb{A}(W)$. In this context, it would also be useful to define $I(W; z)$ when W is not a $\mathbb{Q}$ -homology cobordism, e.g. when $b_1(W) > 0$; to do so, we would also need to make use of homology orientations to remove the sign ambiguity. In general, the main obstruction to defining this extension is the existence of reducible ASD connections on W : One can still define a subspace $V(W; \alpha_1, \alpha_2; z)_0$ of $\mathcal{M}(W; \alpha_1, \alpha_2)_d$ in the case that $\deg(z) = d$, but $V(W; \alpha_1, \alpha_2; z)_0$ might not be compact because of the existence of reducible connections. Thus one cannot proceed easily, as in (6.3), to define $I(W; z)$. In the case that $b^+(W) > 1$, the cobordism map $I(W; z): I(Y_1) \rightarrow I(Y_2)$ is defined for any $z \in \mathbb{A}(W)$; see [Don02, Chapter 6]. For our purposes, we need to consider the case where $b^+(W) = 0$ and the degree of z is sufficiently small. The following compactness result provides the essential analytical input to define $I(W; z)$ in this context.

Lemma 6.4. *Let Y_1 and Y_2 be $\mathbb{Z}$ -homology spheres, and let $\alpha_1 \in \mathfrak{S}(Y_1)$ and $\alpha_2 \in \mathfrak{S}(Y_2)$. Suppose that $W: Y_1 \rightarrow Y_2$ is a cobordism with $b_1(W) = m$ and $b^+(W) = 0$, and that $\{A_i\}_{i=1}^\infty$ is a sequence of connections on W each representing an element of $\mathcal{M}(W; \alpha_1, \alpha_2)_d$, where $d \leq 3m + 4$. Then there are $\alpha'_1 \in \mathfrak{S}(Y_1) \cup \{\emptyset\}$, $\alpha'_2 \in \mathfrak{S}(Y_2) \cup \{\emptyset\}$, a finite set of points $\{p_1, \dots, p_\ell\} \subset W$, and an irreducible connection A_0 on W representing an element of $\mathcal{M}(W; \alpha'_1, \alpha'_2)_{d'}$, such that*

- (1) $0 \leq d' \leq d - 8\ell$; and

- (2) after possibly passing to a subsequence and changing each connection A_i by an action of the gauge group, the sequence of connections $\{A_i\}$ converges in C^∞ -norm to A_0 on any compact subspace of the complement of $\{p_1, \dots, p_\ell\}$.

Proof. This is a consequence of the standard compactness theorem for the solutions of the ASD equation on manifolds with cylindrical ends (see, for example, [Don02, Chapter 5]), together with the following observation. If the chosen perturbations of the Chern–Simons functionals of Y_1 and Y_2 and of the ASD equation on W are small enough, then any reducible ASD connection on W is a (singular) element of a moduli space of the form $\mathcal{M}(W; \Theta, \Theta)_e$, where Θ is the trivial connection, and $e \geq 3m - 3$. A straightforward index computation shows that such reducible connections do not appear as limits of a sequence in $\mathcal{M}(W; \alpha_1, \alpha_2)_d$ when $d \leq 3m + 4$. $\square$

Suppose that W is a cobordism as in the statement of Lemma 6.4. We equip W with a homology orientation by fixing an orientation for the vector space $H^1(W; \mathbb{Q})$. Suppose also that $z \in \mathbb{A}(W)$ has degree at most $3m + 3$. Lemma 6.4 together with a standard counting argument shows that the moduli space $V(W; \alpha_1, \alpha_2; z)_0$ is compact. Thus we may use a formula similar to (6.3) to define the cobordism map $I(W; z): I(Y_1) \rightarrow I(Y_2)$. A standard argument shows that this map is independent of the choice of metric, perturbation, and representative submanifold for the cohomology class associated to z .

6.3. Surgery and cobordism maps in instanton Floer theory. We first start with two basic propositions, in which we will relate certain cobordism maps associated to two cobordisms X and Z , where Z is the result of surgery on X along a loop γ . First, we have a surgery formula for instanton Floer homology of admissible pairs.

Proposition 6.5. *Let (Y_1, w_1) and (Y_2, w_2) be admissible pairs, and let $X: Y_1 \rightarrow Y_2$ be a cobordism. Suppose that $\gamma \subset \text{Int}(X)$ is a loop with neighborhood $\nu(\gamma) \cong \gamma \times D^3$, and denote by Z the result of surgery on X along γ . Fix a properly embedded surface $S \subset \text{Int}(X)$ supported away from $\nu(\gamma)$, such that the cohomology class $c_X \in H^2(X; \mathbb{Z}/2)$ dual to $[S]$ restricts to w_1 and w_2 on Y_1 and Y_2 respectively, and denote by c_Z the class in $H^2(Z; \mathbb{Z}/2)$ determined by $[S]$. Suppose that $z_X \in \mathbb{A}(X)$ admits representatives for its homology classes that are supported away from $\nu(\gamma)$, and denote by z_Z the class in $\mathbb{A}(Z)$ determined by these representatives. Then up to a sign,*

$$I(X, c_X; [\gamma] \cdot z_X) = I(Z, c_Z; z_Z).$$

Proof. This is essentially [Don02, Theorem 7.16], and the same proof works in this set up. $\square$

Similarly, we have a surgery formula for instanton Floer homology of $\mathbb{Z}$ -homology spheres.

Proposition 6.6. *Let Y_1 and Y_2 be $\mathbb{Z}$ -homology spheres, and suppose that $X: Y_1 \rightarrow Y_2$ is a cobordism with $b_1(X) = m$ and $b^+(X) = 0$. Suppose that $\gamma \in \text{Int}(X)$ is a loop with neighborhood $\nu(\gamma) \cong \gamma \times D^3$, and denote by Z the result of surgery on X along γ . Suppose that $z_X \in \mathbb{A}(X)$ has degree at most $3m - 3$ and admits representatives for its homology classes that are supported away from $\nu(\gamma)$, and denote by z_Z the class in $\mathbb{A}(Z)$ determined by these representatives. Then up to a sign,*

$$I(X; [\gamma] \cdot z_X) = I(Z; z_Z).$$

Proof. This is again essentially [Don02, Theorem 7.16]. $\square$

Remark 6.7. While we do not provide a proof, we expect that it is possible to remove the sign ambiguities in Proposition 6.5 and Proposition 6.6, which would then remove the sign ambiguities in Theorem 4.1, Theorem 4.2, Theorem 4.4, and Corollary 4.5. In the case that $[\gamma] = 0$, both sides of the equation vanish. In the case that $[\gamma] \neq 0$, we would have to choose homology orientations.

Note that, in this case, $H^+(X; \mathbb{Q}) \cong H^+(Z; \mathbb{Q})$ and $H_1(X; \mathbb{Q}) \cong \langle [\gamma] \rangle \oplus H_1(Z; \mathbb{Q})$. Fix a homology orientation $\mathfrak{o}_Z$ on Z ; we may set $\mathfrak{o}_X = \omega \wedge \mathfrak{o}$, where $\omega \in H^1(X; \mathbb{Q})$ is determined by $[\gamma]$. With this choice, we expect the equations in [Proposition 6.5](#) and [Proposition 6.6](#) to hold without a sign adjustment.

We now use the propositions above to study ribbon homology cobordisms. First, we verify an analogue of [Theorem 4.1](#) for admissible pairs, which we will use in the following subsections.

Theorem 6.8. *Let (Y_-, w_-) and (Y_+, w_+) be admissible pairs, and suppose that $W: Y_- \rightarrow Y_+$ is a ribbon $\mathbb{Q}$ -homology cobordism. Fix a properly embedded surface $S \subset \text{Int}(W)$ supported away from the cocores in a ribbon handle decomposition of W , such that the cohomology class $c_W \in H^2(W; \mathbb{Z}/2)$ dual to $[S]$ restricts to w_- and w_+ on Y_- and Y_+ respectively, and denote by $c_{D(W)} \in H^2(D(W); \mathbb{Z}/2)$ and $c_{Y_- \times I} \in H^2(Y_- \times I; \mathbb{Z}/2)$ the cohomology classes determined by $D(S)$. Then up to a sign, the cobordism map $I(D(W), c_{D(W)}): I(Y_-, w_-) \rightarrow I(Y_+, w_+)$ satisfies*

$$I(D(W), c_{D(W)}) = |H_1(W, Y_-)| \cdot I(Y_- \times I, c_{Y_- \times I}).$$

In particular, if $c_{Y_- \times I}$ is the pull-back of w_- , then up to a sign,

$$I(D(W), c_{D(W)}) = |H_1(W, Y_-)| \cdot \mathbb{I}_{I(Y_-, w_-)},$$

and $I(W, c_W)$ includes $I(Y_-, w_-)$ into $I(Y_+, w_+)$ as a summand.

Proof. By [Proposition 5.1](#), $D(W)$ is described by surgery on $X \cong (Y_- \times I) \# m(S^1 \times S^3)$ along m disjoint circles $\gamma_1, \dots, \gamma_m$, with

$$[\gamma_1] \wedge \dots \wedge [\gamma_m] = \det(c_{ij}) \cdot \alpha_1 \wedge \dots \wedge \alpha_m \in (\Lambda^*(H_1(X)/\text{Tors})/\langle H_1(Y_-)/\text{Tors} \rangle) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where $\alpha_i \in H_1(X)$ is the homology class of the core of the i^{th} $S^1 \times S^3$ summand, c_{ij} is the multiplicity of α_i in $[\gamma_j]$, and $|\det(c_{ij})| = |H_1(W, Y_-)|$. Applying [Proposition 6.5](#) with $Z = D(W)$, we have that, up to a sign,

$$I(X, c_X; [\gamma_1] \wedge \dots \wedge [\gamma_m]) = I(D(W), c_{D(W)}).$$

We claim that

$$(6.9) \quad I(X, c_X; [\gamma_1] \wedge \dots \wedge [\gamma_m]) = \det(c_{ij}) \cdot I(X, c_X; \alpha_1 \wedge \dots \wedge \alpha_m),$$

where we are using $\mathfrak{o}_{X, \gamma}$ on both sides of the equation; indeed, by the linearity of I , it suffices to show that $I(X, c_X; \xi) = 0$ for $\xi \in \Lambda^m(H_1(X; \mathbb{Q})) \cap \langle H_1(Y_-; \mathbb{Q}) \rangle$. To see this, we may apply [Proposition 6.5](#) in the opposite direction to see that $I(X, c_X; \xi) = I(Z', c_{Z'})$ for some cobordism Z' with at least one $S^1 \times S^3$ connected summand; the general vanishing theorem for connected sums implies that this map is zero. (The interested reader may compare this argument with the penultimate paragraph of the proof of [Theorem 4.10](#) in [Section 7.1](#).) Note that $\det(c_{ij}) = |H_1(W, Y_-)|$ up to a sign.

Applying [Proposition 6.5](#) again with $Z = Y_- \times I$, we see that up to a sign,

$$I(X, c_X; \alpha_1 \wedge \dots \wedge \alpha_m) = I(Y_- \times I, c_{Y_- \times I}).$$

This completes our proof. □

Similarly, we prove [Theorem 4.1](#).

Proof of Theorem 4.1. The proof is completely analogous to that of [Theorem 6.8](#), without the need to keep track of the cohomology classes or consider elements of $H_1(Y_-)$. □

Proof of Corollary 1.11. A standard gluing argument shows that the signed count of elements in the moduli space of index-0 (perturbed) ASD connections on $\overline{D(W)}$ is equal to $2 \text{Lef}(I(D(W): I(Y_-) \rightarrow I(Y_-))$. (See [Don02, Theorem 6.7] for a similar gluing result.) By definition, the former count is equal to $4\lambda_{\text{FO}}(\overline{D(W)})$, and by Theorem 4.1, the Lefschetz number $\text{Lef}(I(D(W)))$ is the Euler characteristic of $I(Y_-)$, which is precisely twice the Casson invariant of Y_- . $\square$

6.4. Framed instanton Floer theory. Instanton Floer homology of admissible pairs can be used to define a 3-manifold invariant called framed instanton Floer homology [KM11b]. First, by a *framed manifold*, we mean a closed 3-manifold with a framed basepoint. Fix (T^3, u) to be the admissible pair of the 3-dimensional torus and the element of $H^2(T^3; \mathbb{Z}/2)$ given by the dual of $S^1 \times \{q\} \subset T^3$ for some point $q \in T^2$. Let Y be a framed manifold with a framed basepoint $p \in Y$. Then define $Y^\sharp$ to be $Y \sharp T^3$, where the connected sum takes place in a neighborhood of p , and let $w^\sharp \in H^2(Y^\sharp; \mathbb{Z}/2)$ be the class induced by the trivial class in Y and u in T^3 . Let $x \in \mathbb{A}(Y^\sharp \times I)$ be the class of degree 4 determined by the homology class of a point in $Y^\sharp \times I$. The operator $\mu(x) = I(Y^\sharp \times I, \pi_1^*(w^\sharp); x)$ acts on the $\mathbb{Z}/8$ -graded vector space $I(Y^\sharp, w^\sharp)$, and satisfies $\mu(x)^2 = 4 \cdot \mathbb{I}_{I(Y^\sharp, w^\sharp)}$ [KM10, Corollary 7.2]. The *framed instanton Floer homology* of Y , denoted by $I^\sharp(Y)$, is defined to be the kernel of $\mu(x) - 2$; it inherits a $\mathbb{Z}/4$ -grading from $I(Y^\sharp, w^\sharp)$. This flavor of instanton Floer homology is conjectured to agree with the hat flavor of Heegaard Floer homology, when both are computed over $\mathbb{Q}$.

Framed instanton Floer homology is functorial with respect to cobordisms of framed manifolds. Given framed 3-manifolds Y_1 and Y_2 with framed basepoints p_1 and p_2 respectively, a *framed cobordism* $W: Y_1 \rightarrow Y_2$ is a cobordism together with a choice of an embedded framed path in W between p_1 and p_2 . A framed cobordism $W: Y_1 \rightarrow Y_2$ can be used to define a cobordism $W^\sharp: Y_1^\sharp \rightarrow Y_2^\sharp$ by taking the sum with $T^3 \times I$ along a regular neighborhood of the framed path in W . A homology orientation on W induces a homology orientation on $W^\sharp$ in an obvious way. Moreover, the dual of $S^1 \times \{q\} \times I \subset T^3 \times I$ defines a cohomology class $c \in H^2(W^\sharp; \mathbb{Z}/2)$ that restricts to $w_1^\sharp$ and $w_2^\sharp$ on $Y_1^\sharp$ and $Y_2^\sharp$ respectively. The functoriality of instanton Floer homology of admissible pairs implies that

$$I(W^\sharp, c) \circ I(Y_1^\sharp \times I, \pi_1^*(w_1^\sharp); x_1) = I(Y_2^\sharp \times I, \pi_1^*(w_2^\sharp); x_2) \circ I(W^\sharp, c).$$

In particular, $I(W^\sharp, c)$ gives rise to a homomorphism $I^\sharp(W): I^\sharp(Y_1) \rightarrow I^\sharp(Y_2)$.

Proof of Theorem 4.2. Let $W: Y_- \rightarrow Y_+$ be a ribbon $\mathbb{Q}$ -homology cobordism of framed 3-manifolds. We also denote by $w_-^\sharp$ and $w_+^\sharp$ the cohomology classes in $Y_-^\sharp$ and $Y_+^\sharp$ induced by u respectively. Then $(Y_-^\sharp, w_-^\sharp)$, $(Y_+^\sharp, w_+^\sharp)$, $W^\sharp$, and $S^1 \times \{q\} \times I \subset W^\sharp$ satisfy the conditions of Theorem 6.8, and we can thus apply it to conclude that, up to a sign, $I(D(W)^\sharp, c)$ is equal to multiplication by $|H_1(W, Y_-)|$. Since this map clearly respects the eigenspace decomposition of $\mu(x)$, we obtain the analogous statement for $I^\sharp(D(W))$. $\square$

6.5. Sutured instanton Floer theory. We first define what we mean by a cobordism of sutured manifolds. Note that this definition is narrower than the one used by Juhász [Juh16].

Definition 6.10. Let (M_1, η_1) and (M_2, η_2) be sutured manifolds. A *cobordism* $N: (M_1, \eta_1) \rightarrow (M_2, \eta_2)$ is a 4-manifold N obtained by a sequence of interior handle attachments on $M_1 \times I$. In particular, there is a natural diffeomorphism of ∂M_1 and ∂M_2 that identifies η_1 with η_2 .

If Y is a framed 3-manifold, then we can define a sutured manifold (M, η) , where M is the complement of a regular neighborhood of the basepoint diffeomorphic to the 3-ball, and α is the equator in ∂M . A framed cobordism $W: Y_1 \rightarrow Y_2$ of framed 3-manifolds then induces a cobordism of the sutured manifolds associated to Y_1 and Y_2 .

More generally, the theory of instanton Floer homology of admissible pairs can be also used to define a functorial invariant of sutured manifolds, generalizing the framed instanton Floer construction. Instanton homology of sutured manifolds is defined using *closures* of sutured manifolds, which we now recall.

Let (M, η) be a sutured manifold whose set of sutures η has d elements. Denote by $F_{g,d}$ the genus- g surface with d boundary components. Fix an arbitrary $g \geq 0$; we glue (M, η) to the product sutured manifold $F_{g,d} \times [-1, 1]$ by identifying $A(\eta)$ with $(\partial F_{g,d}) \times [-1, 1]$. The resulting space has two boundary components $\widehat{R}_\pm \cong R_\pm(\eta) \cup (F_{g,d} \times \{\pm 1\})$, which are closed surfaces of the same genus; we choose a diffeomorphism ϕ of these two boundary components that fixes some point $p \in F_{g,d}$, and glue $\widehat{R}_\pm$ together by ϕ to obtain a closed 3-manifold $\widehat{M}$. Then $\{p\} \times [-1, 1] \subset F_{g,d} \times [-1, 1]$ determines a closed curve in $\widehat{M}$, and we write $w \in H^2(\widehat{M}; \mathbb{Z}/2)$ for its Poincaré dual. The image of $\widehat{R}_\pm$ gives rise to an embedded oriented surface R of a certain genus g' in $\widehat{M}$ with $g' \geq g$, and $(\widehat{M}, w)$ is an admissible pair because the pairing of w with R is not trivial. At this point, we require $g' \geq 1$; this could be ensured by the sufficient (but not necessary) condition that we choose $g \geq 1$. Then, R induces an endomorphism

$$\mu(R) = \mathrm{I}(\widehat{M} \times I, \pi_1^*(w); R) : \mathrm{I}(\widehat{M}, w) \rightarrow \mathrm{I}(\widehat{M}, w).$$

If $g' > 1$, then the instanton homology of (M, η) is defined by

$$\mathrm{SHI}(M, \eta) = \ker(\mu(R) - (2g' - 2))$$

In the case that $g' = 1$, the operator $\mu(R)$ acts trivially and the definition of $\mathrm{SHI}(M, \eta)$ should be modified using the operator $\mu(x) = \mathrm{I}(\widehat{M} \times I, \pi_1^*(w); x)$, where $x \in \mathbb{A}(\widehat{M} \times I)$ is the class given by a point. Thus, if $g' = 1$, we define

$$\mathrm{SHI}(M, \eta) = \ker(\mu(x) - 2)$$

In any case, the key fact is that this construction of $\mathrm{SHI}(M, \eta)$ above is independent of all choices made in the process. (The interested reader may compare the above with the proof of [Theorem 4.12](#) in [Section 7.3](#), in the context of sutured Heegaard Floer theory.)

