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3D shear flows driven by Lévy noise at the boundary

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Dedicated to Professor Alain Bensoussan on the occasion of his 80th birthday.

Abstract This paper is concerned with the stochastic incompressible Navier–Stokes equations in a layer of fluid between two flat no-slip boundaries. The fluid is driven by the noisy movement of the bottom boundary, where the noise is given by a Lévy process. After establishing existence of a martingale solution, we use the background flow method to derive an upper bound on the turbulent energy dissipation rate. Our estimate recovers one of the basic scaling ideas of turbulence theory, namely, that the dissipation rate is independent of the viscosity at high Reynolds number.

Keywords Existence, Lévy noise, Navier–Stokes equations, 3D shear flows, Upper bound

2020 Mathematics Subject Classification 35Q30, 76D05

1. Introduction

We consider the stochastic 3D Navier–Stokes equations in the box $\mathrm{D} = [0, L]^2 \times [0, h]$

$$\begin{aligned} &\mathrm{d}\mathbf{u} + (\mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p)\,\mathrm{d}t = 0, \\ &\nabla\cdot\mathbf{u} = 0, \end{aligned} \tag{1.1}$$

which is only driven by the random motion of the bottom wall (so its body force is $\mathbf{f} = 0$). A random boundary condition is given by

$$\mathbf{u}(x_1, x_2, 0, t) = (\mathbb{X}_t, 0, 0)^\top \quad \text{and} \quad \mathbf{u}(x_1, x_2, h, t) = (0, 0, 0)^\top , \tag{1.2}$$

for all time $t \in \mathbb{R}_+$ and $(x_1, x_2) \in (0, L)^2$, while $\mathbb{X} = (\mathbb{X}_t)_{t\in\mathbb{R}_+}$ is the given real-valued, square-integrable Lévy process described below. In addition, L-periodic boundary conditions in the x_1 and x_2 directions are imposed. In the above, the stochastic processes $\mathbf{u}$ and p are the velocity and pressure, respectively, and the kinematic viscosity is denoted by $\nu > 0$.

With the above boundary condition (1.2), when a fluid is enclosed between two plates and the bottom plate is moved in one direction, a shear flow results. Heuristically, the flow near the bottom plate is faster than the top one, therefore vorticity is not negligible. When the fluid's

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vorticity becomes large enough, the flow becomes swirly and turbulent [26]. This flow problem is very close to flow between rotating cylinders, which is one of the most classical problems in experimental fluid dynamics [16].

The shear flow problem with constant velocity[1] is well studied in the literature [10, 21, 25, 28, 31, 41]. However, in practice, the velocity of the shear wall cannot be kept constant, due to the randomness of the background movement. This randomness can be caused by unavoidable perturbations in the boundary conditions, or material properties [5, 34]. It is therefore natural to add noise to the velocity of the shear wall aiming to model this randomness. For the stochastic Navier–Stokes equations, most works focus on the motions which are derived by a stochastic force, which dates back to the early 1970s with (as far as we know) the paper of Bensoussan and Temam [2]. Other than [4] and [13], to the best of our knowledge, there are not many works rigorously studying the equations of the motion with stochastic boundary conditions. The objective in this paper is to first study the existence of global martingale solutions to the stochastic Navier–Stokes equations (1.1) driven by the random boundary condition (1.2). We then study the effect of the noise on key characteristics of turbulence (dissipation rate) as manifested by these solutions with $\mathbb{X}_t$ considered to be the Lévy noise.

1.1 Kolmogorv dissipation law

In turbulent flows, it is not feasible to obtain a detailed description of the fluid velocity since the state of motion is too complex. Experimental or numerical measurements of instantaneous system variables appear chaotic, disorganized, and unpredictable [34]. When averaged, however, certain quantities obey robust laws. One such quantity, the energy dissipation rate, carries important information about the structure and statistical properties of a turbulent flow. It is well known that the statistical properties are much more important, physically relevant, and stable than single trajectories [15, 38]. Based on Kolmogorov's conventional turbulence theory at large Reynolds number, dissipation appears to exist independently of viscosity, see [18, 22]. Hence, by a dimensional consideration, the energy dissipation rate per unit volume, ε, scales as

$$\varepsilon := \limsup_{T\to\infty} \frac{1}{|\mathrm{D}|}\,\frac{1}{T}\int_0^T \nu\,\|\nabla \mathbf{u}\|^2_{L^2(\mathrm{D})} \simeq C_\varepsilon \frac{U^3}{h},$$

where U and h are global velocity and length scales, with C_ε as the asymptotic constant (Kolmogorov 1941). This result is fundamental to an understanding of turbulence [26, 36], and confirmed by measurements (e.g. [19, 24, 35]).

The energy dissipation rate has been widely studied in the literature in the deterministic case, see, e.g. [3, 9, 11, 12, 18, 21, 25, 27, 28, 31–33, 41]. In the theory of turbulence, upper estimates of energy dissipation rates are useful for, in particular, determining the overall complexities of turbulent flow simulations. It also determines the smallest persistent length scales and the dimension of any global attractor (if it exists) [7, 16, 34, 37]. Doering and Constantin in [10] proved rigorous asymptotic upper bounds for the deterministic shear driven turbulence. Their bound is of the form

$$\varepsilon \lesssim U^3/h \quad \text{as} \quad \mathrm{Re}\to\infty, \tag{1.3}$$

[1] In the deterministic case when the bottom wall moves with a constant velocity, i.e., $\mathbb{X}_t = U$ in (1.2), one can show that

$$\mathbf{u} = ((h - x_3)\,U/h, 0, 0)^\top, \quad 0 \leqslant x_3 \leqslant h, \quad \text{and} \quad p = \text{constant},$$

is a solution of the steady Navier-Stokes equations for every Reynolds number. However, for higher velocities U, the solution is not unique anymore (see, e.g. [39]) and this flow becomes unstable and it is no longer observed in physical experiments.

where $\mathrm{Re} = Uh/\nu$. Recently the authors in [13] considered a shear turbulence flow when the boundary moves at the random speed as an Ornstein–Uhlenbeck process. They could quantify the effect of the noise by upper bounds on the first moment of the dissipation rate as

$$\varepsilon = \limsup_{T\to\infty} \mathbb{E}\left[\frac{1}{|\mathrm{D}|}\,\frac{1}{T}\int_0^T \nu\|\nabla \mathbf{u}(t,\cdot,\omega)\|^2_{L^2(\mathrm{D})}\,\mathrm{d}t\right] \lesssim \frac{U^3}{h} + C_{\mathrm{Re},\theta}\,\sigma^2, \tag{1.4}$$

where $C_{\mathrm{Re},\theta} = \mathcal{O}\left(1/\mathrm{Re}, 1/\theta\right)$. A potential overdissipation is observed if the O.U. process were replaced by the Wiener process, that is, if the dissipation coefficient θ in (1.5) goes to 0, the bound in the right-hand side of (1.8) goes to infinity.

1.2 Assumptions and setup

This paper generalizes the results in [13] by allowing the process $\mathbb{X}$ to have jumps. More precisely, we take $\mathbb{X}_t$ to be a stochastic process that satisfies the equation

$$\mathrm{d}\mathbb{X}_t = \theta(U - \mathbb{X}_t)\mathrm{d}t + \mathrm{d}L_t, \tag{1.5}$$

where $\theta, U > 0$ are constants, and L is a square-integrable Lévy martingale given by

$$L_t = \sigma W_t + \int_{E_0\times(0,t]} K(\xi,s)\mathrm{d}\hat{\pi}(\xi,s), \tag{1.6}$$

where $W = (W_t)_{t\in\mathbb{R}_+}$ is a Wiener process, $\sigma\in\mathbb{R}$ is a constant, and π is a Poisson random measure on $E\times(0,\infty)$ arising from a stationary Poisson point process Π on a measurable space $(E,\mathcal{E})$. We assume that the intensity measure of π has the form $\mathrm{d}\lambda\otimes\mathrm{d}t$ for some σ-finite measure λ on $(E,\mathcal{E})$. In equation (1.6) we fix $E_0\in\mathcal{E}$ such that $\lambda(E_0)<\infty$. We denote by $\hat{\pi}$ the associated compensated Poisson random measure, i.e. $\hat{\pi}(A\times(0,t]) = \pi(A\times(0,t]) - \lambda(A)\,t$ for $A\in\mathcal{E}$ and $t>0$. Furthermore, we assume that W is independent of π. Under this condition it is possible to construct a right-continuous filtration $(\mathcal{F}_t)_{t\geqslant 0}$ on $(\Omega,\mathcal{F},\mathbb{P})$ such that $(\Omega,\mathcal{F}_0,\mathbb{P})$ is complete and W and Π are both $(\mathcal{F}_t)_{t\geqslant 0}$-adapted. Furthermore the filtration can be chosen such that $W(t)-W(s)$ and $\Pi(t)-\Pi(s)$ are each independent of $\mathcal{F}_s$, for all $t\geqslant s\geqslant 0$.

We will assume that the noise coefficient function $K\colon[0,T]\times E\to\mathbb{R}$ is a bounded Borel measurable function; $K\in L^\infty([0,T]\times E)$ for any $T>0$. The existence of a unique strong solution to (1.5) is well known (see, e.g. [1, Section 6.3]) and it is given by

$$\mathbb{X}_t = e^{-\theta t}\mathbb{X}_0 \;+\; U(1-e^{-\theta t}) \;+\; e^{-\theta t}\int_{(0,t]} e^{\theta s}\,\mathrm{d}L_s. \tag{1.7}$$

We assume that the initial condition $\mathbb{X}_0$ has finite p-th moment for all $p\in(0,\infty)$ and $\mathbb{E}[\mathbb{X}_0] = U$.

To make sure that the results are all dimensionally consistent throughout the paper, it is worth mentioning that with U being the mean velocity of the bottom wall, $\mathbb{X}_t$ has the dimension of velocity. Therefore, θ scales as $\dfrac{1}{\text{time}}$, and σ has dimension $\dfrac{\text{velocity}}{\sqrt{\text{time}}}$.

1.3 Results of this paper

Beside establishing existence of a martingale (weak) solution for (1.1) and (1.2), we derive an upper bound (see Theorem 1.1) on the expected value of the energy dissipation rate ε in terms of characteristics of the randomly moving bottom wall. Our estimate recovers (1.3) in the limit when the variance of the noise tends to 0.

