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Completing an Operator Matrix and the Free Joint Numerical Radius

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Abstract

Ando's classical characterization of the unit ball in the numerical radius norm was generalized by Farenick, Kavruk, and Paulsen using the free joint numerical radius of a tuple of Hilbert space operators $(X_1, \dots, X_m)$. In particular, the characterization leads to a positive definite completion problem. In this paper, we study various aspects of Ando's result in this generalized setting. Among other things, this leads to the study of finding a positive definite solution L to the equation

$$L = I + \sum_{j=1}^{m} \left[\left(L^{\frac{1}{2}} X_j^* L X_j L^{\frac{1}{2}} + \frac{1}{4} I \right)^{\frac{1}{2}} + \left(L^{\frac{1}{2}} X_j L X_j^* L^{\frac{1}{2}} + \frac{1}{4} I \right)^{\frac{1}{2}} \right],$$

which may be viewed as a fixed point equation. Once such a fixed point is identified, the desired positive definite completion is easily obtained. Along the way we derive other related results including basic properties of the free joint numerical radius and an easy way to determine the free joint numerical radius of a tuple of generalized permutations. Finally, we present some open problems.

Keywords Free joint numerical radius · Matrix completion · Fixed point · Generalized permutation

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1 Introduction

For a bounded Hilbert space operator $X \in B(\mathcal{H})$, the *numerical radius* is defined by

$$w(X) = \sup\{|\langle Xh, h\rangle| : \ h \in \mathcal{H}, \ \|h\| = 1\}.$$

The numerical radius corresponds to the radius of the smallest circle centered at 0 that contains the *numerical range*

$$W(X) = \{\langle Xh, h\rangle : \ h \in \mathcal{H}, \ \|h\| = 1\}.$$

Ando's [1] well known characterization of operators whose numerical radius is at most 1 states that $w(X) \leq 1$ if and only if there exists $Z = Z^* \in B(\mathcal{H})$ so that

$$\begin{bmatrix} I - Z & X \\ X^* & I + Z \end{bmatrix} \geq 0,$$

where $T \geq 0$ is shorthand for T being a positive semidefinite operator. Equivalently, $w(X) \leq 1$ if and only if there exist $A_1, A_2 \in B(\mathcal{H})$ with $A_1 + A_2 = I$ so that

$$\begin{bmatrix} A_1 & \frac{X}{2} \\ \frac{X^*}{2} & A_2 \end{bmatrix} \geq 0. \tag{1}$$

One way to prove Ando's result is to observe that $w(X) \leq 1$ if and only if

$$Q(e^{i\theta}) = I - \mathrm{Re}(e^{i\theta}X) \geq 0, \text{ for all } \theta \in [0, 2\pi],$$

and subsequently use Fejér-Riesz factorization

$$I - z\frac{X}{2} - \overline{z}\frac{X^*}{2} = Q(z) = (P_0 + P_1 z)^*(P_0 + P_1 z), \ |z| = 1.$$

Now $P_0^* P_0 + P_1^* P_1 = I$ and $P_0^* P_1 = -\frac{X}{2}$ and thus

$$0 \leq \begin{bmatrix} P_0^* \\ -P_1^* \end{bmatrix} \begin{bmatrix} P_0 & -P_1 \end{bmatrix} =: \begin{bmatrix} A_1 & \frac{X}{2} \\ \frac{X^*}{2} & A_2 \end{bmatrix}$$

where $A_1 = P_0^* P_0$ and $A_2 = P_1^* P_1$ satisfy $A_1 + A_2 = I$.

There are different ways to find A_1 and A_2 so that (1) holds. In finite dimensions, one can find A_1 and A_2 numerically by using semidefinite programming, as a block matrix (1) is in the intersection of the cone of positive semidefinite matrices and the affine space

$$\left\{ \begin{bmatrix} I & \frac{X}{2} \\ \frac{X^*}{2} & 0 \end{bmatrix} + \begin{bmatrix} -Z & 0 \\ 0 & Z \end{bmatrix} : Z = Z^* \right\}.$$

Semidefinite programming is exactly designed to handle such a situation.

An alternative process to arrive at (1), which was used by Ando in his original paper, is to consider $Z_k = Z_k^*$ defined via

$$\langle Z_k h, h \rangle = \inf_{h_1,\ldots,h_k} \left\langle \begin{bmatrix} I & \frac{X}{2} & \cdots & 0 \\ \frac{X^*}{2} & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{X}{2} \\ 0 & \cdots & \frac{X^*}{2} & I \end{bmatrix} \begin{bmatrix} h \\ h_1 \\ \vdots \\ h_k \end{bmatrix}, \begin{bmatrix} h \\ h_1 \\ \vdots \\ h_k \end{bmatrix} \right\rangle.$$

Then Z_k converges decreasingly to Z, say; and we obtain

$$\begin{bmatrix} I - Z & \frac{X}{2} \\ \frac{X^*}{2} & Z \end{bmatrix} \geq 0 \tag{2}$$

yielding representation (1). In fact, this process yields the maximal Z in (2) (and gives a co-outer factorization of $Q(z)$). In the case when $w(X) < 1$, this leads to the iterative scheme

$$Z_1 = I \text{ and } Z_{k+1} = I - \frac{X}{2} Z_k^{-1} \frac{X^*}{2} \text{ for } k \in \mathbb{N},$$

which monotonically decreases; see Algorithm 4.1 in [2].

In [3], Ando's result was generalized to the multivariable setting as follows.

Theorem 1.1 [3, Theorem 3.4] *Let $X_1, \ldots, X_m \in B(\mathcal{H})$. The following are equivalent:*

(i) $w(X_1, \ldots, X_m) < \frac{1}{2}$.

(ii) *There exist $A_1, \ldots, A_{m+1} \in B(\mathcal{H})$ so that $\sum_{j=1}^{m+1} A_j = I$ and*

$$\Gamma(A_1, \ldots, A_{m+1}) := \begin{bmatrix} A_1 & X_1 & 0 & \cdots & 0 \\ X_1^* & A_2 & X_2 & & \vdots \\ 0 & X_2^* & \ddots & \ddots & 0 \\ \vdots & & \ddots & A_m & X_m \\ 0 & \cdots & 0 & X_m^* & A_{m+1} \end{bmatrix} > 0. \tag{3}$$

In (3), $T > 0$ is shorthand for T being a positive definite operator. Condition (i) in Theorem 1.1 concerns the *free joint numerical radius* of a tuple of m Hilbert space operators $X_1, \ldots, X_m \in B(\mathcal{H})$, defined as

$$w(X_1, \ldots, X_m) = \sup\{w(X_1 \otimes U_1 + \cdots + X_m \otimes U_m)\},$$

where the supremum is taken over every Hilbert space $\mathcal{K}$, every choice of m unitaries $U_1, \ldots, U_m \in B(\mathcal{K})$, and the tensor product is *spatial*, which can be defined as follows. Consider an inner product on the algebraic tensor of $\mathcal{H}$ and $\mathcal{K}$ by letting $\langle h_1 \otimes k_1, h_2 \otimes k_2\rangle := \langle h_1, h_2\rangle_{\mathcal{H}} \cdot \langle k_1, k_2\rangle_{\mathcal{K}}$ for all $h_1, h_2 \in \mathcal{H}$, $k_1, k_2 \in \mathcal{K}$, and then extending linearly. Denote by $\mathcal{H} \otimes \mathcal{K}$ the resulting Hilbert space after completion. For $R \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$, consider defining a map $(R \otimes S)(h \otimes k) := (Rh) \otimes (Sk)$ for all $h \in \mathcal{H}$ and $k \in \mathcal{K}$, and then extending linearly. The resulting operator $R \otimes S$ has the property that $\|R \otimes S\| = \|R\| \cdot \|S\|$. Hence, the algebraic tensor of $B(\mathcal{H})$ and $B(\mathcal{K})$ naturally inherits a norm (called the *spatial tensor norm*) as a subset of $B(\mathcal{H} \otimes \mathcal{K})$. Taking the closure with respect to the spatial tensor norm yields a C^*-subalgebra of $B(\mathcal{H} \otimes \mathcal{K})$.

The free joint numerical radius coincides with the classical numerical radius when there is only one operator ($m = 1$), and Theorem 1.1 reduces to Ando's classical result. The objective of this paper is to pursue the different aspects of Ando's result in this more general setting. This includes (i) finding a solution using semidefinite programming; (ii) finding a solution via an iterative scheme (which may have the potential to generalize to the infinite dimensional case); and (iii) exploring the connection with factorization. As we will see, along the way we derive other related results including basic properties of the free joint numerical radius and an easy way to determine the free joint numerical radius of a tuple of generalized permutations.

Our approach to solve for $A_1, \ldots, A_{m+1}$ in (3) will be different than Ando's. We will show, in finite dimensions, that a solution $A_1, \ldots, A_{m+1}$ in (3) exists exactly when the function $f_{X_1,\ldots,X_m}$ defined below has a positive definite fixed point. For a given tuple $X_1, \ldots, X_m \in B(\mathcal{H})$ and for any $Z \geq 0$, define

$$f_{X_1,\ldots,X_m}(Z) := I + \sum_{j=1}^{m} \left[\left(Z^{\frac{1}{2}} X_j^* Z X_j Z^{\frac{1}{2}} + \frac{1}{4} I \right)^{\frac{1}{2}} + \left(Z^{\frac{1}{2}} X_j Z X_j^* Z^{\frac{1}{2}} + \frac{1}{4} I \right)^{\frac{1}{2}} \right]. \tag{4}$$

Operator monotonicity of $t^{\frac{1}{2}}$ implies $f_{X_1,\ldots,X_m}(Z) \geq (m+1)I > 0$ for any $Z \geq 0$.

Theorem 1.2 *Let $X_1, \ldots, X_m \in B(\mathcal{H})$. Consider the following statements:*

(i) *$f_{X_1,\ldots,X_m}$ as defined in (4) has a positive definite fixed point, i.e., there exists positive definite $L \in B(\mathcal{H})$ for which*

$$L = I + \sum_{j=1}^{m} \left[\left(L^{\frac{1}{2}} X_j^* L X_j L^{\frac{1}{2}} + \frac{1}{4} I \right)^{\frac{1}{2}} + \left(L^{\frac{1}{2}} X_j L X_j^* L^{\frac{1}{2}} + \frac{1}{4} I \right)^{\frac{1}{2}} \right], \tag{5}$$

(ii) $w(X_1, \dots, X_m) < \frac{1}{2}$.

Then (i)→(ii). If $\dim(\mathcal{H}) < \infty$*, then (ii)→(i).*

Corollary 1.3 *Let* $X_1, \dots, X_m \in B(\mathcal{H})$ *with* $\dim(\mathcal{H}) < \infty$*. Then (i) and (ii) in Theorem 1.2 are equivalent.*

We prove Theorem 1.2 using matrix completion techniques (Sect. 2). A discussion of difficulties encountered in generalizing $(ii) \to (i)$ to the infinite dimensional case is presented in Remarks 2.6 and 4.11. Once a positive definite fixed point for $f_{X_1,\dots,X_m}$ is identified, there is an easy construction for the unknowns $A_1, \dots, A_{m+1}$ in (3) (which works in all dimensions); see Proposition 2.5.

In order to find a solution L to (5), one can use well known iterative schemes to find such a fixed point, with the iterative scheme $L_{k+1} = f_{X_1,\dots,X_m}(L_k)$ being the standard choice. The choice of a starting point is of course important, and we have found that the choice $L_1 = (m+1)I$ (which is the fixed point when $X_1 = \dots = X_m = 0$) works perfectly numerically, and in fact we find that the corresponding sequence $\{L_k\}_{k\in\mathbb{N}}$ is monotonically nondecreasing in the Loewner partial ordering. Recall that the Loewner partial ordering on Hermitian operators is given by $R \le S$ if and only if $S - R \ge 0$. This leads to the following conjecture.

Conjecture 1.4 *Let* $X_1, \dots, X_m \in B(\mathcal{H})$*. Consider the recurrence*

$$L_1 = (m+1)I \text{ and } L_{k+1} = f_{X_1,\dots,X_m}(L_k) \text{ for } k \in \mathbb{N}, \tag{6}$$

where $f_{X_1,\dots,X_m}$ *is defined in* (4)*. Then*

(i) $L_k \le L_{k+1}$ *for all* $k \in \mathbb{N}$*.*
(ii) *If* $w(X_1, \dots, X_m) < \frac{1}{2}$*, then* $\{L_k\}_{k\in\mathbb{N}}$ *converges in the weak operator topology to a fixed point* $L \in B(\mathcal{H})$ *of* $f_{X_1,\dots,X_m}$*.*

In general, $L_1 = (m+1)I \le f(L_1) = L_2$. We will prove Conjecture 1.4 in the case when $X_1, \dots, X_m$ are generalized permutations, i.e., each X_j is the product of a permutation matrix and a diagonal matrix; see Theorems 4.6 and 4.13. It is worthwhile to observe that our iterative scheme has a different origin than the iterative scheme from Ando's work. Indeed, in Ando's approach one maximizes A_2 in (1) (in the Loewner partial order) while our approach is based on maximizing the determinant of (1). Even though our approach is based on finite dimensional considerations, the iteration scheme can also be defined in infinite dimensional settings. It is our hope that a convergence proof for that case can be obtained in the future.

Aside from the results mentioned above, we will also cover the following. In Sect. 3, we will show some basic properties of the free joint numerical radius. In Sect. 4, we will prove a closed formula for the free joint numerical radius of a tuple of n-by-n generalized permutations. In Sect. 5, we will describe how to use semidefinite programming to numerically compute $w(X_1, \dots, X_m)$ for a tuple of n-by-n matrices. In Sect. 6, we will prove a limit formula for the free joint numerical radius of a tuple of generalized permutations on infinite dimensional separable Hilbert spaces. In Sect. 7, we will discuss the connection with factorization of Hermitian pencils, and in the final section we will highlight some open problems.

2 Existence of a Fixed Point Using Matrix Completions

When $\dim(\mathcal{H}) < \infty$, we will show first that if a solution $A_1, \ldots, A_{m+1}$ exists such that $\sum_{j=1}^{m+1} A_j = I$ and $\Gamma(A_1, \ldots, A_{m+1}) > 0$ in (3), then there is a unique one that maximizes the determinant among all positive definite completions. This unique maximal determinant solution has the property that the diagonal blocks in $\Gamma(A_, \ldots, A_{m+1})^{-1}$ are all the same. This diagonal block in the inverse is the L that appears in Conjecture 1.4(ii).

Proposition 2.1 *Given are $X_1, \ldots, X_m \in B(\mathcal{H})$ with $\dim(\mathcal{H}) < \infty$. Suppose that $A_1, \ldots, A_{m+1}$ exist so that $\sum_{j=1}^{m+1} A_j = I$ and $\Gamma(A_1, \ldots, A_{m+1}) > 0$ as defined in (3). Then there exist unique solution $A_1, \ldots, A_{m+1}$ with the additional property that*

$$\Gamma(A_1, \ldots, A_{m+1})^{-1} = \begin{bmatrix} L & * & * \\ * & \ddots & * \\ * & * & L \end{bmatrix}.$$

This unique solution may be found by maximizing the determinant of $\Gamma(A_1, \ldots, A_{m+1})$ among all possible $A_1, \ldots, A_{m+1}$ with $\sum_{j=1}^{m+1} A_j = I$ and $\Gamma(A_1, \ldots, A_{m+1}) > 0$.