We also have an analogous construction for a cobordism of sutured manifolds $N : (M_1, \eta_1) \rightarrow (M_2, \eta_2)$. First, fix $g \geq 0$, and glue the product of an interval and the product sutured manifold $F_{g,d} \times [-1, 1]$ to N to obtain a cobordism of manifolds with boundary, where the induced cobordism of the boundary components is the trivial cobordism $(\widehat{R}_+ \times I) \sqcup (\widehat{R}_- \times I)$ to itself. Using the diffeomorphism ϕ of $\widehat{R}_+$ and $\widehat{R}_-$, we identify $\widehat{R}_+ \times I$ with $\widehat{R}_- \times I$ to obtain a cobordism $\widehat{N}$ from a closure $\widehat{M}_1$ of (M_1, η_1) to a closure $\widehat{M}_2$ of (M_2, η_2) . (As before, we require that the image R of $\widehat{R}_\pm$ has genus $g' \geq 1$.) Also, the product of an interval and $\{p\} \times [-1, 1]$ determines a properly embedded surface in $\widehat{N}$, whose Poincaré dual $c \in H^2(\widehat{N}; \mathbb{Z}/2)$ restricts to $w_i \in H^2(\widehat{M}_i; \mathbb{Z}/2)$ for $i \in \{1, 2\}$. Thus, we obtain a cobordism map $\mathrm{I}(\widehat{N}, c) : \mathrm{I}(\widehat{M}_1, w_1) \rightarrow \mathrm{I}(\widehat{M}_2, w_2)$ of admissible pairs. It turns out that $\mathrm{I}(\widehat{N}, c)$ respects the eigenspace decompositions of $\mathrm{I}(\widehat{M}_1, w_1)$ and $\mathrm{I}(\widehat{M}_2, w_2)$, and so we obtain a homomorphism $\mathrm{SHI}(N) : \mathrm{SHI}(M_1, \eta_1) \rightarrow \mathrm{SHI}(M_2, \eta_2)$ simply by restricting to the $(+2)$ -eigenspace.

Proof of Theorem 4.4. This follows directly from [Theorem 6.8](#) together with the description of sutured instanton Floer homology as the eigenspace of the instanton Floer homology for an admissible pair. $\square$

The sutured instanton homology of the sutured manifold associated to a framed 3-manifold Y is isomorphic to $\mathrm{I}^\sharp(Y)$. In fact, the manifold $Y^\sharp$ can be obtained as a closure of the sutured manifold

associated to Y , where we use the product sutured manifold $F_{1,1} \times I$ in the construction of the closure.

Proof of Corollary 4.5. The main idea of this proof is that the known isomorphism between KHI and $\text{I}^\natural$ is natural with respect to cobordism maps. To simplify the exposition, we focus on the cobordism maps associated to $(D(W), D(C))$ below.

To make this precise, we first recall an explicit description of $\text{KHI}(Y, K)$, as contained in [KM10, Section 5.1 and Section 7.6]. Let K be a knot in a closed, oriented 3-manifold Y ; first, we associate to the pair (Y, K) the sutured manifold $(Y \setminus \nu(K), \eta)$, where $Y \setminus \nu(K)$ is the exterior of Y , and $\eta \subset \partial(Y \setminus \nu(K))$ consists of two sutures that are oppositely oriented meridians. Then $\text{KHI}(Y, K)$ is defined as $\text{SHI}(Y \setminus \nu(K), \eta)$. As described earlier in this subsection, $\text{SHI}(Y \setminus \nu(K), \eta)$ is in turn defined by taking a closure; we choose to work with the closure associated to $F_{0,2}$, the genus-0 surface with 2 boundary components, and denote this closure $\widehat{M}$ by T_K^3 . According to [KM10, Section 5.1], the closed 3-manifold T_K^3 admits an equivalent description. It is formed by gluing $F_{1,1} \times S^1$ to $Y \setminus \nu(K)$, with $\partial F_{1,1} \times \{p\}$ being identified with a longitude of K on $\partial \nu(K)$, and $\{q\} \times S^1$ being identified with the meridian of K on $\partial \nu(K)$. In this new description, the element $w \in H^2(T_K^3; \mathbb{Z}/2)$ is the Poincaré dual of the oriented loop $\gamma \times \{p\} \subset F_{1,1} \times S^1 \subset T_K^3$, where $\gamma \subset F_{1,1}$ is some oriented, non-separating loop. The embedded oriented surface R is then $\gamma' \times S^1 \subset F_{1,1} \times S^1 \subset T_K^3$, where γ' is another non-separating loop in $F_{1,1}$ that intersects γ at exactly one point. This, in particular, means that R has genus $g' = 1$, and so

$$\text{KHI}(Y, K) = \text{SHI}(Y \setminus \nu(K), \eta) = \ker(\mu(x) - 2),$$

where $\mu(x): \text{I}(T_K^3, w) \rightarrow \text{I}(T_K^3, w)$ is a degree-4 operator determined by a point $x \in T_K^3$. By [KM10, Corollary 7.2], one can see that $\mu(x)$ has eigenvalues ± 2 (so that $\mu(x)^2 = 4 \cdot \mathbb{I}_{\text{I}(T_K^3, w)}$), each of whose eigenspace has half the dimension of $\text{I}(T_K^3, w)$. In particular, one concludes that the dimension of $\text{KHI}(Y, K)$ is half that of $\text{I}(T_K^3, w)$.

Next, we recall the isomorphism between KHI and $\text{I}^\natural$. In [KM11a, Section 5], a degree-4 involution $\psi_K: \text{I}(T_K^3, w) \rightarrow \text{I}(T_K^3, w)$ is constructed, whose associated quotient is denoted $\text{I}(T_K^3, w)^\psi$; then, using a version of Floer's Excision Theorem, it is shown that there is an isomorphism $\Phi_{(Y, K)}: \text{I}^\natural(Y, K) \rightarrow \text{I}(T_K^3, w)^\psi$. From this, one again concludes that the dimension of $\text{I}^\natural(Y, K)$ is half that of $\text{I}(T_K^3, w)$, and thus that $\text{I}^\natural(Y, K)$ is isomorphic to $\text{KHI}(Y, K)$.

Let $(Y_\pm, K_\pm)$ and $(W, C): (Y_-, K_-) \rightarrow (Y_+, K_+)$ be as in the statement. We now argue that the isomorphism between KHI and $\text{I}^\natural$ is natural with respect to cobordism maps associated to $(D(W), D(C))$. To begin, let $N: (Y_- \setminus \nu(K_-), \eta_-) \rightarrow (Y_+ \setminus \nu(K_+), \eta_+)$ be the cobordism of sutured manifolds, in the sense of Definition 6.10, obtained by removing a regular neighborhood of C from W ; obviously, N is a ribbon $\mathbb{Q}$ -homology cobordism. Then, the cobordism map

$$\text{KHI}(D(W), D(C)): \text{KHI}(Y_-, K_-) \rightarrow \text{KHI}(Y_+, K_+)$$

is defined as the cobordism map

$$\text{SHI}(D(N)): \text{SHI}(Y_- \setminus \nu(K_-), \eta_-) \rightarrow \text{SHI}(Y_+ \setminus \nu(K_+), \eta_+).$$

By Theorem 4.4, we know that, up to a sign,

$$\text{SHI}(D(N)) = |H_1(N, Y_- \setminus \nu(K_-))| \cdot \mathbb{I}_{\text{SHI}(Y_- \setminus \nu(K_-), \eta_-)} = |H_1(W, Y_-)| \cdot \mathbb{I}_{\text{SHI}(Y_- \setminus \nu(K_-), \eta_-)},$$

which in particular implies that it is a degree-0 map. Passing to the closure, this homomorphism is in turn induced by a cobordism map

$$\text{I}(D(\widehat{N}), c): \text{I}(T_{K_-}^3, w_-) \rightarrow \text{I}(T_{K_+}^3, w_+)$$

of admissible pairs that commutes with $\mu(x_-): I(T_{K_-}^3, w_-) \rightarrow I(T_{K_-}^3, w_-)$, where $x_- \in T_{K_-}^3$. In particular, $\text{SHI}(D(N))$ is the restriction of $I(D(\hat{N}), c)$ to the $(+2)$ -eigenspace of $\mu(x_-)$. Taking into account the facts that $I(T_{K_-}^3, w_-)$ is a $\mathbb{Z}/8$ -graded vector space and that $\mu(x_-)$ is a degree-4 map, we conclude that $I(D(\hat{N}), c)$ itself must satisfy

$$I(D(\hat{N}), c) = |H_1(W, Y_-)| \cdot \mathbb{I}_{I(T_{K_-}^3, w_-)}$$

up to a sign.

Now $I(D(\hat{N}), c)$ commutes with the degree-4 involution $\psi_{K_-}: I(T_{K_-}^3, w_-) \rightarrow I(T_{K_-}^3, w_-)$ [KM11a, Section 5], and thus induces a map

$$I(D(\hat{N}), c)^\psi: I(T_{K_-}^3, w_-)^\psi \rightarrow I(T_{K_-}^3, w_-)^\psi$$

on the quotients. Clearly, this must also satisfy

$$I(D(\hat{N}), c)^\psi = |H_1(W, Y_-)| \cdot \mathbb{I}_{I(T_{K_-}^3, w_-)^\psi}$$

up to a sign. Finally, we claim that $\Phi_{(Y_-, K_-)}: I^\natural(Y_-, K_-) \rightarrow I(T_{K_-}^3, w_-)^\psi$ intertwines $I(D(\hat{N}), c)^\psi$ with $I^\natural(D(W), D(C))$: $I^\natural(Y_-, K_-) \rightarrow I^\natural(Y_-, K_-)$:

$$\Phi_{(Y_-, K_-)} \circ I(D(\hat{N}), c)^\psi = I^\natural(D(W), D(C)) \circ \Phi_{(Y_-, K_-)}.$$

Indeed, this claim follows from the fact that the excision map $\Phi_{(Y_-, K_-)}$ is itself a cobordism map, meaning that the two sides of the identity above can be interpreted as two homomorphisms associated to diffeomorphic cobordisms. This implies that the desired result that

$$I^\natural(Y_-, K_-) = |H_1(W, Y_-)| \cdot \mathbb{I}_{I^\natural(Y_-, K_-)}$$

holds, up to a sign. $\square$

6.6. Equivariant instanton Floer theory. For a $\mathbb{Z}$ -homology sphere Y , one can define a stronger invariant that contains the information of $I(Y)$ and $I^\natural(Y)$. Let $(C(Y), d)$ be the instanton Floer chain complex whose homology is equal to $I(Y)$. We consider a larger chain complex $(\tilde{C}(Y), \tilde{d})$ defined by $\tilde{C}(Y) = C(Y) \oplus \mathbb{Q} \oplus C(Y)[3]$, where $C(Y)[3]$ denotes the complex $C(Y)$ with the $\mathbb{Z}/8$ -grading shifted up by 3. The complex $\tilde{C}(Y)$ is equipped with a $\mathbb{Z}/8$ -grading on $\tilde{C}(Y)$ by assigning degree 0 to the summand $\mathbb{Q}$. With respect to the direct sum decomposition of $\tilde{C}(Y)$ above, the differential $\tilde{d}$, which has degree -1 , has the matrix form

$$(6.11) \quad \tilde{d} = \begin{pmatrix} d & 0 & 0 \\ D_1 & 0 & 0 \\ U & D_2 & -d \end{pmatrix},$$

where $U: C(Y) \rightarrow C(Y)[4]$ is a degree-preserving map, D_1 is a functional on $C(Y)$ that is not zero only on elements of degree 1, and $D_2(1)$ is a degree-4 element in $C(Y)$. We refer the reader to [Don02, Frø02] for more details on the definition of U , D_1 , and D_2 . Here we use the same conventions as in [Dae20], where an exposition of the definition of $(\tilde{C}(Y), \tilde{d})$ is given. The characterizing feature of the special form of $\tilde{d}$ in (6.11) is that it anti-commutes with the endomorphism of $\tilde{C}(Y)$ given by

$$\chi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We call a chain complex $(\tilde{C}, \tilde{d})$ over $\mathbb{Q}$ whose differential has the form in (6.11) an $\mathcal{SO}$ -complex.¹²

The chain complex $(\tilde{C}(Y), \tilde{d})$ depends on some auxiliary choices, namely the Riemannian metric on Y and the perturbation of the Chern–Simons functional of Y . However, the chain homotopy type of $(\tilde{C}(Y), \tilde{d})$ is an invariant of Y in an appropriate sense. Suppose $(\tilde{C}'(Y), \tilde{d}')$ is the chain complex that is obtained from another set of auxiliary choices. Then there is a degree-0 chain map $\tilde{\lambda}: \tilde{C}(Y) \rightarrow \tilde{C}'(Y)$, such that

$$(6.12) \quad \tilde{\lambda} = \begin{pmatrix} \lambda & 0 & 0 \\ \Delta_1 & 1 & 0 \\ \mu & \Delta_2 & \lambda \end{pmatrix}.$$

Notice that the map $\tilde{\lambda}$ commutes with χ . Similarly, there is a degree-0 chain map $\tilde{\lambda}': \tilde{C}'(Y) \rightarrow \tilde{C}(Y)$ with the same form as (6.12), with λ' playing the role of λ . Moreover, there are degree-1 maps $\tilde{K}: \tilde{C}(Y) \rightarrow \tilde{C}(Y)$ and $\tilde{K}': \tilde{C}'(Y) \rightarrow \tilde{C}'(Y)$ that anti-commute with χ , such that

$$\tilde{K} \circ \tilde{d} + \tilde{d} \circ \tilde{K} = \tilde{\lambda}' \circ \tilde{\lambda} - \mathbb{I}_{\tilde{C}(Y)}, \quad \tilde{K}' \circ \tilde{d}' + \tilde{d}' \circ \tilde{K}' = \tilde{\lambda} \circ \tilde{\lambda}' - \mathbb{I}_{\tilde{C}'(Y)}.$$

As is customary in Floer theories, the existence of the maps $\tilde{\lambda}$ and $\tilde{\lambda}'$ is a consequence of a more general functoriality of the theory. In fact, for any $\mathbb{Z}$ -homology cobordism $W: Y_1 \rightarrow Y_2$, there is a chain map $\tilde{\lambda}(W): \tilde{C}(Y_1) \rightarrow \tilde{C}(Y_2)$ of the form in (6.12). In particular, this morphism contains in its data a chain map $\lambda(W): C(Y_1) \rightarrow C(Y_2)$, which induces the cobordism map $I(W): I(Y_1) \rightarrow I(Y_2)$ on the level of homology.

For general $\mathcal{SO}$ -complexes, a chain map of the form (6.12), with the number 1 possibly replaced by a non-zero rational number, is called an $\mathcal{SO}$ -morphism. An $\mathcal{SO}$ -homotopy from an $\mathcal{SO}$ -morphism $\tilde{\lambda}_1$ to another $\mathcal{SO}$ -morphism $\tilde{\lambda}_2$ is given by a map $\tilde{K}$ of degree 1 that anti-commutes with χ , such that

$$\tilde{K} \circ \tilde{d} + \tilde{d} \circ \tilde{K} = \tilde{\lambda}_2 - \tilde{\lambda}_1,$$

and we say that two $\mathcal{SO}$ -complexes $\tilde{C}$ and $\tilde{C}'$ are $\mathcal{SO}$ -homotopy equivalent if there are $\mathcal{SO}$ -morphisms $\tilde{\lambda}: \tilde{C} \rightarrow \tilde{C}'$ and $\tilde{\lambda}': \tilde{C}' \rightarrow \tilde{C}$ such that $\tilde{\lambda}' \circ \tilde{\lambda}$ and $\tilde{\lambda} \circ \tilde{\lambda}'$ are $\mathcal{SO}$ -homotopic to identity maps. In other words, the discussion above shows that the $\mathcal{SO}$ -homotopy type of $(\tilde{C}(Y), \tilde{d})$ is an invariant of Y .

The $\mathcal{SO}$ -homotopy type of the complex $(\tilde{C}(Y), \tilde{d})$ contains the information of the instanton homology groups $I(Y)$ and $I^\sharp(Y)$. It is clear from the definition that $I(Y)$ is the homology of the quotient complex $(C(Y), d)$, whose chain homotopy type can be recovered from the $\mathcal{SO}$ -homotopy type of $(\tilde{C}(Y), \tilde{d})$. The homology of the chain complex $(\tilde{C}(Y), \tilde{d} + 4\chi)$ is also isomorphic to $I^\sharp(Y)$ [Sca15].

One could extract from $(\tilde{C}(Y), \tilde{d})$ several other homologies, which are analogous to HF^- , HF^+ , and HF^∞ in Heegaard Floer theory respectively. Following [Don02, Dae20], consider the $\mathbb{Z}/8$ -graded chain complexes $(\hat{C}(Y), \hat{d})$ and $(\check{C}(Y), \check{d})$ defined by

$$\begin{aligned} \hat{C}(Y) &= C(Y)[3] \oplus \mathbb{Q}[x], & \hat{d} \left(\alpha, \sum_{i=0}^N a_i x^i \right) &= \left(d\alpha - \sum_{i=0}^N U^i D_2(a_i), 0 \right), \\ \check{C}(Y) &= C(Y) \oplus (\mathbb{Q}[[x^{-1}, x]]/\mathbb{Q}[x]), & \check{d} \left(\alpha, \sum_{i=-\infty}^{-1} a_i x^i \right) &= \left(d\alpha, \sum_{i=-\infty}^{-1} D_1 U^{-i-1}(\alpha) x^i \right). \end{aligned}$$

¹²For a topological space with an $\text{SO}(3)$ -action that has a unique fixed point, one can form an $\mathcal{SO}$ -complex whose homology is the homology of the space. This justifies the terminology $\mathcal{SO}$ -complex.