Theorem 1.1 *Suppose* $\{(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, \mathbb{P}), \mathbb{X}, u\}$ *is a martingale solution to* (1.1)–(1.2) *in the sense of Defintition 2.2. Assume that* $\mathrm{Re} = \dfrac{U\,h}{\nu} > 1$ *and* $\mathbb{E}[\mathbb{X}_0] = U$*. Assume also that the initial condition* $\mathbb{X}_0$ *has finite p-th moment for all* $p \in (0,\infty)$ *and that the initial condition* $u(0)$ *is such that* $\mathbb{E}[\|u(0)\|^2] < \infty$ *. Then the energy dissipation rate* (2.13) *can be bounded as follows:*

$$\begin{aligned}\varepsilon = \limsup_{T\to\infty} \mathbb{E}[\langle \epsilon\rangle_T] \leqslant\; & 110\frac{U^3}{h} + 12\frac{\sigma^2}{\mathrm{Re}} + 48\frac{1}{\mathrm{Re}^2}\frac{h\sigma^4}{U^3} \\ & + 48\frac{1}{\mathrm{Re}^2}\frac{h\theta^2}{U}\left(3U^2 + \frac{3}{2\theta}\left(\sigma^2 + \|K\|_\infty^2\lambda(E_0)\right)\right) \\ & + \frac{20}{hU}\left(\frac{\sigma^4 + 2\|K\|_\infty\lambda(E_0) + \|K\|_\infty^4\lambda^2(E_0) + \|K\|_\infty\lambda(E_0)\theta}{\theta^2}\right) \\ & + \frac{8}{\mathrm{Re}}\|K\|_\infty^2\left(27 + \frac{\|K\|_\infty^2}{8U^2} + \frac{50\|K\|_\infty^4}{3U^4}\right)\lambda(E_0) \\ & + 16\frac{1}{\mathrm{Re}^2}\frac{h^2\,\lambda(E_0)}{3U}\left(9\|K\|_\infty^4 + \frac{\|K\|_\infty^6}{U^4}\right). \end{aligned} \tag{1.8}$$

2. Mathematical preliminaries

Stochastic Background Flow. Our analysis here critically uses a construction of *background flow*, which was initially introduced by Hopf [17] in the deterministic case. The key idea is to decompose the flow variables into a stochastic incompressible background field and a fluctuating incompressible part and extend the nonhomogeneous boundary conditions into Ω. As constructed in [13], consider the **stochastic background flow** $\Phi = \Phi_t(x_1,x_2,x_3;\omega)$ given by

$$\Phi_t(x_1,x_2,x_3;\omega) := \big(\phi(x_3,\mathbb{X}_t(\omega)),\ 0,\ 0\big)^\top, \tag{2.1}$$

and

$$\phi(x_3,\mathbb{X}_t(\omega)) = \begin{cases} \left(1 - \dfrac{x_3}{\delta_t}\right)\mathbb{X}_t(\omega), & \text{if } 0 \leqslant x_3 \leqslant \delta_t, \\ 0, & \text{if } \delta_t \leqslant x_3 \leqslant h, \end{cases} \tag{2.2}$$

where $\delta : \mathbb{R} \to (0,\infty)$ is the function $\delta(z) = \dfrac{A}{z^2+B}$, and we choose[2] the boundary layer thickness δ_t in the background flow to be a random process

$$\delta_t = \delta(\mathbb{X}_t(\omega)) = \frac{A}{|\mathbb{X}_t(\omega)|^2 + B}. \tag{2.3}$$

Based on the need of analysis in Lemma 3.3, we later choose $A = \nu U$ and $B = U^2$, so δ_t has the dimension of a length and $\delta_t \in (0,h)$ if $\mathrm{Re} = \dfrac{U\,h}{\nu} > 1$. Moreover, the **boundary layer** is denoted by $D_\delta = (0,L)^2 \times (0,\delta_t)$.

Before proceeding to the main analysis, we gather some basic calculations as follows. With $\phi : [0,h]\times\mathbb{R} \to \mathbb{R}$ given as

$$\phi(a,z) = \left(1 - \frac{a}{\delta(z)}\right) z\, 1_{\{0\leqslant a\leqslant \delta(z)\}},$$

and $\delta(z) = \dfrac{A}{z^2+B}$, we let $\phi(x_3,\mathbb{X}_t(\omega)) = f(\mathbb{X}_t(\omega))$, where $f : \mathbb{R}\to\mathbb{R}$ is the smooth function

$$f(z) = f_{x_3}(z) = \left(1 - \frac{x_3}{\delta(z)}\right) z. \tag{2.4}$$

[2] There can be other choices for the function δ_t, and our choice in (2.3) is motivated by the general analysis in (3.13).

Itô's rule asserts that $\mathbb{P}$-a.s. we have

$$\begin{aligned} \mathrm{d}f(\mathbb{X}_t) &= f'(\mathbb{X}_t)\,\mathrm{d}\mathbb{X}_t + \frac{\sigma^2}{2}f''(\mathbb{X}_t)\,\mathrm{d}t + \int_{E_0}(f(\mathbb{X}_{t-}+K(\xi,t)) - f(\mathbb{X}_t) - f'(\mathbb{X}_t)K)\,\mathrm{d}\pi \\ &= \mathcal{L}f(\mathbb{X}_t)\,\mathrm{d}t + \sigma f'(\mathbb{X}_t)\mathrm{d}W_t + \int_{E_0}(f(\mathbb{X}_t+K(\xi,t)) - f(\mathbb{X}_t))\,\mathrm{d}\hat{\pi}, \end{aligned} \tag{2.5}$$

for $t \geqslant 0$, where

$$\mathcal{L}f(z(t)) = f'(z)\,\theta(U-z) + \frac{\sigma^2}{2}f''(z) + \int_{E_0}(f(z+K(\xi,t)) - f(z) - f'(z)K(\xi,t))\,\mathrm{d}\lambda(\xi). \tag{2.6}$$

And from (2.4), we have

$$\begin{aligned} f'(z) &= 1 - x_3\frac{\delta - z\delta'}{\delta^2}, \\ f''(z) &= x_3\frac{z\delta^2\delta'' + 2\delta^2\delta' - 2z\delta(\delta')^2}{\delta^4}. \end{aligned} \tag{2.7}$$

Lemma 2.1 *Consider* $\delta(z) = \dfrac{A}{z^2+B}$ *and* $f(z) = f_{x_3}(z) = \left(1 - \dfrac{x_3}{\delta(z)}\right)z$ *as above. Then,*

$$\delta'(z) = \frac{-2Az}{(z^2+B)^2} \quad \text{and} \quad \delta''(z) = \frac{2A(3z^2-B)}{(z^2+B)^3}. \tag{2.8}$$

Hence from (2.7) *we have*

$$f'(z) = 1 - x_3\frac{3z^2+B}{A} \quad \text{and} \quad f''(z) = -x_3\,\frac{6z}{A}. \tag{2.9}$$

Throughout this manuscript, the $L^2(D)$ norm and inner product will be denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. Concerning the nonhomogeneous boundary conditions, we consider the following velocity spaces

$$\begin{aligned} H &= \{v \in [L^2(D)]^3 : \nabla\cdot v = 0,\ v_3(x_1,x_2,0) = v_3(x_1,x_2,h) = 0,\ v\cdot n|_{\partial D} \text{ is periodic in } x_1,x_2\}, \\ V &= \{v \in [H^1(D)]^3 : \nabla\cdot v = 0,\ v(x_1,x_2,0) = v(x_1,x_2,h) = 0,\ v \text{ is periodic in } x_1,x_2\}, \\ C^\infty_{\mathrm{div}} &= \{v \in [C^\infty(D)]^3 : \nabla\cdot v = 0,\ v(x_1,x_2,0) = v(x_1,x_2,h) = 0,\ v \text{ is periodic in } x_1,x_2\}. \end{aligned}$$

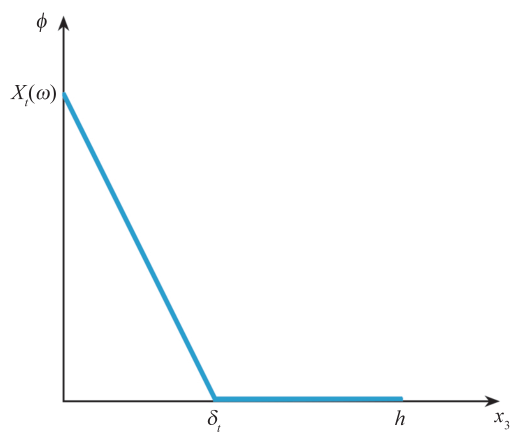

Figure 1 The graph of $x_3 \mapsto \phi(x_3,\mathbb{X}_t(\omega))$, where $\delta_t = \delta(\mathbb{X}_t(\omega))$ is the boundary layer thickness

Martingale solutions. We follow the standard notion of martingale solutions for stochastic Navier–Stokes equations as initiated by Viot in [40] and further developed in Flandoli and Gatarek [14, Definition 3.1] or Debussche, Glatt-Holtz, and Temam [8]. We define a martingale solution for our system (1.1)–(1.2). This notion is the probabilistically weak analogue of the Leray–Hopf weak solution to the deterministic Navier–Stokes equations.

Definition 2.2 *(Martingale solution on compact intervals) Let* $T \in [0,\infty)$. *A martingale solution to (1.1)–(1.2) on* $[0,T]$ *consists of a stochastic basis* $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}, W, \pi\big)$ *with a complete right-continuous filtration* $(\mathcal{F}_t)_{t\in[0,T]}$, *an* $(\mathcal{F}_t)_{t\in[0,T]}$*-adapted process* $(\mathbb{X}_t)_{t\in[0,T]}$, *and an* $(\mathcal{F}_t)_{t\in[0,T]}$*-adapted stochastic process* u *with càdlàg sample paths in* $D(A)'$ *a.s. such that*

- $u-\Phi$ *has sample paths in* $L^\infty(0,T;\, H)\cap L^2(0,T;\, V)$ *almost surely,*
- *for all* $t\in[0,T]$ *and all* $\varphi\in C^\infty_{\mathrm{div}}$, *the following identity holds almost surely,*

$$(u(t),\varphi)+\nu\int_0^t(\nabla u(s),\nabla\varphi)\,\mathrm{d}s+\int_0^t(u(s)\cdot\nabla u(s),\varphi)\,\mathrm{d}s=(u(0),\varphi), \tag{2.10}$$

- *the following holds*

$$\mathbb{E}\left[\sup_{s\in[0,T]}\|u(s)\|^2+\int_0^T\|\nabla u(s)\|^2\,\mathrm{d}t\right]<\infty. \tag{2.11}$$

Definition 2.3 *(Energy Dissipation Rate) The time-averaged energy dissipation as a random variable is given as*

$$\langle\epsilon\rangle_T:=\frac{1}{|D|}\frac{1}{T}\int_0^T\nu\|\nabla u(t,\cdot,\omega)\|^2_{L^2}\,\mathrm{d}t. \tag{2.12}$$

We also define the time-averaged expected energy dissipation rate for a martingale solution u *of (1.1)–(1.2) by setting*

$$\varepsilon:=\limsup_{T\to\infty}\mathbb{E}[\langle\epsilon\rangle_T]=\limsup_{T\to\infty}\mathbb{E}\left[\frac{1}{|D|}\frac{1}{T}\int_0^T\nu\|\nabla u(t,\cdot,\omega)\|^2_{L^2}\,\mathrm{d}t\right]. \tag{2.13}$$

3. Existence of martingale solutions

In this section we prove existence of (weak) martingale solutions to (1.1)–(1.2) on $[0,T]$.

Theorem 3.1 *Assume that a given law* μ_0 *satisfies*

$$\int_H|\phi|^2\mathrm{d}\mu_0(\phi)<\infty. \tag{3.1}$$

Assume also that $\mathbb{E}[\mathbb{X}_0]^p$ *is finite for any* $p\in(0,\infty)$ *and that* $\mathbb{E}[X_0]=U$. *Then there exists a stochastic basis* $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geqslant 0},\mathbb{P},W,\pi)$, *a predictable process* $\mathbb{X}_t$, *and an* $(\mathcal{F}_t)_{t\in[0,T]}$*-adapted process* u *with càdlàg sample paths in* $D(A)'$ *such that* $u\in L^2(\Omega;L^\infty(0,T;H))\cap L^2(\Omega\times[0,T],\mathrm{d}t\otimes\mathrm{d}\mathbb{P};V)$, $u(0)$ *has law* μ_0 *and such that* $\big\{\big(\Omega,\,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\,\mathbb{P},W,\pi\big),\,(\mathbb{X}_t)_{t\in[0,T]},\,u\big\}$ *is a martingale solution to* (1.1)–(1.2) *on* $[0,T]$.