Proof Let $n = \dim(\mathcal{H})$ and consider $\Gamma(A_1, \ldots, A_{m+1})$ as a Hermitian matrix of size $(m+1)n$. Let $\mathcal{S}_{(m+1)n}$ denote the real vector space of Hermitian matrices of size $(m+1)n$, with the inner product $\langle Y, Z\rangle = \text{trace}(ZY)$. Let $\mathcal{W}$ be the subspace of $\mathcal{S}_{(m+1)n}$ consisting of block diagonal Hermitian matrices defined by

$$\mathcal{W} = \left\{ W_1 \oplus \cdots \oplus W_{m+1} \in \mathcal{S}_{(m+1)n} : \sum_{j=1}^{m+1} W_j = 0 \right\}.$$

Adopting the setup of [4], we are now interested in the positive definite elements in the affine space $\Gamma(I, 0, \ldots, 0) + \mathcal{W}$. By the main result in [4], since the subspace $\mathcal{W}$ contains no nonzero positive semidefinite matrix, among all positive definite elements in the affine space $\Gamma(I, 0, \ldots, 0) + \mathcal{W}$, there is a unique one with maximal determinant (call it Γ_0), and the optimality conditions yield $\Gamma_0^{-1} \in \mathcal{W}^\perp$. Since

$$\mathcal{W}^\perp = \{[Y_{jk}]_{j,k=1}^{m+1} \in \mathcal{S}_{(m+1)n} : Y_{11} = \cdots = Y_{m+1,m+1}\},$$

the result follows. □

Proposition 2.1 suggests we look at operator matrices with block tridiagonal inverses. Let us start with the 3-by-3 case. First, we recall some useful facts about 2-by-2 invertible operator matrices. If A is invertible, then $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is invertible if and only if $D - CA^{-1}B$ is invertible. In that case,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \left(\begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix} \right)^{-1}$$
$$= \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & (D - CA^{-1}B)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix}$$
$$= \begin{bmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}.$$

In particular, if A is positive definite, then $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is positive definite if and only if $D - CA^{-1}B$ is positive definite.

Now let $P = [P_{ij}]_{i,j=1}^3$ be an operator matrix with P_{22} and $\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ invertible. Then

$$P = \begin{bmatrix} I & P_{12}P_{22}^{-1} & 0 \\ 0 & I & 0 \\ 0 & P_{32}P_{22}^{-1} & I \end{bmatrix} \begin{bmatrix} Q & 0 & R \\ 0 & P_{22} & 0 \\ S & 0 & T \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ P_{22}^{-1}P_{21} & I & P_{22}^{-1}P_{23} \\ 0 & 0 & I \end{bmatrix} \tag{7}$$

where $Q := P_{11} - P_{12}P_{22}^{-1}P_{21}$, $R := P_{13} - P_{12}P_{22}^{-1}P_{23}$, $S := P_{31} - P_{32}P_{22}^{-1}P_{21}$, and $T := P_{33} - P_{32}P_{22}^{-1}P_{23}$. By assumption, Q is invertible. Because of their structure, the first and third factor on the right-hand side of (7) are also invertible. Hence, P is invertible if and only if $\begin{bmatrix} Q & R \\ S & T \end{bmatrix}$ is invertible, or equivalently, $T - SQ^{-1}R$ is invertible.

Suppose P is invertible. By (7),

$$P^{-1} = \begin{bmatrix} I & 0 & 0 \\ -P_{22}^{-1}P_{21} & I & -P_{22}^{-1}P_{23} \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} Q & 0 & R \\ 0 & P_{22} & 0 \\ S & 0 & T \end{bmatrix}^{-1} \begin{bmatrix} I & -P_{12}P_{22}^{-1} & 0 \\ 0 & I & 0 \\ 0 & -P_{32}P_{22}^{-1} & I \end{bmatrix}. \tag{8}$$

The middle factor simplifies to

$$\begin{bmatrix} Q^{-1} + Q^{-1}R(T - SQ^{-1}R)^{-1}SQ^{-1} & 0 & -Q^{-1}R(T - SQ^{-1}R)^{-1} \\ 0 & P_{22}^{-1} & 0 \\ -(T - SQ^{-1}R)^{-1}SQ^{-1} & 0 & (T - SQ^{-1}R)^{-1} \end{bmatrix}. \tag{9}$$

Assume further that P^{-1} is block tridiagonal. By (8)–(9), the $(1, 3)$ entry of P^{-1} is 0 implies $0 = R = P_{13} - P_{12}P_{22}^{-1}P_{23}$, equivalently, $P_{13} = P_{12}P_{22}^{-1}P_{23}$. Similarly, the $(3, 1)$ entry of P^{-1} is 0 implies $S = 0$, equivalently, $P_{31} = P_{32}P_{22}^{-1}P_{21}$. Moreover, $T - SQ^{-1}R = T = P_{33} - P_{32}P_{22}^{-1}P_{23}$ is invertible, and thus $\begin{bmatrix} P_{22} & P_{23} \\ P_{32} & P_{33} \end{bmatrix}$ is invertible.

We also obtain the following decomposition of P^{-1} from (8)–(9):

$$P^{-1} = \begin{bmatrix} Q^{-1} & -Q^{-1}P_{12}P_{22}^{-1} & 0 \\ -P_{22}^{-1}P_{21}Q^{-1} & P_{22}^{-1}P_{21}Q^{-1}P_{12}P_{22}^{-1} + P_{22}^{-1}P_{23}T^{-1}P_{32}P_{22}^{-1} + P_{22}^{-1} & -P_{22}^{-1}P_{23}T^{-1} \\ 0 & -T^{-1}P_{32}P_{22}^{-1} & T^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}^{-1} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \begin{bmatrix} P_{22} & P_{23} \\ P_{32} & P_{33} \end{bmatrix}^{-1} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & P_{22}^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{10}$$

Conversely, suppose $P_{13} = P_{12}P_{22}^{-1}P_{23}$ and $P_{31} = P_{32}P_{22}^{-1}P_{21}$. If $\begin{bmatrix} P_{22} & P_{23} \\ P_{32} & P_{33} \end{bmatrix}$ is invertible, then T is invertible and so is $\begin{bmatrix} Q & R \\ S & T \end{bmatrix} = \begin{bmatrix} Q & 0 \\ 0 & T \end{bmatrix}$. In particular, P is also invertible. The operator P^{-1} satisfies (8)–(9), and hence P^{-1} is block tridiagonal.

Similar assertions about a matrix with a banded inverse have been considered in [5] for the matrix case and in [6] for the block matrix case. Furthermore, if P_{22}, $\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$, $\begin{bmatrix} P_{22} & P_{23} \\ P_{32} & P_{33} \end{bmatrix} > 0$, then (8)–(9) implies that $P > 0$. The above 3-by-3 case generalizes to the $(m+1)$-by-$(m+1)$ case.

Theorem 2.2 *Consider an operator matrix $P = [P_{ij}]_{i,j=1}^{m+1}$ with $m \geq 2$. Assume that the following are invertible:*

(i) *P_{ii} for $i = 2, \ldots, m$ and*
(ii) $\begin{bmatrix} P_{ii} & P_{i,i+1} \\ P_{i+1,i} & P_{i+1,i+1} \end{bmatrix}$ *for $i = 1, \ldots, m$.*

Then P is invertible and P^{-1} is block tridiagonal if and only if

$$P_{ij} = P_{i,i+1}P_{i+1,i+1}^{-1}P_{i+1,j} \text{ and } P_{ji} = P_{j,i+1}P_{i+1,i+1}^{-1}P_{i+1,i} \text{ for } j \geq i+2. \tag{11}$$

In that case, P^{-1} equals

$$\begin{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}^{-1} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & 0 \end{matrix} \end{bmatrix} + \cdots + \begin{bmatrix} \begin{matrix} 0 & \cdots \\ \vdots & \ddots \end{matrix} & \begin{matrix} 0 & 0 \\ \vdots & \vdots \end{matrix} \\ \begin{matrix} 0 & \cdots \\ 0 & \cdots \end{matrix} & \begin{bmatrix} P_{mm} & P_{m,m+1} \\ P_{m+1,m} & P_{m+1,m+1} \end{bmatrix}^{-1} \end{bmatrix} - \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \bigoplus_{j=2}^{m} P_{jj}^{-1} & \vdots \\ 0 & \cdots & 0 \end{bmatrix}. \tag{12}$$

Moreover, if the operators in assumption (ii) are positive definite, then so is P.

Proof The $m = 2$ case, which corresponds to the 3-by-3 case, follows from the calculations before the theorem.

Consider the $(m+1)$-by-$(m+1)$ case viewed as a 3-by-3:

$$\left[\begin{array}{c|c|ccc} P_{11} & P_{12} & P_{13} & \cdots & P_{1,m+1} \\ \hline P_{21} & P_{22} & P_{23} & \cdots & P_{2,m+1} \\ \hline P_{31} & P_{32} & P_{33} & \cdots & P_{3,m+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{m+1,1} & P_{m+1,2} & P_{m+1,3} & \cdots & P_{m+1,m+1} \end{array}\right]. \tag{13}$$

Suppose the statement is true for operator matrices of size at most m.

Assume P is invertible and P^{-1} is block tridiagonal. By applying the 3-by-3 case to (13), we get

$$\begin{bmatrix} P_{13} & \cdots & P_{1,m+1} \end{bmatrix} = P_{12}P_{22}^{-1}\begin{bmatrix} P_{23} & \cdots & P_{2,m+1} \end{bmatrix}$$

and

$$\begin{bmatrix} P_{31} \\ \vdots \\ P_{m+1,1} \end{bmatrix} = \begin{bmatrix} P_{32} \\ \vdots \\ P_{m+1,2} \end{bmatrix} P_{22}^{-1}P_{21}.$$

In particular, (11) holds when $i = 1$. By the 3-by-3 case, $[P_{ij}]_{i,j=2}^{m+1}$ is invertible and its inverse is block tridiagonal due to (10).

Conversely, assume the expressions in (11) hold. By assumption, P_{22} and $\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ are invertible. The induction hypothesis applies to $[P_{ij}]_{i,j=2}^{m+1}$, and so it is invertible whose inverse is block tridiagonal. The 3-by-3 case guarantees that the conditions $P_{1j} = P_{12}P_{22}^{-1}P_{2j}$ and $P_{j1} = P_{j2}P_{22}^{-1}P_{21}$ for $j \geq 3$ are equivalent to the assertion that P is invertible and P^{-1} is block tridiagonal conformal to the partitioning in (13). Also, the 3-by-3 case gives

$$P^{-1} = \begin{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}^{-1} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & ([P_{ij}]_{i,j=2}^{m+1})^{-1} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & P_{22}^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{14}$$

By (14) and the induction hypothesis, P^{-1} is block tridiagonal conformal to the original partitioning of P as $[P_{ij}]_{i,j=1}^{m+1}$ and that (12) holds. If the operators in assumption (ii) are positive definite, then the 3-by-3 case and the induction hypothesis guarantees that P is also positive definite. □

Given $X_1, \dots, X_m \in B(\mathcal{H})$, let us now consider the operator matrix completion problem

$$\begin{bmatrix} P_{11} & ? & ? \\ ? & \ddots & ? \\ ? & ? & P_{m+1,m+1} \end{bmatrix}^{-1} = \begin{bmatrix} ? & X_1 & 0 & \cdots & 0 \\ X_1^* & ? & X_2 & & \vdots \\ 0 & X_2^* & \ddots & \ddots & 0 \\ \vdots & & \ddots & ? & X_m \\ 0 & \cdots & 0 & X_m^* & ? \end{bmatrix}, \tag{15}$$

where we would like the completion to be positive definite. Let us fill in the unknowns in (15) with symbols P_{ij} for $i \neq j$ and $A_1, \dots, A_{m+1}$:

$$\left([P_{ij}]_{i,j=1}^{m+1}\right)^{-1} = \begin{bmatrix} A_1 & X_1 & 0 & \cdots & 0 \\ X_1^* & A_2 & X_2 & & \vdots \\ 0 & X_2^* & \ddots & \ddots & 0 \\ \vdots & & \ddots & A_m & X_m \\ 0 & \cdots & 0 & X_m^* & A_{m+1} \end{bmatrix}. \tag{16}$$

Due to the right-hand side being block tridiagonal, Theorem 2.2 guarantees that $\left([P_{ij}]_{i,j=1}^{m+1}\right)^{-1}$ can be written in the form (12). Combining (16) with (12) leads to the following 2-by-2 operator matrix completion problem for $i = 1, \dots, m$:

$$\begin{bmatrix} P_{ii} & ? \\ ? & P_{i+1,i+1} \end{bmatrix}^{-1} = \begin{bmatrix} ? & X_i \\ X_i^* & ? \end{bmatrix}, \tag{17}$$

where we would like the completion to be positive definite. The matrix version of this completion problem has been considered in [7] where the prescribed blocks are not necessarily square matrices.

We will show in Theorem 2.4 that there is a unique positive definite completion in (17). We need the following result.

Lemma 2.3 *If* $S \in B(\mathcal{H})$, *then* $S\left[\left(S^*S + \frac{1}{4}I\right)^{\frac{1}{2}} + \frac{1}{2}I\right]^{-1} S^* + \frac{1}{2}I = \left(SS^* + \frac{1}{4}I\right)^{\frac{1}{2}}$.