Here, the degree of x is defined to be -4 . The homology of $(\widehat{C}(Y), \widehat{d})$ and $(\check{C}(Y), \check{d})$ are denoted by $\widehat{I}(Y)$ and $\check{I}(Y)$ respectively. They are modules over the polynomial ring $\mathbb{Q}[x]$, with the action of x given by the endomorphisms

$$\begin{aligned} x: \widehat{C}(Y) &\rightarrow \widehat{C}(Y), & x \cdot \left(\alpha, \sum_{i=0}^N a_i x^i \right) &= \left(U\alpha, D_1(\alpha) + \sum_{i=0}^N a_i x^{i+1} \right), \\ x: \check{C}(Y) &\rightarrow \check{C}(Y), & x \cdot \left(\alpha, \sum_{i=-\infty}^{-1} a_i x^i \right) &= \left(U\alpha + D_2(a_{-1}), \sum_{i=-\infty}^{-2} a_i x^{i+1} \right). \end{aligned}$$

We define $(\overline{C}(Y), \overline{d})$ to be the $\mathbb{Q}[x]$ -module $\mathbb{Q}[[x^{-1}, x]]$ with the trivial differential; although it is independent of Y , it is convenient to consider it and its homology $\overline{I}(Y) \cong \mathbb{Q}[[x^{-1}, x]]$. Together, $\widehat{I}(Y)$, $\check{I}(Y)$, and $\overline{I}(Y)$ are called the *equivariant instanton Floer homologies* of Y .

The modules $\widehat{I}(Y)$, $\check{I}(Y)$ and $\overline{I}(Y)$ fit into an exact triangle

$$(6.13) \quad \begin{array}{ccc} \check{I}(Y) & \xrightarrow{j_*} & \widehat{I}(Y) \\ & \swarrow p_* & \searrow i_* \\ & \overline{I}(Y) & \end{array}$$

where the module homomorphisms are induced by the maps

$$\begin{aligned} i: \widehat{C}(Y) &\rightarrow \overline{C}(Y), & i \left(\alpha, \sum_{i=0}^N a_i x^i \right) &= \sum_{i=-\infty}^{-1} D_1 U^{-i-1}(\alpha) x^i + \sum_{i=0}^N a_i x^i, \\ j: \check{C}(Y) &\rightarrow \widehat{C}(Y), & j \left(\alpha, \sum_{i=-\infty}^{-1} a_i x^i \right) &= (-\alpha, 0), \\ p: \overline{C}(Y) &\rightarrow \check{C}(Y), & p \left(\sum_{i=-\infty}^N a_i x^i \right) &= \left(\sum_{i=0}^N U^i D_2(a_i), \sum_{i=-\infty}^{-1} a_i x^i \right). \end{aligned}$$

As is apparent from the definitions, the construction of the equivariant instanton homologies and the exact triangle (6.13) from $(\check{C}(Y), \check{d})$ is completely algebraic and does not require any additional geometric input. In particular, for any $\mathcal{SO}$ -complex $(\tilde{C}, \tilde{d})$, one can define the chain complexes $(\widehat{C}, \widehat{d})$, $(\check{C}, \check{d})$, their homologies $\widehat{I}$, $\check{I}$, and the analogue of the exact triangle (6.13). These constructions are functorial; given an $\mathcal{SO}$ -morphism $\tilde{\lambda}: \tilde{C} \rightarrow \tilde{C}'$, there are corresponding chain maps $\widehat{\lambda}: \widehat{C} \rightarrow \widehat{C}'$, $\check{\lambda}: \check{C} \rightarrow \check{C}'$, and $\overline{\lambda}: \overline{C} \rightarrow \overline{C}'$, which induce module homomorphisms $\widehat{\lambda}_*: \widehat{I} \rightarrow \widehat{I}'$, $\check{\lambda}_*: \check{I} \rightarrow \check{I}'$, and $\overline{\lambda}_*: \overline{I} \rightarrow \overline{I}'$ that commute with the exact triangles associated to $\tilde{C}$ and $\tilde{C}'$, as explained in [Dae20, Section 2.3]. An $\mathcal{SO}$ -homotopy between two $\mathcal{SO}$ -morphisms $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ induces a homotopy between $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$, a homotopy between $\check{\lambda}_1$ and $\check{\lambda}_2$, and a homotopy between $\overline{\lambda}_1$ and $\overline{\lambda}_2$. Moreover, the maps corresponding to the composition $\tilde{\lambda}' \circ \tilde{\lambda}$ of two $\mathcal{SO}$ -morphisms are equal to the compositions of the maps corresponding to $\tilde{\lambda}'$ and $\tilde{\lambda}$. As a consequence of this functoriality, the equivariant instanton homologies $\widehat{I}(Y)$, $\check{I}(Y)$, $\overline{I}(Y)$ and the exact triangle (6.13) are invariants of Y , and do not depend on the auxiliary choices in the definition of $(\check{C}(Y), \check{d})$.

We now turn our attention to the behavior of equivariant instanton Floer homologies under ribbon $\mathbb{Q}$ -homology cobordisms. (Recall from Remark 1.13 that $\mathbb{Q}$ -homology cobordisms between

$\mathbb{Z}$ -homology spheres are in fact $\mathbb{Z}$ -homology cobordisms.) The key statement is the following proposition about the associated $\mathcal{SO}$ -complexes.

Proposition 6.14. *Let Y_- and Y_+ be $\mathbb{Z}$ -homology spheres, and suppose that $W: Y_- \rightarrow Y_+$ is a ribbon $\mathbb{Q}$ -homology cobordism. Then the $\mathcal{SO}$ -morphism $\tilde{\lambda}(D(W)): \tilde{C}(Y_-) \rightarrow \tilde{C}(Y_-)$ is $\mathcal{SO}$ -homotopic to an $\mathcal{SO}$ -isomorphism.*

Proof. Write the differential $\tilde{d}$ of the $\mathcal{SO}$ -complex $(\tilde{C}(Y_-), \tilde{d})$ as in (6.11), with the maps d, U, D_1 , and D_2 , and write the $\mathcal{SO}$ -morphism $\tilde{\lambda}(D(W))$ as in (6.12), with the maps $\lambda(D(W)), \Delta_1, \Delta_2$, and μ . Since we are working with chain complexes over a field, our argument in Section 6.3 shows that there is a chain homotopy $K: C(Y_-) \rightarrow C(Y_-)$ such that $K \circ d + d \circ K = \mathbb{I}_{C(Y_-)} - \lambda(D(W))$. Defining the map $\tilde{K}: \tilde{C}(Y_-) \rightarrow \tilde{C}(Y_-)$ by

$$\tilde{K} = \begin{pmatrix} K & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -K \end{pmatrix},$$

we immediately see that $\tilde{K}$ anti-commutes with χ ; moreover, we can compute that

$$\tilde{K} \circ \tilde{d} + \tilde{d} \circ \tilde{K} + \tilde{\lambda}(D(W)) = \begin{pmatrix} \mathbb{I}_{C(Y_-)} & 0 & 0 \\ * & 1 & 0 \\ * & * & \mathbb{I}_{C(Y_-)} \end{pmatrix},$$

and so $\tilde{K}$ is an $\mathcal{SO}$ -homotopy between $\tilde{\lambda}(D(W))$ and $\tilde{q} = \tilde{K} \circ \tilde{d} + \tilde{d} \circ \tilde{K} + \tilde{\lambda}(D(W))$, which is clearly invertible over $\mathbb{Q}$. $\square$

Proof of Theorem 4.8. Proposition 6.14 and the discussion above it together imply that $\hat{I}(D(W)) = \hat{q}_*$, $\check{I}(D(W)) = \check{q}_*$, and $\bar{I}(D(W)) = \bar{q}_*$, where $\hat{q}: \hat{C}(Y_-) \rightarrow \hat{C}(Y_-)$, $\check{q}: \check{C}(Y_-) \rightarrow \check{C}(Y_-)$, and $\bar{q}: \bar{C}(Y_-) \rightarrow \bar{C}(Y_-)$ are the chain maps corresponding to some $\mathcal{SO}$ -isomorphism $\tilde{q}: \tilde{C}(Y_-) \rightarrow \tilde{C}(Y_-)$. It is clear that $\hat{q}_*$, $\check{q}_*$, and $\bar{q}_*$ are $\mathbb{Q}[x]$ -module isomorphisms. $\square$

6.7. A character variety approach to Theorem 4.1. In this subsection, we sketch a different approach to prove Theorem 4.1 and Theorem 4.2. For simplicity, we focus on the proof of Theorem 4.1. In particular, let Y_- and Y_+ be $\mathbb{Z}$ -homology spheres, and suppose that $W: Y_- \rightarrow Y_+$ is a ribbon $\mathbb{Q}$ -homology cobordism. (See Remark 1.13.) Our approach in this section is based on the relationship between the character varieties of Y_- and Y_+ . A key component of our proof is an energy argument that also appears in [BD95, Fuk96, DFL21].

Fix a Riemannian metric on Y_- and a cylindrical metric on $D(W)$ that is compatible with the metric on Y_- . For simplicity, we first assume that these metrics allow us to define the instanton Floer homology $I(Y_-)$ and the cobordism map $I(D(W))$ without perturbing the Chern–Simons functional of Y_- or the ASD equation on $D(W)$. In particular, $I(Y_-)$ is the homology of a chain complex $(C(Y_-), d)$, where $C(Y_-)$ is generated by gauge equivalence classes of non-trivial flat connections, or equivalently, non-trivial elements of the character variety of Y_- . The cobordism map $I(D(W))$ is defined using the moduli spaces $\mathcal{M}(D(W); \alpha_1, \alpha_2)$, i.e. gauge equivalence classes of solutions of the (unperturbed) ASD equation

$$(6.15) \quad F(A)^+ = 0,$$

where A is an $SU(2)$ -connection on $D(W)$ asymptotic to the non-trivial flat $SU(2)$ -connections α_1 and α_2 on the ends of $D(W)$.

Let $\mathcal{B}(D(W); \alpha_1, \alpha_2)$ be the space of gauge equivalence classes of connections on $D(W)$ that are asymptotic to α_1 and α_2 on the ends (that may or may not satisfy (6.15)). For a connection A

representing an element of $\mathcal{B}(D(W); \alpha_1, \alpha_2)$, the topological energy of A , given by the Chern–Weil integral

$$\mathcal{E}(A) = \frac{1}{8\pi^2} \int_{D(W)} \text{tr}(F(A) \wedge F(A)),$$

can easily be verified to be invariant under the action of the gauge group, and also under continuous deformation of A . Moreover, (6.15) implies that, for connections A that represent an element in $\mathcal{M}(D(W); \alpha_1, \alpha_2)$, we always have $\mathcal{E}(A) \geq 0$, and $\mathcal{E}(A) = 0$ if and only if A is a flat connection. We will also need the following fact, which says that the topological energy of A determines the dimension of the component of the moduli space $\mathcal{M}(D(W); \alpha_1, \alpha_2)$ that contains A .

Lemma 6.16. *There exists a function ϵ that associates to each non-trivial (i.e. irreducible) flat connection α on Y_- a real number $\epsilon(\alpha)$, such that the equality*

$$d = 8\mathcal{E}(A) + \epsilon(\alpha_1) - \epsilon(\alpha_2)$$

holds whenever $[A] \in \mathcal{M}(D(W); \alpha_1, \alpha_2)_d$.

Proof. To each connection A representing an element of $\mathcal{M}(D(W); \alpha_1, \alpha_2)$, we may associate the ASD operator $\mathcal{D}_A$, which is an elliptic operator; if A represents an element of $\mathcal{M}(D(W); \alpha_1, \alpha_2)_d$, then d is equal to the index of $\mathcal{D}_A$. Therefore, it suffices to show that, for some choice of ϵ ,

$$(6.17) \quad \text{ind}(\mathcal{D}_A) = 8\mathcal{E}(A) + \epsilon(\alpha_1) - \epsilon(\alpha_2).$$

We first verify this formula when $\alpha_1 = \alpha_2$. Since index and topological energy are invariant under continuous deformation, we may assume without loss of generality that the connection A is the pull-back of a fixed flat connection α_1 on the cylindrical ends of $D(W)$. In particular, A induces a connection $\bar{A}$ on the closed 4-manifold $\overline{D(W)}$ obtained by gluing the incoming and the outgoing ends of $D(W)$ by the identity. Clearly, the topological energy of A and $\bar{A}$ are equal to each other. Moreover, the additive property of indices with respect to gluing (see [Don02, Chapter 3]) implies that $\text{ind}(\mathcal{D}_A) = \text{ind}(\mathcal{D}_{\bar{A}})$. Since $\overline{D(W)}$ has the same $\mathbb{Z}$ -homology as $S^1 \times S^3$, the standard index theorems for the ASD operator on closed 4-manifolds imply that

$$\text{ind}(\mathcal{D}_{\bar{A}}) = 8\mathcal{E}(\bar{A}).$$

This shows that (6.17) holds in the case that $\alpha_1 = \alpha_2$.

In the more general case, we fix an arbitrary irreducible flat connection α_1 on Y_- , and set $\epsilon(\alpha_1) = 0$. For any other irreducible flat connection α , we take an arbitrary connection B on $Y_- \times \mathbb{R}$ that is equal to the pull-backs of representatives of α_1 and α on $(-\infty, -1] \times Y_-$ and $[1, \infty) \times Y_-$ respectively, and define

$$\epsilon(\alpha) = \text{ind}(\mathcal{D}_B) - \frac{1}{\pi^2} \int_{Y_- \times \mathbb{R}} \text{tr}(F(B) \wedge F(B)).$$

One can check that ϵ is well defined, and it only depends on the gauge equivalence class of α . Another application of the additive property of the index of ASD operators with respect to gluing completes the proof of the lemma. $\square$

Lemma 6.18. *If the moduli space $\mathcal{M}(D(W); \alpha_1, \alpha_2)_0$ is not empty, then either $\epsilon(\alpha_2) > \epsilon(\alpha_1)$, or $\alpha_1 = \alpha_2$. Moreover, the moduli space $\mathcal{M}(D(W); \alpha_1, \alpha_1)_0$ consists of an odd number of flat connections.*

Proof. Lemma 6.16 implies that, if there is an element $[A]$ in $\mathcal{M}(D(W); \alpha_1, \alpha_2)_0$, then $\epsilon(\alpha_2) \geq \epsilon(\alpha_1)$. Moreover, if $\epsilon(\alpha_1) = \epsilon(\alpha_2)$, then the connection A has to be flat, which is to say that A represents an element of the character variety $\mathcal{X}_{\text{SU}(2)}(D(W))$. In particular, Proposition 2.1 implies that $\alpha_1 = \alpha_2$. By assumption, any element of $\mathcal{M}(D(W); \alpha_1, \alpha_1)_0$ is cut out regularly, and we do not need to

perform any perturbation. Regularity of a flat connection A on $D(W)$ is equivalent to the property that $H^1(D(W); \text{Ad}_A)$ is trivial. The proof of [Proposition 2.1](#) implies that the $\text{SU}(2)$ -representations of $\pi_1(D(W))$ that extend a given representation of $\pi_1(Y_-)$ is the set of solutions of $K(g_1, \dots, g_m) = 1$, where $K: \text{SU}(2)^m \rightarrow \text{SU}(2)^m$ is a map of degree ± 1 . Since the solutions of these equations are cut out transversely, the number of solutions of this extension problem is an odd integer. $\square$

[Lemma 6.18](#) implies that if we sort flat connections on Y_- based on their ϵ -values, then the chain map $C(D(W))$ is upper triangular with non-zero diagonal entries. In particular, $I(D(W))$ is an isomorphism.

In general, we need to consider perturbations of the Chern–Simons functional of Y_- and the ASD equation on $D(W)$. There are standard functions on the space of connections on Y_- that give rise to perturbed Chern–Simons functionals of Y_- (see [\[Don02, Chapter 5\]](#)) that are sufficient to define the instanton Floer homology $I(Y_-)$. Any such perturbation can be extended to a perturbation of the ASD equation on $D(W)$ that is *time independent* in the sense defined by Braam and Donaldson [\[BD95\]](#). The main point of considering such perturbations is that, even after we slightly modify the definition of topological energy, the solutions of the perturbed ASD equation will still have non-negative topological energy. Having fixed the above, another technical issue would be to know whether the solutions of the perturbed ASD equation with vanishing topological energy are cut out regularly. If we happen to know that our chosen perturbation has this additional property, then we can proceed as above to show that the map $I(D(W))$ is an isomorphism. However, the authors have not checked whether there is a time-independent perturbation with this property.

7. HEEGAARD FLOER HOMOLOGY

7.1. Surgery and cobordism maps in Heegaard Floer theory. In light of [Proposition 5.1](#), our strategy to prove [Theorem 4.10](#) will be to show that the cobordism map for $D(W)$ is actually just determined by that for $X \cong (Y_- \times I) \# m(S^1 \times S^3)$ and the homology classes of the γ_i 's, and hence must agree with that of $Y_- \times I$. We will first focus on $\widehat{\text{HF}}$; it will be shown later in the proof of [Theorem 4.10](#) that this is sufficient to recover the result for the other flavors. The necessary tool is [Proposition 7.2](#) below, which shows the behavior of the Heegaard Floer cobordism maps under surgery along circles, and is the counterpart of [Proposition 6.5](#) and [Proposition 6.6](#) for Heegaard Floer homology. This statement is known to experts, and can be derived from the link cobordism TQFT of Zemke; see [Remark 7.4](#) below. A closely related result is also already established in [\[KLS20, Example 1.4\]](#). For completeness, we provide a proof in this subsection. Note that we do not assume 3- and 4-manifolds to be connected in this subsection.

Recall that given a connected Spin^c -cobordism $(W, \mathfrak{t}): (Y_1, \mathfrak{s}_1) \rightarrow (Y_2, \mathfrak{s}_2)$ between closed, connected 3-manifolds, Ozsváth and Szabó [\[OSz06\]](#) define cobordism maps

$$F_{W, \mathfrak{t}}^\circ: \text{HF}^\circ(Y_1, \mathfrak{s}_1) \otimes (\Lambda^*(H_1(W)/\text{Tors}) \otimes \mathbb{Z}/2) \rightarrow \text{HF}^\circ(Y_2, \mathfrak{s}_2).$$

These maps have the property that

$$(7.1) \quad F_{W, \mathfrak{t}}^\circ(x \otimes \xi) = F_{W, \mathfrak{t}}^\circ(\xi_1 \cdot x) + \xi_2 \cdot F_{W, \mathfrak{t}}^\circ(x),$$

whenever $\xi \in H_1(W)/\text{Tors}$ satisfies $\xi = \iota_1(\xi_1) - \iota_2(\xi_2)$, where $\xi_i \in H_1(Y_i)/\text{Tors}$ and ι_i is induced by inclusion; see [\[OSz03a, p. 186\]](#). We may also sum over all Spin^c -structures on W , and obtain a total map

$$F_W^\circ: \text{HF}^\circ(Y_1) \otimes (\Lambda^*(H_1(W)/\text{Tors}) \otimes \mathbb{Z}/2) \rightarrow \text{HF}^\circ(Y),$$

satisfying a property analogous to [\(7.1\)](#). We are now ready to state:

Proposition 7.2. *Let Y_1 and Y_2 be closed, connected 3-manifolds, and let $X: Y_1 \rightarrow Y_2$ be a connected cobordism. Suppose that $\gamma_1, \dots, \gamma_\ell \subset \text{Int}(X)$ are loops with disjoint neighborhoods $\nu(\gamma_i) \cong \gamma_i \times D^3$, and denote by Z the result of surgery on X along $\gamma_1, \dots, \gamma_\ell$. Then for $x \in \widehat{\text{HF}}(Y_1)$,*

$$(7.3) \quad \widehat{F}_X(x \otimes ([\gamma_1] \wedge \dots \wedge [\gamma_\ell])) = \widehat{F}_Z(x).$$

Thus, $\widehat{F}_Z$ depends only on X and $[\gamma_1] \wedge \dots \wedge [\gamma_\ell] \in \Lambda^(H_1(X)/\text{Tors}) \otimes \mathbb{Z}/2$.*

Remark 7.4. A surgery formula for link cobordisms and link Floer homology, similar to [Proposition 7.2](#), is provided by Zemke [[Zem19a](#), Proposition 5.4]. One may obtain [Proposition 7.2](#) via an identification, also provided by Zemke [[Zem19d](#), Theorem C], of link cobordism maps with maps induced by cobordisms between 3-manifolds. In this paper, we instead provide a direct proof without mentioning any link cobordism theory, in the interest of providing a self-contained discussion.