Proof The key idea, used in this proof and in what follows, is to consider $u-\Phi$ which satisfies homogeneous boundary conditions, where Φ is the stochastic, incompressible background field (2.1), carrying the inhomogeneities of the problem.

We present the rest of the analysis based on $v = u - \Phi$, where v is a fluctuating incompressible field which is unforced and hence of arbitrary amplitude. Making the substitution $u = v + \Phi$ in (1.1), we find that the stochastic process v satisfies

$$\begin{aligned} &\mathrm{d}v + \mathrm{d}\Phi = -(v\cdot\nabla v + v\cdot\nabla\Phi + \Phi\cdot\nabla v + \Phi\cdot\nabla\Phi - \nu\Delta v - \nu\Delta\Phi + \nabla p)\,\mathrm{d}t,\\ &\nabla\cdot v = 0,\end{aligned} \tag{3.2}$$

in the weak sense. The boundary conditions for v are periodic in the x_1 and x_2 directions, while in the x_3 direction,

$$v(x_1, x_2, 0, t) = v(x_1, x_2, h, t) = 0. \tag{3.3}$$

We begin by proving the existence of a martingale solution to the equation (3.2) subject to the boundary conditions (3.3). That is, we prove the existence of a basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P}, W, \pi)$, an $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted process $\mathbb{X}_t$, and an $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted process $v \in L^2(\Omega; L^\infty(0,T;H)) \cap L^2(\Omega\times [0,T], \mathrm{d}t\otimes\mathrm{d}\mathbb{P}; V)$ with a.s. càdlàg sample paths in $D(A)'$ such that

$$\begin{aligned} v(s) - v(0) = &-\int_0^s (B(v(t), v(t)+\Phi(t)) + B(\Phi(t), v(t)) - \nu A v(t) - \nu A\Phi(t))\,\mathrm{d}t\\ &+ \mathscr{P}\left(\int_0^s \mathcal{L}f(\mathbb{X}_t)\,\mathrm{d}t + \int_0^s \sigma f'(\mathbb{X}_t)\mathrm{d}W_t + \int_{(0,s]}\int_{E_0} (f(\mathbb{X}_t + K(\xi,t)) - f(\mathbb{X}_t))\mathrm{d}\hat{\pi}(\xi,t), 0, 0\right)^T \end{aligned} \tag{3.4}$$

holds $\mathbb{P}$-a.s. in $D(A)'$ for all $s \in [0,T]$. Here, $B(u,w) = \mathscr{P}(u\cdot\nabla w)$, where $\mathscr{P}$ is the Leray-projector and A is the Stokes operator and $\mathcal{L}f(\mathbb{X}_t)$ is defined in (2.6).

In what follows, we provide detailed calculations to obtain energy estimates and then provide a brief outline for the rest of the procedure.

To derive a priori estimates, we begin by applying Itô's formula to (3.4). To be precise, we use Theorem 2.19 in [6] with the special case in which $\psi(u) = |u|^2$ as stated in Corollary 1 in [6] (see also Theorem I.3.1 in [23] and [29] for the Itô formula in the general context of semi-martingales), and see that almost surely and for any $r \in [0,T]$ we have

$$\begin{aligned} &\frac{1}{2}\|v(r)\|^2 + \int_0^r \nu\|\nabla v\|^2\,\mathrm{d}t\\ &= |v(0)|^2 + \int_0^r \Big(-\underbrace{(v,\mathrm{d}\Phi)}_{\mathrm{I}} + \underbrace{(v\cdot\nabla v, v)}_{\mathrm{II}} + \underbrace{(v\cdot\nabla\Phi, v)}_{\mathrm{III}} + \underbrace{(\Phi\cdot\nabla v, v)}_{\mathrm{IV}} + \underbrace{(\Phi\cdot\nabla\Phi, v)}_{\mathrm{V}} + \underbrace{\nu(\nabla v, \nabla\Phi)}_{\mathrm{VI}} \Big)\,\mathrm{d}t\\ &\quad + \int_0^r\int_D \frac{\sigma^2}{2}(f'(\mathbb{X}_t))^2\,\mathrm{d}t + \frac{1}{2}\int_{(0,r]}\int_{E_0}\int_D |f(\mathbb{X}_t + K(\xi,t)) - f(\mathbb{X}_t)|^2\mathrm{d}\hat{\pi}(\xi,t). \end{aligned} \tag{3.5}$$

Here we also used the fact that that the boundary layer $D_\delta = (0,L)^2\times(0,\delta_t)$. We continue our analysis by estimating each of the underlined terms in the above equation.

Term I. From (2.5) it follows that

$$\begin{aligned} \int_0^r\int_D v\,\mathrm{d}\Phi\,\mathrm{d}x &= \int_0^r\int_{D_\delta} v_1\,\mathrm{d}f(\mathbb{X}_t)\,\mathrm{d}x\mathrm{d}t\\ &= \int_0^r\int_{D_\delta} v_1\,\mathcal{L}f(\mathbb{X}_t)\mathrm{d}x\mathrm{d}t + \int_0^r\int_{D_\delta}\sigma v_1 f'(\mathbb{X}_t)\mathrm{d}x\,\mathrm{d}W_t\\ &\quad + \int_{(0,r]}\int_{E_0}\int_{D_\delta} v_1(f(\mathbb{X}_t + K(\xi,t)) - f(\mathbb{X}_t))\mathrm{d}x\mathrm{d}\hat{\pi}(\xi,t). \end{aligned} \tag{3.6}$$

Term II. Using the incompressibility of v, along with integration by parts, we get

$$(v \cdot \nabla v,\, v) = 0.$$

Term III. Application of the fundamental theorem of calculus and the Schwarz inequality shows that the $x_1\, x_2$ integral of the product $v_1 v_3$, is bounded uniformly in x_3 according to

$$\left| \int_0^L \int_0^L v_1\, v_3 \mathrm{d}x_1 \mathrm{d}x_2 \right| = \left| \int_0^L \int_0^L \int_0^{x_3} \frac{\partial v_1}{\partial \xi}(x_1, x_2, \xi)\, \mathrm{d}\xi \int_0^{x_3} \frac{\partial v_3}{\partial \eta}(x_1, x_2, \eta)\, \mathrm{d}\eta\, \mathrm{d}x_1 \mathrm{d}x_2 \right|$$
$$\leqslant x_3 \| \frac{\partial v_1}{\partial x_3} \| \, \| \frac{\partial v_3}{\partial x_3} \| \, .$$

The quadratic source term is then estimated in terms of noise, δ_t, and the dissipation (for more details see [13])

$$\int_0^r |(v \cdot \nabla \Phi, v)| \mathrm{d}t \leqslant \int_0^r \frac{\delta_t}{2}\, |\mathbb{X}_t|\, \|\nabla v\|^2 \mathrm{d}t. \tag{3.7}$$

Term IV. Since $\Phi \cdot \nabla v = \phi(x_3, \mathbb{X}_t) \dfrac{\partial v}{\partial x_1}$, using the periodicity of v, one can show that

$$\begin{aligned}(\Phi \cdot \nabla v, v) &= \frac{1}{2} \int_{D_\delta} \phi(x_3, \mathbb{X}_t) \frac{\partial}{\partial x_1} |v|^2\, \mathrm{d}x \\ &= \frac{1}{2} \int_0^{\delta_t} \phi(x_3, \mathbb{X}_t) \int_0^L \left(\int_0^L \frac{\partial}{\partial x_1} |v|^2\, \mathrm{d}x_1 \right) \mathrm{d}x_2\, \mathrm{d}x_3 = 0.\end{aligned} \tag{3.8}$$

Term V. A pointwise calculation leads to $\Phi \cdot \nabla \Phi = 0$, hence,

$$(\Phi \cdot \nabla \Phi, v) = 0.$$

Term VI. Direct calculation shows that $\dfrac{\partial \phi(x_3, z)}{\partial x_3} = \dfrac{-z}{\delta(z)}$ for $0 < x_3 < \delta(z)$. Hence

$$\| \frac{\partial \phi}{\partial x_3} \| = \frac{L}{\delta_t^{1/2}}\, |\mathbb{X}_t|. \tag{3.9}$$

Therefore, using the Cauchy–Schwarz inequality and Young's inequality, we find

$$\int_0^r |\nu\, (\nabla v, \nabla \Phi)| \mathrm{d}t \leqslant \int_0^r \frac{\nu}{\delta_t} L^2 |\mathbb{X}_t|^2 + \frac{\nu}{4} \|\nabla v\|^2 \mathrm{d}t. \tag{3.10}$$

Using the estimates for all seven terms above in (3.5) yields, $\mathbb{P}$-a.s. and for any $r \in [0, T]$,

$$\begin{aligned}&\frac{1}{2} \|v(r)\|^2 + \int_0^r \frac{3\nu}{4} \|\nabla v\|^2\, \mathrm{d}t \\ &\quad + \int_0^r \int_{D_{\delta_t}} v_1 \sigma f'(\mathbb{X}_t) \mathrm{d}x \mathrm{d}W_t + \int_{(0,r]} \int_{E_0} \int_{D_{\delta_t}} v_1 (f(\mathbb{X}_t + K(\xi, t)) - f(\mathbb{X}_t)) \mathrm{d}x\, \mathrm{d}\hat{\pi}(t, \xi) \\ &\leqslant \frac{1}{2} \|v(0)\|^2 + \int_0^r \left| \int_{D_\delta} v_1\, \mathcal{L} f(\mathbb{X}_t)\, \mathrm{d}x \right| \mathrm{d}t + \int_0^r \left[\frac{\delta_t}{2} |\mathbb{X}_t|\, \|\nabla v\|^2 + \nu L^2 \frac{|\mathbb{X}_t|^2}{\delta_t} \right] \mathrm{d}t \\ &\quad + \frac{\sigma^2}{2} \int_0^r \int_{D_\delta} (f'(\mathbb{X}_t))^2\, \mathrm{d}x\, \mathrm{d}t + \frac{1}{2} \int_{(0,r]} \int_{E_0} \int_{D_\delta} |f(\mathbb{X}_t + K(\xi, t)) - f(\mathbb{X}_t)|^2 \mathrm{d}x\, \mathrm{d}\pi(\xi, t),\end{aligned} \tag{3.11}$$

where we recall that $\delta_t = \delta(\mathbb{X}_t)$, the function $f(z) = f_{x_3}(z) = \left(1 - \dfrac{x_3}{\delta(z)}\right) z$ is defined in (2.4) and therefore has derivatives given by (2.7).