Proof Since both sides are positive semidefinite, it suffices to show equality when we square both sides. Squaring the left hand side gives

$$\frac{1}{4}I + S\left\{\left[\left(S^*S + \frac{1}{4}I\right)^{\frac{1}{2}} + \frac{1}{2}I\right]^{-1} + \left[\left(S^*S + \frac{1}{4}I\right)^{\frac{1}{2}} + \frac{1}{2}I\right]^{-1} S^*S \left[\left(S^*S + \frac{1}{4}I\right)^{\frac{1}{2}} + \frac{1}{2}I\right]^{-1}\right\} S^*$$

which simplifies to

$$\frac{1}{4}I + S\left[\left(S^*S+\frac{1}{4}I\right)^{\frac{1}{2}}+\frac{1}{2}I\right]^{-1}$$
$$\left\{\left[\left(S^*S+\frac{1}{4}I\right)^{\frac{1}{2}}+\frac{1}{2}I\right]^{2}\right\}\left[\left(S^*S+\frac{1}{4}I\right)^{\frac{1}{2}}+\frac{1}{2}I\right]^{-1}S^* = \frac{1}{4}I + SS^*.$$

□

Theorem 2.4 *Let $P, Q, X \in B(\mathcal{H})$ be given such that P, Q are positive definite. If*

$$\begin{bmatrix} P & ? \\ ? & Q \end{bmatrix}^{-1} = \begin{bmatrix} ? & X \\ X^* & ? \end{bmatrix} \tag{18}$$

has a positive definite completion, then it is unique. In particular, if we set

$$W = -PX\left[Q^{-\frac{1}{2}}\left(Q^{\frac{1}{2}}X^*PXQ^{\frac{1}{2}}+\tfrac{1}{4}I\right)^{\frac{1}{2}}Q^{-\frac{1}{2}}+\tfrac{1}{2}Q^{-1}\right]^{-1}, \tag{19}$$

then $\begin{bmatrix} P & W \\ W^* & Q \end{bmatrix}^{-1}$ *is positive definite and is equal to*

$$\begin{bmatrix} P^{-\frac{1}{2}}(P^{\frac{1}{2}}XQX^*P^{\frac{1}{2}}+\frac{1}{4}I)^{\frac{1}{2}}P^{-\frac{1}{2}}+\frac{1}{2}P^{-1} & X \\ X^* & Q^{-\frac{1}{2}}(Q^{\frac{1}{2}}X^*PXQ^{\frac{1}{2}}+\frac{1}{4}I)^{\frac{1}{2}}Q^{-\frac{1}{2}}+\frac{1}{2}Q^{-1} \end{bmatrix}. \tag{20}$$

Conversely, (19) *and* (20) *define a positive definite completion of* (18)*.*

Proof Suppose

$$\begin{bmatrix} P & W \\ W^* & Q \end{bmatrix}^{-1} = \begin{bmatrix} Y & X \\ X^* & Z \end{bmatrix}. \tag{21}$$

In particular, (21) implies $QZ+W^*X = I$ and $X^*P+ZW^* = 0$. Hence, $ZQZ-Z = Z(QZ-I) = -ZW^*X = X^*PX$. Moreover,

$$\left(Q^{\frac{1}{2}}ZQ^{\frac{1}{2}}-\tfrac{1}{2}I\right)^{2} = Q^{\frac{1}{2}}X^*PXQ^{\frac{1}{2}}+\tfrac{1}{4}I. \tag{22}$$

We claim that $Q^{\frac{1}{2}}ZQ^{\frac{1}{2}}-\frac{1}{2}I \geq \frac{1}{2}I \geq 0$. Indeed,

$$Q^{\frac{1}{2}}ZQ^{\frac{1}{2}}-I = Q^{-\frac{1}{2}}(QZ-I)Q^{\frac{1}{2}} = -Q^{-\frac{1}{2}}W^*XQ^{\frac{1}{2}}.$$

Now, $YW + XQ = 0$ follows from (21), and so

$$Q^{\frac{1}{2}}ZQ^{\frac{1}{2}} - I = -Q^{-\frac{1}{2}}W^*(-YWQ^{-1})Q^{\frac{1}{2}} = Q^{-\frac{1}{2}}W^*YWQ^{-\frac{1}{2}} \geq 0.$$

Hence, $Q^{\frac{1}{2}}ZQ^{\frac{1}{2}} - \frac{1}{2}I$ is the unique positive semidefinite square root of the right-hand side of (22). Solving for Z, we obtain

$$Z = Q^{-\frac{1}{2}}\left(Q^{\frac{1}{2}}X^*PXQ^{\frac{1}{2}} + \tfrac{1}{4}I\right)^{\frac{1}{2}}Q^{-\frac{1}{2}} + \tfrac{1}{2}Q^{-1}.$$

Since $PX + WZ = 0$ from (21), we have $W = -PXZ^{-1}$ and (19) holds. To see that the formula for Y in (20) holds, we apply the argument above to the permutation of (21) as $\begin{bmatrix} Q & W^* \\ W & P \end{bmatrix}^{-1} = \begin{bmatrix} Z & X^* \\ X & Y \end{bmatrix}$. For the uniqueness, suppose there exist $\widetilde{W}$, $\widetilde{Y}$, $\widetilde{Z}$ for which (21) is positive definite. In particular, $\widetilde{Z}$ satisfies (22), and so $\left(Q^{\frac{1}{2}}\widetilde{Z}Q^{\frac{1}{2}} - \frac{1}{2}I\right)^2 = Q^{\frac{1}{2}}X^*PXQ^{\frac{1}{2}} + \frac{1}{4}I = \left(Q^{\frac{1}{2}}ZQ^{\frac{1}{2}} - \frac{1}{2}I\right)^2$. Since $Q^{\frac{1}{2}}\widetilde{Z}Q^{\frac{1}{2}} - \frac{1}{2}I \geq 0$, we obtain $\widetilde{Z} = Z$ due to the uniqueness of the positive semidefinite square root. Similarly, $\widetilde{Y} = Y$, and $\widetilde{W} = -PX\widetilde{Z}^{-1} = -PXZ^{-1} = W$.

Conversely, suppose Y and Z are the $(1, 1)$ and $(2, 2)$ entries in (20) and W is defined as (19). Direct computation reveals that (21) holds. Hence, $Y - XZ^{-1}X^* = P^{-1}$, which by assumption is positive definite. Since Z is also positive definite, the completion defined by (19) and (20) is positive definite. □

For each $i = 1, \ldots, m$, the unique positive definite completion of (17) is given by (20) where we take $P = P_{ii}$, $Q = P_{i+1,i+1}$, and $X = X_i$ in Theorem 2.4. Thus, (12) and (20) imply that the unknowns $A_1, \ldots, A_{m+1}$ in (16) can be taken to be

$$\begin{aligned}
A_1 &= P_{11}^{-\frac{1}{2}}\left(P_{11}^{\frac{1}{2}}X_1P_{22}X_1^*P_{11}^{\frac{1}{2}} + \frac{1}{4}I\right)^{\frac{1}{2}}P_{11}^{-\frac{1}{2}} + \frac{1}{2}P_{11}^{-1}\\
A_j &= P_{jj}^{-\frac{1}{2}}\left(P_{jj}^{\frac{1}{2}}X_{j-1}^*P_{j-1,j-1}X_{j-1}P_{jj}^{\frac{1}{2}} + \frac{1}{4}I\right)^{\frac{1}{2}}P_{jj}^{-\frac{1}{2}}\\
&\quad + P_{jj}^{-\frac{1}{2}}\left(P_{jj}^{\frac{1}{2}}X_j^*P_{j+1,j+1}X_jP_{jj}^{\frac{1}{2}} + \frac{1}{4}I\right)^{\frac{1}{2}}P_{jj}^{-\frac{1}{2}}, \text{ for } j = 2, \ldots, m, \text{ and}\\
A_{m+1} &= P_{m+1,m+1}^{-\frac{1}{2}}\left(P_{m+1,m+1}^{\frac{1}{2}}X_m^*P_{mm}X_mP_{m+1,m+1}^{\frac{1}{2}} + \frac{1}{4}I\right)^{\frac{1}{2}}P_{m+1,m+1}^{-\frac{1}{2}} + \frac{1}{2}P_{m+1,m+1}^{-1}.
\end{aligned} \tag{23}$$

If for some $L > 0$, we insist $P_{11} = \cdots = P_{m+1,m+1} = L$ in the considerations above and $\sum_{j=1}^{m+1} A_j = I$ (like the setting in Proposition 2.1), then adding (23) gives $I = \sum_{j=1}^{m+1} A_j = L^{-\frac{1}{2}}f_{X_1,\ldots,X_m}(L)L^{-\frac{1}{2}}$ or $f_{X_1,\ldots,X_m}(L) = L$.

Conversely, if there exists $L > 0$ such that $f_{X_1,\ldots,X_m}(L) = L$, then setting $P_{11} = \cdots = P_{m+1,m+1} = L$ in (23) suggests the following formula for $A_1, \ldots, A_{m+1}$.

Proposition 2.5 *Given are $X_1, \ldots, X_m \in B(\mathcal{H})$. Suppose $f_{X_1,\ldots,X_m}$ has a positive definite fixed point $L \in B(\mathcal{H})$. Let*

$$A_1 = L^{-\frac{1}{2}}\left[\left(L^{\frac{1}{2}}X_1LX_1^*L^{\frac{1}{2}} + \frac{1}{4}I\right)^{\frac{1}{2}} + \frac{1}{2}I\right]L^{-\frac{1}{2}},$$

$$A_j = L^{-\frac{1}{2}}\left(L^{\frac{1}{2}}X_{j-1}^*LX_{j-1}L^{\frac{1}{2}} + \frac{1}{4}I\right)^{\frac{1}{2}}L^{-\frac{1}{2}}$$
$$+L^{-\frac{1}{2}}\left(L^{\frac{1}{2}}X_jLX_j^*L^{\frac{1}{2}} + \frac{1}{4}I\right)^{\frac{1}{2}}L^{-\frac{1}{2}}, \text{ for } j = 2, \ldots, m, \text{ and}$$

$$A_{m+1} = L^{-\frac{1}{2}}\left[\left(L^{\frac{1}{2}}X_m^*LX_mL^{\frac{1}{2}} + \frac{1}{4}I\right)^{\frac{1}{2}} + \frac{1}{2}I\right]L^{-\frac{1}{2}}.$$

Then $\sum_{j=1}^{m+1} A_j = I$ and $\Gamma(A_1, \ldots, A_{m+1}) > 0$ as defined in (3).

Proof Note that $\sum_{j=1}^{m+1} A_j = L^{-\frac{1}{2}} f_{X_1,\ldots,X_m}(L)L^{-\frac{1}{2}} = I$. Let $P_{11} = \cdots = P_{m+1,m+1} = L$. For each $i = 1, \ldots, m$, Theorem 2.4 guarantees that there exists $P_{i,i+1}$ given by (19) such that $\begin{bmatrix} P_{ii} & P_{i,i+1} \\ P_{i,i+1}^* & P_{i+1,i+1} \end{bmatrix}$ solves the positive definite completion problem (17). If we define P_{ij} as in (11), then Theorem 2.2 implies that $P := [P_{ij}]_{i,j=1}^{m+1}$ is positive definite and P^{-1} is block tridiagonal. In particular, P^{-1} satisfies (12) and hence $\Gamma(A_1, \ldots, A_{m+1}) = P^{-1}$ is positive definite. □

We now prove Theorem 1.2.

Proof of Theorem 1.2 The first statement is Proposition 2.5 in conjunction with Theorem 1.1. The second statement follows from Proposition 2.1 and Theorem 1.1. □

Remark 2.6 The finite dimensionality of the underlying Hilbert space is used in Proposition 2.1. In particular, the argument to prove the main result in [4] uses the compactness of the closed unit ball. This guarantees the existence of a maximizer for the determinant. In the following example, $\mathcal{H}$ is not necessarily finite dimensional but we show that if a completion exists satisfying analogous assumptions in Proposition 2.1, then it has to be unique. What is left to show then is the existence of a completion with the desired properties.

Example 2.7 Let $\mathcal{R}$ be a C^*-algebra with faithful normal tracial state τ. Consider the analogous problem of Proposition 2.1 in $M_{m+1}(\mathcal{R})$, i.e., if $\Gamma(A_1, \ldots, A_{m+1}) \in M_{m+1}(\mathcal{R})_+^{-1}$ for some $A_1, \ldots, A_{m+1} \in \mathcal{R}_+^{-1}$ with $\sum_{j=1}^{m+1} A_j = I$, can we find $A_1, \ldots, A_{m+1} \in \mathcal{R}_+^{-1}$ satisfying the same properties and such that $\Gamma(A_1, \ldots, A_{m+1})^{-1}$ has the same block diagonal entries? (Here, $\mathcal{A}_+^{-1}$ denotes the positive and invertible elements of a C^*-algebra $\mathcal{A}$.)

To show uniqueness, we first prove a generalization of Fiedler's inequality [8]:

$$\tau[(A - B)(B^{-1} - A^{-1})] \geq \frac{\tau[(A - B)^2]}{\|A\| \cdot \|B\|} \text{ for all } A, B \in \mathcal{R}_+^{-1}.$$

Indeed, the argument is similar to the one in [8]. First, observe that for any $X, Y \in \mathcal{R}$,

$$\tau(XYY^*X^*) \leq \|X\|^2 \tau(YY^*). \tag{24}$$

Now, write $A = X^*X$ and $B = Y^*Y$ for some invertible $X, Y \in \mathcal{R}$. Note that $\|A\| = \|X^*\|^2$ and $\|B\| = \|Y^*\|^2$. Then

$$\begin{aligned}\tau[(A-B)(B^{-1}-A^{-1})] &= \tau[(A-B)B^{-1}(A-B)A^{-1}] \\ &= \tau[(A-B)Y^{-1}Y^{-*}(A-B)X^{-1}X^{-*}] \\ &= \tau[X^{-*}(A-B)Y^{-1}Y^{-*}(A-B)X^{-1}] \\ &\geq \frac{\tau[(A-B)Y^{-1}Y^{-*}(A-B)]}{\|X^*\|^2}\end{aligned}$$

by taking $X_1 = X^*$ and $Y_1 = X^{-*}(A-B)Y^{-1}$ in (24). Rewriting the last expression in the inequality and using $X_2 = Y^*$ and $Y_2 = Y^{-*}(A-B)$ in (24), we obtain

$$\tau[(A-B)(B^{-1}-A^{-1})] \geq \frac{\tau[Y^{-*}(A-B)^2Y^{-1}]}{\|A\|} \geq \frac{\tau[(A-B)^2]}{\|A\| \cdot \|Y^*\|^2} = \frac{\tau[(A-B)^2]}{\|A\| \cdot \|B\|}.$$

Going back to the uniqueness assertion, suppose $\Gamma(A_1^{(k)}, \ldots, A_{m+1}^{(k)}) \in M_{m+1}(\mathcal{R})_+^{-1}$ whose inverse has the same diagonal entry equal to L_k and $\sum_{j=1}^{m+1} A_j^{(k)} = I$ for $k = 1, 2$. By using the faithful normal trace $\widetilde{\tau}([x_{ij}]) = \frac{1}{m+1}\sum_{j=1}^{m+1} \tau(x_{jj})$ on $M_{m+1}(\mathcal{R})$ and Fiedler's inequality, we see that for some $s > 0$

$$\begin{aligned}&\widetilde{\tau}\left\{\left[\Gamma(A_1^{(1)}, \ldots, A_{m+1}^{(1)}) - \Gamma(A_1^{(2)}, \ldots, A_{m+1}^{(2)})\right]^2\right\} \\ &\leq s\widetilde{\tau}\left\{\left[\sum_{j=1}^{m+1} A_j^{(1)} - \sum_{j=1}^{m+1} A_j^{(2)}\right](L_2 - L_1)\right\} = 0.\end{aligned}$$

Since $\widetilde{\tau}$ is faithful, $\Gamma(A_1^{(1)}, \ldots, A_{m+1}^{(1)}) = \Gamma(A_1^{(2)}, \ldots, A_{m+1}^{(2)})$, i.e., the completion with equal diagonal blocks is unique.