Before giving the proof, we describe the idea informally. Surgery on γ_i is the result of removing a copy of $S^1 \times D^3$ and replacing it with $D^2 \times S^2$. The cobordism map for $D^2 \times S^2$ agrees with that of $S^1 \times D^3$ if one contracts the latter map by the generator of H_1 . Composing with the cobordism map for $X \setminus (\bigcup \nu(\gamma_i))$, the result follows. However, to prove this carefully, we must cut and re-glue several different codimension-0 submanifolds, and thus need to use the graph TQFT framework by Zemke [[Zem21b](#)]. Below, we give a brief review of the necessary elements.

Let Y be a possibly disconnected 3-manifold, and let $\mathbf{p}$ be a set of points in Y with at least one point in each component. Let $W: Y_1 \rightarrow Y_2$ be a cobordism, and let Γ be a graph embedded in W with $\partial\Gamma = \mathbf{p}_1 \cup \mathbf{p}_2$. Then, Zemke [[Zem21b](#)] constructs Heegaard Floer homology groups $\widehat{\text{HF}}(Y_i, \mathbf{p}_i)$ and cobordism maps $\widehat{F}_{W,\Gamma}: \widehat{\text{HF}}(Y_1, \mathbf{p}_1) \rightarrow \widehat{\text{HF}}(Y_2, \mathbf{p}_2)$.

In a later paper, Zemke [[Zem19b](#)] constructs cobordism maps

$$F_{W,\Gamma,\mathbf{t}}^A: \text{HF}^\circ(Y_1, \mathbf{p}_1, \mathbf{t}|_{Y_1}) \rightarrow \text{HF}^\circ(Y_2, \mathbf{p}_2, \mathbf{t}|_{Y_2}), \quad F_{W,\Gamma,\mathbf{t}}^B: \text{HF}^\circ(Y_1, \mathbf{p}_1, \mathbf{t}|_{Y_1}) \rightarrow \text{HF}^\circ(Y_2, \mathbf{p}_2, \mathbf{t}|_{Y_2})$$

for each $\mathbf{t} \in \text{Spin}^c(W)$, for various flavors HF° of Heegaard Floer homology groups. One may also take the sum over all $\mathbf{t} \in \text{Spin}^c(W)$ to obtain maps $F_{W,\Gamma}^A$ and $F_{W,\Gamma}^B$. In this theory, for HF^- , the graph Γ needs to be equipped with a cyclic ordering of the edges adjacent to each vertex; however, for $\widehat{\text{HF}}$, the map is independent of this choice of a cyclic ordering [[Zem19b](#), Lemma 4.5]. Furthermore, for $\widehat{\text{HF}}$, the maps $F_{W,\Gamma,\mathbf{t}}^A$ and $F_{W,\Gamma,\mathbf{t}}^B$ coincide, as can be seen by combining [[Zem19b](#), Lemma 5.7] and the definitions of the type- A and type- B graph action maps [[Zem19b](#), Equation (7.1) and Equation (7.2)].

As pointed out to the authors by Ian Zemke, for $\widehat{\text{HF}}$, the maps $F_{W,\Gamma}^A = F_{W,\Gamma}^B$ in fact agree with $\widehat{F}_{W,\Gamma}$. Indeed, it suffices to check this for maps associated to 4-dimensional 1-, 2-, and 3-handles, as well as maps associated to three elementary graph cobordisms: free-stabilization cobordisms, free-destabilization cobordisms, and wye-shaped cobordisms [[Zem21b](#), Figure 1.1 (Γ -1) and (Γ -2)]. For handles, the definitions of $\widehat{F}_{W,\Gamma}$ [[Zem21b](#), Section 2.4 and Section 3] and $F_{W,\Gamma}^A = F_{W,\Gamma}^B$ [[Zem19b](#), Section 8 and Section 9] coincide, as they are ultimately equal to the maps described by Ozsváth and Szabó [[OSz06](#)]. For graph cobordisms, $\widehat{F}_{W,\Gamma}$ is computed in [[Zem21b](#), Section 4], while $F_{W,\Gamma}^A = F_{W,\Gamma}^B$ are computed in [[Zem21a](#), Section 4]. This equivalence between the two graph TQFTs helps us establish some of the properties for $\widehat{F}_{W,\Gamma}$ in the following theorem.

Theorem 7.5 (Zemke [[Zem21b](#), [Zem19b](#)]). *The cobordism maps $\widehat{F}_{W,\Gamma}$ satisfy the following.*

- (1) *Under disjoint union, we have that $\widehat{\text{HF}}(Y_1 \sqcup Y_2, \mathbf{p}_1 \sqcup \mathbf{p}_2) = \widehat{\text{HF}}(Y_1, \mathbf{p}_1) \otimes \widehat{\text{HF}}(Y_2, \mathbf{p}_2)$, and $\widehat{F}_{(W_1,\Gamma_1) \sqcup (W_2,\Gamma_2)} = \widehat{F}_{(W_1,\Gamma_1)} \otimes \widehat{F}_{(W_2,\Gamma_2)}$.*

- (2) Given $(W, \Gamma): (Y_1, \mathbf{p}_1) \rightarrow (Y_2, \mathbf{p}_2)$ and $(W', \Gamma'): (Y_2, \mathbf{p}_2) \rightarrow (Y_3, \mathbf{p}_3)$, then $\widehat{F}_{W', \Gamma'} \circ \widehat{F}_{W, \Gamma} = \widehat{F}_{W \cup W', \Gamma \cup \Gamma'}$; see [Zem21b, Theorem 1.2 (2)].
- (3) $\widehat{F}_{W, \Gamma}$ admits a decomposition by Spin^c -structures in the usual way. In particular, $\widehat{F}_{W, \Gamma} = \sum_{\mathbf{t} \in \text{Spin}^c(W)} \widehat{F}_{W, \Gamma, \mathbf{t}}$, and

$$\widehat{F}_{W', \Gamma', \mathbf{t}_{W'}} \circ \widehat{F}_{W, \Gamma, \mathbf{t}_W} = \sum_{\substack{\mathbf{t} \in \text{Spin}^c(W \cup W') \\ \mathbf{t}|_W = \mathbf{t}_W, \mathbf{t}|_{W'} = \mathbf{t}_{W'}}} \widehat{F}_{W \cup W', \Gamma \cup \Gamma', \mathbf{t}};$$

see [Zem19b, Theorem C]. (We take the convention that this equation remains valid when $\mathbf{t}_W|_{Y_2} \neq \mathbf{t}_{W'}|_{Y_2}$, in which case both sides of the equation are identically zero.)

- (4) If λ is an arc from the boundary of some $B^4 \subset W$ to Γ , then $\widehat{F}_{W, \Gamma}(x) = \widehat{F}_{W \setminus B^4, \Gamma \cup \lambda}(x \otimes y)$, where y is the generator of $\widehat{\text{HF}}(\partial B^4)$; see [Zem19b, Proposition 11.1].
- (5) Suppose that Y_1 and Y_2 are connected, $\mathbf{p}_1$ and $\mathbf{p}_2$ each consist of a single point, and Γ is a path. Then $\widehat{F}_W(x) = \widehat{F}_{W, \Gamma}(x)$, where $\widehat{F}_W$ is the original Ozsváth–Szabó cobordism map; see [Zem21b, Theorem 1.2 (1)]. (Implicitly, the Ozsváth–Szabó cobordism map requires a choice of basepoints and a choice of path, but the injectivity statement in Theorem 4.10 is independent of both choices.)
- (6) Suppose again that Y_1 and Y_2 are connected, $\mathbf{p}_1$ and $\mathbf{p}_2$ each consist of a single point, and Γ is a path. Let γ be a simple closed loop in $\text{Int}(W)$ that intersects Γ at a single point. Then $\widehat{F}_W(x \otimes [\gamma]) = \widehat{F}_{W, \Gamma \cup \gamma}(x)$, where the left-hand side is the Ozsváth–Szabó cobordism map defined above; see [Zem21b, Lemma 4.3].
- (7) As a special case of (6), Let Y be connected and let $\mathbf{p}$ consist of a single point. Consider $\Gamma = \mathbf{p} \times I \subset Y \times I$. Choose a simple closed loop γ in Y based at $\mathbf{p}$ and let Γ_γ be the graph obtained by appending $\gamma \times \{1/2\}$ to Γ . Denote the cobordism map $\widehat{F}_{Y \times I, \Gamma_\gamma}$ by $\mathcal{F}(\gamma)$. Then, $\mathcal{F}(\gamma)$ depends only on $[\gamma] \in H_1(Y)$. Furthermore, $\mathcal{F}(\gamma * \gamma') = \mathcal{F}(\gamma) + \mathcal{F}(\gamma')$ and $\mathcal{F}(\gamma) \circ \mathcal{F}(\gamma) = 0$. Here, $\gamma * \gamma'$ is a simple closed loop in the based homotopy class of the concatenation.

We now need a slight generalization of Theorem 7.5 (6), i.e. [Zem21b, Lemma 4.3], which will allow us to analyze the effect on the cobordism map of appending multiple loops to a path. We begin with the identity cobordism.

Lemma 7.6. *Suppose that Y is connected, and that $\mathbf{p}$ consists of a single point. Suppose that Γ is a graph obtained by taking $\mathbf{p} \times I \subset Y \times I$ and appending to it ℓ disjoint simple closed curves $\gamma_1, \dots, \gamma_\ell$, which each intersect $\mathbf{p} \times I$ only at a single point. Then*

$$\widehat{F}_{Y \times I}(x \otimes ([\gamma_1] \wedge \dots \wedge [\gamma_\ell])) = \widehat{F}_{Y \times I, \Gamma}(x),$$

where the left-hand side is the Ozsváth–Szabó cobordism map.

Proof. This is implicit in the work of Zemke [Zem21b], but we give the proof for completeness. By a homotopy, and hence isotopy, in $Y \times I$, we may arrange that $\gamma_i \subset Y \times \{i/(\ell + 1)\}$. Therefore, using Theorem 7.5 (2), we can write $\widehat{F}_{Y \times I, \Gamma}$ as a composition of the maps $\mathcal{F}(\gamma_i)$. Viewing $\mathcal{F}$ as a function from $H_1(Y)$ to $\text{End}_{\mathbb{Z}/2}(\widehat{\text{HF}}(Y))$, Theorem 7.5 (7) implies that this descends to the exterior algebra. $\square$

We move on to more general cobordisms.

Lemma 7.7. *Suppose that Y_1 and Y_2 are connected, and that $\mathbf{p}_1$ and $\mathbf{p}_2$ each consist of a single point. Let $W: Y_1 \rightarrow Y_2$ be a connected cobordism. Suppose that Γ is a graph obtained by taking*

a path α from $\mathbf{p}_1$ to $\mathbf{p}_2$ and appending to it ℓ disjoint simple closed loops $\gamma_1, \dots, \gamma_\ell$, which each intersect α only at a single point. Then

$$\widehat{F}_W(x \otimes ([\gamma_1] \wedge \dots \wedge [\gamma_\ell])) = \widehat{F}_{W, \Gamma}(x),$$

where the left-hand side is the Ozsváth–Szabó cobordism map.

Proof. We may decompose (W, Γ) as a composition of three cobordisms: (W_1, Γ_1) , where W_1 consists only of 1-handles and Γ_1 is a path; $(\partial W_1 \times I, \Gamma_*)$, where Γ_* consists of a graph in $\partial W_1 \times I$ as in the statement of Lemma 7.6; and (W_2, Γ_2) , where W_2 consists of 2- and 3-handles, and Γ_2 is again a path. The result now follows from Lemma 7.6 together with Theorem 7.5 (2). $\square$

With this generalization, we may now complete the proof of Proposition 7.2.

Proof of Proposition 7.2. Let $\nu(\gamma_i) \cong \gamma_i \times D^3$ be a neighborhood of γ_i , and let $P = X \setminus (\coprod_i \nu(\gamma_i))$. Let $X' = X \setminus (B_1^4 \sqcup \dots \sqcup B_\ell^4)$, where $B_i^4 \subset \text{Int}(\nu(\gamma_i))$. We construct a properly embedded graph $\Gamma_{X'}$ in X' as follows; see Figure 4.

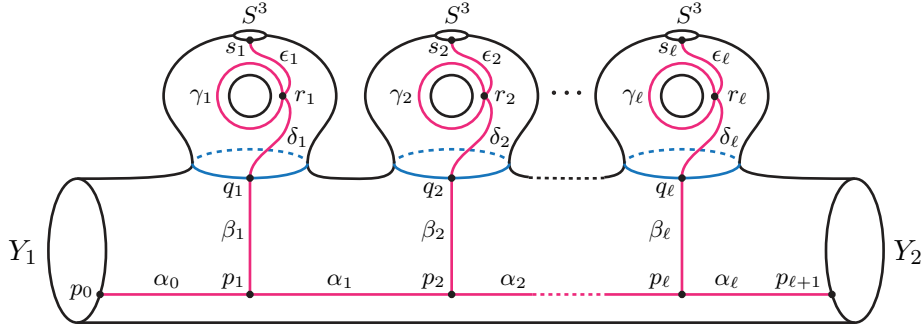


FIGURE 4. The embedded graph $\Gamma_{X'}$ in X' .

We begin with the vertex set. Choose ℓ points $p_1, \dots, p_\ell$ in the interior of P , and points p_0 and $p_{\ell+1}$ in Y_1 and Y_2 respectively. Choose ℓ points $q_1, \dots, q_\ell$ with $q_i \in \partial \nu(\gamma_i)$, which are copies of $S^1 \times S^2$. Choose ℓ points $r_1, \dots, r_\ell$ with $r_i \in \gamma_i$. Finally, let s_i be a point in $S_i^3 = \partial B_i^4$ for each i .

Now we define the edge sets. Choose any collection of embedded arcs $\alpha_0, \dots, \alpha_\ell$ with $\alpha_i \subset P$ connecting p_i and p_{i+1} . Let $\beta_i \subset P$ be an arc from p_i to q_i . Connect q_i and r_i by arcs δ_i , and r_i and s_i by arcs ϵ_i , in $\nu(\gamma_i) \setminus B_i^4$. We may choose the edges above in such a way that their interiors are mutually disjoint, avoid the γ_i , and are contained in the interior of X' . Then, the edge set of $\Gamma_{X'}$ consists of the edges $\alpha_i, \beta_i, \gamma_i, \delta_i$, and ϵ_i . In accordance with Theorem 7.5 (1), we view the cobordism map for $(X', \Gamma_{X'})$ as a map

$$\widehat{F}_{X', \Gamma_{X'}}: \widehat{\text{HF}}(Y_1) \otimes \left(\bigotimes_{i=1}^{\ell} \widehat{\text{HF}}(S_i^3) \right) \rightarrow \widehat{\text{HF}}(Y_2).$$

It follows from Lemma 7.7 as well as Theorem 7.5 (1) and (4) that

$$\widehat{F}_X(x \otimes ([\gamma_1] \wedge \dots \wedge [\gamma_\ell])) = \widehat{F}_{X', \Gamma_{X'}}(x \otimes y_1 \otimes \dots \otimes y_\ell),$$

where y_i is the generator of $\widehat{\text{HF}}(S_i^3)$. (We can first contract the homology elements, and then contract the arcs $\beta_i \cup \delta_i \cup \epsilon_i$.) Let Γ_P be the intersection of $\Gamma_{X'}$ with P , which can alternatively be obtained by excising the γ_i, δ_i , and ϵ_i arcs.

Note that $Z = P \cup (\coprod_i (D^2 \times S^2)_i)$. Here, we suppress the choice of gluing from the notation. Similarly, we let $Z' = Z \setminus (B_1^4 \sqcup \dots \sqcup B_\ell^4)$ where $B_i^4 \subset (D^2 \times S^2)_i$; then $Z' = P \cup (\coprod_i R_i)$, where

each R_i is a punctured $D^2 \times S^2$. Let ζ_i be an arc in R_i that connects q_i and s_i ; then we define Γ_{R_i} in R_i to be ζ_i , and define $\Gamma_{Z'}$ in Z' as the union of the arcs $\alpha_i, \beta_i, \zeta_i$. See Figure 5 for an illustration of $(Z', \Gamma_{Z'})$.

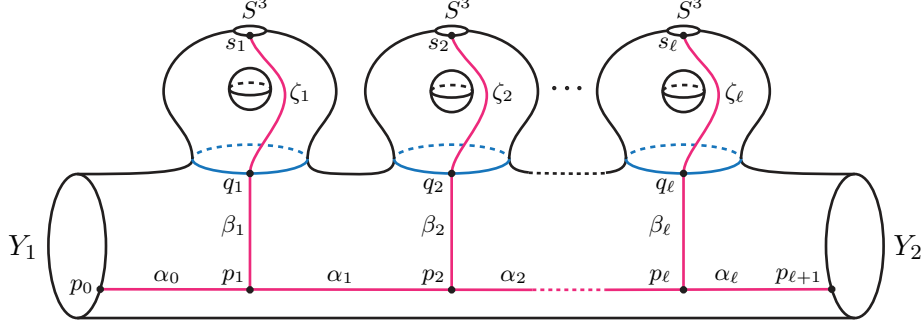


FIGURE 5. The embedded graph $\Gamma_{Z'}$ in Z' .

Viewing the cobordism map for $(Z', \Gamma_{Z'})$ as a map $\widehat{F}_{Z', \Gamma_{Z'}}: \widehat{\text{HF}}(Y_1) \otimes (\bigotimes_i \widehat{\text{HF}}(S^3_i)) \rightarrow \widehat{\text{HF}}(Y_2)$, we have

$$\widehat{F}_Z(x) = \widehat{F}_{Z', \Gamma_{Z'}}(x \otimes y_1 \otimes \cdots \otimes y_\ell),$$

again by Theorem 7.5 (4). Thus, (7.3) will follow if we can show

$$\widehat{F}_{X', \Gamma_{X'}}(x \otimes y_1 \otimes \cdots \otimes y_\ell) = \widehat{F}_{Z', \Gamma_{Z'}}(x \otimes y_1 \otimes \cdots \otimes y_\ell).$$

To do so, let $Q_i = \nu(\gamma_i) \setminus B_i^4$, and let Γ_{Q_i} be the intersection of $\Gamma_{X'}$ with Q_i . Both (Q_i, Γ_{Q_i}) and (R_i, Γ_{R_i}) are cobordisms from (S^3, s_i) to $(S^1 \times S^2, q_i)$; see Figure 6.