The second term on the right side of (3.11) can be bounded from above using the following lemma, which is proved in the Appendix of [13]. □

Lemma 3.2 (Lemma 4.2 [13]) *Let* $G=(G_t)_{t\in\mathbb{R}_+}$ *be a stochastic process defined on the probability space* $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geqslant 0},\mathbb{P})$. *Then* $\mathbb{P}$*-a.s., we have for all* $t\in\mathbb{R}_+$,

$$\left|\int_{D_\delta} v_1\, G_t\,\mathrm{d}x\right| \leqslant \|\nabla v(t)\|\,\delta_t\, L\left(\int_0^{\delta_t}|G_t|^2\,\mathrm{d}x_3\right)^{\frac{1}{2}}.$$

Applying Lemma 3.2 with $G_t=\mathcal{L}f(\mathbb{X}_t)$ and then using Young's inequality, we have

$$\begin{aligned}\int_0^r\left|\int_{D_\delta} v_1\,\mathcal{L}f(\mathbb{X}_t)\,\mathrm{d}x\right|\mathrm{d}t &\leqslant \int_0^r\|\nabla v\|\,\delta_t\,L\left(\int_0^{\delta_t}|\mathcal{L}f(\mathbb{X}_t)|^2\,\mathrm{d}x_3\right)^{\frac{1}{2}}\mathrm{d}t\\ &\leqslant \int_0^r\frac{\nu}{4}\|\nabla v\|^2\mathrm{d}t+\int_0^r\frac{1}{\nu}\delta_t^2\,L^2\left(\int_0^{\delta_t}|\mathcal{L}f(\mathbb{X}_t)|^2\,\mathrm{d}x_3\right)\mathrm{d}t. \qquad (3.12)\end{aligned}$$

Hence inserting estimate (3.12) in (3.11), collecting terms that involve $\|\nabla v\|$, and integrating in time from 0 to T, we have the following stochastic inequality that holds $\mathbb{P}$ a.s.:

$$\begin{aligned}&\frac{1}{2}\|v(r)\|^2+\int_0^r\left(\frac{1}{2}-\frac{\delta_t\,|\mathbb{X}_t|}{2\nu}\right)\nu\|\nabla v\|^2\mathrm{d}t\\ &+\int_0^r\int_{D_{\delta_t}}\sigma v_1\,f'(\mathbb{X}_t)\mathrm{d}x\mathrm{d}W_t+\int_{(0,r]}\int_{E_0}\int_{D_{\delta_t}}v_1\,(f(\mathbb{X}_t+K(\xi,t))-f(\mathbb{X}_t))\mathrm{d}x\,\mathrm{d}\hat{\pi}(t,\xi)\\ &\leqslant\frac{1}{2}\|v(0)\|^2+\frac{\sigma^2}{2}\int_0^r\int_{D_\delta}(f'(\mathbb{X}_t))^2\,\mathrm{d}x\mathrm{d}t+\int_0^r\frac{1}{\nu}\delta_t^2\,L^2\int_0^{\delta_t}|\mathcal{L}f(\mathbb{X}_t)|^2\,\mathrm{d}x_3\,\mathrm{d}t\\ &+\int_0^r\nu L^2\frac{|\mathbb{X}_t|^2}{\delta_t}\mathrm{d}t+\frac{1}{2}\int_{(0,r]}\int_{E_0}\int_{D_\delta}|f(\mathbb{X}_t+K(\xi,t))-f(\mathbb{X}_t)|^2\mathrm{d}x\,\mathrm{d}\pi(\xi,t). \qquad (3.13)\end{aligned}$$

We note that the calculations up to and including (3.13) work for a general C^2 function $\delta=\delta(z)$. For δ as in (2.3) it is crucial to choose A and B such that $\left(\frac{1}{2}-\frac{\delta_t\,|\mathbb{X}_t|}{2\nu}\right)$ in the second term of (3.13) is positive. Such conditions are summarized in the following lemma borrowed from [13].

Lemma 3.3 *Let* $\delta_t=\delta(\mathbb{X}_t)$, *where* $\mathbb{X}_t$ *is a stochastic process in* $\mathbb{R}$ *and* $\delta(z)=\dfrac{A}{z^2+B}$. *Suppose* A *and* B *are positive numbers such that* $\dfrac{A}{B}<h$ *and* $A\leqslant\nu\sqrt{B}$. *Then with probability one, for all* $t\geqslant 0$, *we have* $\delta_t<h$ *and*

$$\frac{1}{4}\leqslant\frac{1}{2}-\frac{\delta_t\,|\mathbb{X}_t|}{2\nu}\leqslant\frac{1}{2}. \qquad (3.14)$$

These hold if, for instance, $A=\nu U$ *and* $B=U^2$ *and* $\frac{U\,h}{\nu}>1$.

We derive upper bounds for the terms on the right side of (3.13) and summarize our findings on the almost sure upper bound for the energy dissipation in the following lemma.

Lemma 3.4 *Suppose* A *and* B *are positive constants such that* $\dfrac{A}{B}<h$ *and* $A\leqslant\nu\sqrt{B}$. *Then with probability one, the following inequality holds a.s. for all* $r\in[0,T]$, $T>0$ *:*

$$2\|v(r)\|^2 + \int_0^r \nu\|\nabla v\|^2 \mathrm{d}t + 4M_r \leqslant 2\|v(0)\|^2 + Y_r, \tag{3.15}$$

where for any $r \in [0,T]$ we have the following definitions

$$M_r := \int_0^r \int_{D_{\delta_t}} \sigma v_1\, f'(\mathbb{X}_t)\mathrm{d}x\mathrm{d}W_t + \int_{(0,r]}\int_{E_0}\int_{D_{\delta_t}} v_1\,(f(\mathbb{X}_t + K(\xi,t)) - f(\mathbb{X}_t))\mathrm{d}x\,\mathrm{d}\hat{\pi}(t,\xi), \tag{3.16}$$

and

$$\begin{aligned}Y_r := {}& 4\sigma^2 L^2 \left[\frac{3}{2}\frac{A}{B} + \frac{6}{\nu}\left(\frac{A}{B}\right)^3 \frac{\sigma^2}{B}\right] r + 4\,L^2 \int_0^r \left(\nu\frac{|\mathbb{X}_t|^2}{\delta_t} + \frac{6}{\nu}\left(\frac{A}{B}\right)^3 \theta^2\,|U - \mathbb{X}_t|^2\right)\mathrm{d}t \\ &+ 4L^2\|K\|_\infty^2 \pi((0,r],E_0)\left(\frac{27A}{2B} + \frac{\|K\|_\infty^2 A}{8B^2} + \frac{50\|K\|_\infty^4 A}{3B^3}\right) \\ &+ \frac{8\lambda(E_0)}{3\nu}\left(\frac{A}{B}\right)^3\left(9\|K\|_\infty^4 + \frac{\|K\|_\infty^6}{B^2}\right) r.\end{aligned} \tag{3.17}$$

Proof We estimate the terms on the right side of (3.13). For the first term, we write $\int_{D_\delta}(f'(\mathbb{X}_t))^2\,\mathrm{d}x = L^2\int_0^{\delta_t}(f'(\mathbb{X}_t))^2\,\mathrm{d}x_3$ and thus observe that

$$\begin{aligned}\sigma^2\int_0^r\int_{D_\delta}(f'(\mathbb{X}_t))^2\,\mathrm{d}x\mathrm{d}t &= \sigma^2 L^2\int_0^r\int_0^{\delta_t}\left(1 - x_3\frac{3\mathbb{X}_t^2 + B}{A}\right)^2\mathrm{d}x_3\mathrm{d}t \\ &= \sigma^2 L^2\int_0^r \delta_t - \delta_t^2\frac{3\mathbb{X}_t^2 + B}{A} + \frac{\delta_t^3}{3}\left(\frac{3\mathbb{X}_t^2 + B}{A}\right)^2\mathrm{d}t \\ &\leqslant \sigma^2 L^2\int_0^r \delta_t - \delta_t^2\frac{\mathbb{X}_t^2 + B}{A} + \frac{\delta_t^3}{3}\,\frac{9}{\delta_t^2}\,\mathrm{d}t \\ &\leqslant \frac{3A}{B}\sigma^2 L^2 r.\end{aligned} \tag{3.18}$$

Here, we used the fact that $\dfrac{3\mathbb{X}_t^2 + B}{A} \leqslant \dfrac{3}{\delta_t}$ and $\delta(z) \leqslant \dfrac{A}{B}$ for all $z \in \mathbb{R}$. Similarly, we also obtain

$$\begin{aligned}&\int_{(0,r]}\int_{E_0}\int_{D_\delta}|f(\mathbb{X}_t + K(\xi,t)) - f(\mathbb{X}_t)|^2\mathrm{d}x\,\mathrm{d}\pi(\xi,t) \\ &= L^2\int_{(0,r]}\int_{E_0}\int_0^{\delta_t} K^2(\xi,t)\left(1 - x_3\frac{2\mathbb{X}_t^2 + 3\mathbb{X}_t K(\xi,t) + K^2(\xi,t) + B}{A}\right)^2\mathrm{d}x_3\,\mathrm{d}\pi(\xi,t).\end{aligned}$$

Observe that

$$\frac{3\mathbb{X}_t^2 + 3\mathbb{X}_t K(\xi,t) + K^2(\xi,t) + B}{A} = \frac{\mathbb{X}_t^2 + 2(\mathbb{X}_t^2 + \frac{3}{4}K(\xi,t))^2 - \frac{K^2(\xi,t)}{8} + B}{A} \geqslant \frac{1}{\delta_t} - \frac{K^2(\xi,t)}{8A}$$

and

$$\frac{3\mathbb{X}_t^2 + 3\mathbb{X}_t K(\xi,t) + K^2(\xi,t) + B}{A} \leqslant \frac{\frac{9}{2}\mathbb{X}_t^2 + \frac{5}{2}K^2(\xi,t) + B}{A} \leqslant \frac{9}{2\delta_t} + \frac{5K^2(\xi,t)}{A}.$$

Thus, we obtain that

$$\begin{aligned}&\int_{(0,r]}\int_{E_0}\int_{D_\delta}|f(\mathbb{X}_t + K(\xi,t)) - f(\mathbb{X}_t)|^2\mathrm{d}x\,\mathrm{d}\pi(\xi,t) \\ &\leqslant L^2\|K\|_\infty^2\int_{(0,r]}\int_{E_0}\delta_t - \delta_t^2\left(\frac{1}{\delta_t} - \frac{K^2(\xi,t)}{8A}\right) + \frac{\delta_t^3}{3}\left(\frac{81}{2\delta_t^2} + \frac{50K^4(\xi,t)}{A^2}\right)\mathrm{d}\pi(\xi,t) \\ &\leqslant L^2\|K\|_\infty^2\pi((0,r],E_0)\left(\frac{27A}{2B} + \frac{\|K\|_\infty^2 A}{8B^2} + \frac{50\|K\|_\infty^4 A}{3B^3}\right).\end{aligned} \tag{3.19}$$

Now we consider the term involving $\mathcal{L}f(\mathbb{X}_t)$. By the definition (2.6) of $\mathcal{L}$ and the elementary inequality $(a+b)^2 \leqslant 2(a^2+b^2)$, we obtain

$$\mathcal{L}f(\mathbb{X}_t) = f'(\mathbb{X}_t)\theta(U-\mathbb{X}_t) + \frac{\sigma^2}{2}f''(\mathbb{X}_t) + \int_{E_0} (f(\mathbb{X}_t+K) - f(\mathbb{X}_t) - f'(\mathbb{X}_t)K(\xi,t))\,\mathrm{d}\lambda(\xi),$$

$$|\mathcal{L}f(\mathbb{X}_t)|^2 \leqslant 2|f'(\mathbb{X}_t)|^2\theta^2(U-\mathbb{X}_t)^2 + \sigma^4\left(f''(\mathbb{X}_t)\right)^2 + 4\int_{E_0} |f(\mathbb{X}_t+K) - f(\mathbb{X}_t) - f'(\mathbb{X}_t)K(\xi,t)|^2\mathrm{d}\lambda(\xi).$$

Observe that $|f(\mathbb{X}_t+K) - f(\mathbb{X}_t) - f'(\mathbb{X}_t)K|^2 = \left(\frac{x_3}{A}\right)^2 (3\mathbb{X}_t K^2 + K^3)^2$.