3 Basic Properties of the Free Joint Numerical Radius

Recall that an *isometry* is a distance-preserving map between two metric spaces. By [9, Proposition 5.2], a *linear* isometry between Hilbert spaces can be characterized as follows: for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, a linear map $V : \mathcal{H} \to \mathcal{K}$ is an *isometry* if and only if $V^*V = I_{\mathcal{H}}$. From now on, whenever we mention isometry, we assume it is a linear isometry between Hilbert spaces. A surjective linear isometry $U : \mathcal{H} \to \mathcal{K}$ is called *unitary*. Equivalently, U is unitary if and only if both $U^*U = I_{\mathcal{H}}$ and $UU^* = I_{\mathcal{K}}$ hold.

The following proposition is well-known and can be verified easily.

Proposition 3.1 *Let $V : \mathcal{H} \to \mathcal{K}$ be an isometry and $T \in B(\mathcal{K})$. Then $W(V^*TV) \subseteq W(T)$ and $w(V^*TV) \leq w(T)$. In particular, if V is unitary, then equality holds in both assertions.*

Suppose $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces. If $\mathcal{H}$ has an orthonormal basis $\{e_i\}_{i\in\Lambda}$ with $\Lambda \subseteq \mathbb{N}$, then $\mathcal{H} \otimes \mathcal{K}$ and $\bigoplus_{i\in\Lambda} \mathcal{K}$ are isomorphic as Hilbert spaces. In particular, we can take the unitary extension of the mapping defined by $e_i \otimes k \mapsto [\overbrace{0 \ \cdots \ 0}^{i-1} \ k \ 0 \ \cdots]^t$ for all $i \in \Lambda$ and $k \in \mathcal{K}$ (here we write elements of $\bigoplus_{i\in\Lambda} \mathcal{K}$ as column vectors and the superscript "t" indicates transpose).

Proposition 3.2 *Suppose $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces such that $\mathcal{H}$ is separable. If $X = [x_{ij}]$ is the matrix representation of $R \in B(\mathcal{H})$ with respect to a countable orthonormal basis $\{e_i\}_{i\in\Lambda}$ of $\mathcal{H}$ and $S \in B(\mathcal{K})$, then there exists unitary U such that $U(R \otimes S)U^* = [x_{ij}S] =: X \otimes S$ and $\|X \otimes S\| = \|R\| \cdot \|S\|$.*

Proof There exists a unitary U defined by the mapping $e_i \otimes k \mapsto [\overbrace{0 \ \cdots \ 0}^{i-1} \ k \ 0 \ \cdots]^t$ for all $i \in \Lambda$ and $k \in \mathcal{K}$. Let $h = \sum_{j\in\Lambda} h_j e_j \in \mathcal{H}$ and $k \in \mathcal{K}$. Then $U(h \otimes k) = [h_1 k \ h_2 k \ \cdots]^t$. Since $[U(R \otimes S)U^*](U(h \otimes k)) = U(Rh \otimes Sk)$, note that

$$
\begin{aligned}
U(Rh \otimes Sk) &= U\left(\sum_{j\in\Lambda} h_j Re_j \otimes Sk\right) = U\left(\sum_{j\in\Lambda}\sum_{i\in\Lambda} h_j x_{ij} e_i \otimes Sk\right) \\
&= U\left(\sum_{i\in\Lambda} e_i \otimes \sum_{j\in\Lambda} h_j x_{ij} Sk\right) = \sum_{i\in\Lambda} U\left(e_i \otimes \sum_{j\in\Lambda} h_j x_{ij} Sk\right) \\
&= [x_{ij}S](U(h \otimes k)).
\end{aligned}
$$

Thus, $\|X \otimes S\| = \|U(R \otimes S)U^*\| = \|R \otimes S\| = \|R\| \cdot \|S\|$. □

For $X \in B(\mathcal{H})$, the invariance $w(X) = w(UXU^*)$ for any unitary U and the basic inequality $\frac{1}{2}\|X\| \leq w(X) \leq \|X\|$ have analogues in the free joint numerical radius context, as stated in Proposition 3.3. The first assertion below follows from the fact that the numerical radius is unchanged under conjugation by a unitary (in particular, $U \otimes I$) while the second one can be deduced from the subadditivity of the numerical radius, the definition of the free joint numerical radius, and the fact that $w(X \otimes U) = w(X)$ for any unitary $U \in B(\mathcal{H})$ [10, Proposition 2.4].

Proposition 3.3 *Let $X_1, \ldots, X_m \in B(\mathcal{H})$.*

(i) $w(X_1, \ldots, X_m) = w(UX_1U^*, \ldots, UX_mU^*)$ *for any unitary U.*

(ii) $\frac{1}{2}\left\|\sum_{j=1}^m X_j\right\| \leq w\left(\sum_{j=1}^m X_j\right) \leq w(X_1, \ldots, X_m) \leq \sum_{j=1}^m w(X_j) \leq \sum_{j=1}^m \|X_j\|$.

(iii) *The free joint numerical radius is a norm on $\bigoplus_{j=1}^m B(\mathcal{H})$.*

Let $X, Y \in B(\mathcal{H})$. Note that $A = X \otimes I$ and $B = I \otimes Y$ *doubly commute*, i.e., $AB = BA$ and $AB^* = B^*A$. By [11, Theorem 3.4],

$$\begin{aligned} w(X \otimes Y) &= w(AB) \leq \min\{w(A)\|B\|, \|A\|w(B)\} \\ &= \min\{w(X)\|Y\|, \|X\|w(Y)\}. \end{aligned} \tag{25}$$

Hence, if $a_1, \ldots, a_m \in \mathbb{C}$ and $U_1, \ldots, U_m \in B(\mathcal{K})$ are unitary, then

$$\begin{aligned} w\left(\sum_{j=1}^{m} a_j X \otimes U_j\right) &= w\left[X \otimes \left(\sum_{j=1}^{m} a_j \otimes U_j\right)\right] \\ &\leq w(X)\left\|\sum_{j=1}^{m} a_j U_j\right\| \leq w(X)\sum_{j=1}^{m}|a_j|, \end{aligned}$$

and so $w(a_1X, \ldots, a_mX) \leq \sum_{j=1}^{m}|a_j|w(X)$. Equality holds by letting the unitaries be $e^{-i\arg(a_j)}$, for $j = 1, \ldots, m$ (if $a_j = 0$, take $\arg(a_j) = 0$).

Proposition 3.4 *Let* $X \in B(\mathcal{H})$ *and* $a_1, \ldots, a_m \in \mathbb{C}$. *Then* $w(a_1X, \ldots, a_mX) = \sum_{j=1}^{m}|a_j|w(X)$.

Recall that $C \in B(\mathcal{H})$ is called a *contraction* if $\|C\| \leq 1$.

Proposition 3.5 *If* $X_1, \ldots, X_m \in B(\mathcal{H})$, *then*

$$w(X_1, \ldots, X_m) = \sup\{w\,(X_1 \otimes C_1 + \cdots + X_m \otimes C_m)\}$$

where the supremum is taken over every Hilbert space $\mathcal{K}$, *every choice of m contractions* $C_1, \ldots, C_m \in B(\mathcal{K})$, *and the tensor product is spatial.*

Proof Let ℓ be the left-hand side and ρ be the right hand side of the desired equality. Note that $\ell \leq \rho$ since unitaries have norm equal to 1. To prove the reverse inequality, let $\mathcal{K}$ be a Hilbert space and $C_1, \ldots, C_m \in B(\mathcal{K})$ be contractions. By Halmos dilation theorem [12, 13], each C_j has a unitary dilation $U_j \in B(\mathcal{K} \oplus \mathcal{K})$ of the form

$$U_j = \begin{bmatrix} C_j & \left(I - C_jC_j^*\right)^{\frac{1}{2}} \\ \left(I - C_j^*C_j\right)^{\frac{1}{2}} & -C_j^* \end{bmatrix}.$$

The isometry $V : \mathcal{K} \to \mathcal{K} \oplus \mathcal{K}$ with $Vx = [x\ 0]^t$ for all $x \in \mathcal{K}$ has the property that $C_j = V^*U_jV$ for each $j = 1, \ldots, m$. Since $I \otimes V$ is an isometry, we have that

$$w\left(\sum_{j=1}^{m} X_j \otimes C_j\right) \leq w\left(\sum_{j=1}^{m} X_j \otimes U_j\right) \leq w(X_1, \ldots, X_m) = \ell.$$

Taking the supremum over all contractions in $B(\mathcal{K})$ and over all Hilbert spaces $\mathcal{K}$ yields $\rho \leq \ell$. □

The *spectral radius* of $X \in B(\mathcal{H})$ is defined as $\rho(X) = \sup\{|\lambda| : \lambda \in \sigma(X)\}$ where $\sigma(X)$ is the spectrum of X. It is known that the spectral radius is monotone on nonnegative matrices, i.e., if $A = [a_{ij}]$ and $B = [b_{ij}] \in \mathbb{R}^{n\times n}$ with $0 \le a_{ij} \le b_{ij}$ for all i, j, then $\rho(A) \le \rho(B)$ [14, Theorem 8.1.18]. In [15], the authors showed that the classical numerical radius is monotone on nonnegative matrices. Indeed, A and B having nonnegative entries guarantees that $w(A) = \rho(\mathrm{Re}(A))$ and $w(B) = \rho(\mathrm{Re}(B))$ due to [16, Theorem 1]. Since $\mathrm{Re}(A)$ and $\mathrm{Re}(B)$ have entries that satisfy $0 \le \dfrac{a_{ij} + a_{ji}}{2} \le \dfrac{b_{ij} + b_{ji}}{2}$ for all i, j, it follows that $w(A) = \rho(\mathrm{Re}(A)) \le \rho(\mathrm{Re}(B)) = w(B)$. The authors of [15] attributes this observation to Panayiotis Psarrakos, and it will be used to give another upper bound for the free joint numerical radius in terms of the matrix of absolute values of $X_1, \ldots, X_m$.

Definition 3.6 For $X = [x_{ij}] \in \mathbb{C}^{n\times n}$, denote its matrix of absolute values as $\mathrm{abs}(X) := [|x_{ij}|]$.

Proposition 3.7 *Let $X_1, \ldots, X_m \in \mathbb{C}^{n\times n}$. Then*

$$w(X_1, \ldots, X_m) \le w\left(\mathrm{abs}(X_1) + \cdots + \mathrm{abs}(X_m)\right).$$

Proof Let $\mathcal{K}$ be a Hilbert space and let $U_1, \ldots, U_m \in B(\mathcal{K})$ be unitary. If $X_k = [x_{ij}^{(k)}]$ for $k = 1, \ldots, m$, then $\sum_{k=1}^m X_k \otimes U_k = \left[\sum_{k=1}^m x_{ij}^{(k)} U_k\right]$. By [17, Theorem 1.1(i)],

$$w\left(\sum_{k=1}^m X_k \otimes U_k\right) \le w\left(\left[\left\|\sum_{k=1}^m x_{ij}^{(k)} U_k\right\|\right]\right).$$

Now, $\left\|\sum_{k=1}^m x_{ij}^{(k)} U_k\right\| \le \sum_{k=1} |x_{ij}^{(k)}| \, \|U_k\| = \sum_{k=1} |x_{ij}^{(k)}|$ which is the (i, j)-entry of $\sum_{k=1}^m \mathrm{abs}(X_k)$. By the monotonicity of the numerical radius on nonnegative matrices, we have

$$w\left(\sum_{k=1}^m X_k \otimes U_k\right) \le w\left(\left[\left\|\sum_{k=1}^m x_{ij}^{(k)} U_k\right\|\right]\right) \le w\left(\sum_{k=1}^m \mathrm{abs}(X_k)\right).$$

Since the unitaries and $\mathcal{K}$ are arbitrary, the assertion holds. □

4 The Free Joint Numerical Radius of Generalized Permutations

Definition 4.1 A matrix $X \in \mathbb{C}^{n\times n}$ is an n-by-n *generalized permutation* if it has at most one nonzero entry in each row and column.

If $X \in \mathbb{C}^{n\times n}$ is a generalized permutation, then $X = DP$ where $P \in \mathbb{R}^{n\times n}$ is a permutation matrix and $D \in \mathbb{C}^{n\times n}$ is diagonal (see [18]). We do not require that D is nonsingular. If $X \in \mathbb{C}^{n\times n}$ is a generalized permutation, then so is X^*. Generalized permutations include important classes of matrices like the permutation matrices, diagonal matrices, and weighted shift matrices.

Suppose $A \in \mathbb{C}^{n\times n}$ is a weighted shift whose entries along the first superdiagonal are $a_1, \ldots, a_{n-1}$ and whose $(n, 1)$ entry is a_n. Then [19, Lemma 2(2)] guarantees that A is unitarily similar to $e^{i\theta}\text{abs}(A)$ for some $\theta \in \mathbb{R}$. Hence, $W(A) = e^{i\theta}W(\text{abs}(A))$, and so $w(A) = w(\text{abs}(A))$. However, an arbitrary generalized permutation may not be unitarily similar to a rotation of its absolute value matrix. For example, consider $X = \text{diag}(-1, 1)$ where $W(X) = [-1, 1] \neq \{e^{i\theta}\} = e^{i\theta}W(\text{abs}(X))$ for any $\theta \in \mathbb{R}$. Despite this, the assertion $w(X) = w(\text{abs}(X))$ still holds. We will show that this equality of numerical radii remains true for the free joint numerical radii of a tuple of generalized permutations. We need the following lemma.

Lemma 4.2 *Let $X \in \mathbb{C}^{n\times n}$ be a generalized permutation. There exists unitary $U \in \mathbb{C}^{n\times n}$ such that* $\text{abs}(X)$ *is a principal submatrix of $X \otimes U$ corresponding to the rows and columns* $1, 2+n, 3+2n, \dots, n+(n-1)n$.

Proof Let $D = \text{diag}(x_1, \dots, x_n)$ and $P = [p_{ij}] \in \mathbb{R}^{n\times n}$ a permutation matrix such that $X = DP$. Define $U = \overline{E}P$ where $E = \text{diag}(e^{i\arg(x_1)}, \dots, e^{i\arg(x_n)})$ (if $z = 0$, take $\arg(z) = 0$). Let $V = [e_1\ e_{2+n}\ \cdots\ e_{n+(n-1)n}] \in \mathbb{C}^{n^2\times n}$ where e_j is the j^{th} standard basis vector in $\mathbb{C}^{n^2}$. We show that $V^*(X \otimes U)V = \text{abs}(X)$. Indeed, note that

$$V^*(D \otimes \overline{E}) = [|x_1|e_1\ |x_2|e_{2+n}\ \cdots\ |x_n|e_{n+(n-1)n}]^t.$$

On the other hand, there exists permutation τ of $\{1, \dots, n\}$ such that the nonzero entries of P are at row $\tau(j)$ for each column j. Let $j = 1, \dots, n$. Column $j + (j-1)n$ of $P \otimes P$ is from the block $p_{ij}P$, and we know

$$p_{ij}P = \begin{cases} P, & i = \tau(j) \\ 0, & \text{otherwise.} \end{cases}$$

Hence, the only nonzero entry of column $j+(j-1)n$ of $P \otimes P$ is at row $\tau(j)+(\tau(j)-1)n$. It follows that $(P \otimes P)V = [e_{\tau(1)+(\tau(1)-1)n}\ \cdots\ e_{\tau(n)+(\tau(n)-1)n}]$ and so

$$\begin{aligned} V^*(X \otimes U)V &= [V^*(D \otimes \overline{E})][(P \otimes P)V] \\ &= \begin{bmatrix} |x_1|e_1^t \\ |x_2|e_{2+n}^t \\ \vdots \\ |x_n|e_{n+(n-1)n}^t \end{bmatrix} [e_{\tau(1)+(\tau(1)-1)n}\ \cdots\ e_{\tau(n)+(\tau(n)-1)n}] \\ &= \text{abs}(X). \end{aligned}$$

□

We are now ready to prove an alternative formula for $w(X_1, \dots, X_m)$ when $X_1, \dots, X_m \in \mathbb{C}^{n\times n}$ are generalized permutations.