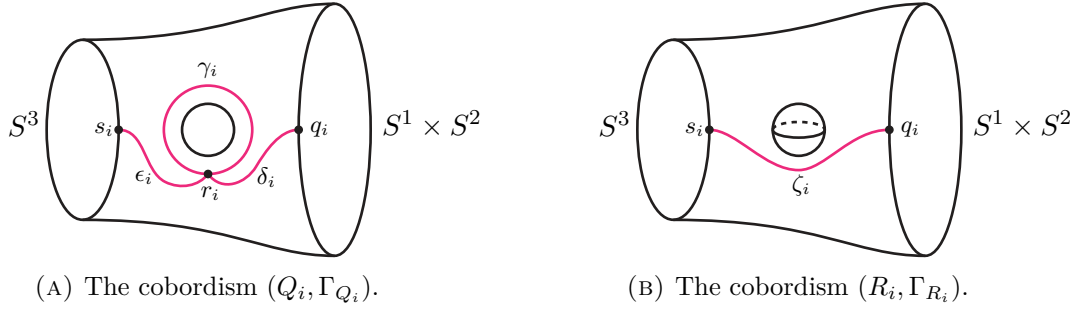


FIGURE 6. The cobordisms (Q_i, Γ_{Q_i}) and (R_i, Γ_{R_i}) .

Viewing (P, Γ_P) as a cobordism from $(Y_1, p_0) \sqcup (\bigsqcup_i (S^1 \times S^2)_i, q_i) \rightarrow (Y_2, p_{\ell+1})$, by Theorem 7.5 (1) and (2), we have that

$$\widehat{F}_{X', \Gamma_{X'}} = \widehat{F}_{P, \Gamma_P} \circ \left(\mathbb{I}_{\widehat{\text{HF}}(Y_1)} \otimes \widehat{F}_{Q_1, \Gamma_{Q_1}} \otimes \cdots \otimes \widehat{F}_{Q_\ell, \Gamma_{Q_\ell}} \right)$$

and

$$\widehat{F}_{Z', \Gamma_{Z'}} = \widehat{F}_{P, \Gamma_P} \circ \left(\mathbb{I}_{\widehat{\text{HF}}(Y_1)} \otimes \widehat{F}_{R_1, \Gamma_{R_1}} \otimes \cdots \otimes \widehat{F}_{R_\ell, \Gamma_{R_\ell}} \right).$$

Thus, we need only to show that $\widehat{F}_{Q_i, \Gamma_{Q_i}} = \widehat{F}_{R_i, \Gamma_{R_i}}$ for each i . On the one hand, Theorem 7.5 (6) together with (7.1) imply that

$$\widehat{F}_{Q_i, \Gamma_{Q_i}}(y_i) = \widehat{F}_{Q_i}(y_i \otimes [\gamma_i]) = [\gamma_i] \cdot \widehat{F}_{Q_i}(y_i).$$

Since Q_i is simply a 1-handle attachment to S^3 , its cobordism map, by Ozsváth and Szabó's definition, sends y_i to the topmost generator of $\widehat{\text{HF}}(S^1 \times S^2)$ (see [OSz06, p. 364]), and the action by $[\gamma_i]$ sends this to the bottommost generator (see [OSz04b, Proposition 6.4]). On the other hand, Theorem 7.5 (5) implies that

$$\widehat{F}_{R_i, \Gamma_{R_i}}(y_i) = \widehat{F}_{R_i}(y_i).$$

Since R_i is simply a 0-framed 2-handle attachment along the unknot in S^3 , its cobordism map sends y_i to the bottommost generator of $\widehat{\text{HF}}(S^1 \times S^2)$. (One could directly compute the map from the definition on [OSz06, pp. 356–357] using the standard Heegaard triple diagram for R_i . Alternatively, one could observe that $\widehat{F}_{R_i}$ must not be zero because of the exact triangle for surgery along the unknot; since the cobordism map respects the H_1 -action, and the H_1 -action on $\widehat{\text{HF}}(S^3)$ is trivial, this means that $\widehat{F}_{R_i}(y_i)$ is in the kernel of the H_1 -action on $\widehat{\text{HF}}(S^1 \times S^2)$.) Consequently, $\widehat{F}_{Q_i, \Gamma_{Q_i}}(y_i) = \widehat{F}_{R_i, \Gamma_{R_i}}(y_i)$ as desired. $\square$

Proof of Theorem 4.10. We consider the hat flavor first. Consider the double $D(W)$ of W . Then, by Proposition 5.1, $D(W)$ is described by surgery on $X \cong (Y_- \times I) \sharp m(S^1 \times S^3)$ along m circles $\gamma_1, \dots, \gamma_m$, where $[\gamma_1] \wedge \dots \wedge [\gamma_m] = \alpha_1 \wedge \dots \wedge \alpha_m \in (\Lambda^*(H_1(X)/\text{Tors})/\langle H_1(Y_-)/\text{Tors} \rangle) \otimes_{\mathbb{Z}} \mathbb{Z}/2$, and α_i is the homology class of the core of the i^{th} $S^1 \times S^3$ summand. Note that the same description is true of $Y_- \times I$; in this case, the surgery is performed along the core circles γ'_i of the $(S^1 \times S^3)$'s themselves.

Applying Proposition 7.2 with $Z = D(W)$, we have that

$$\widehat{F}_X(x \otimes ([\gamma_1] \wedge \dots \wedge [\gamma_m])) = \widehat{F}_{D(W)}(x) = \widehat{F}_{-W} \circ \widehat{F}_W(x).$$

Now consider $Y_- \times I$ as surgery on X along the cores γ'_i . Applying Proposition 7.2 again, this time with $Z = Y_- \times I$, we have

$$\widehat{F}_X(x \otimes ([\gamma'_1] \wedge \dots \wedge [\gamma'_m])) = \widehat{F}_{Y_- \times I}(x) = \mathbb{I}_{\widehat{\text{HF}}(Y_-)}.$$

Since $[\gamma_1] \wedge \dots \wedge [\gamma_m] = [\gamma'_1] \wedge \dots \wedge [\gamma'_m]$ in $\Lambda^*(H_1(X)/\text{Tors})/\langle H_1(Y_-)/\text{Tors} \rangle \otimes_{\mathbb{Z}} \mathbb{Z}/2$, by the linearity of $\widehat{F}$, it suffices to show that $\widehat{F}_X(x \otimes \xi) = 0$ for $x \in \widehat{\text{HF}}(Y_-)$ and $\xi \in \Lambda^m(H_1(X)/\text{Tors}) \cap \langle H_1(Y_-)/\text{Tors} \rangle \otimes_{\mathbb{Z}} \mathbb{Z}/2$. Indeed, this will imply that $\widehat{F}_{-W} \circ \widehat{F}_W = \mathbb{I}_{\widehat{\text{HF}}(Y_-)}$, and we have the desired result for $\widehat{\text{HF}}$.

Note that $\Lambda^m(H_1(X)/\text{Tors}) \cap \langle H_1(Y_-)/\text{Tors} \rangle$ is generated by elements of the form $\omega \wedge (\bigwedge_{i \in I} \alpha_i)$, where ω is a wedge of elements in $H_1(Y_-)/\text{Tors}$ and $I \subsetneq \{1, \dots, m\}$; we would like to show that if $\xi \in \langle H_1(Y_-)/\text{Tors} \rangle \otimes_{\mathbb{Z}} \mathbb{Z}/2$ is of this form, then $\widehat{F}(x \otimes \xi) = 0$ for $x \in \widehat{\text{HF}}(Y_-)$. Therefore, let $\xi = \omega \wedge (\bigwedge_{i \in I} \alpha_i)$ be of this form. The idea is that ξ misses at least one $S^1 \times S^3$ summand, and the cobordism map associated to a twice punctured $S^1 \times S^3$, without an H_1 -action, is identically zero. Concretely, choose $j \in \{1, \dots, m\} \setminus I$, and write $X = T_j \cup_{S^3} V$, where T_j is the j^{th} $S^1 \times S^3$ summand punctured once, and $V = ((Y_- \times I) \sharp (m-1)(S^1 \times S^3)) \setminus B_j^4$; then ξ determines a graph Γ_ξ in X such that $\widehat{F}_X(x \otimes \xi) = \widehat{F}_{X, \Gamma_\xi}(x)$, and we may assume that $\Gamma_\xi \cap T_j = \emptyset$. Let $X' = X \setminus B_j^4$, where $B_j^4 \subset \text{Int}(T_j)$. As in Theorem 7.5 (4), choose an arc λ from ∂B_j^4 to Γ_ξ that intersects $\partial T_j = \partial V$ once, and let $\Gamma'_\xi = \Gamma_\xi \cup \lambda$; then we have

$$\widehat{F}_{X, \Gamma_\xi}(x) = \widehat{F}_{X', \Gamma'_\xi}(x \otimes y_j),$$

where y_j is the generator of $\widehat{\text{HF}}(\partial B_j^4)$. Writing $T'_j = T_j \setminus B_j^4$, it is also clear that

$$\widehat{F}_{X', \Gamma'_\xi} = \widehat{F}_{V, \Gamma'_\xi \cap V} \circ \left(\mathbb{I}_{\widehat{\text{HF}}(Y_-)} \otimes \widehat{F}_{T'_j, \lambda \cap T'_j} \right).$$

Since $\lambda \cap T'_j$ is simply a path, $\widehat{F}_{T'_j, \lambda \cap T'_j} = \widehat{F}_{T'_j}$. Note that $T'_j \cong (S^3 \times I) \sharp (S^1 \times S^3)$ is obtained by adding a 1-handle and a 3-handle to $S^3 \times I$, and so a direct computation shows that $\widehat{F}_{T'_j}(y_j) = 0$; thus, $\widehat{F}_X(x \otimes \xi) = 0$, as desired.

To obtain the analogous result for the other flavors of Heegaard Floer homology, we use that the long exact sequences relating the various flavors are natural with respect to cobordism maps. It is straightforward to see that only an isomorphism on HF^+ can induce the identity map on $\widehat{\text{HF}}$, and similarly for HF^- . Finally, only an isomorphism on HF^∞ can induce an isomorphism on both HF^+ and HF^- . $\square$

7.2. A Spin^c -refinement of Theorem 4.10. We now provide a Spin^c -refinement of Theorem 4.10. First, observe that any Spin^c -structure $\mathfrak{t}$ on a cobordism $W: Y_1 \rightarrow Y_2$ can be extended to a Spin^c -structure $D(\mathfrak{t})$ on $D(W)$, since $\mathfrak{t}$ on W and $\mathfrak{t}$ on $-W$ coincide on the intersection $W \cap -W = Y_2$. We now have the following observation when W is a ribbon $\mathbb{Q}$ -homology cobordism.

Lemma 7.8. *Let Y_- and Y_+ be closed 3-manifolds, and suppose that $W: Y_- \rightarrow Y_+$ is a ribbon $\mathbb{Q}$ -homology cobordism. If a Spin^c -structure $\mathfrak{s}_+$ on Y_+ can be extended to a Spin^c -structure $\mathfrak{t}$ on W , then the extension is unique; moreover, in this case, $D(\mathfrak{t})$ is the unique Spin^c -structure on $D(W)$ that restricts to $\mathfrak{s}_+$ on Y_+ .*

Proof. For the first statement, consider

$$H^2(W, Y_+) \rightarrow H^2(W) \rightarrow H^2(Y_+)$$

from the long exact sequence of the pair (W, Y_+) . By the Poincaré Duality, $H^2(W, Y_+) \cong H_2(W, Y_-)$. Take a ribbon handle decomposition of W ; since W is a $\mathbb{Q}$ -homology cobordism, the numbers m of 1- and 2- handles are the same, and the differential $\partial_2: C_2(W, Y_-) \rightarrow C_1(W, Y_-)$ in the cellular chain complex is given by a homomorphism $R: \mathbb{Z}^m \rightarrow \mathbb{Z}^m$ such that $R \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism. This means that R , and hence ∂_2 , are injective, and so $H_2(W, Y_-) = 0$. Thus, the map $H^2(W) \rightarrow H^2(Y_+)$ induced by inclusion is injective, proving that any extension $\mathfrak{t}$ of $\mathfrak{s}_+$ is unique.

For the second statement, consider

$$H^1(W) \oplus H^1(-W) \rightarrow H^1(Y_+) \rightarrow H^2(D(W)) \rightarrow H^2(W) \oplus H^2(-W)$$

from the Mayer–Vietoris exact sequence; we wish to prove the first map is surjective. In fact, we will prove that the map $H^1(W) \rightarrow H^1(Y_+)$ is an isomorphism. To do so, consider the map $H_1(Y_+) \rightarrow H_1(W)$. Since W is a $\mathbb{Q}$ -homology cobordism, we have $\text{rk}_{\mathbb{Z}} H_1(Y_+) = \text{rk}_{\mathbb{Z}} H_1(W)$, and we denote this number by k ; then the map in question is given by some homomorphism $R': \mathbb{Z}^k \oplus T_1 \rightarrow \mathbb{Z}^k \oplus T_2$, where T_1 and T_2 are torsion, with matrix

$$R' = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}.$$

Viewing W upside down, it is built from Y_+ by adding 2- and 3-handles, which implies that R' is surjective; in particular, $A: \mathbb{Z}^k \rightarrow \mathbb{Z}^k$ is also surjective, and thus an isomorphism. By the Universal Coefficient Theorem, the map $H^1(W) \rightarrow H^1(Y_+)$ is exactly given by the transpose $A^T: \mathbb{Z}^k \rightarrow \mathbb{Z}^k$, which is also an isomorphism. Returning to the exact sequence, we see that the third map is injective, showing that the extension from $\mathfrak{t}$ to $D(\mathfrak{t})$ is unique. $\square$

We are now ready to state the following refinement of Theorem 4.10.

Theorem 7.9. *Let Y_- and Y_+ be closed 3-manifolds, and suppose that $W: Y_- \rightarrow Y_+$ is a ribbon $\mathbb{Z}/2$ -homology cobordism. Fix a Spin^c -structure $\mathfrak{s}$ on Y_- . Then the sum of cobordism maps*

$$\left(\sum_{\substack{\mathfrak{t} \in \text{Spin}^c(W) \\ \mathfrak{t}|_{Y_-} = \mathfrak{s}}} F_{W,\mathfrak{t}}^\circ \right) : \text{HF}^\circ(Y_-, \mathfrak{s}) \rightarrow \bigoplus_{\substack{\mathfrak{t} \in \text{Spin}^c(W) \\ \mathfrak{t}|_{Y_-} = \mathfrak{s}}} \text{HF}^\circ(Y_+, \mathfrak{t}|_{Y_+})$$

includes $\text{HF}^\circ(Y_-, \mathfrak{s})$ into the codomain as a summand. In fact,

$$\left(\sum_{\substack{\mathfrak{t} \in \text{Spin}^c(W) \\ \mathfrak{t}|_{Y_-} = \mathfrak{s}}} \widehat{F}_{D(W), D(\mathfrak{t})} \right) : \widehat{\text{HF}}(Y_-, \mathfrak{s}) \rightarrow \widehat{\text{HF}}(Y_-, \mathfrak{s})$$

is the identity map.

Proof. The assertion that the first map is injective is a direct consequence of [Theorem 4.10](#), since it is simply the restriction of F_W° to the summand $\text{HF}^\circ(Y_-, \mathfrak{s})$ of $\text{HF}^\circ(Y_-)$. (However, in writing the codomain as the direct sum above, we have implicitly used the fact that for distinct $\mathfrak{t}_1, \mathfrak{t}_2 \in \text{Spin}^c(W)$, their restrictions $\mathfrak{t}_1|_{Y_+}, \mathfrak{t}_2|_{Y_+} \in \text{Spin}^c(Y_+)$ are distinct, which is a consequence of [Lemma 7.8](#).) The second assertion is obtained by restricting the identity map $\widehat{F}_{D(W)}$ in [Theorem 4.10](#) to the summand $\widehat{\text{HF}}(Y_-, \mathfrak{s})$, and observing that all Spin^c -structures on $D(W)$ are of the form $D(\mathfrak{t})$, which follows from [Lemma 7.8](#). $\square$

With the additional condition that W is a $\mathbb{Z}$ -homology cobordism, a Spin^c structure $\mathfrak{s}_-$ on Y_- determines a unique $\mathfrak{t}$ on W , and hence a unique $\mathfrak{s}_+$ on Y_+ . We have:

Corollary 7.10. *Let Y_- and Y_+ be 3-manifolds, and suppose that $W: Y_- \rightarrow Y_+$ is a ribbon $\mathbb{Z}$ -homology cobordism. Fix a Spin^c -structure $\mathfrak{s}_-$ on Y_- , and let $\mathfrak{t}$ and $\mathfrak{s}_+$ be the corresponding Spin^c structures on W and Y_+ respectively. Then the cobordism map $F_{W,\mathfrak{t}}^\circ: \text{HF}^\circ(Y_-, \mathfrak{s}_-) \rightarrow \text{HF}^\circ(Y_+, \mathfrak{s}_+)$ includes $\text{HF}^\circ(Y_-, \mathfrak{s}_-)$ into $\text{HF}^\circ(Y_+, \mathfrak{s}_+)$ as a summand.* $\square$

7.3. Sutured Heegaard Floer theory. First, we mention that the definition of a cobordism of sutured manifolds is given in [Definition 6.10](#).

We now use [Theorem 7.9](#) to prove the sutured analogue.