So using (3.18) and the expression $f''(\mathbb{X}_t) = -6x_3\frac{\mathbb{X}_t}{A}$, we see that for the second term on the right of (3.13) we have

$$\begin{aligned}
\int_0^{\delta_t} |\mathcal{L}f(\mathbb{X}_t)|^2\,\mathrm{d}x_3 &\leqslant 2\,\theta^2\,(U-\mathbb{X}_t)^2 \int_0^{\delta_t} (f'(\mathbb{X}_t))^2\,\mathrm{d}x_3 + \frac{\sigma^4}{2}\int_0^{\delta_t} \left(f''(\mathbb{X}_t)\right)^2\,\mathrm{d}x_3 \\
&\quad + \int_0^{\delta_t}\int_{E_0}\left(\frac{x_3}{A}\right)^2 (3\mathbb{X}_t K^2(\xi,t) + K^3(\xi,t))^2\mathrm{d}\lambda(\xi)\mathrm{d}x_3 \\
&\leqslant 6\,\frac{A}{B}\,\theta^2\,(U-\mathbb{X}_t)^2 + 6\sigma^4\,\frac{\mathbb{X}_t^2}{A^2}\,\delta_t^3 + 2\lambda(E_0)\int_0^{\delta_t}\left(\frac{x_3}{A}\right)^2\left(9\mathbb{X}_t^2\|K\|_\infty^4 + \|K\|_\infty^6\right)\mathrm{d}x_3 \\
&\leqslant 6\,\frac{A}{B}\,\theta^2\,(U-\mathbb{X}_t)^2 + 6\sigma^4\,\frac{\delta_t^2}{A} + 2\lambda(E_0)\frac{\delta_t^3}{3A^2}\left(9\left(\frac{A}{\delta_t}\right)^2\|K\|_\infty^4 + \|K\|_\infty^6\right) \\
&\leqslant 6\,\frac{A}{B}\,\theta^2\,(U-\mathbb{X}_t)^2 + 6\sigma^4\,\frac{A}{B^2} + \frac{2\lambda(E_0)}{3}\left(\frac{9A}{B}\|K\|_\infty^4 + \frac{A}{B^3}\|K\|_\infty^6\right).
\end{aligned} \tag{3.20}$$

In the above, we used the fact that $|\mathbb{X}_t|^2 \leqslant \frac{A}{\delta_t}$ and $\delta_t \leqslant \frac{A}{B}$.

Applying (3.14) to the second term on the left of (3.13), and (3.18), (3.19), (3.20) to the right of (3.13), we obtain

$$\begin{aligned}
&\frac{1}{2}\|v(r)\|^2 - \frac{1}{2}\|v(0)\|^2 + \frac{1}{4}\int_0^r \nu\|\nabla v\|^2\mathrm{d}t + M_r \\
&\leqslant \frac{3}{2}\frac{A}{B}L^2\,\sigma^2 r + L^2\|K\|_\infty^2\pi((0,r],E_0)\left(\frac{27A}{2B} + \frac{\|K\|_\infty^2 A}{8B^2} + \frac{50\|K\|_\infty^4 A}{3B^3}\right) \\
&\quad + L^2\int_0^r\left(\nu\frac{|\mathbb{X}_t|^2}{\delta_t} + \frac{6}{\nu}\left(\frac{A}{B}\right)^3\theta^2\,(U-\mathbb{X}_t)^2\right)\mathrm{d}t \\
&\quad + \frac{6}{\nu}\left(\frac{A}{B}\right)^3\frac{\sigma^4}{B}r + \frac{2\lambda(E_0)}{3\nu}\left(\frac{A}{B}\right)^3\left(9\|K\|_\infty^4 + \frac{\|K\|_\infty^6}{B^2}\right)r.
\end{aligned} \tag{3.21}$$

Next we take $\sup_{r\in[0,T]}$ and then $\mathbb{E}$ on both sides of (3.15). This gives us

$$\mathbb{E}\sup_{r\in[0,T]}\|v(r)\|^2 + \frac{\nu}{2}\mathbb{E}\int_0^T \|\nabla v(s)\|^2\mathrm{d}s \leqslant \mathbb{E}\|v(0)\|^2 + 2\mathbb{E}\sup_{r\in[0,T]}|M_r| + \mathbb{E}\frac{1}{2}[Y_T]. \tag{3.22}$$

We now estimate the expectation of the integral term in Y_T defined in (3.17). Assuming that $\mathbb{E}[\mathbb{X}_0] = U$, we have $\mathbb{E}\left[|U-\mathbb{X}_t|^2\right] = \mathbb{E}[\mathbb{X}_t^2] - U^2$. Hence, using the moment bound (5.12), from the Appendix, we observe that

$$\begin{aligned}
&\mathbb{E}\int_0^T \left(\nu\frac{|\mathbb{X}_t|^2}{\delta_t}+\frac{6}{\nu}\left(\frac{A}{B}\right)^3\theta^2\,|U-\mathbb{X}_t|^2\right)\mathrm{d}t\\
&=\mathbb{E}\int_0^T \left(\nu\frac{\mathbb{X}_t^4+B\mathbb{X}_t^2}{A}+\frac{6}{\nu}\left(\frac{A}{B}\right)^3\theta^2\,|U-\mathbb{X}_t|^2\right)\mathrm{d}t\\
&=\mathbb{E}\int_0^T \left(\frac{\nu B}{A}+\frac{6\theta^2}{\nu}\left(\frac{A}{B}\right)^3\right)\mathbb{E}[\mathbb{X}_t^2]-\frac{6\theta^2U^2}{\nu}\left(\frac{A}{B}\right)^3+\frac{\nu\mathbb{E}[\mathbb{X}_t^4]}{A}\mathrm{d}t\\
&\leqslant\left(\frac{\nu B}{A}+\frac{6\theta^2}{\nu}\left(\frac{A}{B}\right)^3\right)\left(\frac{3}{2\theta}\mathbb{E}[\mathbb{X}_0^2]+3U^2\left(T-\frac{2}{\theta}+\frac{1}{2\theta}\right)+\frac{3T}{2\theta}\left(\sigma^2+\|K\|_\infty^2\lambda(E_0)\right)\right)\\
&\quad+\frac{\nu}{A}\left(\frac{2}{\theta}\mathbb{E}[\mathbb{X}_0^4]+8U^4T+8T\left(\frac{\sigma^4+2\|K\|_\infty\lambda(E_0)+\|K\|_\infty^4\,\lambda^2(E_0)+\|K\|_\infty\lambda(E_0)\theta}{4\theta^2}\right)\right). \qquad (3.23)
\end{aligned}$$

Thus thanks to (3.23), we have

$$\begin{aligned}
\frac{1}{2}\mathbb{E}[Y_T]&\leqslant 2\sigma^2L^2\left[\frac{3}{2}\frac{A}{B}+\frac{6}{\nu}\left(\frac{A}{B}\right)^3\frac{\sigma^2}{B}\right]T+\frac{4\lambda(E_0)}{3}\left(\frac{\nu^2}{U^3}\right)\left(9\|K\|_\infty^4+\frac{\|K\|_\infty^6}{B^2}\right)T\\
&\quad+2L^2\left(U+6\theta^2\left(\frac{\nu^2}{U^3}\right)^3\right)\left(\frac{3}{2\theta}\mathbb{E}[\mathbb{X}_0^2]+3U^2\left(T-\frac{3}{2\theta}\right)+\frac{3T}{2\theta}\left(\sigma^2+\|K\|_\infty^2\lambda(E_0)\right)\right)\\
&\quad+\frac{2L^2}{U}\left(\frac{2}{\theta}\mathbb{E}[\mathbb{X}_0^4]+8U^4T+8T\left(\frac{\sigma^4+2\|K\|_\infty\lambda(E_0)+\|K\|_\infty^4\,\lambda^2(E_0)+\|K\|_\infty\lambda(E_0)\theta}{4\theta^2}\right)\right)\\
&\quad+2L^2\|K\|_\infty^2\lambda(E_0)T\left(\frac{27\nu}{2U}+\frac{\|K\|_\infty^2\nu}{8U^3}+\frac{50\|K\|_\infty^4\nu}{3U^5}\right)\\
&=:K_1, \qquad (3.24)
\end{aligned}$$

where $K_1>0$ depends on the given data $A,B,U,\nu,T,\sigma,\lambda(E_0),\mathbb{E}[\mathbb{X}_0^k],\|K\|_\infty$.

Next we use the Burkholder–Davis–Gundy (BDG) inequality to treat the two terms appearing in the martingale M_r. First,

$$\begin{aligned}
\mathbb{E}\sup_{r\in[0,T]}\left|\int_0^r\int_{D_{\delta_s}}\sigma v_1\,f'(\mathbb{X}_s)\mathrm{d}x\mathrm{d}W(s)\right|&\leqslant\sigma\mathbb{E}\left(\int_0^T\|v(s)\|^2\left(\int_{D_\delta}|f'(\mathbb{X}_s)|^2\right)\mathrm{d}s\right)^{\frac{1}{2}}\\
&\leqslant\sigma\mathbb{E}\left(\int_0^T\sup_{r\in[0,s]}\|v(r)\|\left(\int_{D_\delta}|f'(\mathbb{X}_s)|^2\right)\mathrm{d}s\right)^{\frac{1}{2}}\\
&\leqslant\sigma^2L^2\left(\frac{3\nu}{U}\right)+C\mathbb{E}\int_0^T\sup_{r\in[0,s]}\|v(r)\|^2\mathrm{d}s. \qquad (3.25)
\end{aligned}$$

Here we have also used the argument in (3.18). Similarly, using (3.19), we obtain for some $C>0$ that

$$\begin{aligned}
&\mathbb{E}\sup_{r\in[0,T]}\left|\int_{(0,r]}\int_{E_0}\int_{D_{\delta_t}}v_1\left(f(\mathbb{X}_t+K(\xi,t))-f(\mathbb{X}_t)\right)\mathrm{d}x\,\mathrm{d}\hat{\pi}(t,\xi)\right|\\
&\leqslant\mathbb{E}\left(\sup_{r\in[0,T]}\|v(r)\|^2\int_{(0,T]}\int_{E_0}\int_{D_\delta}|f(\mathbb{X}_t+K(\xi,t))-f(\mathbb{X}_t)|^2\mathrm{d}x\mathrm{d}\pi(\xi,t)\right)^{\frac{1}{2}}\\
&\leqslant\frac{1}{4}\mathbb{E}\left(\sup_{r\in[0,T]}\|v(r)\|^2\right)+\mathbb{E}\int_{(0,T]}\int_{E_0}\int_{D_\delta}|f(\mathbb{X}_t+K(\xi,t))-f(\mathbb{X}_t)|^2\mathrm{d}x\mathrm{d}\lambda(\xi)\mathrm{d}t\\
&\leqslant\frac{1}{4}\mathbb{E}\sup_{r\in[0,T]}\|v(r)\|^2+L^2\|K\|_\infty^2\lambda(E_0)T\left(\frac{27\nu}{2U}+\frac{\|K\|_\infty^2\nu}{8U^3}+\frac{50\|K\|_\infty^4\nu}{3U^5}\right). \qquad (3.26)
\end{aligned}$$

Hence, combining (3.25) and (3.26) and substituting in (3.22), we obtain

$$\mathbb{E}\sup_{r\in[0,T]}\|v(r)\|^2+\frac{\nu}{2}\mathbb{E}\int_0^T\|\nabla v(t)\|^2\mathrm{d}t\leqslant\mathbb{E}\|v(0)\|^2+K_2+C\mathbb{E}\int_0^T\sup_{r\in[0,s]}\|v(r)\|^2\mathrm{d}s,\tag{3.27}$$

where $K_2>0$ depends on K_1 given in (3.24) and other terms appearing in (3.25) and (3.26); and $C>0$ depends on the given data.