Theorem 4.3 *Let $X_1, \dots, X_m \in \mathbb{C}^{n\times n}$ be generalized permutations. Then*

$$w(X_1, \dots, X_m) = w(\text{abs}(X_1), \dots, \text{abs}(X_m)) = w\left(\text{abs}(X_1) + \cdots + \text{abs}(X_m)\right).$$

Proof Since $\text{abs}(X_j)$ is a generalized permutation, it suffices to prove $w(X_1, \dots, X_m) = w\left(\text{abs}(X_1) + \ldots + \text{abs}(X_m)\right)$. Proposition 3.7 implies the inequality $w(X_1, \dots, X_m) \leq$

$w\,(\mathrm{abs}(X_1)+\cdots+\mathrm{abs}(X_m))$. By Lemma 4.2, there exist unitaries $U_1,\ldots,U_m\in\mathbb{C}^{n\times n}$ such that $\mathrm{abs}(X_1)+\cdots+\mathrm{abs}(X_m)$ is a principal submatrix of $X_1\otimes U_1+\cdots+X_m\otimes U_m$. Then

$$w(\mathrm{abs}(X_1)+\cdots+\mathrm{abs}(X_m))\le w(X_1\otimes U_1+\cdots+X_m\otimes U_m)\le w(X_1,\ldots,X_m).$$

□

When X is diagonal, $\mathrm{abs}(X)$ coincides with the standard definition of $|X|$ for operators, viz. $|X|=(X^*X)^{\frac{1}{2}}$. We obtain the following consequence of Theorem 4.3.

Corollary 4.4 *If* $\{X_1,\ldots,X_m\}\subseteq\mathbb{C}^{n\times n}$ *is a commuting family of normal matrices, then* $w(X_1,\ldots,X_m)=w(|X_1|,\ldots,|X_m|)=w\,(|X_1|+\cdots+|X_m|)$.

Proof By [14, Theorem 2.5.5], there exists unitary $U\in\mathbb{C}^{n\times n}$ such that $UX_jU^*=D_j$ where D_j is diagonal for each $j=1,\ldots,m$. Note that $\mathrm{abs}(D_j)=|D_j|=U|X_j|U^*$. By Proposition 3.3(i), $w(X_1,\ldots,X_m)$ is equal to

$$w(UX_1U^*,\ldots,UX_mU^*)=w(D_1,\ldots,D_m)=w(\mathrm{abs}(D_1)+\cdots+\mathrm{abs}(D_m))$$

where the last equality is due to Theorem 4.3. Since $U^*\left[\sum_{j=1}^m\mathrm{abs}(D_j)\right]U=\sum_{j=1}^m U^*|D_j|U=\sum_{j=1}^m|X_j|$, we obtain

$$w(X_1,\ldots,X_m)=w(U^*\left[\mathrm{abs}(D_1)+\cdots+\mathrm{abs}(D_m)\right]U)=w(|X_1|+\cdot+|X_m|).$$

□

For the remainder of this section, we will prove Conjecture 1.4 for a tuple of n-by-n generalized permutations. One preliminary step is to show that $f_{X_1,\ldots,X_m}(Z)$ remains diagonal when evaluated at a diagonal matrix Z. As a consequence, the corresponding recurrence (6) $\{L_k\}_{k\in\mathbb{N}}$ is a sequence of diagonal matrices since the initial value $L_1=(m+1)I$ is diagonal.

Lemma 4.5 *If* $X_1,\ldots,X_m\in\mathbb{C}^{n\times n}$ *are generalized permutations, then* $f_{X_1,\ldots,X_m}(Z)$ *is diagonal for all diagonal* $Z\ge 0$.

Proof Let $X\in\mathbb{C}^{n\times n}$ be a generalized permutation. Then $X=DP$ and $X^*=EQ$ where $D,E\in\mathbb{C}^{n\times n}$ are diagonals and $P,Q\in\mathbb{R}^{n\times n}$ are permutation matrices. Let $Z\in\mathbb{C}^{n\times n}$ be diagonal. Note that $X^*ZX=P^*D^*ZDP=P^*Z|D|^2P$ is diagonal since Z and D are diagonals and P is a permutation matrix. Similarly, $XZX^*=Q^*Z|E|^2Q$ is diagonal. In particular, if $Z\ge 0$, then $Z^{\frac{1}{2}}X^*ZXZ^{\frac{1}{2}}$ and $Z^{\frac{1}{2}}X_jZX_j^*Z^{\frac{1}{2}}$ are diagonals.

Suppose $X_1,\ldots,X_m\in\mathbb{C}^{n\times n}$ are generalized permutations and $Z\ge 0$ is diagonal. Then $Z^{\frac{1}{2}}X_j^*ZX_jZ^{\frac{1}{2}}$ and $Z^{\frac{1}{2}}X_jZX_j^*Z^{\frac{1}{2}}$ are diagonals for all $j=1,\ldots,m$. Since $f_{X_1,\ldots,X_m}(Z)$ is a sum of the identity and terms of the form $\left(Z^{\frac{1}{2}}X_j^*ZX_jZ^{\frac{1}{2}}+\frac{1}{4}I\right)^{\frac{1}{2}}$ and $\left(Z^{\frac{1}{2}}X_jZX_j^*Z^{\frac{1}{2}}+\frac{1}{4}I\right)^{\frac{1}{2}}$, the assertion follows. □

For a tuple of generalized permutations, we now prove Conjecture 1.4(i).

Theorem 4.6 *Let $X_1, \ldots, X_m \in \mathbb{C}^{n\times n}$ be generalized permutations. Then Conjecture 1.4(i) holds, i.e., the recurrence (6) is monotonically increasing.*

Proof By Lemma 4.5, the recurrence (6) is a sequence $\{L_k\}_{k\in\mathbb{N}}$ of positive semidefinite diagonal matrices where $L_1 \le L_2$ as observed before. Hence, it suffices to prove: if $0 \le Y \le Z$ where Y and Z are diagonals, then $f_{X_1,\ldots,X_m}(Y) \le f_{X_1,\ldots,X_m}(Z)$. Let $j = 1, \ldots, m$ and $0 \le Y \le Z$ be diagonals. Then $0 \le X_j^* Y X_j \le X_j^* Z X_j$ are diagonals due to Lemma 4.5. Since the matrices involved are positive semidefinite diagonals, we have

$$0 \le Y^{\frac{1}{2}}(X_j^* Y X_j)Y^{\frac{1}{2}} = (X_j^* Y X_j)Y \le (X_j^* Y X_j)Z = Z^{\frac{1}{2}}(X_j^* Z X_j)Z^{\frac{1}{2}}.$$

By operator monotonicity of $t^{\frac{1}{2}}$ on $[0, \infty)$ [20, Proposition V.1.8],

$$\left(Y^{\frac{1}{2}} X_j^* Y X_j Y^{\frac{1}{2}} + \frac{1}{4}I\right)^{\frac{1}{2}} \le \left(Z^{\frac{1}{2}} X_j^* Z X_j Z^{\frac{1}{2}} + \frac{1}{4}I\right)^{\frac{1}{2}}.$$

Similarly,

$$\left(Y^{\frac{1}{2}} X_j Y X_j^* Y^{\frac{1}{2}} + \frac{1}{4}I\right)^{\frac{1}{2}} \le \left(Z^{\frac{1}{2}} X_j Z X_j^* Z^{\frac{1}{2}} + \frac{1}{4}I\right)^{\frac{1}{2}}.$$

Thus, $f_{X_1,\ldots,X_m}(Y) \le f_{X_1,\ldots,X_m}(Z)$. □

Remark 4.7 The Proof of Theorem 4.6 shows that for a tuple of generalized permutations $f_{X_1,\ldots,X_m}$ is monotone on positive semidefinite diagonals, a stronger statement than the monotonicity of the recurrence (6). Numerical experiments show that the analogous statement for a tuple of *general* matrices is false, i.e., $0 \le Y \le Z$ does not imply $f_{X_1,\ldots,X_m}(Y) \le f_{X_1,\ldots,X_m}(Z)$. However, numerical experiments suggest that (6) is always monotonically increasing.

The inner product on $\mathbb{C}^{m\times n}$ defined by $\langle Y, Z\rangle = \text{trace}(Z^*Y)$ for all $Y, Z \in \mathbb{C}^{m\times n}$ induces the *Frobenius norm* $\|X\|_2 = \langle X, X\rangle^{\frac{1}{2}} = [\text{trace}(X^*X)]^{\frac{1}{2}}$. Consider the map vec : $\mathbb{C}^{m\times n} \to \mathbb{C}^{mn}$ defined by

$$\text{vec}(X) = [x_{11}, \ldots, x_{m1}, x_{12}, \ldots, x_{m2}, \ldots, x_{1n}, \ldots, x_{mn}]^t, \; X = [x_{ij}] \in \mathbb{C}^{m\times n}.$$

Note that $\langle Y, Z\rangle = \langle \text{vec}(Y), \text{vec}(Z)\rangle$ for any $Y, Z \in \mathbb{C}^{m\times n}$, i.e., the vec map is an isometry with respect to the Frobenius norm [21, Exercise on p. 244]. The *trace norm* of $X \in \mathbb{C}^{m\times n}$ is defined by $\|X\|_1 = \text{trace}[(X^*X)^{\frac{1}{2}}]$. Some straightforward properties of the trace norm are listed in the following lemma.

Lemma 4.8 (i) *If $X, Y \ge 0$, then* $\|X\|_1 = \text{trace}(X) = \|\text{vec}(X^{\frac{1}{2}})\|^2$ *and* $\|X + Y\|_1 = \|X\|_1 + \|Y\|_1$.
(ii) $\|(Y^*Y)^{\frac{1}{2}}\|_1 = \|Y\|_1$.
(iii) *If* $0 \le X \le Y$, *then* $\|X\|_1 \le \|Y\|_1$.

We relate the trace norm and the free joint numerical radius.

Lemma 4.9 *Let* $Z, X_1, \ldots, X_m \in \mathbb{C}^{n\times n}$ *and* $Z \geq 0$. *Then*

$$\sum_{j=1}^{m} \|Z^{\frac{1}{2}} X_j Z^{\frac{1}{2}}\|_1 \leq \|Z\|_1 w(X_1, \ldots, X_m).$$

Proof The trace norm of $X \in \mathbb{C}^{n\times n}$ has the following variational characterization [21, Problem 4 on p.199]:

$$\|X\|_1 = \max\{|\langle X, U\rangle| : U \in \mathbb{C}^{n\times n} \text{ is unitary}\}.$$

Hence, there exist unitaries $U_1, \ldots, U_m \in \mathbb{C}^{n\times n}$ such that $\|Z^{\frac{1}{2}} X_j Z^{\frac{1}{2}}\|_1 = \langle Z^{\frac{1}{2}} X_j Z^{\frac{1}{2}}, U_j\rangle$ for each $j = 1, \ldots, m$.

Consider:

$$\|Z^{\frac{1}{2}} X_j Z^{\frac{1}{2}}\|_1 = \langle Z^{\frac{1}{2}} X_j Z^{\frac{1}{2}}, U_j\rangle = \langle X_j Z^{\frac{1}{2}}, Z^{\frac{1}{2}} U_j\rangle = \langle \text{vec}(X_j Z^{\frac{1}{2}}), \text{vec}(Z^{\frac{1}{2}} U_j)\rangle.$$

In general, $\text{vec}(AXB) = (B^t \otimes A)\text{vec}(X)$ for any $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$, and $X \in \mathbb{C}^{n\times p}$ [21, Lemma 4.3.1]. We use this conversion formula to simplify

$$\begin{aligned}\langle \text{vec}(X_j Z^{\frac{1}{2}}), \text{vec}(Z^{\frac{1}{2}} U_j)\rangle &= \langle (I \otimes X_j)\text{vec}(Z^{\frac{1}{2}}), (U_j^t \otimes I)\text{vec}(Z^{\frac{1}{2}})\rangle \\ &= \langle (\overline{U_j} \otimes X_j)\text{vec}(Z^{\frac{1}{2}}), \text{vec}(Z^{\frac{1}{2}})\rangle.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{j=1}^{m} \|Z^{\frac{1}{2}} X_j Z^{\frac{1}{2}}\|_1 &= \sum_{j=1}^{m} \langle (\overline{U_j} \otimes X_j)\text{vec}(Z^{\frac{1}{2}}), \text{vec}(Z^{\frac{1}{2}})\rangle \\ &= \left\langle \sum_{j=1}^{m} (\overline{U_j} \otimes X_j)\text{vec}(Z^{\frac{1}{2}}), \text{vec}(Z^{\frac{1}{2}}) \right\rangle \\ &\leq w\left(\sum_{j=1}^{m} \overline{U_j} \otimes X_j\right) \langle \text{vec}(Z^{\frac{1}{2}}), \text{vec}(Z^{\frac{1}{2}})\rangle \\ &= w\left(\sum_{j=1}^{m} \overline{U_j} \otimes X_j\right) \|Z\|_1.\end{aligned}$$

by definition of the numerical radius and Lemma 4.8(i). Observe that $\sum_{j=1}^{m} \overline{U_j} \otimes X_j$ is permutation similar to $\sum_{j=1}^{m} X_j \otimes \overline{U_j}$ due to [21, Corollary 4.3.10]. Since each $\overline{U_j}$ is unitary,

$$\sum_{j=1}^{m} \|Z^{\frac{1}{2}} X_j Z^{\frac{1}{2}}\|_1 \leq w\left(\sum_{j=1}^{m} X_j \otimes \overline{U_j}\right) \|Z\|_1 \leq w(X_1, \ldots, X_m)\|Z\|_1$$

by definition of $w(X_1, \ldots, X_m)$. □

For any $r \geq 0$ and $X \in B(\mathcal{H})$ with $X \geq 0$, $(X + rI)^{\frac{1}{2}} \leq X^{\frac{1}{2}} + \sqrt{r} I$ due to operator monotonicity of $t^{\frac{1}{2}}$ on $[0, \infty)$ [20, Proposition V.1.8]. Hence, for a given tuple $X_1, \ldots, X_m \in B(\mathcal{H})$, we get

$$
\begin{aligned}
0 < f_{X_1,\ldots,X_m}(Z) &\leq (m+1)I \\
&+ \sum_{j=1}^{m} \left[\left(Z^{\frac{1}{2}} X_j^* Z X_j Z^{\frac{1}{2}} \right)^{\frac{1}{2}} + \left(Z^{\frac{1}{2}} X_j Z X_j^* Z^{\frac{1}{2}} \right)^{\frac{1}{2}} \right] \qquad (26)
\end{aligned}
$$

for any $Z \geq 0$.