Proof of Theorem 4.12. Recall from Lekili's work [[Lek13](#), Theorem 24] that the sutured Floer homology of a sutured manifold (M, η) can be described in terms of the Heegaard Floer homology of the sutured closure $\widehat{M} = M \cup (F_{g,d} \times [-1, 1])$ and a closed surface R in $\widehat{M}$ obtained from $F_{g,d}$, where $F_{g,d}$ is a surface of genus $g \geq 2$ and d boundary components. For more details on the construction of $\widehat{M}$ and R , see [Section 6.5](#). Then we have the isomorphism

$$\text{SFH}(M, \eta) \cong \bigoplus_{\langle c_1(\mathfrak{s}), R \rangle = 2g-2} \text{HF}^+(\widehat{M}, \mathfrak{s}).$$

Now, given a ribbon $\mathbb{Z}/2$ -homology cobordism $N: (M_-, \eta_-) \rightarrow (M_+, \eta_+)$ between sutured manifolds, we can attach $F_{g,d} \times [-1, 1] \times I$ to obtain a ribbon $\mathbb{Z}/2$ -homology cobordism $\widehat{N}$ between the sutured closures. Furthermore, for any Spin^c -structure $\mathfrak{t}$ on $\widehat{N}$,

$$\langle c_1(\mathfrak{t}|_{\widehat{M}_-}), [R_{\widehat{M}_-}] \rangle = \langle c_1(\mathfrak{t}|_{\widehat{M}_+}), [R_{\widehat{M}_+}] \rangle.$$

Consequently, the desired result follows from [Theorem 7.9](#). $\square$

7.4. Involutive Heegaard Floer theory. We now extend our work in [Section 7.1](#) to prove [Theorem 4.15](#). Recall that $\widehat{\text{HFI}}(Y)$ is defined as the homology of the mapping cone of $1 + \iota$, where ι is a chain homotopy equivalence on $\widehat{\text{CF}}(Y)$ coming from Spin^c -conjugation. Since we are working over $\mathbb{Z}/2$, we have that $\widehat{\text{HFI}}(Y)$ is in fact isomorphic to the homology of the mapping cone of $1 + \iota_*: \widehat{\text{HF}}(Y) \rightarrow \widehat{\text{HF}}(Y)$. Unfortunately, the theory of cobordism maps is not fully developed in the theory, but we can still compare the involutive Heegaard Floer homologies under ribbon homology cobordisms.

Proof of Theorem 4.15. Fix a self-conjugate Spin^c -structure $\mathfrak{s}_-$ on Y_- , which determines a unique Spin^c -structure $\mathfrak{t}$ on W and a unique $\mathfrak{s}_+$ on Y_+ . Then we have the commutative diagram

$$\begin{array}{ccccc} \widehat{\text{HF}}(Y_-, \mathfrak{s}_-) & \xrightarrow{\widehat{F}_{W, \mathfrak{t}}} & \widehat{\text{HF}}(Y_+, \mathfrak{s}_+) & \xrightarrow{\widehat{F}_{-W, \mathfrak{t}}} & \widehat{\text{HF}}(Y_-, \mathfrak{s}_-) \\ \downarrow 1 + \iota_* & & \downarrow 1 + \iota_* & & \downarrow 1 + \iota_* \\ \widehat{\text{HF}}(Y_-, \mathfrak{s}_-) & \xrightarrow{\widehat{F}_{W, \mathfrak{t}}} & \widehat{\text{HF}}(Y_+, \mathfrak{s}_+) & \xrightarrow{\widehat{F}_{-W, \mathfrak{t}}} & \widehat{\text{HF}}(Y_-, \mathfrak{s}_-). \end{array}$$

The result now follows from [Theorem 7.9](#). □

8. SOME SPECIFIC OBSTRUCTIONS

In this section, we derive some obstructions to ribbon homology cobordisms from our results on character varieties ([Section 2](#)) and Floer homologies ([Section 4](#)).

8.1. Ribbon cobordisms between Seifert fibered homology spheres. First, we prove [Theorem 1.5](#), a statement on ribbon $\mathbb{Q}$ -homology cobordisms between Seifert fibered homology spheres. Since [Theorem 1.5](#) (1) and (2) will follow easily from instanton or Heegaard Floer homology, our main goal is to prove the following, which is a restatement of [Theorem 1.5](#) (3). The complete proofs of [Theorem 1.5](#) and [Corollary 1.6](#) are given at the end of this subsection.

Theorem 8.1. *Suppose that there exists a ribbon $\mathbb{Z}$ -homology cobordism from the Seifert fibered homology sphere $\Sigma(a_1, \dots, a_n)$ to $\Sigma(a'_1, \dots, a'_m)$. Then the numbers of fibers satisfy $n \leq m$.*

Our strategy will be to use [Proposition 1.18](#) with $G = \text{SU}(2)$. We begin by mentioning a basic fact about $\text{SU}(2)$ -representations. Every representation $\rho: \pi \rightarrow \text{SU}(2)$ is either trivial, Abelian, or irreducible, and $\dim_{\mathbb{R}} H^0(\pi; \text{Ad}_{\rho})$ is respectively 3, 1, or 0, according to this trichotomy.

We now review some useful facts from the work of Fintushel and Stern [[FS90](#)] on $\text{SU}(2)$ -representations for Seifert fibered homology spheres (see also the work of Boyer [[Boy88](#)]). To fix our notation, the Seifert fibered homology sphere $\Sigma(a_1, \dots, a_n)$ has base orbifold S^2 and presentation $(b; (a_1, b_1), \dots, (a_n, b_n))$, where we do not require that $0 < b_i < a_i$, but do require that a_i and b_i are relatively prime, and that

$$-b + \sum_{i=1}^n \frac{b_i}{a_i} = \frac{1}{a_1 \cdots a_n}.$$

Then the fundamental group of $\Sigma(a_1, \dots, a_n)$ is given by

$$\pi_1(\Sigma(a_1, \dots, a_n)) \cong \langle x_1, \dots, x_n, h \mid h \text{ central}, x_i^{a_i} = h^{-b_i}, x_1 \cdots x_n = h^{-b} \rangle.$$

Theorem 8.2 (Fintushel and Stern [[FS90](#)]). *Suppose that $\rho \in \mathcal{R}(\Sigma(a_1, \dots, a_n))$ is irreducible. Then*

- (1) [[FS90](#), Lemma 2.1] $\rho(h) = \pm 1$;
- (2) [[FS90](#), Lemma 2.2] $\rho(x_i) \neq \pm 1$ for at least three values of $i \in \{1, \dots, n\}$; and

(3) [FS90, Proposition 2.5] $\dim_{\mathbb{R}} H^1(\Sigma(a_1, \dots, a_n); \text{Ad}_{\rho}) = 2t - 6$, where t is the number of x_i 's such that $\rho(x_i) \neq \pm 1$.

We also recall their recipe for constructing conjugacy classes of irreducible $\text{SU}(2)$ -representations. For this, it will be useful to think of elements of $\text{SU}(2)$ as unit quaternions. After choosing the sign of $\rho(h)$, we may choose an integer ℓ_1 and define $\rho(x_1) = e^{i\pi\ell_1/a_1}$, as long as $0 \leq \ell_1 \leq a_1$ and $(-1)^{\ell_1} = (\rho(h))^{-b_1}$. Next, for each $q \in \{2, \dots, n\}$, we choose an integer ℓ_q with analogous constraints and consider $e^{i\pi\ell_q/a_q}$; we will eventually define $\rho(x_q)$ to be some element conjugate to this. (The choice of the integers ℓ_q is also subject to Theorem 8.2 (2).) Note that once we choose $\rho(x_2), \dots, \rho(x_{n-1})$, they will determine $\rho(x_n)$ by the equation

$$(8.3) \quad \rho(x_1) \cdots \rho(x_n) = (\rho(h))^{-b};$$

the difficulty, however, lies in ensuring that $\rho(x_n)$ is also conjugate to $e^{i\pi\ell_n/a_n}$ for some integer ℓ_n with analogous constraints. Plugging this last condition into (8.3), we see that $\rho(x_2), \dots, \rho(x_{n-1})$ must satisfy

$$(8.4) \quad (\rho(x_1) \cdots \rho(x_{n-1}))^{a_n} = (\rho(h))^{-ba_n+b_n} = \pm 1.$$

To fulfill this condition, let S_q denote the set of elements in $\text{SU}(2)$ conjugate to $e^{i\pi\ell_q/a_q}$, for each $q \in \{2, \dots, n-1\}$. If $e^{i\pi\ell_q/a_q} \neq \pm 1$, then S_q is a copy of S^2 . In any case, consider the map $\phi: S_2 \times \cdots \times S_{n-1} \rightarrow [0, \pi]$ given by

$$\phi(s_2, \dots, s_{n-1}) = \text{Arg}(\rho(x_1)s_2 \cdots s_{n-1}),$$

where $\text{Arg}(z)$ is defined to be the value $\theta \in [0, \pi]$ such that z is conjugate to $e^{i\theta}$. If $\pi\ell'_n/a_n$ is in the image of ϕ , then, for each integer ℓ'_n such that $0 \leq \ell'_n \leq a_n$ and $(-1)^{\ell'_n} = (\rho(h))^{-ba_n+b_n}$, there exists some choice of $\rho(x_2), \dots, \rho(x_{n-1})$ such that $\rho(x_1) \cdots \rho(x_{n-1})$ is conjugate to $e^{i\pi\ell'_n/a_n}$ (and hence (8.4) holds). This determines a well-defined representation ρ . Finally, note that since the Abelianization of $\pi_1(\Sigma(a_1, \dots, a_n))$ is trivial, there are no non-trivial Abelian $\text{SU}(2)$ -representations, and every non-trivial representation is irreducible.

We now proceed towards the proof of Theorem 8.1. The main technical proposition we will prove is the following. While this is well known, we include a direct proof for completeness.

Proposition 8.5. *Suppose that Y is the Seifert fibered homology sphere $\Sigma(a_1, \dots, a_n)$. Then there exists an irreducible $\rho \in \mathcal{R}(Y)$ such that $H^1(Y; \text{Ad}_{\rho})$ has the maximal dimension possible, i.e. $2n - 6$.*

We now briefly describe our strategy to prove this proposition. By Theorem 8.2 (3), we would like to show that $\pi_1(\Sigma(a_1, \dots, a_n))$ admits an irreducible $\text{SU}(2)$ -representation that does not send any x_i to ± 1 . The idea is to reduce to the case where there are exactly three singular fibers; in other words, we will construct such a representation from an irreducible $\text{SU}(2)$ -representation of $\pi_1(\Sigma(a_1, a_2, a_3 \cdots a_n))$, by a pinching argument. A subtlety here is that for this pinching argument to work, we will require the representation of $\pi_1(\Sigma(a_1, a_2, a_3 \cdots a_n))$ to be of a certain form; to show that this exists, we will assume the primality of the a_i 's. Thus, we begin by reducing to the case that all the a_i 's are prime.

Lemma 8.6. *Let $r \in \mathbb{Z}_{\geq 0}$. Suppose that $\pi_1(Y)$ admits an irreducible representation ρ such that $\dim_{\mathbb{R}} H^1(Y; \text{Ad}_{\rho}) = r$ for $Y \cong \Sigma(a_1, \dots, a_n)$. Then the same holds for $Y \cong \Sigma(a_1, \dots, a_{n-1}, ka_n)$, where k is relatively prime to $a_1, \dots, a_{n-1}$.*

Proof. Fix a presentation $(b; (a_1, b_1), \dots, (a_{n-1}, b_{n-1}), (ka_n, b_n))$ for $\Sigma(a_1, \dots, a_{n-1}, ka_n)$; then

$$\Sigma(a_1, \dots, a_n) = S^2(kb; (a_1, kb_1), \dots, (a_{n-1}, kb_{n-1}), (a_n, b_n)).$$

We denote by x_i and y_i the respective generators of $\pi_1(\Sigma(a_1, \dots, a_{n-1}, ka_n))$ and $\pi_1(\Sigma(a_1, \dots, a_n))$ associated to the singular fibers, but we abusively write h for the central generator in both groups. Consider the homomorphism $\phi: \pi_1(\Sigma(a_1, \dots, a_{n-1}, ka_n)) \rightarrow \pi_1(\Sigma(a_1, \dots, a_n))$ defined by

$$\phi(h) = h^k, \quad \phi(x_i) = y_i \text{ for all } i \in \{1, \dots, n\},$$

which can be easily checked to be well-defined. (For completeness, we note that ϕ is induced by the k -fold cover of $\Sigma(a_1, \dots, a_n)$ branched over the singular fiber of order a_n .)

Since $\rho \in \mathcal{R}(\Sigma(a_1, \dots, a_n))$ is irreducible, so is $\phi^*\rho \in \mathcal{R}(\Sigma(a_1, \dots, a_{n-1}, ka_n))$, and the number of x_i 's such that $\phi^*\rho(x_i) = \pm 1$ is exactly the same as the number of y_i 's such that $\rho(y_i) = \pm 1$. Therefore, by [Theorem 8.2 \(3\)](#), we conclude that $\dim_{\mathbb{R}} H^1(\Sigma(a_1, \dots, a_{n-1}, ka_n); \text{Ad}_{\phi^*\rho}) = \dim_{\mathbb{R}} H^1(\Sigma(a_1, \dots, a_n); \text{Ad}_{\rho}) = r$. $\square$

Next, we reduce to the case where there are exactly three singular fibers. Given $\Sigma(a_1, \dots, a_n)$, let $p = a_3 \cdots a_n$. Denote the generators for $\pi_1(\Sigma(a_1, \dots, a_n))$ corresponding to the singular fibers by $x_1, \dots, x_n$, and those for $\pi_1(\Sigma(a_1, a_2, p))$ by y_1, y_2 , and z respectively; we continue to write h for the central generator. Note that the a_i 's are not assumed to be prime in the following lemma; their primality will instead be used later.

Lemma 8.7. *Suppose that $n \geq 4$. Then there exists a surjective homomorphism*

$$f: \pi_1(\Sigma(a_1, \dots, a_n)) \rightarrow \pi_1(\Sigma(a_1, a_2, p))$$

such that, for every irreducible $\rho \in \mathcal{R}(\Sigma(a_1, a_2, p))$ where $\rho(z)$ is conjugate to $e^{i\pi\ell_z/p}$ with ℓ_z relatively prime to p , we have that $f^\rho \in \mathcal{R}(\Sigma(a_1, \dots, a_n))$ is irreducible and $f^*\rho(x_i) \neq \pm 1$ for all i ; in other words, $\dim_{\mathbb{R}} H^1(\Sigma(a_1, \dots, a_n); \text{Ad}_{f^*\rho})$ is maximal.*

Recall that there exists a degree-1 map from $\Sigma(a_1, \dots, a_n)$ to $\Sigma(a_1, a_2, a_3 \cdots a_n)$ given by pinching along a suitable vertical torus in the Seifert fibration. The homomorphism f above is induced by this map.

Proof. Fix a presentation $(b; (a_1, b_1), \dots, (a_n, b_n))$ for $\Sigma(a_1, \dots, a_n)$, and let $q = \sum_{i=3}^n pb_i/a_i$. Then

$$\Sigma(a_1, a_2, p) = S^2(b; (a_1, b_1), (a_2, b_2), (p, q)).$$

(Since the a_i 's are pairwise relatively prime, p and q are relatively prime.) The two fundamental groups are

$$\begin{aligned} \pi_1(\Sigma(a_1, \dots, a_n)) &\cong \langle x_1, \dots, x_n, h \mid h \text{ central}, x_i^{a_i} = h^{-b_i}, x_1 \cdots x_n = h^{-b} \rangle, \\ \pi_1(\Sigma(a_1, a_2, p)) &\cong \langle y_1, y_2, z, h \mid h \text{ central}, y_i^{a_i} = h^{-b_i}, z^p = h^{-q}, y_1 y_2 z = h^{-b} \rangle. \end{aligned}$$

With these presentations, we now define f . Since the a_i 's are pairwise relatively prime, for each $i \geq 3$, we may choose an integer η_i such that $\eta_i p/a_i \equiv 1 \pmod{a_i}$. Clearly, η_i and a_i are relatively prime for each i . We define

$$f(x_1) = y_1, \quad f(x_2) = y_2, \quad f(x_i) = z^{\alpha_i} h^{\beta_i} \text{ for } i \geq 3, \quad f(h) = h,$$

where $\alpha_i = \eta_i p/a_i$ and $\beta_i = (\eta_i q - b_i)/a_i$. (Note that f does not depend on the choice of η_i .) Observe that $\sum_{j=3}^n \alpha_j \equiv 1 \pmod{a_i}$ for each $i \geq 3$, which implies that $\sum_{j=3}^n \alpha_j \equiv 1 \pmod{p}$; using this fact, it is straightforward to check that f is a well-defined group homomorphism.

We now claim that, for an irreducible $\rho \in \mathcal{R}(\Sigma(a_1, a_2, p))$ satisfying the conditions in the lemma, we have $f^*\rho(x_i) \neq \pm 1$ for $i = 1, \dots, n$. This is clear for $i = 1$ and $i = 2$. For $i \geq 3$, suppose that $f^*\rho(x_i) = \pm 1$ for some $i \geq 3$; then $\rho(z)^{\eta_i p/a_i} = \pm 1$, implying that $e^{i\pi\ell_z\eta_i/a_i} = \pm 1$, which is a contradiction since η_i is relatively prime to a_i . $\square$

We now demonstrate the existence of an irreducible $\rho \in \mathcal{R}(\Sigma(a_1, a_2, p))$ satisfying the conditions of [Lemma 8.7](#), in the case that the a_i 's are pairwise prime.

Proposition 8.8. *Suppose that $n \geq 4$, and that $a_1 < \cdots < a_n$ are positive prime numbers. There exists an irreducible $\rho \in \mathcal{R}(\Sigma(a_1, a_2, p))$ such that $\rho(z)$ is conjugate to $e^{i\pi\ell_z/p}$, where ℓ_z is relatively prime to p .*

Proof. We continue to write $\Sigma(a_1, a_2, p) = S^2(b; (a_1, b_1), (a_2, b_2), (p, q))$, and use the same presentation for $\pi_1(\Sigma(a_1, a_2, p))$ as before. First, we make some general observations. Recall the construction of irreducible $SU(2)$ -representations in the paragraph after [Theorem 8.2](#). Observe that $\rho(z)$ is conjugate to $e^{i\pi\ell_z/p}$ for some ℓ_z relatively prime to p if and only if $\rho(y_1)\rho(y_2)$ is conjugate to $e^{i\pi\ell'_z/p}$ for some ℓ'_z relatively prime to p ; thus, the goal is to show that after picking appropriate values for $\rho(h)$, ℓ_1 , and ℓ_2 satisfying $(-1)^{\ell_j} = (\rho(h))^{-b_j}$ (for us to define $\rho(y_1) = e^{i\pi\ell_1/a_1}$ and decree $\rho(y_2)$ to be conjugate to $e^{i\pi\ell_2/a_2}$), there exists an integer ℓ'_z such that

- (1) ℓ'_z is relatively prime to p ;
- (2) $(-1)^{\ell'_z} = (\rho(h))^{-bp+q}$; and
- (3) $\pi\ell'_z/p$ is in the image of $\phi: S_2 \rightarrow [0, \pi]$.

Since ϕ is continuous, to satisfy (3), we simply need to exhibit choices $s_2, s'_2 \in S_2$ (i.e. elements s_2 and s'_2 that are conjugate to $e^{i\pi\ell_2/a_2}$) such that

$$(8.9) \quad \text{Arg}(\rho(y_1)s_2) \leq \frac{\pi\ell'_z}{p} \leq \text{Arg}(\rho(y_1)s'_2).$$

Our strategy will be to find s_2, s'_2 , and two values of ℓ'_z of opposite parities satisfying (1) and (3); then exactly one of them will satisfy (2). Finally, by construction, ρ is not trivial, and thus is irreducible.