Next we apply the Grönwall inequality to (3.27) to obtain that for some constant $C>0$ depending on the given data,

$$\mathbb{E}\sup_{t\in[0,T]}\|v(t)\|^2+\frac{\nu}{2}\mathbb{E}\int_0^T\|\nabla v(t)\|^2\mathrm{d}t\leqslant C.\tag{3.28}$$

To obtain compactness and keeping in mind that v is not expected to be differentiable, we look for bounds on the fractional time derivative of v. To see this, we first note that, for any $\alpha\in\left(0,\frac{1}{2}\right)$, the following bounds can be obtained (see e.g Lemma 2.1 [14]):

$$\mathbb{E}\|\int_0^{\cdot}\sigma f'(\mathbb{X}_t)\mathrm{d}W_t+\int_{(0,\cdot]}\int_{E_0}(f(\mathbb{X}_t+K(\xi,t))-f(\mathbb{X}_t))\mathrm{d}\hat{\pi}(\xi,t)\|_{H^\alpha([0,T];L^2)}\leqslant C.$$

Also observe that $\|B(v,v)\|_{D(A)'}\leqslant C\|v\|\|\nabla v\|$ (see e.g. [39]). Thus, for some $C>0$, we have

$$\mathbb{E}\|\int_0^{\cdot}B(v,v)\|_{H^1(0,T;D(A)')}\leqslant C\mathbb{E}\sup_{t\in[0,T]}\|v(t)\|^2\ \mathbb{E}\int_0^T\|\nabla v(t)\|^2\mathrm{d}t\leqslant C.$$

Similarly, we can see that

$$\mathbb{E}\|\int_0^{\cdot}(B(v,v+\Phi)+B(v,\Phi)-\nu\Delta v-\nu\Delta\Phi)\mathrm{d}t\|_{H^1(0,T;D(A)')}\leqslant C$$

and that

$$\mathbb{E}\|\int_0^{\cdot}\mathcal{L}f(\mathbb{X}_t)\|^2_{H^1(0,T;L^2)}\leqslant C.$$

Thus, we conclude that for some $C>0$,

$$\mathbb{E}\|v\|_{H^\alpha(0,T;D(A)')}\leqslant C.\tag{3.29}$$

Having the above estimates in hand, we employ the Galerkin approximation scheme to obtain the existence of martingale solutions using standard arguments as in [6, 8, 14, 30]. We refer the reader to [6] for an analysis of the Navier–Stokes equations, driven by a general multiplicative Lévy noise and specifically for a detailed argument for the passage of limit. □

4. Estimation of the mean value

In this section, we prove an almost sure upper bound for the energy dissipation.

To derive the estimate on $\mathbb{E}[\langle\epsilon\rangle_T]$, we take the expected value of (3.15) with respect to $\mathbb{P}$, then average it over $[0,T]$, and finally take the limit superior as $T\to\infty$. Since $u=v+\Phi$, we obtain

$$\int_0^T\|\nabla u\|^2\mathrm{d}t=\int_0^T\|\nabla v+\nabla\Phi\|^2\mathrm{d}t\leqslant2\int_0^T\|\nabla v\|^2+\|\nabla\Phi\|^2\mathrm{d}t.\tag{4.1}$$

The second term in the integrand is, from (3.9),

$$\|\nabla\Phi\|^2=\|\frac{\partial\phi}{\partial x_3}\|^2=\frac{L^2}{\delta_t}\mathbb{X}_t^2=L^2\,\frac{\mathbb{X}_t^4+B\mathbb{X}_t^2}{A}.\tag{4.2}$$

Hence,

$$\mathbb{E}\left[\int_0^T \|\nabla\Phi\|^2 \mathrm{d}t\right] = \frac{L^2}{A}\int_0^T \mathbb{E}\left[\mathbb{X}_t^4 + B\mathbb{X}_t^2\right]\mathrm{d}t, \tag{4.3}$$

which can be bounded using the moment bounds for $\mathbb{X}_t$ in (5.12) in the Appendix. We now estimate the first term on the right of (4.1). We know that M_T defined in (3.16) is a martingale and hence $\mathbb{E}[M_T] = 0$ for all $T \in [0, \infty)$. Therefore, taking the expectation $\mathbb{E}$ of both sides of (3.15) gives

$$\mathbb{E}\int_0^T \nu\|\nabla v\|^2 \mathrm{d}t \leqslant \mathbb{E}\left[2\|v(0)\|^2 + Y_T\right]. \tag{4.4}$$

We use the expected value of Y_T calculated in (3.23).

Now we continue from (4.4); we divide both sides by T and $|D| = L^2h$ and use (3.24). We know that, for a fixed set E_0, $\pi((0,T], E_0)$ is a Poisson random variable with intensity $\mathbb{E}\frac{\pi((0,T],E_0)}{T} = \lambda(E_0)$. Thus, for any $T > 0$, we obtain

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{TL^2h}\mathbb{E}\int_0^T \nu\|\nabla v\|^2 \mathrm{d}t \leqslant \limsup_{T\to\infty} \frac{1}{TL^2h}\,\mathbb{E}[Y_T] \\
&\leqslant \frac{4}{h}\left[\frac{3}{2}\frac{A}{B} + \frac{6}{\nu}\left(\frac{A}{B}\right)^3 \frac{\sigma^2}{B}\right]\sigma^2 \\
&\quad + \frac{4}{h}\left(\frac{\nu B}{A} + \frac{6\theta^2}{\nu}\left(\frac{A}{B}\right)^3\right)\left(3U^2 + \frac{3}{2\theta}\left(\sigma^2 + \|K\|_\infty^2\lambda(E_0)\right)\right) \\
&\quad + \frac{4}{h}\frac{\nu}{A}\left(8U^4 + 8\left(\frac{\sigma^4 + 2\|K\|_\infty\lambda(E_0) + \|K\|_\infty^4\,\lambda^2(E_0) + \|K\|_\infty\lambda(E_0)\theta}{4\theta^2}\right)\right) \\
&\quad + \frac{4}{h}\|K\|_\infty^2\left(\frac{27A}{2B} + \frac{\|K\|_\infty^2 A}{8B^2} + \frac{50\|K\|_\infty^4 A}{3B^3}\right)\lambda(E_0) \\
&\quad + \frac{8\lambda(E_0)}{3\nu}\left(\frac{A}{B}\right)^3\left(9\|K\|_\infty^4 + \frac{\|K\|_\infty^6}{B^2}\right).
\end{aligned} \tag{4.5}$$

Similar calculations and an application of (5.12) lead us to

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \|\nabla\Phi\|^2 \mathrm{d}t\right] = \frac{L^2}{A}\int_0^T \mathbb{E}\left[\mathbb{X}_t^4 + B\mathbb{X}_t^2\right]\mathrm{d}t \\
&\leqslant T\frac{BL^2}{A}\left(3U^2 + \frac{3}{2\theta}\left(\sigma^2 + \|K\|_\infty^2\lambda(E_0)\right)\right) \\
&\quad + T\frac{L^2}{A}\left(8U^4 + 8\left(\frac{\sigma^4 + 2\|K\|_\infty\lambda(E_0) + \|K\|_\infty^4\,\lambda^2(E_0) + \|K\|_\infty\lambda(E_0)\theta}{4\theta^2}\right)\right).
\end{aligned}$$

Thus, we infer that

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{\nu}{TL^2h}\mathbb{E}\left[\int_0^T \|\nabla\Phi\|^2 \mathrm{d}t\right] \\
&\leqslant \frac{B\nu}{Ah}\left(3U^2 + \frac{3}{2\theta}\left(\sigma^2 + \|K\|_\infty^2\lambda(E_0)\right)\right) \\
&\quad + \frac{\nu}{Ah}\left(8U^4 + 8\left(\frac{\sigma^4 + 2\|K\|_\infty\lambda(E_0) + \|K\|_\infty^4\,\lambda^2(E_0) + \|K\|_\infty\lambda(E_0)\theta}{4\theta^2}\right)\right).
\end{aligned} \tag{4.6}$$

Finally, by (4.1), (4.5), and (4.6), and taking $A = \nu U$, $B = U^2$, one obtains the estimate

$$\begin{aligned}\varepsilon &\leqslant \limsup_{T\to\infty}\frac{2}{TL^2h}\,\mathbb{E}\int_0^T \nu\|\nabla v\|^2\mathrm{d}t+\limsup_{T\to\infty}\frac{2}{TL^2h}\,\mathbb{E}\int_0^T \nu\|\nabla\Phi\|^2\mathrm{d}t\\
&\leqslant \frac{8}{h}\left[\frac{3}{2}\frac{\nu}{U}+\frac{6\nu^2\sigma^2}{U^5}\right]\sigma^2+\frac{8}{h}\left(\frac{6\theta^2\nu^2}{U^3}+\frac{5U}{4}\right)\left(3U^2+\frac{3}{2\theta}\left(\sigma^2+\|K\|_\infty^2\lambda(E_0)\right)\right)\\
&\quad+\frac{10}{hU}\left(8U^4+8\left(\frac{\sigma^4+2\|K\|_\infty\lambda(E_0)+\|K\|_\infty^4\,\lambda^2(E_0)+\|K\|_\infty\lambda(E_0)\theta}{4\theta^2}\right)\right)\\
&\quad+\frac{8}{h}\|K\|_\infty^2\left(\frac{27\nu}{2U}+\frac{\|K\|_\infty^2\nu}{8U^3}+\frac{50\|K\|_\infty^4\nu}{3U^5}\right)\lambda(E_0)\\
&\quad+\frac{16\lambda(E_0)}{3\nu}\left(\frac{\nu}{U}\right)^3\left(9\|K\|_\infty^4+\frac{\|K\|_\infty^6}{B^2}\right).\end{aligned}\tag{4.7}$$

Remark 4.1 *(Higher moments) One can obtain upper bounds for higher moments of $\langle\epsilon\rangle_T$ by following our method, using the general moment bounds (5.12), and Lemma 5.1 in the Appendix. We expect that for all integer $k\geqslant 1$, the $2k$-th moment of $\langle\epsilon\rangle_T$ satisfies*

$$\limsup_{T\to\infty}\mathbb{E}[\langle\epsilon\rangle_T^{2k}]\lesssim\frac{U^{6k}}{h^{2k}}+P_k\big(\sigma^2,\,\|K\|_\infty^2,\,\lambda(E_0)\big),\tag{4.8}$$

where $P_k(x,y,z)$ is a polynomial in (x,y,z) with $P_k(0,0,0)=0$, so that (4.8) recovering an upper bound in [10] when there is no noise. The coefficients of P_k are explicit functions of k, U,ν, and θ.