Theorem 4.10 *Let $X_1, \ldots, X_m \in \mathbb{C}^{n \times n}$. For $R \geq 0$, let $B_R = \{Z \in \mathbb{C}^{n\times n} : Z \geq 0 \text{ and } \|Z\|_1 \leq R\}$. If $w(X_1, \ldots, X_m) < \frac{1}{2}$, then $f_{X_1,\ldots,X_m}$ as defined in (4) satisfies $f_{X_1,\ldots,X_m}(B_R) \subseteq B_R$ where $R = n(m+1)\,[1 - 2w(X_1, \ldots, X_m)]^{-1}$.*

Proof Applying Lemma 4.8(i) and (iii) to (26), we get

$$
\begin{aligned}
\|f_{X_1,\ldots,X_m}(Z)\|_1 &\leq (m+1)n + \left\| \sum_{j=1}^{m} \left[\left(Z^{\frac{1}{2}} X_j^* Z X_j Z^{\frac{1}{2}} \right)^{\frac{1}{2}} + \left(Z^{\frac{1}{2}} X_j Z X_j^* Z^{\frac{1}{2}} \right)^{\frac{1}{2}} \right] \right\|_1 \\
&= (m+1)n + \sum_{j=1}^{m} \left[\left\| \left(Z^{\frac{1}{2}} X_j^* Z X_j Z^{\frac{1}{2}} \right)^{\frac{1}{2}} \right\|_1 + \left\| \left(Z^{\frac{1}{2}} X_j Z X_j^* Z^{\frac{1}{2}} \right)^{\frac{1}{2}} \right\|_1 \right]
\end{aligned}
$$

Let $j = 1, \ldots, m$. Since $\left(Z^{\frac{1}{2}} X_j^* Z X_j Z^{\frac{1}{2}} \right)^{\frac{1}{2}} = \left[(Z^{\frac{1}{2}} X_j Z^{\frac{1}{2}})^* (Z^{\frac{1}{2}} X_j Z^{\frac{1}{2}}) \right]^{\frac{1}{2}}$ and $\left(Z^{\frac{1}{2}} X_j Z X_j^* Z^{\frac{1}{2}} \right)^{\frac{1}{2}} = \left[(Z^{\frac{1}{2}} X_j^* Z^{\frac{1}{2}})^* (Z^{\frac{1}{2}} X_j^* Z^{\frac{1}{2}}) \right]^{\frac{1}{2}}$, Lemma 4.8(ii) implies $\left\| \left(Z^{\frac{1}{2}} X_j^* Z X_j Z^{\frac{1}{2}} \right)^{\frac{1}{2}} \right\|_1 = \|Z^{\frac{1}{2}} X_j Z^{\frac{1}{2}}\|_1$ and $\left\| \left(Z^{\frac{1}{2}} X_j Z X_j^* Z^{\frac{1}{2}} \right)^{\frac{1}{2}} \right\|_1 = \|Z^{\frac{1}{2}} X_j^* Z^{\frac{1}{2}}\|_1$ $= \|Z^{\frac{1}{2}} X_j Z^{\frac{1}{2}}\|_1$. Due to Lemma 4.9,

$$
\begin{aligned}
\|f_{X_1,\ldots,X_m}(Z)\|_1 &\leq (m+1)n + 2 \sum_{j=1}^{m} \|Z^{\frac{1}{2}} X_j Z^{\frac{1}{2}}\|_1 \\
&\leq (m+1)n + 2\|Z\|_1 w(X_1, \ldots, X_m).
\end{aligned}
$$

Now it is easy to check that $Z \in B_R$ implies $f_{X_1,\ldots,X_m}(Z) \in B_R$. □

Remark 4.11 By using Theorem 4.10 and Brouwer's fixed point theorem, one gets an alternative proof of (ii)→(i) in Theorem 1.2. One key step in the Proof of Theorem 4.10 is when Lemma 4.9 is used. In the proof of this lemma, we used $\text{vec}(AXB) = (B^t \otimes A)\text{vec}(X)$ for any $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$, and $X \in \mathbb{C}^{n\times p}$. It is not immediately clear how to extend this formula to the infinite dimensional case.

Corollary 4.12 *Let $X_1, \ldots, X_m \in \mathbb{C}^{n\times n}$. If $w(X_1, \ldots, X_m) < \frac{1}{2}$, then the recurrence $\{L_k\}_{k\in\mathbb{N}}$ defined in (6) is bounded with respect to the trace-norm. In particular, $\{L_k\}_{k\in\mathbb{N}}$ has a convergent subsequence.*

Proof Let R and B_R be as in Theorem 4.10. Note that $\|L_1\|_1 = n(m+1) \leq R$. Hence, $L_2 = f_{X_1,\ldots,X_m}(L_1) \in B_R$ since $f_{X_1,\ldots,X_m}(B_R) \subseteq B_R$. Inductively, $\|L_k\|_1 \leq R$ for all $k \in \mathbb{N}$. The remaining assertion follows from Bolzano-Weierstrass Theorem. □

We are now ready to prove Conjecture 1.4(ii) for a tuple of generalized permutations.

Theorem 4.13 *Let $X_1, \ldots, X_m \in \mathbb{C}^{n\times n}$ be generalized permutations. If $w(X_1, \ldots, X_m) < \frac{1}{2}$, then Conjecture 1.4(ii) holds, i.e., the recurrence (6) converges.*

Proof Let $\{L_k\}_{k\in\mathbb{N}}$ be the recurrence (6). Theorem 4.6 guarantees that $\{L_k\}_{k\in\mathbb{N}}$ is monotonically increasing. By Corollary 4.12, $\{L_k\}_{k\in\mathbb{N}}$ is bounded with respect to the trace-norm. Since the dimension is finite, the recurrence $\{L_k\}_{k\in\mathbb{N}}$ also converges. □

Corollary 4.14 *If $\{X_1, \ldots, X_m\} \subseteq \mathbb{C}^{n\times n}$ is a commuting family of normal matrices, then $w(X_1, \ldots, X_m) = w(|X_1|, \ldots, |X_m|) = w(|X_1| + \cdots + |X_m|)$.*

Proof By [14, Theorem 2.5.5], there exists unitary $U \in \mathbb{C}^{n\times n}$ such that $UX_jU^* = D_j$ where D_j is diagonal for each $j = 1, \ldots, m$. For $Z \geq 0$, note that $f_{X_1,\ldots,X_m}(Z) = U^* f_{D_1,\ldots,D_m}(UZU^*)U$. The assertions follow by applying Theorems 4.6 and 4.13 to $f_{D_1,\ldots,D_m}$. □

Remark 4.15 It is worth noting that each L_k lies in the unital C^*-subalgebra of $B(\mathcal{H})$, generated by general $X_1, \ldots, X_m$. The latter may be a useful observation, as the monotonicity seems to require more than a simple operator monotonicity argument and it also depends on the initial value. As noted in Remark 4.7, numerical experiments show that $f_{X_1,\ldots,X_m}$ is not necessarily monotone on the cone of positive semidefinite operators.

Example 4.16 In the scalar case, it is easy to see why the convergence works so well. Indeed,

$$\begin{aligned} f_{x_1,\ldots,x_m}(z) &= 1 + 2\sqrt{z^2|x_1|^2 + \frac{1}{4}} + \cdots + 2\sqrt{z^2|x_m|^2 + \frac{1}{4}} \\ &\approx m + 1 + 2(|x_1| + \cdots + |x_m|)z \end{aligned}$$

as $z \to \infty$. If $|x_1| + \cdots + |x_m| = w(x_1, \ldots, x_m) < \frac{1}{2}$, then f has a unique positive fixed point as its graph intersects the half line $y = x$ $(x \geq 0)$ in a single point. Next, one finds for $z > 0$ that

$$0 < f'(z) = \sum_{j=1}^{m} \frac{4|x_j|^2 z}{\sqrt{4|x_j|^2 z^2 + 1}} < \sum_{j=1}^{m} \frac{4|x_j|^2 z}{2|x_j| z} = 2(|x_1| + \cdots + |x_m|) < 1.$$

Thus, the unique fixed point is a stable one, and the sequence is monotone.

5 Semidefinite Programming

Let $X_1, \ldots, X_m \in B(\mathcal{H})$ be given. For Hermitian $H_1, \ldots, H_m \in B(\mathcal{H})$, consider $\Phi(H_1, \ldots, H_m) := \Gamma\left(H_1, \ldots, H_m, -\sum_{j=1}^{m} H_j\right)$. Observe that

$$s_{H_1,\ldots,H_m} := \sup\{\lambda : \Phi(H_1, \ldots, H_m) - \lambda I \geq 0\} \in (-\infty, 0].$$

Proposition 5.1 *Let $X_1, \dots, X_m \in B(\mathcal{H})$. Then*

$$w(X_1, \dots, X_m) = -\frac{m+1}{2} \sup\{s_{H_1,\dots,H_m}\}$$

where the supremum is taken over all Hermitian $H_1, \dots, H_m \in B(\mathcal{H})$.

Proof Let ℓ be the left-hand side and ρ be the right hand side of the desired equality. If s denotes the supremum in the formula, then $s = -\frac{2\rho}{m+1}$.

Let $\varepsilon > 0$. By definition of the supremum, there exist Hermitian $H_1, \dots, H_m$ such that $s_{H_1,\dots,H_m} > s - \frac{2\varepsilon}{m+1} = -\frac{2\rho + 2\varepsilon}{m+1}$, and so $\Phi(H_1, \dots, H_m) + \left(\frac{2\rho + 2\varepsilon}{m+1}\right) I \geq 0$. The diagonal blocks add up to $(2\rho + 2\varepsilon)I$. Thus, $\ell = w(X_1, \dots, X_m) \leq \frac{1}{2}(2\rho + 2\varepsilon) = \rho + \varepsilon$ due to Theorem 1.1. Taking $\varepsilon \to 0$, we have $\ell \leq \rho$.

For the reverse inequality, let $\varepsilon > 0$. Since $w\left(\frac{X_1}{2\ell + 2\varepsilon}, \dots, \frac{X_m}{2\ell + 2\varepsilon}\right) = \frac{\ell}{2\ell + 2\varepsilon} \leq \frac{1}{2}$, there exist $A_1, \dots, A_{m+1} \in B(\mathcal{H})$ with $\sum_{j=1}^{m+1} A_j = (2\ell + 2\varepsilon)I$ so that

$$\begin{bmatrix} A_1 & \frac{X_1}{2\ell+2\varepsilon} & 0 & \cdots & 0 \\ \frac{X_1^*}{2\ell+2\varepsilon} & A_2 & \frac{X_2}{2\ell+2\varepsilon} & & \vdots \\ 0 & \frac{X_2^*}{2\ell+2\varepsilon} & \ddots & \ddots & 0 \\ \vdots & & \ddots & A_m & \frac{X_m}{2\ell+2\varepsilon} \\ 0 & \cdots & 0 & \frac{X_m^*}{2\ell+2\varepsilon} & A_{m+1} \end{bmatrix} \geq 0$$

due to Theorem 1.1. Put $H_j = A_j - \frac{1}{m+1} I$ for $j = 1, \dots, m$. Then $-\sum_{j=1}^{m} H_j = A_{m+1} - \frac{1}{m+1} I$. By the positivity of the operator above,

$$s \geq s_{(2\ell+2\varepsilon)H_1,\dots,(2\ell+2\varepsilon)H_m} \geq -\frac{2\ell + 2\varepsilon}{m+1}.$$

Then $\rho - \varepsilon = -\frac{m+1}{2} s - \varepsilon \leq \ell$. Taking $\varepsilon \to 0$, we obtain $\rho \leq \ell$. □

Let $X_1, \dots, X_m \in \mathbb{C}^{n \times n}$ be given. As a consequence of Theorem 1.1 (see also [3, Corollary 3.5]), we can numerically compute $w(X_1, \dots, X_m)$ by considering the following semidefinite program.

Primal problem:

$$\begin{aligned} &\text{minimize} \quad \rho \\ &\text{subject to} \quad \begin{bmatrix} A_1 & \frac{X_1}{2} & 0 & \cdots & 0 \\ \frac{X_1^*}{2} & A_2 & \frac{X_2}{2} & & \vdots \\ 0 & \frac{X_2^*}{2} & \ddots & \ddots & 0 \\ \vdots & & \ddots & A_m & \frac{X_m}{2} \\ 0 & \cdots & 0 & \frac{X_m^*}{2} & A_{m+1} \end{bmatrix} \geq 0 \\ &\text{and} \quad A_1 + \cdots + A_{m+1} = \rho I. \end{aligned}$$

To numerically solve the primal problem above, we used CVX, a package for solving convex programs [22, 23].

Dual problem:

$$\text{maximize} \quad -\text{Re}\left[\sum_{j=1}^{m} \text{trace}\left(X_j Z_{j,j+1}^*\right)\right]$$

$$\begin{aligned} \text{subject to} \quad & Z = [Z_{ij}]_{i,j=1}^{m+1} \geq 0 \\ \text{where} \quad & Z_{ij} \in \mathbb{C}^{n\times n} \\ & Z_{11} = \cdots = Z_{m+1,m+1} \\ & \text{trace}(Z_{11}) = 1. \end{aligned}$$

We claim that when $m = 1$, *strong duality* occurs, i.e., the dual optimal value is equal to the primal optimal value. To see this, we identify the extreme points of the convex feasible region of the dual problem. Suppose $Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{11} \end{bmatrix} \geq 0$. By [14, Theorem 7.7.9], we can write $Z_{12} = Z_{11}^{\frac{1}{2}} G Z_{11}^{\frac{1}{2}}$ for some contraction G. Since G is a square contraction, G is a convex combination of unitaries U_j. Next, $U_j = V_j \Delta_j V_j^*$, where V_j is unitary and $\Delta_j = \text{diag}(e^{i\theta_1^{(j)}}, \ldots, e^{i\theta_n^{(j)}})$. Hence, Z is a convex combination of

$$\begin{bmatrix} Z_{11}^{\frac{1}{2}} V_j & 0 \\ 0 & Z_{11}^{\frac{1}{2}} V_j \end{bmatrix} \begin{bmatrix} e_k \\ e^{-i\theta_k^{(j)}} e_k \end{bmatrix} \begin{bmatrix} e_k \\ e^{-i\theta_k^{(j)}} e_k \end{bmatrix}^* \begin{bmatrix} Z_{11}^{\frac{1}{2}} V_j & 0 \\ 0 & Z_{11}^{\frac{1}{2}} V_j \end{bmatrix}^*$$

where e_k is the k^{th} standard basis vector in $\mathbb{C}^n$. Thus, the extreme points of $\left\{ Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{11} \end{bmatrix} \geq 0 : \ \text{trace}(Z_{11}) = 1 \right\}$ are rank one matrices of the form

$$\begin{bmatrix} u \\ e^{-i\theta} u \end{bmatrix} \begin{bmatrix} u \\ e^{-i\theta} u \end{bmatrix}^*$$

where $u \in \mathbb{C}^n$ is a unit vector.