First, we consider the special case that $a_1 = 2$. In this case, we may choose a presentation where $b_1 = 1$ and b_2 is odd (at the expense of changing b). We take $\rho(h) = -1$, $\ell_1 = 1$, and $\ell_2 = 1$. We claim that the image of ϕ contains $\pi r_{\pm}/p$, where $\ell'_z = r_{\pm} = (p \pm 1)/2$. Note that these two numbers are *both* integers relatively prime to p , and have opposite parities; thus, if $\pi r_{\pm}/p$ are both in the image of ϕ , the proof will be complete in this case. To prove our claim, note that if we choose $s_2 = e^{-i\pi/a_2}$ and $s'_2 = e^{i\pi/a_2}$, which are both conjugate to $e^{i\pi/a_2}$, then since $a_2 < p$, we have

$$\text{Arg}(e^{i\pi/2}e^{-i\pi/a_2}) = \frac{\pi(a_2 - 2)}{2a_2} \leq \frac{\pi(p \pm 1)}{2p} \leq \frac{\pi(a_2 + 2)}{2a_2} = \text{Arg}(e^{i\pi/2}e^{i\pi/a_2});$$

in other words, (8.9) is satisfied.

We may now assume that all the a_i 's are odd. Next, we consider the special case that $a_1 = 3$ and $a_2 = 5$. Again, we may choose a presentation where b_1 and b_2 are both odd, and take $\rho(h) = -1$ and $\ell_1 = \ell_2 = 1$. We again claim that the image of ϕ contains $\pi r_{\pm}/p$, where $\ell'_z = r_{\pm} = (p \pm 1)/2$. Indeed, if we choose $s_2 = e^{-i\pi/5}$ and $s'_2 = e^{i\pi/5}$, both of which are conjugate to $e^{i\pi/5}$, then since $p > 15$, we have

$$\text{Arg}(e^{i\pi/3}e^{-i\pi/5}) = \frac{2\pi}{15} < \frac{\pi(p \pm 1)}{2p} < \frac{8\pi}{15} = \text{Arg}(e^{i\pi/3}e^{i\pi/5}),$$

and (8.9) is satisfied.

By dispensing with the two cases above, we may assume that all the a_i 's are odd, and further that $1/a_1 + 1/a_2 < 1/2$. In this case, there are several choices we could take for ℓ_j and $\rho(h)$; for concreteness, we choose a presentation where b_1 and b_2 are both even, and take $\ell_1 = \ell_2 = 2$ and

$\rho(h) = 1$. We choose $s_2 = e^{-i\pi/2a_2}$ and $s'_2 = e^{i\pi/2a_2}$, and compute the arguments to be

$$0 < \text{Arg}(e^{i\pi/2a_1} e^{\pm i\pi/2a_2}) = 2\pi \left(\frac{1}{a_1} \pm \frac{1}{a_2} \right) < \pi.$$

As before, we now wish to find two values r and r' for ℓ'_z , of opposite parities, each relatively prime to p , such that (8.9) is satisfied, i.e.

$$2 \left(\frac{1}{a_1} - \frac{1}{a_2} \right) \leq \frac{r}{p}, \frac{r'}{p} \leq 2 \left(\frac{1}{a_1} + \frac{1}{a_2} \right).$$

Let I denote the interval governed by the above inequality. Note that the length of this interval is $4/a_2$, and $4/a_3 < 4/a_2 < 1$. Therefore, we may choose an integer k with $0 < k < k+2 < a_3$, such that $[k/a_3, (k+2)/a_3] \subset I$; we can rewrite this as

$$\left[\frac{k \frac{p}{a_3}}{p}, \frac{(k+2) \frac{p}{a_3}}{p} \right] \subset I.$$

In fact, since $p/a_3 > a_3$, we have

$$\frac{k \frac{p}{a_3} + a_3}{p}, \frac{k \frac{p}{a_3} + 2a_3}{p} \in I.$$

Let $r = kp/a_3 + a_3$ and $r' = kp/a_3 + 2a_3$. Note that r and r' are between 0 and p , and have opposite parities since a_3 is odd. It remains to see that r and r' are relatively prime to p . First, since $0 < k < a_3$ and a_3 is prime, we see that k is relatively prime to a_3 ; thus, $r, r' \equiv kp/a_3 \not\equiv 0 \pmod{a_3}$. At the same time, for $i > 3$, we observe that $r \equiv a_3 \pmod{a_i}$ and $r' \equiv 2a_3 \pmod{a_i}$; since the a_i 's are odd, prime, and greater than a_3 , we have that a_3 and $2a_3$ are also relatively prime to a_i . Putting this together, we conclude that r, r' are relatively prime to p , which completes the proof. $\square$

Proof of Proposition 8.5. Since the Casson invariant of any Seifert fibered homology sphere is never zero, we have that the result trivially holds for $n = 3$. For $n \geq 4$, the result follows from combining Lemma 8.6, Lemma 8.7, and Proposition 8.8. $\square$

Proof of Theorem 8.1. Theorem 8.2 (3) says that the Zariski tangent space to the $\text{SU}(2)$ -character variety of $\Sigma(a_1, \dots, a_n)$ has dimension less than or equal to $2n - 6$. By Proposition 8.5, the equality is always realized at some irreducible representation. The result then follows from Proposition 1.18. $\square$

Proof of Theorem 1.5. (1) This follows from Theorem 4.1, since $2|\lambda(Y)| = |\chi(I(Y))| = \dim I(Y)$ for a Seifert fibered homology sphere Y [Sav92].

(2) The only Seifert fibered homology sphere with trivial Casson invariant is S^3 , which bounds both positive- and negative-definite plumbings. Again by [Sav92], $I(Y_+)$ is supported in one $\mathbb{Z}/2$ -grading; this $\mathbb{Z}/2$ -grading determines the sign of $\lambda(Y_+)$ and hence the definiteness of the plumbing Y_+ bounds. Theorem 4.1 implies that $I(Y_-)$ is supported in the same $\mathbb{Z}/2$ -grading.

(3) This is Theorem 8.1. $\square$

Note that the first two items above can also be proved using Heegaard Floer homology, by [OSz03a, Theorem 1.3] and [OSz03b, Corollary 1.4].

Remark 8.10. While the conclusions in Theorem 1.5 seem strong, the authors do not know of any ribbon $\mathbb{Q}$ -homology cobordisms between two Seifert fibered homology spheres distinct from S^3 , or from a non-Seifert fibered space to a Seifert fibered space. For comparison, given any closed 3-manifold Y_- , one can always construct a ribbon $\mathbb{Q}$ -homology cobordism from Y_- to a hyperbolic 3-manifold, and one to a 3-manifold with non-trivial JSJ decomposition, as explained below.

To construct a ribbon $\mathbb{Q}$ -homology cobordism to a hyperbolic 3-manifold, first attach a 1-handle to Y_- , so that the positive end is $Y_- \# S^1 \times S^2$. Let $K \subset Y_- \# S^1 \times S^2$ be a hyperbolic knot that is homotopic to $S^1 \times \{p\}$; such a knot exists by a result of Myers [Mye93, Theorem 1.1]. Attaching a 2-handle along K with any framing will then yield a ribbon $\mathbb{Z}$ -homology cobordism. By Thurston's Hyperbolic Dehn Surgery Theorem, all but finitely many surgeries along K will yield a hyperbolic 3-manifold. In other words, for any choice among all but finitely many surgery slopes, we have constructed a ribbon $\mathbb{Z}$ -homology cobordism from Y_- to a hyperbolic 3-manifold.

To construct a ribbon $\mathbb{Q}$ -homology cobordism to a 3-manifold with non-trivial JSJ decomposition, recall that the exterior of a hyperbolic knot has incompressible boundary. Again, we attach a 1-handle to Y_- , and choose a hyperbolic knot $K \subset Y_- \# S^1 \times S^2$ that is homotopic to $S^1 \times \{p\}$. This time, we will attach a 2-handle along a satellite knot with K as the companion; for any framing, this will give a ribbon $\mathbb{Z}$ -homology cobordism as long as the pattern P of the satellite knot $P(K)$ has winding number 1, since K and $P(K)$ will be homologous. To carry this out, take a hyperbolic knot $P \subset S^1 \times D^2$ with winding number 1, such as the Mazur pattern; note that Thurston's Hyperbolic Dehn Surgery Theorem again implies that all but finitely many surgeries along P will give rise to a hyperbolic 3-manifold. A 3-manifold obtained via surgery along the satellite $P(K) \subset Y_- \# S^1 \times S^2$ can be expressed as the union of $Y_- \# S^1 \times S^2 \setminus K$ and a surgery along $P \subset S^1 \times D^2$. In other words, for any choice among all but finitely many surgery slopes, the positive end of the ribbon $\mathbb{Z}$ -homology cobordism we have constructed is obtained by gluing two hyperbolic 3-manifolds with incompressible torus boundary, which is exactly a 3-manifold with non-trivial JSJ decomposition.

Theorem 1.5 (3) immediately implies a statement on Montesinos knots.

Proof of Corollary 1.6. Let $C: K_- \rightarrow K_+$ be a strongly homotopy-ribbon concordance from K_- to K_+ . The branched double cover of C gives a ribbon $\mathbb{Q}$ -homology cobordism between Seifert fibered homology spheres $Y_{\pm}$, where the number of exceptional fibers in $Y_{\pm}$ is precisely the number of rational tangles in the Montesinos knot $K_{\pm}$ with denominator at most 2. $\square$

8.2. Ribbon homology cobordisms and L -spaces. We now utilize the U -action on HF^- to derive two obstructions to ribbon homology cobordisms involving L -spaces.

Recall that the *reduced Heegaard Floer homology* $\mathrm{HF}^{\mathrm{red}}$ is the U -torsion submodule of HF^- , and a $\mathbb{Q}$ -homology sphere Y is an L -space if $\mathrm{HF}^{\mathrm{red}}(Y) = 0$.¹³

Corollary 8.11. *Suppose that Y_- and Y_+ are $\mathbb{Q}$ -homology spheres, and that Y_+ is an L -space while Y_- is not. Then there does not exist a ribbon $\mathbb{Z}/2$ -homology cobordism from Y_- to Y_+ .*

Note that this applies whenever Y_- is a toroidal $\mathbb{Z}$ -homology sphere, since such a manifold is necessarily not an L -space [Eft18, Theorem 1.1] (see also [HRW17, Corollary 10]).

Proof. Suppose that $W: Y_- \rightarrow Y_+$ is a ribbon $\mathbb{Z}/2$ -homology cobordism; then Theorem 4.10 implies that $F_W^-: \mathrm{HF}^-(Y_-) \rightarrow \mathrm{HF}^-(Y_+)$ is injective. Under this map, U -torsion elements must be mapped to U -torsion elements; thus, we obtain an injection on $\mathrm{HF}^{\mathrm{red}}$ as well. $\square$

Corollary 8.12. *Suppose that Y_1 and Y_2 are $\mathbb{Q}$ -homology spheres that are not L -spaces. Then there does not exist a ribbon $\mathbb{Z}/2$ -homology cobordism from $Y_1 \# Y_2$ to a Seifert fibered space.*

Proof. If Y_1 and Y_2 both have non-trivial $\mathrm{HF}^{\mathrm{red}}$, then in both $\mathbb{Z}/2$ -gradings, $\mathrm{HF}^{\mathrm{red}}(Y_1 \# Y_2)$ is not trivial. Indeed, by the Künneth formula [OSz04b, Theorem 1.5], $\mathrm{HF}^{\mathrm{red}}(Y_1 \# Y_2)$ contains a summand isomorphic to two copies of $\mathrm{HF}^{\mathrm{red}}(Y_1) \otimes \mathrm{HF}^{\mathrm{red}}(Y_2)$, with one copy shifted in grading by 1. (One

¹³Technically, Y should be called a $\mathbb{Z}/2$ -Heegaard L -space. One could also define L -spaces with other coefficients, or with instanton Floer homology. However, we never consider these concepts of L -spaces in the present article.

comes from the tensor product and one from the Tor term.) Meanwhile, Seifert fibered spaces have $\mathrm{HF}^{\mathrm{red}}$ supported in a single $\mathbb{Z}/2$ -grading [OSz03b, Corollary 1.4]. $\square$

8.3. Ribbon homology cobordisms from connected sums. Corollary 8.12 concerns the existence of ribbon homology cobordisms from a connected sum. The following corollary also concerns connected sums, but is proved using $\mathcal{R}_G$.

Corollary 8.13. *Let Y and N be compact 3-manifolds, and suppose that $N \not\cong S^3$. Then there does not exist a ribbon $\mathbb{Q}$ -homology cobordism from $Y \sharp N$ to Y .*

Proof of Corollary 8.13. First, we fix some notation. Let π be a group, and let G be a compact, connected Lie group. Fix a presentation $\langle a_1, \dots, a_g \mid w_1, \dots, w_r \rangle$ of π . For each $\rho \in \mathcal{R}_G(\pi)$, the words w_i give a smooth map $\Phi: G^g \rightarrow G^r$; denote by ϕ_ρ the derivative of Φ at $(\rho(a_1), \dots, \rho(a_g))$, and we define $\omega_G: \mathcal{R}_G(\pi) \rightarrow \mathbb{Z}_{\geq 0}$ by

$$\omega_G(\rho) = \dim_{\mathbb{R}} \ker(\phi_\rho).$$

(The reader may wish to compare ϕ_ρ here with the map ϕ in (2.3).) It is not difficult to check that ω_G is independent of the presentation of π . We also define

$$\omega_G(\pi) = \max_{\rho \in \mathcal{R}_G(\pi)} \omega_G(\rho),$$

and write $\omega_G(X)$ for $\omega_G(\pi_1(X))$, for a path-connected space X .

Suppose that Y_- , Y_+ , W , ρ_- , ρ_+ , and ρ_W are as in Proposition 1.15. By Proposition 2.1, we have

$$\omega_G(\rho_-) \leq \omega_G(\rho_W) \leq \omega_G(\rho_+).$$

Indeed, since $\pi_1(W)$ is obtained from $\pi_1(Y_+)$ by adding relations, the matrix for $\omega_G(\rho_W)$ contains that for $\omega_G(\rho_+)$ as a block, with additional rows; similarly, the matrix for ϕ_{ρ_W} contains that for ϕ_{ρ_-} as a block, with m additional rows and columns. Consequently, we see that

$$(8.14) \quad \omega_G(Y_-) \leq \omega_G(W) \leq \omega_G(Y_+).$$

Now suppose that there exists a ribbon $\mathbb{Q}$ -homology cobordism $W: Y \sharp N \rightarrow Y$. Homology considerations show that N must be a $\mathbb{Z}$ -homology sphere (cf. Remark 1.13 and Lemma 3.1). This means that $\pi_1(N)$ cannot be solvable, since the Abelianization of a solvable group is trivial. Thus, the residual finiteness of 3-manifold groups implies the existence of a non-trivial, finite quotient H of $\pi_1(N)$. Since $\pi_1(N)$ is perfect, the quotient H is also perfect. Take any non-trivial, irreducible representation of H in $\mathrm{GL}_n(\mathbb{C})$; by Weyl's trick, we may turn it into a non-trivial, irreducible, unitary representation from H to $\mathrm{U}(n)$; since $\mathrm{U}(1)$ is Abelian and H is perfect, we may assume $n \geq 2$. (Of course, the possible choices for n depend on H .) Let $\eta: \pi_1(N) \rightarrow \mathrm{U}(n)$ be the associated representation, and choose $\rho \in \mathcal{R}_{\mathrm{U}(n)}(Y)$ that maximizes $\omega_{\mathrm{U}(n)}$. Consider the free product representation $\rho * \eta: \pi_1(Y \sharp N) \rightarrow \mathrm{U}(n)$. It follows from the definitions that

$$\omega_{\mathrm{U}(n)}(\rho * \eta) = \omega_{\mathrm{U}(n)}(\rho) + \omega_{\mathrm{U}(n)}(\eta).$$

It is easy to see that since η is irreducible, we have $\omega_{\mathrm{U}(n)}(\eta) > 0$. We see that $\omega_{\mathrm{U}(n)}(Y \sharp N) > \omega_{\mathrm{U}(n)}(Y)$, which contradicts (8.14). $\square$

Note that Corollary 8.13 can alternatively be proved if N has non-trivial $\mathrm{HF}^{\mathrm{red}}$, by an application of the Künneth formula, as in the proof of Corollary 8.12. However, such an argument does not work for $N = \Sigma(2, 3, 5) \sharp (-\Sigma(2, 3, 5))$, since this is an L -space. (In fact, in this case, $Y \sharp N$ is even $\mathbb{Z}$ -homology cobordant to Y .) The same issue arises for framed instanton Floer homology $\mathcal{I}^\sharp$. For $\mathcal{I}$, it is difficult to study the instanton Floer homology of connected sums in general.

[Corollary 8.13](#) can be viewed as obstructing ribbon homology cobordisms from a 3-manifold with an essential sphere to one without. We now turn to proving [Corollary 1.7](#), which is a statement for knots with a similar flavor: It states that there are no strongly homotopy-ribbon concordances from a connected sum $K_1 \sharp K_2$ to a knot without a closed, non-boundary-parallel, incompressible surface in its exterior.

Proof of [Corollary 1.7](#). Write $K_- \cong K_1 \sharp K_2$. For a knot $K \subset S^3$, denote a fixed meridian by μ , and a fixed longitude by λ . Also, write $\mathcal{R}(K)$ for $\mathcal{R}_{\mathrm{SU}(2)}(S^3 \setminus K)$ and $\mathcal{X}(K)$ for $\mathcal{X}_{\mathrm{SU}(2)}(S^3 \setminus K)$. First, recall from the proof of [\[Kla91, Proposition 12\]](#) that if $\rho_1 \in \mathcal{R}(K_1)$ and $\rho_2 \in \mathcal{R}(K_2)$ satisfy $\rho_1(\mu) = \rho_2(\mu)$, then we can amalgamate ρ_1 and ρ_2 into a 1-parameter family of representations $\rho \in \mathcal{R}(K_1 \sharp K_2)$, by fixing ρ on one summand and conjugating it on the other by the $\mathrm{U}(1)$ -stabilizer of the peripheral subgroup.

Now by [\[SZ22, Theorem 4.1\]](#), for a non-trivial knot $K \subset S^3$, at least one of the following holds:

- (1) There is a smooth 1-parameter family of irreducible $\rho_t \in \mathcal{R}(K)$, such that $\rho_t(\mu) = \mathrm{diag}(i, -i)$ and $\rho_t(\lambda) = \mathrm{diag}(e^{it}, e^{-it})$, for t in some interval (t_0, t_2) ; or
- (2) There is a smooth 1-parameter family of irreducible $\rho_s \in \mathcal{R}(K)$, such that $\rho_s(\mu) = \mathrm{diag}(e^{is}, e^{-is})$, where $s \in [\pi/2, \pi/2 + \epsilon)$, for some $\epsilon > 0$.