To give some details, by (5.12) and (5.5), for all even integers k, we have

$$\limsup_{t\to\infty}\mathbb{E}\left[X_t^{2k}\right]\leqslant 9^kU^{2k}+9^kC_{2k}\left(\frac{\left[\sigma^2+\|K\|_\infty^2\,\lambda(E_0)\right]^k}{\theta^k}+2^k\|K\|_\infty^{2k}\,\frac{\lambda(E_0)}{4\theta}\right),\tag{4.9}$$

where C_{2k} is the absolute constant in the Burkholder–Davis–Gundy inequality (5.4). By (4.1), for all $p\in[1,\infty)$,

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^T\|\nabla u\|^2\mathrm{d}t\right|^p\right]&\leqslant 2^p\,\mathbb{E}\left[\left|\int_0^T\|\nabla v\|^2+\|\nabla\Phi\|^2\mathrm{d}t\right|^p\right]\\
&\leqslant 4^p\left(\mathbb{E}\left[\left|\int_0^T\|\nabla v\|^2\mathrm{d}t\right|^p\right]+\mathbb{E}\left[\left|\int_0^T\|\nabla\Phi\|^2\mathrm{d}t\right|^p\right]\right).\end{aligned}\tag{4.10}$$

To bound the second term on the right of (4.10), from Hölder's inequality we have

$$\mathbb{E}\left[\left|\int_0^T\|\nabla\Phi\|^2\mathrm{d}t\right|^p\right]\leqslant T^{p-1}\,\mathbb{E}\left[\int_0^T\|\nabla\Phi\|^{2p}\mathrm{d}t\right],\tag{4.11}$$

which can be bounded in terms of the $4p$-th moment of $\mathbb{X}_t$, according to (4.2). The first term on the right of (4.10) can also be bounded in terms of the $4p$-th moment of $\mathbb{X}_t$ by applying the almost sure upper bound (3.15) and then Doob's L^p-inequality to the martingale term.

5. Appendix

5.1 OU process driven by Lévy noise

We obtain moment estimates for the process $\mathbb{X}$ solving (1.5)–(1.6).

Recall from (1.6) that L is a square-integrable Lévy martingale given by

$$L_t = \sigma W_t + \int_{E_0\times(0,t]} K(\xi,s)\mathrm{d}\hat{\pi}(\xi,s).$$

The Laplace functional of the Poisson random measure π is given by

$$\mathbb{E}\left[\exp\left\{-\int_{E_0\times(0,\infty)} F(\xi,s)\,\mathrm{d}\pi(\xi,s)\right\}\right] = \exp\left\{-\int_{E_0\times(0,\infty)} 1-e^{-F(\xi,s)}\,\mathrm{d}\lambda(\xi)\,\mathrm{d}s\right\}. \tag{5.1}$$

Let $[L]$ be the quadratic variation of L and let $[L]^c$ be the continuous part of $[L]$. Then $[M]^c_t = \sigma^2 t$ and

$$\begin{aligned}\int_{(0,t]} \phi(s)\,\mathrm{d}[L]_s &= \int_{(0,t]} \phi(s)\,\mathrm{d}[L]^c_s + \sum_{s\in(0,t]} \phi(s)\,|\Delta L_s|^2 \\ &= \sigma^2 \int_{(0,t]} \phi(s)\,\mathrm{d}s + \int_{E_0\times(0,t]} \phi(s)\,K^2(\xi,s)\,\mathrm{d}\pi(\xi,s)\end{aligned} \tag{5.2}$$

for all $t\in(0,\infty)$ and for any continuous function $\phi:(0,\infty)\to\mathbb{R}$.

We take $\phi(s)=e^{\theta s}$ and define the stochastic integral

$$I_t := \int_{(0,t]} e^{\theta s}\,\mathrm{d}L_s, \tag{5.3}$$

which is a local martingale with quadratic variation $[I]_t = \int_{(0,t]} e^{2\theta s}\,\mathrm{d}[L]_s$. By the Burkholder–Davis–Gundy inequality for general local martingales (see, for instance, [20, Theorem 26.12]), for all $p\geqslant 1$, there exists a constant $C_p\in(0,\infty)$ such that

$$C_p^{-1}\mathbb{E}\left[[I]_t^{p/2}\right] \leqslant \mathbb{E}\left[\left(\sup_{s\in[0,t]}|I_s|\right)^p\right] \leqslant C_p\mathbb{E}\left[[I]_t^{p/2}\right] \quad t\geqslant 0. \tag{5.4}$$

To get a bound for the $2k$-th moment of I, we take $p=2k$ in (5.4) and apply the lemma below.

Lemma 5.1 $\mathbb{E}\left[[I]_t^k\right]$ *has the following upper bound for* $k\in\mathbb{N}$. *When* $k=1$, *for all* $t\in[0,\infty)$,

$$\mathbb{E}\left[[I]_t\right] \leqslant \left(\frac{e^{2\theta t}-1}{2\theta}\right)\left[\sigma^2 + \|K\|^2_\infty\,\lambda(E_0)\right].$$

When $k=2$ *and* C_2 *is the constant in* (5.4), *for all* $t\in[0,\infty)$,

$$\mathbb{E}\left[[I]_t^2\right] \leqslant 4\left\{\left(\frac{e^{2\theta t}-1}{2\theta}\right)^2\left[\sigma^2+\|K\|^2_\infty\,\lambda(E_0)\right]^2 + C_2\,\|K\|^4_\infty\,\lambda(E_0)\,\frac{e^{4\theta t}-1}{4\theta}\right\}.$$

For all even integers $k\geqslant 2$, *there exists a constant* $\widetilde{C}_k\in(0,\infty)$ *such that, for* $t>\dfrac{1}{2\theta}$,

$$\mathbb{E}\left[[I]_t^k\right] \leqslant 2^k\left\{\left(\frac{e^{2\theta t}-1}{2\theta}\right)^k\left[\sigma^2+\|K\|^2_\infty\,\lambda(E_0)\right]^k + \widetilde{C}_k\,\|K\|^{2k}_\infty\,e^{2\theta k\,t}\,\frac{\lambda(E_0)}{4\theta}\right\}.$$

An immediate consequence of Lemma 5.1 is the uniform bound

$$\limsup_{t\to\infty}\frac{\mathbb{E}\left[[I]_t^k\right]}{e^{2\theta k t}} \leqslant \frac{\left[\sigma^2+\|K\|^2_\infty\,\lambda(E_0)\right]^k}{\theta^k} + 2^k\|K\|^{2k}_\infty\,\frac{\lambda(E_0)}{4\theta} < \infty \tag{5.5}$$

for all even integers k.

Proof of Lemma 5.1 Let M be the process $M_t := \int_{(0,t]}\int_{E_0} e^{2\theta s}\,K^2(\xi,s)\,\mathrm{d}\hat{\pi}(\xi,s)$, which is a local martingale. Then

$$\begin{aligned}
0 \leqslant [I]_t =& \int_{(0,t]} e^{2\theta s}\, \mathrm{d}[L]_s \\
=& \frac{\sigma^2}{2\theta}(e^{2\theta t}-1) + \int_{E_0\times(0,t]} e^{2\theta s}\, K^2(\xi,s)\, \mathrm{d}\pi(\xi,s) \\
\leqslant& \frac{\sigma^2}{2\theta}(e^{2\theta t}-1) + \|K\|_\infty^2\, \lambda(E_0) \left(\frac{e^{2\theta t}-1}{2\theta}\right) + M_t \\
=& \left(\frac{e^{2\theta t}-1}{2\theta}\right) \left(\sigma^2 + \|K\|_\infty^2\, \lambda(E_0)\right) + M_t, \qquad t\in[0,\infty).
\end{aligned} \tag{5.6}$$

The desired inequality for $k=1$ follows immediately.

For $k\geqslant 2$, using (5.6) again and the simple fact that $(a+b)^k \leqslant 2^k(a^k+b^k)$ for $k\geqslant 1$ and $a,b\geqslant 0$, we obtain

$$\mathbb{E}\left[[I]_t^k\right] \leqslant 2^k \left\{ \left(\frac{e^{2\theta t}-1}{2\theta}\right)^k \left[\sigma^2 + \|K\|_\infty^2\, \lambda(E_0)\right]^k + \mathbb{E}\left[M_t^k\right] \right\}. \tag{5.7}$$

Applying the BDG inequality for the local martingale M,

$$\begin{aligned}
\mathbb{E}\left[M_t^k\right] \leqslant& C_k\, \mathbb{E}\left[[M]_t^{k/2}\right] \\
=& C_k\, \mathbb{E}\left[\left(\int_{E_0\times(0,t]} e^{4\theta s}\, K^4(\xi,s)\, \mathrm{d}\pi(\xi,s)\right)^{k/2}\right] \\
\leqslant& C_k\, \|K\|_\infty^{2k}\, \mathbb{E}\left[\left(\int_{E_0\times(0,t]} e^{4\theta s}\, \mathrm{d}\pi(\xi,s)\right)^{k/2}\right].
\end{aligned} \tag{5.8}$$

When $k=2$ the expectation in the last displayed expression is

$$\mathbb{E}\left[\left(\int_{E_0\times(0,t]} e^{4\theta s}\, \mathrm{d}\pi(\xi,s)\right)\right] = \lambda(E_0)\, \frac{e^{4\theta t}-1}{4\theta},$$

giving the desired inequality for $k=2$.

Since $\lambda(E_0)<\infty$, there exists a countable set $\{(Z_i,t_i)\}\subset E_0\times[0,\infty)$ whose elements are the jump points for π. Then

$$\int_{E_0\times(0,t]} \Phi(\xi,s)\, \mathrm{d}\pi(\xi,s) = \sum_{i:\ t_i\in(0,t]} \Phi(Z_i,t_i)$$

for all bounded measurable function $\Phi: E_0\times(0,t]\to\mathbb{R}$ and $t\in(0,\infty)$. The sets $\{t_i\}\cap(0,t]$ are distributed as the jump times of a Poisson process on $(0,t]$ with intensity $\lambda(E_0)$. Hence

$$\begin{aligned}
\mathbb{E}\left[\left(\int_{E_0\times(0,t]} e^{4\theta s}\, \mathrm{d}\pi(\xi,s)\right)^{k/2}\right] =& \mathbb{E}\left[\left(\sum_{\{t_i\}\cap(0,t]} e^{4\theta t_i}\right)^{k/2}\right] \\
= e^{-\lambda(E_0)\,t} \sum_{m=0}^{\infty} \frac{(\lambda(E_0)\,t)^m}{m!}\, \mathbb{E}\left[\left(\sum_{i=1}^{m} e^{4\theta S_i}\right)^{k/2}\right],
\end{aligned} \tag{5.9}$$

where $\{S_i\}_{i\geqslant 1}$ are i.i.d. uniform on $(0,t]$. By the multinomial theorem, for $k=2\ell$ even integers,