It is well known that the dual optimal value is a lower bound for the primal optimal value, which in this case is $w(X_1)$. At the extreme points of the dual feasible region, the objective function evaluates to

$$-\mathrm{Re}\left[\mathrm{trace}\left(e^{i\theta}X_1uu^*\right)\right] = \mathrm{Re}\left(e^{it}\langle X_1u, u\rangle\right).$$

Maximizing this over all $t \in \mathbb{R}$ and unit vectors $u \in \mathbb{C}^n$, we see that the dual optimal value is $w(X_1)$.

6 Limit Formula When $\mathcal{H}$ is Separable and $\dim(\mathcal{H}) = \infty$

Theorem 6.1 *Let $\mathcal{H}$ be separable with $\dim(\mathcal{H}) = \infty$ and $T_1, \ldots, T_m \in B(\mathcal{H})$. If $\{h_j\}_{j=1}^\infty$ is an orthonormal basis of $\mathcal{H}$ and $T_1^{(n)}, \ldots, T_m^{(n)}$ are the corresponding compressions onto $\mathrm{Span}\{h_j\}_{j=1}^n$ for each $n \in \mathbb{N}$, then*

$$w(T_1, \ldots, T_m) = \lim_{n\to\infty} w(T_1^{(n)}, \ldots, T_m^{(n)}).$$

Proof Let X_j be the matrix representation of T_j with respect to the basis $\{h_j\}_{j=1}^\infty$ for each $j = 1, \ldots, m$. Write $X_j = \begin{bmatrix} X_{11}^{(j,n)} & X_{12}^{(j,n)} \\ X_{21}^{(j,n)} & X_{22}^{(j,n)} \end{bmatrix}$ where $X_{11}^{(j,n)}$ is the n-by-n leading principal submatrix of X_j. By Proposition 3.1, it suffices to compute $\lim_{n\to\infty} w(X_{11}^{(1,n)}, \ldots, X_{11}^{(m,n)})$.

Let $\varepsilon > 0$ be given. Define $M := 2 + 3\sum_{j=1}^m \|X_j\| > 0$. By definition, there exist unitaries $U_1, \ldots, U_m \in B(\mathcal{K})$ for some Hilbert space $\mathcal{K}$ and unit $z = [z_1\ z_2\ \cdots]^t \in \bigoplus_{j=1}^\infty \mathcal{K}$ such that

$$w(T_1, \ldots, T_m) = w(X_1, \ldots, X_m) < \frac{2\varepsilon}{M} + \left|\left\langle \left(\sum_{j=1}^m X_j \otimes U_j\right) z, z\right\rangle\right|. \tag{27}$$

Consider

$$\sum_{j=1}^m X_j \otimes U_j = \begin{bmatrix} \sum_{j=1}^m X_{11}^{(j,n)} \otimes U_j & \sum_{j=1}^m X_{12}^{(j,n)} \otimes U_j \\ \sum_{j=1}^m X_{21}^{(j,n)} \otimes U_j & \sum_{j=1}^m X_{22}^{(j,n)} \otimes U_j \end{bmatrix}. \tag{28}$$

Note that $\|X_{ab}^{(j,n)} \otimes U_j\| \le \|X_j\|$, and hence $\left\|\sum_{j=1}^m X_{ab}^{(j,n)} \otimes U_j\right\| \le \sum_{j=1}^m \|X_j\|$. For each $n \in \mathbb{N}$, let $y_n = [z_1\ \cdots\ z_n]^t$ and $\widetilde{y} = [z_{n+1}\ z_{n+2}\ \cdots]^t$. Observe that $\|y_n\|, \|\widetilde{y}\| \le 1$

due to $1 = \|z\|^2 = \|y_n\|^2 + \|\widetilde{y}\|^2$. Since $z \in \bigoplus_{j=1}^{\infty} \mathcal{K}$, there exists N for which $n \geq N$ implies

$$\|\widetilde{y}\|^2 = \|z\|^2 - \|y_n\|^2 = \|z\|^2 - \sum_{j=1}^{n} \|z_k\|^2 < \frac{\varepsilon^2}{M^2}. \tag{29}$$

Using (28), (29), triangle inequality, and Cauchy-Schwarz, we obtain

$$\begin{aligned}\left|\left\langle \left(\sum_{j=1}^{m} X_j \otimes U_j\right) z, z \right\rangle\right| &\leq w\left(\sum_{j=1}^{m} X_{11}^{(j,n)} \otimes U_j\right) + \frac{3\varepsilon}{M}\sum_{j=1}^{m} \|X_j\| \\ &\leq w(X_{11}^{(1,n)}, \ldots, X_{11}^{(m,n)}) + \frac{3\varepsilon}{M}\sum_{j=1}^{m} \|X_j\|. \end{aligned} \tag{30}$$

By Proposition 3.1 and combining (27) and (30), we get that for all $n \geq N$

$$\begin{aligned}&|w(T_1, \ldots, T_m) - w(T_1^{(n)}, \ldots, T_m^{(n)})| \\ &= w(X_1, \ldots, X_m) - w(X_{11}^{(1,n)}, \ldots, X_{11}^{(m,n)}) < \frac{\varepsilon}{M}\left(2 + 3\sum_{j=1}^{m} \|X_j\|\right) = \varepsilon.\end{aligned}$$

□

Definition 6.2 An infinite matrix $X = [x_{ij}]$ is a *generalized permutation on* ℓ^2 if X has at most one nonzero entry in each row and column.

Observe that any n-by-n leading principal submatrix of X is an n-by-n generalized permutation.

Corollary 6.3 *Let $\mathcal{H}$ be separable with* $\dim(\mathcal{H}) = \infty$ *and* $T_1, \ldots, T_m \in B(\mathcal{H})$ *be simultaneously unitarily similar to* $X_1, \ldots, X_m$ *which are all generalized permutations on* ℓ^2*. If* $X_{11}^{(j,n)}$ *denotes the n-by-n leading principal submatrix of* X_j *for each* $j = 1, \ldots, m$*, then*

$$w(T_1, \ldots, T_m) = \lim_{n\to\infty} w\left(\operatorname{abs}(X_{11}^{(1,n)}) + \cdots + \operatorname{abs}(X_{11}^{(m,n)})\right).$$

In particular, if $X_1, \ldots, X_m$ *are diagonal, then*

$$w(X_1, \ldots, X_m) = w(|X_1|, \ldots, |X_m|) = w(|X_1| + \cdots + |X_m|).$$

Proof The first part follows from Theorems 4.3 and 6.1. For the last part, let $X_1, \ldots, X_m$ be diagonal. Then the absolute value matrix $\operatorname{abs}(X_{11}^{(j,n)})$ of the n-by-n leading principal submatrix of X_j coincides with the n-by-n leading principal submatrix of $|X_j|$.

We now prove an infinite dimensional analogue of Corollary 4.4.

Corollary 6.4 *Let $\mathcal{H}$ be separable with* $\dim(\mathcal{H}) = \infty$. *If* $\{X_1, \ldots, X_m\} \subset B(\mathcal{H})$ *is a commuting family of normal operators, then* $w(X_1, \ldots, X_m) = w(|X_1|, \ldots, |X_m|) = w(|X_1| + \cdots + |X_m|)$.

Proof Define $M := \max_{1\le j\le m} \{\|X_j\|\} + 1$ and let $c > \sqrt{M}$. Let $0 < \varepsilon < 3mc(c - \sqrt{M})$. By Weyl-von Neumann-Berg theorem [24, Theorem 39.4], there exist unitary U, diagonal operators $D_1, \ldots, D_m$, compact operators $K_1, \ldots, K_m$ with $\|K_j\| < \left(\frac{\varepsilon}{3mc}\right)^2$ such that $U^*X_jU = D_j + K_j$ for all $j = 1, \ldots, m$. By Proposition 3.3(i), we may assume that $X_j = D_j + K_j$ for all $j = 1, \ldots, m$. By assumption on c, $x^2 + \sqrt{M}x < cx$ for all $x \in (0, c - \sqrt{M})$. In particular,

$$3m\left(\left(\frac{\varepsilon}{3mc}\right)^2 + \sqrt{M}\left(\frac{\varepsilon}{3mc}\right)\right) < 3m\left(c \cdot \frac{\varepsilon}{3mc}\right) = \varepsilon. \tag{31}$$

Note that for any operators $S_1, \ldots, S_m, T_1, \ldots, T_m$, Proposition 3.3 (ii)-(iii) imply

$$|w(S_1, \ldots, S_m) - w(T_1, \ldots, T_m)| \le w(S_1 - T_1, \ldots, S_m - T_m) \le \sum_{j=1}^{m} \|S_j - T_j\|. \tag{32}$$

In particular,

$$|w(X_1, \ldots, X_m) - w(D_1, \ldots, D_m)| \le \sum_{j=1}^{m} \|K_j\| \tag{33}$$

and

$$|w(|X_1|, \ldots, |X_m|) - w(|D_1|, \ldots, |D_m|)| \le \sum_{j=1}^{m} \||X_j| - |X_j - K_j|\|. \tag{34}$$

By [20, Theorem X.2.1], each term on the right hand side of (34) is bounded above by $\sqrt{2}\|2X_j - K_j\|^{\frac{1}{2}}\|K_j\|^{\frac{1}{2}}$. Hence,

$$|w(|X_1|, \ldots, |X_m|) - w(|D_1|, \ldots, |D_m|)| \le \sum_{j=1}^{m}(2\|X_j\|^{\frac{1}{2}}\|K_j\|^{\frac{1}{2}} + \sqrt{2}\|K_j\|). \tag{35}$$

By Corollary 6.3, $w(D_1, \ldots, D_m) = w(|D_1|, \ldots, |D_m|)$ since $D_1, \ldots, D_m$ are diagonal. Thus, (31)-(35) guarantee

$$\begin{aligned}
|w(X_1, \ldots, X_m) - w(|X_1|, \ldots, |X_m|)| &\le |w(X_1, \ldots, X_m) - w(D_1, \ldots, D_m)| + \\
&\quad\ |w(|D_1|, \ldots, |D_m|) - w(|X_1|, \ldots, |X_m|)| \\
&\le \sum_{j=1}^{m}(\|K_j\| + \sqrt{2}\|K_j\| + 2\|X_j\|^{\frac{1}{2}}\|K_j\|^{\frac{1}{2}}) \\
&\le \sum_{j=1}^{m}(3\|K_j\| + 3\|X_j\|^{\frac{1}{2}}\|K_j\|^{\frac{1}{2}}) \\
&< 3m\left(\left(\frac{\varepsilon}{3mc}\right)^2 + \sqrt{M}\left(\frac{\varepsilon}{3mc}\right)\right) < \varepsilon.
\end{aligned}$$

Note that $w(|D_1|, \ldots, |D_m|) = w(|D_1| + \cdots + |D_m|)$ by Corollary 6.3, and so for the other equality

$$|w(X_1, \ldots, X_m) - w(|X_1| + \cdots + |X_m|)| \leq |w(X_1, \ldots, X_m) - w(D_1, \ldots, D_m)| + |w\left(\sum_{j=1}^m |D_j|\right) - w\left(\sum_{j=1}^m |X_j|\right)|.$$

Since (33) is still an upper bound, the argument in the first part still works if it can be shown that

$$\left| w\left(\sum_{j=1}^m |X_j|\right) - w\left(\sum_{j=1}^m |D_j|\right)\right| \leq \sum_{j=1}^m \| |X_j| - |X_j - K_j| \|.$$

Indeed, this follows from (32).

7 Factorization of Hermitian Pencils

One can relate the free joint numerical radius to a factorization of a certain Hermitian pencil. Consider the operator-valued trigonometric polynomial $Q(z_1, \ldots, z_m) = I + \sum_{j=1}^m z_j X_j + \sum_{j=1}^m z_j^{-1} X_j^*$, where $X_1, \ldots, X_m \in B(\mathcal{H})$. We are interested in the question when an affine operator-valued matrix polynomial $P(z_1, \ldots, z_m) = P_0 + \sum_{j=1}^m P_j z_j$ exists so that $Q(z_1, \frac{z_2}{z_1}, \ldots, \frac{z_m}{z_{m-1}}) = P(z_1, \ldots, z_m)^* P(z_1, \ldots, z_m)$, for all $(z_1, \ldots, z_m) \in \mathbb{T}^m$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Here $P_0, \ldots, P_m \in B(\mathcal{H}, \mathcal{K})$ (= the Banach space of bounded operators acting from Hilbert space $\mathcal{H}$ to Hilbert space $\mathcal{K}$) for some Hilbert space $\mathcal{K}$. Clearly, we need that $Q(z_1, \ldots, z_n) \geq 0$ for all $(z_1, \ldots, z_m) \in \mathbb{T}^m$. The operator-valued one variable Fejér-Riesz Theorem, due to [25] (see also [26]), states that the existence of P is equivalent to $Q(z_1) \geq 0$, $|z_1| = 1$, and in this case one can choose $\mathcal{K} = \mathcal{H}$ and $P(z_1)$ to be outer or co-outer. Recently, Dritschel [27] showed that also in two variables the condition $Q(z_1, z_2) \geq 0$, $(z_1, z_2) \in \mathbb{T}^2$, is necessary and sufficient for the existence of P (in this case, though, $\mathcal{K}$ is not necessarily equal to $\mathcal{H}$ as the example $|z_1 - 1|^2 + |z_2 - 1|^2$ easily shows).

Our result is the following.

Theorem 7.1 *Let $Q(z_1, \ldots, z_m) = I + \sum_{j=1}^m z_j X_j + \sum_{j=1}^m z_j^{-1} X_j^*$, where $X_1, \ldots, X_m \in B(\mathcal{H})$. The following are equivalent:*

(i) $w(X_1, \ldots, X_m) \leq \frac{1}{2}$.
(ii) *For all Hilbert spaces $\mathcal{K}$ and for all unitaries $U_1, \ldots, U_m \in B(\mathcal{K})$, we have*

$$I \otimes I + \sum_{j=1}^m X_j \otimes U_j + \sum_{j=1}^m X_j^* \otimes U_j^* \geq 0.$$

(iii) *There exist $A_1, \ldots, A_{m+1} \in B(\mathcal{H})$ so that $\sum_{j=1}^{m+1} A_j = I$ and $\Gamma(A_1, \ldots, A_{m+1}) \geq 0$.*

(iv) *There exist Hilbert space $\mathcal{K}$ and affine matrix polynomial $P(z_1, \dots, z_m) = P_0 + \sum_{j=1}^m P_j z_j$ where $P_j \in B(\mathcal{H}, \mathcal{K})$ so that*

$$Q\left(z_1, \frac{z_2}{z_1}, \dots, \frac{z_m}{z_{m-1}}\right) = P(z_1, \dots, z_m)^* P(z_1, \dots, z_m) \text{ for all } (z_1, \dots, z_m) \in \mathbb{T}^m. \tag{36}$$

Remark 7.2 In the classical Fejér-Riesz factorization result (see [26] for the one variable case; see [28] for the multivariable case), we only require positive semidefiniteness when we plug scalars of modulus one as the variables. In Theorem 7.1(ii) above, one checks the inequality for all unitaries. In the factorization, this translates into the requirement that P is affine, as opposed to any analytic matrix polynomial.