Note that in either case, there is a representation $\rho' \in \mathcal{R}(K)$ with $\rho'(\mu) = \mathrm{diag}(i, -i)$. (See also [\[KM10, Corollary 7.17\]](#).) If (2) holds for both K_1 and K_2 , then we may amalgamate representations with the same eigenvalue on the meridian to get a 2-parameter family $\rho \in \mathcal{R}(K_1 \sharp K_2)$. If (1) holds for K_1 , then we may amalgamate this 1-parameter family with $\rho' \in \mathcal{R}(K_2)$, again to obtain a 2-parameter family of representations $\rho \in \mathcal{R}(K_1 \sharp K_2)$.

In any case, note that these representations ρ can in fact be conjugated. Thus, we have shown that $\mathcal{R}(K_-)$ has an open submanifold of dimension at least 5 on which the conjugation action by $\mathrm{SO}(3)$ is free. Suppose that there is a strongly homotopy-ribbon concordance $C: K_- \rightarrow K_+$; [Proposition 2.1](#) implies that $\mathcal{R}(K_+)$ has an open submanifold of dimension at least 5 on which $\mathrm{SO}(3)$ acts freely. (Although the representation variety is not a smooth manifold in general, it is a real algebraic variety and hence can be stratified into the union of finitely many smooth manifolds [\[Whi57\]](#); it is easy to see that the maps ι_-^* and ι_+^* in [Proposition 2.1](#) induce smooth maps on an open subset of each stratum.) Thus, $\mathcal{X}(K_+)$ must have a component of dimension at least 2. By [\[Kla91, Proposition 15\]](#), this implies that K_+ is not small, which is a contradiction. $\square$

Remark 8.15. Eliashberg [\[Eli90\]](#) shows that a Stein filling of a connected sum is a boundary sum of Stein fillings. It is interesting to compare this with the two results above.

9. SURGERY OBSTRUCTIONS

In this section, we give some applications of the work above on ribbon homology cobordisms to reducible Dehn surgery problems. We first explain the main idea in this section. Let Y be a $\mathbb{Q}$ -homology sphere, and L a null-homologous link of ℓ components in Y . Denote by $Y_0(L)$ the result of 0-surgery along each component of L . Suppose that $Y_0(L) \cong N \sharp \ell(S^1 \times S^2)$; in this case, we may deduce facts about N or L with the following construction.

Consider the cobordism $W: Y \rightarrow N$ obtained by attaching a 0-framed 2-handle along each of the components of L , and then a 3-handle along some $\{p\} \times S^2$ in each of the $S^1 \times S^2$ summands of $Y_0(L)$. Flipping W upside down and reversing its orientation, we obtain a cobordism $-W: N \rightarrow Y$.

Lemma 9.1. *Suppose that Y is a $\mathbb{Q}$ -homology sphere, L is a null-homologous link of ℓ components in Y , and $Y_0(L) \cong N \sharp \ell(S^1 \times S^2)$. Then the cobordism $-W: N \rightarrow Y$ constructed above is a ribbon $\mathbb{Z}$ -homology cobordism.*

Proof. On the one hand, since $L \subset Y$ is null-homologous, we have $H_1(Y_0(L)) \cong H_1(Y) \oplus (\mathbb{Z}^\ell/M)$, where M is given by the linking matrix. On the other hand, we have $H_1(Y_0(L)) \cong H_1(N) \oplus \mathbb{Z}^\ell$. Since $b_1(Y) = 0$, rank considerations imply that M is trivial. (In particular, the linking matrix of L must be identically zero.) Thus, we have $H_1(Y) \cong H_1(N)$.

Now the cobordism $-W: N \rightarrow Y$ consists of the same number ℓ of 1- and 2-handles and so is ribbon. It is a $\mathbb{Q}$ -homology cobordism if and only if the attaching circles of the 2-handles are linearly independent in $H_1(N \sharp \ell(S^1 \times S^2))/H_1(N)$, which is obviously true since the 3-manifold resulting from the 2-handle attachments is Y , which has $b_1(Y) = 0$. By Lemma 3.2, $-W$ is in fact a ribbon $\mathbb{Z}$ -homology cobordism. $\square$

9.1. Null-homotopic links and reducing spheres. In this subsection, we focus on proving Theorem 1.8, which we recall asserts that when 0-surgery on an ℓ -component null-homotopic link in an irreducible $\mathbb{Q}$ -homology sphere Y produces $N \sharp \ell(S^1 \times S^2)$, then N is orientation-preserving homeomorphic to Y . Recall that a closed 3-manifold Y is aspherical if it is irreducible and has infinite fundamental group.

We begin by relating the fundamental groups of Y and N :

Proposition 9.2. *Suppose that Y is an irreducible $\mathbb{Q}$ -homology sphere, L is a null-homotopic link of ℓ components in Y , and $Y_0(L) \cong N \sharp \ell(S^1 \times S^2)$. Then there is an orientation-preserving degree-1 map from N to Y that induces isomorphisms on π_1 . Moreover, the inclusions of Y and N into the cobordism $W: Y \rightarrow N$ constructed above also induce isomorphisms on π_1 .*

Proof. Consider the $\mathbb{Z}$ -homology cobordism $W: Y \rightarrow N$ constructed above, and decompose W into two cobordisms $W_2: Y \rightarrow N \sharp \ell(S^1 \times S^2)$ and $W_3: N \sharp \ell(S^1 \times S^2) \rightarrow N$, corresponding to the attachment of 2- and 3-handles respectively. We claim that there is a retraction $\rho: W \rightarrow Y$.

Indeed, first observe that since L is null-homotopic, W_2 is homotopy equivalent to $Y \vee \ell S^2$, which retracts onto Y . More precisely, there is a retraction $\rho_2: W_2 \rightarrow Y$ given by deformation retracting the 2-handles to their cores, homotoping the attaching curves of the 2-handles to a point, and collapsing the resulting S^2 summands. Next, we see that ρ_2 extends to W_3 . Indeed, to prove that ρ_2 extends over the 3-handles, it suffices to see that for each $S^1 \times S^2$ summand in $N \sharp \ell(S^1 \times S^2)$, the image $\rho_2(\{p\} \times S^2) \subset Y$ is null-homotopic. (Recall that the 3-handles are attached along the ℓ copies of $\{p\} \times S^2$.) Since Y is irreducible, the Sphere Theorem implies that $\pi_2(Y) = 0$, and ρ_2 extends to a retraction $\rho: W \rightarrow Y$.

We now claim that pre-composing ρ with the inclusion $\iota_N: N \rightarrow W$ results in a map $\rho \circ \iota_N: N \rightarrow Y$ that induces an isomorphism on π_1 . First, we check that the induced map is injective. Indeed, by Lemma 9.1, $-W: N \rightarrow Y$ is a ribbon $\mathbb{Z}$ -homology cobordism, which implies that $(\iota_N)_*: \pi_1(N) \rightarrow \pi_1(W)$ is injective by Theorem 1.14 (1). Turning to $\rho_*: \pi_1(W) \rightarrow \pi_1(Y)$, note that since ρ is a retraction, we have $\rho \circ \iota_Y = \text{id}_Y$. This implies that $(\iota_Y)_*: \pi_1(Y) \rightarrow \pi_1(W)$ is injective; at the same time, Theorem 1.14 (2) states that $(\iota_Y)_*$ is surjective. Thus, $(\iota_Y)_*$, and hence ρ_* , is an isomorphism. This shows that $(\rho \circ \iota_N)_*$ is injective.

Next, observe that $H_3(W) \cong H_3(Y) \cong \mathbb{Z}$, which implies that $\rho_*: H_3(W) \rightarrow H_3(Y)$ is an isomorphism, since ρ is a retraction. In fact, since $\iota_N: N \rightarrow W$ also induces an isomorphism on H_3 , we see that $(\rho \circ \iota_N)_*: H_3(N) \rightarrow H_3(Y)$ sends $[N]$ to $[Y]$ (and not $-[Y]$), which means that $\rho \circ \iota_N: N \rightarrow Y$ is an orientation-preserving degree-1 map.

We claim that such a map must induce a surjection $(\rho \circ \iota_N)_*: \pi_1(N) \rightarrow \pi_1(Y)$, for otherwise we could factor the map through a non-trivial cover $\tilde{Y}$ of Y corresponding to $(\rho \circ \iota_N)_*(\pi_1(N))$, showing that its degree is not 1. We conclude that $(\rho \circ \iota_N)_*$ is an isomorphism. This gives the first claim. For the second claim, since ρ_* is an isomorphism, we see that $(\iota_N)_*$ is also an isomorphism. Since we have already proved that $(\iota_Y)_*$ is an isomorphism, this concludes the proof. $\square$

To deal with the case that Y is a spherical manifold, we will need one additional technical lemma. In what follows, we will fix a basepoint $p_Y \in Y$ and a basepoint $p_N \in N$. We will say that $\hat{Y}$ is an *unoriented* (resp. *oriented*) $\tilde{Y}$ -cover of Y if $\hat{Y}$ corresponds to a concrete subgroup of $\pi_1(Y, p_Y)$ and $\hat{Y}$ is unoriented (resp. orientation-preserving) homeomorphic to $\tilde{Y}$. Here, we do *not* consider subgroups up to conjugacy. Note that, since Y is spherical, $\pi_1(Y, p_Y)$ is finite, and so all covers we consider are finite.

Lemma 9.3. *Suppose that $\tilde{Y}$ does not admit an orientation-reversing homeomorphism, and that both the hypotheses and conclusions of [Theorem 1.8](#) hold for $\tilde{Y}$. Suppose that Y also satisfies the hypotheses of [Theorem 1.8](#). If Y is unoriented homeomorphic to N and has an odd number n of unoriented $\tilde{Y}$ -covers, then Y satisfies the conclusions of [Theorem 1.8](#).*

Proof. By assumption, we know that Y and N are homeomorphic. We assume for contradiction that they are not orientation-preserving homeomorphic, and in particular that $Y \cong -N$.

Suppose that $Y_0(L) \cong N \sharp \ell(S^1 \times S^2)$ and consider the (non-ribbon) $\mathbb{Z}$ -homology cobordism $W: Y \rightarrow N$ constructed in the paragraph preceding [Lemma 9.1](#). Since we are working with covers, we will be a bit pedantic with basepoints for the cautious reader. Pick a path γ in W that starts at $p_Y \in Y$ and ends at $p_N \in N$; this gives rise to a change-of-basepoint isomorphism $\Phi_\gamma: \pi_1(W, p_Y) \rightarrow \pi_1(W, p_N)$. By [Proposition 9.2](#), the inclusions $(\iota_Y)_*: \pi_1(Y, p_Y) \rightarrow \pi_1(W, p_Y)$ and $(\iota_N)_*: \pi_1(N, p_N) \rightarrow \pi_1(W, p_N)$ are isomorphisms. (In this pedantic language, the map on π_1 induced by ρ in [Proposition 9.2](#) is $\rho_*: \pi_1(W, p_Y) \rightarrow \pi_1(Y, p_Y)$, and the isomorphism $(\rho \circ \iota_N)_*$ is really $\rho_* \circ \Phi_\gamma^{-1} \circ (\iota_N)_*$.)

Choose an oriented $\tilde{Y}$ -cover $\hat{Y}$ corresponding to a subgroup $H \leq \pi_1(Y, p_Y)$ of index $[\pi_1(Y, p_Y) : H] = h$. Consider $(\iota_Y)_*(H) \leq \pi_1(W, p_Y)$ and its associated oriented cover $\hat{W}$ of W . This is a cobordism whose incoming end is $\hat{Y}$, and whose outgoing end $\hat{N}$ is the oriented cover of N corresponding to $(\iota_N)_*^{-1} \circ \Phi_\gamma \circ (\iota_Y)_*(H) \leq \pi_1(N, p_N)$. Note that $\hat{N}$ is path connected, because N is path connected and $\hat{N}$ corresponds to a concrete subgroup of $\pi_1(N, p_N)$. Since $(\iota_Y)_*$ and $(\iota_N)_*$ are isomorphisms, each oriented $\tilde{Y}$ -cover $\hat{Y}$ of Y produces a distinct oriented cover $\hat{N}$ of N .

Since W is built out of the same number ℓ of 2- and 3-handles, we see that $\hat{W}$ is built by attaching the same number $h\ell$ of 2- and 3-handles to $\hat{Y}$. (To see this, simply pull back a Morse function on W using the covering map.) Since the attaching curves for the 2-handles in W are null-homotopic, the attaching curves for the 2-handles in $\hat{W}$ form a null-homotopic link $\hat{L} \subset \hat{Y}$. Note also that $\hat{N}$ does not contain any non-separating 2-spheres, since $b_1(\hat{N}) = 0$. Now, since $\hat{N}$ is connected, the result of the surgery along $\hat{L}$ in $\hat{Y}$ must be of the form $\hat{N} \sharp h\ell(S^1 \times S^2)$, since, from the upside-down perspective, it is also the result of attaching 1-handles along $h\ell$ copies of $S^0 \times D^3$. For homology reasons, we see that the surgery coefficients for $\hat{L} \subset \hat{Y}$ must be identically 0. By our assumption, [Theorem 1.8](#) holds for $\tilde{Y}$, and so we see that $\hat{N}$ is orientation-preserving homeomorphic to $\hat{Y}$, and hence to $\tilde{Y}$.

Note that a simple orientation-reversal argument shows that [Theorem 1.8](#) also holds for $-\tilde{Y}$, and so we may also repeat the above arguments for oriented $(-\tilde{Y})$ -covers, where the corresponding covers of N are then orientation-preserving homeomorphic to $-\tilde{Y}$.

Suppose that Y has n_+ (resp. n_-) oriented $\tilde{Y}$ -covers (resp. $(-\tilde{Y})$ -covers); then $n = n_+ + n_-$. Then $N \cong -Y$ has n_+ (resp. n_-) oriented $(-\tilde{Y})$ -covers (resp. $\tilde{Y}$ -covers). However, by the above argument, each oriented $(\pm\tilde{Y})$ -cover of Y induces a distinct oriented $(\pm\tilde{Y})$ -cover of N , which implies that $n_+ = n_-$. This contradicts the fact that n is odd. $\square$

With this, we are ready to prove [Theorem 1.8](#). We use the notation as above.

Proof of Theorem 1.8. First, suppose that Y is aspherical. Since Y is irreducible and π_1 detects irreducibility [Sta59], it follows from Proposition 9.2 that N is also irreducible. Proposition 9.2 also states that there is an orientation-preserving, degree-1 map from N to Y that induces an isomorphism on π_1 . Asphericity implies that this map is an orientation-preserving homotopy equivalence by Whitehead's Theorem. It is thus homotopic to a homeomorphism by the Borel Conjecture in dimension 3; see, for example, [KL09, Theorem 0.7]. Note that this homeomorphism must also preserve orientation, since this property is preserved under homotopy. This concludes the proof when Y is an aspherical $\mathbb{Q}$ -homology sphere.

Therefore, we may assume that $\pi_1(Y)$ is finite, or equivalently, that Y and N have spherical geometry. We first dispense with the case that Y and N are lens spaces. Recall from Lemma 9.1 that N and Y are $\mathbb{Z}$ -homology cobordant. Two $\mathbb{Z}$ -homology cobordant lens spaces are orientation-preserving homeomorphic; see, for example, the discussion in [DW15]. This concludes the proof when Y is a lens space.

It remains to consider the case that Y is spherical but not a lens space. Recall that two spherical 3-manifolds with isomorphic fundamental groups are (possibly orientation-reversing) homeomorphic unless they are lens spaces; therefore, Proposition 9.2 implies that N and Y are homeomorphic. Recall also that a non-lens space spherical 3-manifold Y has π_1 isomorphic to a central extension of a polyhedral group, and in particular, $|\pi_1(Y)| = 2^k m$, where $k \geq 2$ and m is odd; see [AFW15, Section 1.7] and [Ori72, Section 6.2]. In the following, we will provide a proof, first for the case that $m = 1$ by inducting on k , and then generalizing to the case that $m \geq 3$.

Before we proceed, we collect three additional observations here. First, if the fundamental group of a lens space has order 2^k with $k \geq 2$, then it does not admit an orientation-reversing homeomorphism, since -1 is not a square mod 2^k . Second, a non-lens space spherical manifold Y also does not admit an orientation-reversing homeomorphism. Indeed, by considering an order-4 subgroup of $\pi_1(Y)$, we see that such a manifold has an unoriented $L(4, 1)$ -cover. (Recall that there are no 3-manifolds Y with $\pi_1(Y) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, and $L(4, 3) \cong -L(4, 1)$.) An orientation-reversing homeomorphism of Y would induce an orientation-reversing homeomorphism of $L(4, 1)$, which is impossible. Third, the number of index-2 subgroups of any finite group π is odd. Indeed, the index-2 subgroups of π correspond to non-trivial homomorphisms to $\mathbb{Z}/2$, and $\text{Hom}(\pi, \mathbb{Z}/2)$ is a vector space over $\mathbb{Z}/2$, which has even cardinality.

We begin by showing that the theorem holds in the case that $m = 1$. As mentioned, we will induct on k ; to help illustrate the idea of the proof, we will give a more detailed description for small values of k . If $k = 2$, then Y is a lens space $L(4, q)$. If $k = 3$, then $\pi_1(Y)$ has an odd number of index-2 subgroups; in other words, Y has an odd number of unoriented $L(4, 1)$ -covers. We may thus apply Lemma 9.3 with $\tilde{Y} = L(4, 1)$, and conclude that the theorem holds for spherical manifolds with order 8.

Next, if $k = 4$, again $\pi_1(Y)$ has an odd number of index-2 subgroups. This means that the total number of unoriented covers of Y (of possibly distinct unoriented homeomorphism types) whose π_1 has order 8 is odd. Hence, there must be an unoriented homeomorphism type $\tilde{Y}$ of spherical manifolds with $|\pi_1(\tilde{Y})| = 8$, such that Y has an odd number of (2-sheeted) unoriented $\tilde{Y}$ -covers. Again we may apply Lemma 9.3 with this choice of $\tilde{Y}$, and the proof is complete for those spherical manifolds Y with $|\pi_1(Y)| = 16$. (Here, $\tilde{Y}$ may be a lens space $L(8, q)$, or the unique type-**D** manifold with π_1 isomorphic to the quaternion group Q_8 ; in either case, the actual homeomorphism type of $\tilde{Y}$ is irrelevant, and we are simply relying on the fact that there is no orientation-reversing homeomorphism of $\tilde{Y}$, in order to invoke Lemma 9.3.) Similarly, we may now continue this induction on k to complete the proof in the case that $|\pi_1(Y)| = 2^k$ with $k \geq 2$.

Finally, we consider the case that Y is a non-lens space spherical manifold with $|\pi_1(Y)| = 2^k m$, where $k \geq 2$ and $m \geq 3$ is odd. In this case, the Sylow 2-subgroups of $\pi_1(Y)$ have order 2^k , and there are an odd number of them by the Third Sylow Theorem. Hence, there exists an unoriented homeomorphism type $\tilde{Y}$ of spherical manifolds with $|\pi_1(\tilde{Y})| = 2^k$, such that Y has an odd number of finite, unoriented $\tilde{Y}$ -covers. We may now apply a similar argument using [Lemma 9.3](#). This completes the proof. $\square$

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