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{i=1}^m e^{4\theta S_i}\right)^{k/2}\right] &= \frac{1}{t^m}\int_{(0,t]^m}\left(\sum_{i=1}^m e^{4\theta s_i}\right)^{\ell} \mathrm{d}s_1\cdots \mathrm{d}s_m \\
&= \frac{1}{t^m}\sum_{k_1+k_2+\cdots+k_m=\ell}\binom{\ell}{k_1,k_2,\ldots,k_m}\int_{(0,t]^m}\prod_{i=1}^m e^{4\theta\, k_i\, s_i}\,\mathrm{d}s_1\cdots\mathrm{d}s_m \\
&= \frac{1}{t^m}\sum_{z=0}^{m-1}\sum_{j_1+j_2+\cdots+j_{m-z}=\ell,\; j_1,j_2,\cdots,j_{m-z}\geqslant 1}\binom{\ell}{j_1,j_2,\ldots,j_{m-z}} t^z\prod_{i=1}^{m-z}\frac{e^{4\theta j_i\, t}-1}{4\theta j_i} \\
&\leqslant \frac{1}{t^m}\sum_{z=0}^{m-1}\sum_{j_1+j_2+\cdots+j_{m-z}=\ell,\; j_1,j_2,\cdots,j_{m-z}\geqslant 1}\binom{\ell}{j_1,j_2,\ldots,j_{m-z}} t^z\,\frac{e^{4\theta\ell\, t}}{(4\theta)^{m-z}} \\
&\leqslant \frac{1}{t^m}\sum_{z=0}^{m-1} t^z\,\frac{e^{4\theta\ell\, t}}{(4\theta)^{m-z}}\,(m-z)^{\ell} \\
&= e^{4\theta\ell\, t}\, a\sum_{j=1}^m a^{j-1}\, j^{\ell} \quad \text{where } a=1/(4\theta t) \\
&\leqslant e^{4\theta\ell\, t}\, a\,\widetilde{C}_\ell\, m \quad \text{whenever } a\in(0,1/2],
\end{aligned} \tag{5.10}$$

where $\widetilde{C}_\ell \in (0,\infty)$ is a constant that does not depend on $a\in(0,1/2]$ or $m\in\mathbb{Z}_+$. This follows from the fact that $a^{j-1}j^{\ell}\to 0$ uniformly for $a\in(0,1/2]$, as $j\to\infty$. In the third equality above, z is the number of zeros in $\{k_i\}_{i=1}^m$.

Inserting (5.10) into (5.9) gives

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{E_0\times(0,t]} e^{4\theta s}\,\mathrm{d}\pi(\xi,s)\right)^{k/2}\right] \\
&\leqslant e^{4\theta\ell\, t}\, e^{-\lambda(E_0)\, t}\sum_{m=0}^{\infty}\frac{(\lambda(E_0)\, t)^m}{m!}\left[\widetilde{C}_\ell\, m\, a\right] \qquad \text{whenever } a=1/(4\theta t)\in(0,1/2) \\
&= e^{4\theta\ell\, t}\,\widetilde{C}_\ell\,\lambda(E_0)\, t\, a \;=\; e^{4\theta\ell\, t}\,\frac{\widetilde{C}_\ell\,\lambda(E_0)}{4\theta},
\end{aligned} \tag{5.11}$$

Inserting (5.11) into (5.8) and then into (5.7) gives the desired inequality for $k=2\ell\geqslant 2$. □

Now an upper bound for the moments of X_t can be obtained from (1.7) and Lemma 5.1 as follows. Recall that $X_t = e^{-\theta t}X_0 + U(1-e^{-\theta t}) + e^{-\theta t}I_t$ for all $t\in[0,\infty)$. Using the simple fact that $(a+b+c)^k\leqslant 3^p(a^p+b^p+c^p)$ for $p\geqslant 1$ and $a,b,c\geqslant 0$, and taking $p=2k$, we obtain

$$|X_t|^{2k} \leqslant 3^{2k}\,(e^{-2k\theta t}X_0^{2k} + U^{2k}\,(1-e^{-\theta t})^{2k} + e^{-2k\theta t}I_t^{2k}) \quad \text{for } t\geqslant 0, \quad \mathbb{P}\text{-}a.s..$$

Now using the BDG inequality (5.4), we obtain

$$\mathbb{E}[|X_t|^{2k}] \leqslant 9^k\,\left(e^{-2k\theta t}\mathbb{E}[X_0^{2k}] + U^{2k}(1-e^{-\theta t})^{2k} + e^{-2k\theta t}C_{2k}\mathbb{E}\left[[I]_t^k\right]\right) \quad \text{for } t\geqslant 0. \tag{5.12}$$

Combining (5.12) with Lemma 5.1 gives explicit moment bounds for X_t. Furthermore, (5.5) immediately gives the following uniform moment bound.

Corollary 5.2 *Assume that* $\mathbb{E}[|\mathbb{X}_0|^p]<\infty$ *for all* $p>0$. *Then for all* $p>0$,

$$\limsup_{t\to\infty}\mathbb{E}\left[|\mathbb{X}_t|^p\right]<\infty.$$

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References

[1] Applebaum, D., Lévy Processes and Stochastic Calculus, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 116, Cambridge University Press, Cambridge, 2009.

[2] Bensoussan, A. and Temam, R., Equations stochastiques du type Navier-Stokes, J. Funct. Anal., 1973, 13: 195–222.

[3] Busse, F. H., Bounds for turbulent shear flow, Journal of Fluid Mechanics, 1970, 41: 4219–240.

[4] Camassa, R., Kilic, Z. and McLaughlin, R. M., On the symmetry properties of a random passive scalar with and without boundaries, and their connection between hot and cold states, Phys. D, 2019, 400: 132124.

[5] Chorin, A. J., Numerical study of slightly visous flow, J. Fluid Mech., 1973, 57: 785–796.

[6] Cyr, J., Nguyen, P., Tang, S. and Temam, R., Review of local and global existence results for stochastic PDEs with Lévy noise, Discrete Contin. Dyn. Syst., 2020, 40(10): 5639–5710.

[7] Dascaliuc, R., Foias, C. and Jolly, M. S., On the asymptotic behavior of average energy and enstrophy in 3D turbulent flows, Phys. D, 2009, 238: 725–736.

[8] Debussche, A., Glatt-Holtz, N. and Temam, R., Local martingale and pathwise solutions for an abstract fluids model, Phys. D, 2011, 240(14–15): 1123–1144.

[9] DeCaria, V., Layton, W., Pakzad, A., Rong, Y., Sahin, N. and Zhao, H., On the determination of the grad-div criterion, Journal of Mathematical Analysis and Applications, 2018, 467: 1032–1037.

[10] Doering, C. R. and Constantin, P., Energy dissipation in shear driven turbulence, Physical review letters, 1992, 69: 1648.

[11] Doering, C. R. and Foias, C., Energy dissipation in body-forced turbulence, J. Fluid Mech., 2002, 467: 289–306.

[12] Duchon, J. and Robert, R., Inertial energy dissipation for weak solutions of incompressible Euler and Navier–Stokes equations, Nonlinearity, 2000, 13: 249–255.

[13] Fan, W.-T., Jolly, M. and Pakzad, A., Three-dimensional shear driven turbulence with noise at the boundary, Nonlinearity, 2021, 34(7): 4764.

[14] Flandoli, F. and Gatarek, D., Martingale and stationary solutions for stochastic Navier-Stokes equations, Probab, Theory Related Fields, 1995, 102(3): 367–391.

[15] Foias, C., Jolly, M. S., Manley, O. P., Rosa, R. and Temam, R., Kolmogorov theory via finite-time averages, Phys. D, 2005, 212(3–4): 245–270.

[16] Frisch, U., Turbulence: The legacy of A. N. Kolmogorov, Cambridge University Press, Cambridge, 1995.

[17] Hopf, E., Lecture Series of the Symposium on Partial Differential Equations, University of California, Berkeley, 1955.

[18] Howard, L. N., Bounds on flow quantities, Annual Review of Fluid Mechanics, 1972, 4: 473–494.

[19] Hussein, H., Capp, S. and George, W., Velocity measurements in a high-Reynolds-number, momentum-conserving, axisymmetric, turbulent jet, Journal of Fluid Mechanics, 1994, 258: 31–75.

[20] Kallenberg, O., Foundations of Modern Probability, 2nd ed., Probability and its Applications (New York), Springer-Verlag, New York, 2002.

[21] Kerswell, R. R., Variational bounds on shear-driven turbulence and turbulent Boussinesq convection, Physica D,

1997, 100: 355–376.

[22] Kolmogorov, A. N., The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers, Proc. Roy. Soc. London Ser. A, 1991, 1890, 434: 9–13.

[23] Krylov, N. V. and Rozovskii, B. L., Stochastic evolution equations, Current Problems in Mathematics, 1979, 14: 71–147.

[24] Lathrop, D. P., Fineberg, J. and Swinney, H. L., Turbulent flow between concentric rotating cylinders at large Reynolds number, Physical Review Letters, 1992, 68(10): 1515.

[25] Layton, W. J., Energy dissipation bounds for shear flows for a model in large eddy simulation, Math. Comput. Modelling, 2002, 35(13): 1445–1451.

[26] Layton, W. J., Introduction to the Numerical Analysis of Incompressible Viscous Flows, Computational Science & Engineering, vol. 6, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008, With a foreword by Max Gunzburger.

[27] Layton, W. J., Energy dissipation in the Smagorinsky model of turbulence, Appl. Math. Lett., 2016, 59: 56–59.

[28] Marchioro, C., Remark on the energy dissipation in shear driven turbulence, Phys. D, 1994, 74(3-4): 395–398.

[29] Métivier, M., Semimartingales: A course on stochastic processes, de Gruyter Studies in Mathematics, Vol. 2, Walter de Gruyter & Co., Berlin-New York, 1982.

[30] Nguyen, P. and Tawri, K. and Temam, R., Nonlinear stochastic parabolic partial differential equations with a monotone operator of the Ladyzenskaya-Smagorinsky type, driven by a Lévy noise, J. Funct. Anal., 2021, 281(8): 109157.

[31] Pakzad, A., Damping functions correct over-dissipation of the Smagorinsky Model, Mathematical Methods in the Applied Sciences, 2017, 40(16): 5933–5945.

[32] Pakzad, A., Analysis of mesh effects on turbulence statistics, Journal of Mathematical Analysis and Applications, 2019, 475: 839–860.

[33] Pakzad, A., On the long time behavior of time relaxation model of fluids, Phys. D, 2020, 408: 132509.

[34] Pope, S. B., Turbulent Flows, Cambridge University Press, Cambridge, 2000.

[35] Reichardt, H., Gesetzmassigkeiten der geradlinigen turbulenten Couettestromung, MaxPlanck-Institut fur Stromungsforschung und Aerodynamische versuchsanstalt, 1959, No. 22, 1-45

[36] Saffman, P. G., Lectures on Homogeneous Turbulence, In: Zabusky, N. (ed.), Topics in Nonlinear Physics, Springer-Verlag, New York, 1968, 485-614.

[37] Sreenivasan, K. R., An update on the energy dissipation rate in isotropic turbulence, Phys. Fluids, 1998, 10(2): 528–529.

[38] Taylor, G. I., Statistical theory of turbulence, Proc. R. Soc. Lond A, 1935, 151(873): 421–444.

[39] Temam, R., Navier-Stokes Equations: Theory and Numerical Analysis, AMS Chelsea Publishing, Providence, RI, 2001.

[40] Viot, M., Solutions faibles d'équations aux dérivées partielles stochastiques non linéaires, Thèse, Université Pierre et Marie Curie, 1976.

[41] Wang, X., Time-averaged energy dissipation rate for shear driven flows in $\mathbf{R}^n$, Phys. D, 1997, 99(4): 555–563.