Proof of Theorem 7.1 The equivalence of (i), (ii), and (iii) follow from the results in [3].
(iv) → (iii): Assuming (iv) we obtain that

$$\sum_{j=0}^m P_j^* P_j = I, \quad P_{j-1}^* P_j = X_j, \quad P_k^* P_j = 0, k \neq j, j-1.$$

Putting $A_{j+1} = P_j^* P_j$, $j = 0, \dots, m$, we obtain that $\Gamma(A_1, \dots, A_{m+1}) = G^* G$, where $G = \begin{bmatrix} P_0 & \cdots & P_m \end{bmatrix}$. Thus (iii) follows.
(iii) → (iv): Assuming (iii), write $\Gamma(A_1, \dots, A_{m+1}) = G^* G$ with $G \in B(\mathcal{H}^{m+1}, \mathcal{K})$ for some Hilbert space $\mathcal{K}$. Next decompose $G = \begin{bmatrix} P_0 & \cdots & P_m \end{bmatrix}$, with $P_j \in B(\mathcal{H}, \mathcal{K})$. With this choice, (iv) follows. □

The following example illustrates how the requirement that P is affine affects the existence of a solution.

Example 7.3 Let $m = 2$, $X_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $X_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Letting $U_1 = z_1$ and $U_2 = z_2$ be scalars, we obtain that

$$\max_{|z_1|=|z_2|=1} w\left(\begin{bmatrix} z_1 & z_2 \\ z_2 & -z_1 \end{bmatrix}\right) = \max_{|\lambda|=1} w\left(\begin{bmatrix} 1 & \lambda \\ \lambda & -1 \end{bmatrix}\right) = \sqrt{2},$$

where the maximum is achieved at $\lambda = \pm 1$. Since X_1 and X_2 are generalized permutations, Theorem 4.3 guarantees that

$$w(X_1, X_2) = w(\text{abs}(X_1) + \text{abs}(X_2)) = w\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = 2.$$

Thus, for $r \geq 2\sqrt{2}$,

$$Q_r(z_1, z_2) := rI + \sum_{j=1}^2 z_j X_j + \sum_{j=1}^2 z_j^{-1} X_j^* \geq 0, \ |z_1| = |z_2| = 1,$$

but a factorization as in Theorem 7.1(iv) exists only for $r \geq 4$. When $r = 4$, condition Theorem 7.1(iii) is uniquely satisfied for $A_2 = 2I_2$, $A_1 = A_3 = I_2$, leading to the factorization $Q_4(z_1, \frac{z_2}{z_1}) = P(z_1, z_2)^* P(z_1, z_2)$, $|z_1| = |z_2| = 1$, where

$$P(z_1, z_2) = \begin{bmatrix} 0 & -\frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & 0 \\ \frac{1}{2}\sqrt{2} & 0 \\ 0 & \frac{1}{2}\sqrt{2} \end{bmatrix} + z_1 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{bmatrix} + z_2 \begin{bmatrix} -\frac{1}{2}\sqrt{2} & 0 \\ 0 & \frac{1}{2}\sqrt{2} \\ 0 & \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & 0 \end{bmatrix}.$$

It follows from the main result in [27] that $Q_{2\sqrt{2}}$ has a factorization P^*P, where P is analytic matrix polynomial degree of degree (1,1) with $P(z_1, z_2) \in \mathbb{C}^{4\times 2}$. This factorization can be computed by letting

$$K = \frac{\sqrt{2}}{2} I_8 + \begin{bmatrix} 0 & \frac{1}{2}X_1 & \frac{1}{2}X_2 & 0 \\ \frac{1}{2}X_1^* & 0 & 0 & \frac{1}{2}X_2 \\ \frac{1}{2}X_2^* & 0 & 0 & \frac{1}{2}X_1 \\ 0 & \frac{1}{2}X_2^* & \frac{1}{2}X_1^* & 0 \end{bmatrix} = B^*B,$$

where

$$B = \sqrt[4]{2} \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2}\sqrt{2} & 0 \\ -\frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2}\sqrt{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2}\sqrt{2} \end{bmatrix}.$$

Then $K = B^*B \geq 0$ and for $|z_1| = |z_2| = 1$,

$$\left[I_2 \ \overline{z_1} I_2 \ \overline{z_2} I_2 \ \overline{z_1 z_2} I_2 \right] B^*B \begin{bmatrix} I_2 \\ z_1 I_2 \\ z_2 I_2 \\ z_1 z_2 I_2 \end{bmatrix} = P(z_1, z_2)^* P(z_1, z_2) = Q_{2\sqrt{2}}(z_1, z_2),$$

where $P(z_1, z_2)$ equals

$$\sqrt[4]{2} \left(\begin{bmatrix} 0 & 0 \\ -\frac{1}{2}\sqrt{2} & 0 \\ 0 & -\frac{1}{2}\sqrt{2} \\ 0 & 0 \end{bmatrix} + z_1 \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} + z_2 \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} + z_1 z_2 \begin{bmatrix} -\frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{2}\sqrt{2} \end{bmatrix} \right).$$

8 Open Problems

Let $X_1, \ldots, X_m \in B(\mathcal{H})$. In this paper, we related the free joint numerical radius $w(X_1, \ldots, X_m)$ to a fixed point problem involving the operator-valued function

$$f_{X_1,\ldots,X_m}(Z) = I + \sum_{j=1}^{m} \left[\left(Z^{\frac{1}{2}} X_j^* Z X_j Z^{\frac{1}{2}} + \frac{1}{4} I \right)^{\frac{1}{2}} + \left(Z^{\frac{1}{2}} X_j Z X_j^* Z^{\frac{1}{2}} + \frac{1}{4} I \right)^{\frac{1}{2}} \right].$$

In particular, we showed in the matrix case (Corollary 1.3) that the following are equivalent:

(i) $f_{X_1,\ldots,X_m}$ as defined in (4) has a positive definite fixed point, i.e., there exists positive definite $L \in B(\mathcal{H})$ for which

$$L = I + \sum_{j=1}^{m}\left[\left(L^{\frac{1}{2}}X_j^* L X_j L^{\frac{1}{2}} + \frac{1}{4}I\right)^{\frac{1}{2}} + \left(L^{\frac{1}{2}}X_j L X_j^* L^{\frac{1}{2}} + \frac{1}{4}I\right)^{\frac{1}{2}}\right].$$

(ii) $w(X_1, \ldots, X_m) < \frac{1}{2}$.

Theorem 1.2 guarantees that (i)→(ii) holds when $\dim(\mathcal{H}) = \infty$. Remarks 2.6 and 4.11 mention the difficulties encountered for (ii)→(i).

Example 8.1 To illustrate the existence of a fixed point in an infinite dimensional case, consider $\mathcal{H} = L^2([-\pi, \pi])$ where $\langle f, g\rangle := \int_{-\pi}^{\pi} f(t)\overline{g(t)}dt$. Each $x \in L^\infty([-\pi, \pi])$ induces a bounded operator $M_x \in B(\mathcal{H})$ defined by the multiplication operator $(M_x)(h(t)) = x(t)h(t)$ for all $h \in \mathcal{H}$. Note that multiplication operators are normal, and so $w(M_x) = \|M_x\| = \|x\|_\infty$. Let $x \in L^\infty([-\pi, \pi])$ be such that $w(M_x) < \frac{1}{2}$. Let M_z where $z := \dfrac{2}{1 - 4|x|^2}$. Due to $\|x\|_\infty = w(M_x) < \frac{1}{2}$, $z \in L^\infty([-\pi, , \pi])$ and M_z is positive definite in $B(\mathcal{H})$. The value of $f_{M_x}(M_z)$ is the multiplication operator corresponding to $1+2\sqrt{z^2|x|^2 + \frac{1}{4}}$ which simplifies to z. Hence, M_z is a positive definite fixed point of f_{M_x}.

Open problem 1 Assume $\dim(\mathcal{H}) = \infty$. If $w(X_1, \ldots, X_m) < \frac{1}{2}$, show that $f_{X_1,\ldots,X_m}$ has a positive definite fixed point, i.e., there exists positive definite $L \in B(\mathcal{H})$ for which

$$L = I + \sum_{j=1}^{m}\left[\left(L^{\frac{1}{2}}X_j^* L X_j L^{\frac{1}{2}} + \frac{1}{4}I\right)^{\frac{1}{2}} + \left(L^{\frac{1}{2}}X_j L X_j^* L^{\frac{1}{2}} + \frac{1}{4}I\right)^{\frac{1}{2}}\right].$$

We also considered approximating such fixed point by defining the iteration

$$L_1 = (m+1)I \text{ and } L_{k+1} = f_{X_1,\ldots,X_m}(L_k) \text{ for } k \in \mathbb{N}.$$

When each X_j is a generalized permutation, Theorems 4.6 and 4.13 guarantee that the recurrence $\{L_k\}_{k\in\mathbb{N}}$ is monotonically increasing and convergent (provided $w(X_1, \ldots, X_m) < \frac{1}{2}$). For general X_j's, the following remain open problems.

Open problem 2 Show in general that $L_k \leq L_{k+1}$ for all $k \in \mathbb{N}$.

Open problem 3 If $w(X_1, \ldots, X_m) < \frac{1}{2}$, show in general that $\{L_k\}_{k\in\mathbb{N}}$ converges in the weak operator topology to a fixed point $L \in B(\mathcal{H})$ of $f_{X_1,\ldots,X_m}$.

Note that a positive solution to Open problem 3 would imply that if $X_1, \ldots, X_m$ were in a C^*-algebra, then L and subsequently a solution $A_1, \ldots, A_{m+1}$ (as constructed in Proposition 2.5) would also lie in the same C^*-algebra.

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Declarations

Conflict of interest No potential conflict of interest was reported by the authors.

References

1. Ando, T.: Structure of operators with numerical radius one. Acta Sci. Math. (Szeged) **34**, 11–15 (1973)
2. Engwerda, J.C., Ran, A.C.M., Rijkeboer, A.L.: Necessary and Sufficient Conditions for the Existence of a Positive Definite Solution of the Matrix Equation $X + A^*X^{-1}A = Q$. Linear Algebra Appl. **186**, 255–275 (1993)
3. Farenick, D., Kavruk, A.S., Paulsen, V.I.: C*-algebras with the weak expectation property and a multivariable analogue of Ando's theorem on the numerical radius. J. Operator Theory **70**, 573–590 (2013)
4. Bakonyi, M., Woerdeman, H.J.: Maximum Entropy Elements in the Intersection of an Affine Space and the Cone of Positive Definite Matrices. SIAM J. Matrix Anal. Appl. **16**, 369–376 (1995)
5. Barrett, W.W.: A Theorem on Inverses of Tridiagonal Matrices. Linear Algebra Appl. **27**, 211–217 (1979)
6. Dym, H., Gohberg, I.: Extensions of Band Matrices with Band Inverses. Linear Algebra Appl. **36**, 1–24 (1981)
7. Barrett, W.W., Johnson, C.R., Lundquist, M.E., Woerdeman, H.J.: Completing a Block Diagonal Matrix With a Partially Prescribed Inverse. Linear Algebra Appl. **223**(224), 73–87 (1995)
8. Fiedler, M.: A remark on positive definite matrices, Časopis Pěst. Mat. **85**, 75–77 (1960)
9. Conway, J.B.: A Course in Functional Analysis. Springer, New York, USA (1990)
10. Gau, H.-L., Wang, K.-Z., Wu, P.Y.: Numerical Radii for Tensor Products of Operators. Integral Equ. Oper. Theory **78**, 375–382 (2014)
11. Holbrook, J.A.R.: Multiplicative properties of the numerical radius in operator theory. J. Reine Angew. Math. **237**, 166–174 (1969)
12. Halmos, P.R.: Normal dilations and extension of operators. Summa Brasil Math. **2**, 125–134 (1950)
13. Halmos, P.R.: A Hilbert Space Problem Book. Springer, New York, USA (1982)
14. Horn, R.A., Johnson, C.R.: Matrix Analysis. Cambridge University Press, Cambridge (2013)
15. Abu-Omar, A., Kittaneh, F.: Numerical radius inequalities for $n \times n$ operator matrices. Linear Algebra Appl. **468**, 18–26 (2015)
16. Goldberg, M., Tadmor, E., Zwas, G.: Numerical Radius of Positive Matrices. Linear Algebra Appl. **12**, 209–214 (1975)
17. Hou, J.-C., Du, H.-K.: Norm inequalities of positive operator matrices. Integral Equ. Oper. Theory **22**, 281–294 (1995)
18. Kaddoura, I., Mourad, B.: On a class of matrices generated by certain generalized permutation matrices and applications. Linear Multilinear Algebra **67**, 2117–2134 (2019)
19. Tsai, M.C., Wu, P.Y.: Numerical ranges of weighted shift matrices. Linear Algebra Appl. **435**, 243–254 (2011)
20. Bhatia, R.: Matrix Analysis, Springer-Verlag. New York, USA (1997)
21. Horn, R.A., Johnson, C.R.: Topics in Matrix Analysis. Cambridge University Press, Cambridge (1991)

22. Grant, M., Boyd, S.: CVX: Matlab software for disciplined convex programming, version 2.1, http://cvxr.com/cvx (Mar. 2014)
23. Grant, M., Boyd, S.: Graph implementations for nonsmooth convex programs, In: V. Blondel, S. Boyd, H. Kimura (Eds.), Recent Advances in Learning and Control, Lecture Notes in Control and Information Sciences, Springer-Verlag Limited, pp. 95–110 (2008) http://stanford.edu/~boyd/graph_dcp.html
24. Conway, J. B. : A Course in Operator Theory, Amer. Math. Soc., Providence, RI, USA (2000)
25. Rosenblum, M.: Vectorial Toeplitz operators and the Fejér-Riesz theorem. J. Math. Anal. Appl. **23**, 139–147 (1968)
26. Rosenblum, M., Rovnyak, J.: Hardy Classes and Operator Theory. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York (1985)
27. Dritschel, M. A.: Factoring non-negative operator valued trigonometric polynomials in two variables (2018). arXiv:1811.06005
28. Dritschel, M.A.: On factorization of trigonometric polynomials. Integral Equ. Oper. Theory **49**, 11–42 (2004)