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ORIGINAL RESEARCH ARTICLE

A Positive Formula for Type A Peterson Schubert Calculus

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Abstract
The Peterson variety is a special case of a nilpotent Hessenberg variety, a class of subvarieties of G/B that have appeared in the study of quantum cohomology, representation theory and combinatorics. In type A, the Peterson variety Y is a subvariety of $Fl(n;\mathbb{C})$, the set of complete flags in $\mathbb{C}^n$, and comes equipped with an action by a one-dimensional torus subgroup S of a standard torus T that acts on $Fl(n;\mathbb{C})$. Using the *Peterson Schubert basis* introduced in Harada and Tymoczko (Proc Lond Math Soc 103(1):40–72, 2011) and obtained by restricting a specific set of Schubert classes from $H_T^*(Fl(n;\mathbb{C}))$ to $H_S^*(Y)$, we describe the product structure of the equivariant cohomology $H_S^*(Y)$. In particular, we show that the product is *manifestly positive* in an appropriate sense by providing an explicit, positive, combinatorial formula for its structure constants. A key step in our proof requires a new combinatorial identity of binomial coefficients that generalizes Vandermonde's identity, and merits independent interest.

Keywords Peterson · Schubert calculus · Structure constants · Vandermonde

1 Introduction

Let $G = Gl(n,\mathbb{C})$, B upper triangular matrices, and B_- lower triangular matrices. The quotient $G/B = Fl(n;\mathbb{C})$ is the associated *flag variety*. Let T be compact form of the set of diagonal matrices in G, i.e. diagonal matrices in which each entry has norm 1. Then G/B has a left T action with isolated fixed points, $(G/B)^T$. The fixed point set

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may be identified with the Weyl group $W \cong S_n$, the permutation group on n letters. We denote by $\mathfrak{t}$ the Lie algebra of T and by $\mathfrak{t}^*$ its dual. Let x_i be the ith coordinate function on $T \cong (S^1)^n$, for $i = 1, \ldots n$. Finally let $\{\alpha_i := x_i - x_{i+1} : i \in \{1, \ldots, n-1\}\}$ denote a choice of positive simple roots, with the property that the roots spaces of the Lie algebra $\mathfrak{b}$ of B are positive.

The ordinary cohomology and the T-equivariant cohomology of G/B have a linear basis given by *Schubert classes* σ_w as w varies over elements of W. Indeed, they are each free modules over the corresponding ordinary or equivariant cohomology of a point. We use cohomology with complex coefficients throughout, and identify the equivariant cohomology of a point, denoted H_T^*, with the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$.

The products of Schubert classes define coefficients $c_{u,v}^w \in H_T^*$ by expanding in the basis:

$$\sigma_u \sigma_v = \sum_{w \in W} c_{u,v}^w \sigma_w$$

for all $u, v \in W$. The coefficients $c_{u,v}^w$ are polynomials in $\alpha_1, \ldots, \alpha_{n-1}$ with *nonnegative coefficients* [12].

This manuscript describes a similar story with a particular subvariety of $Fl(n; \mathbb{C})$, namely the *Peterson variety* Y. The Peterson variety is a special nilpotent Hessenberg variety first introduced in unpublished work by Peterson [17], in which he proposed a link with the quantum cohomology of $Fl(n; \mathbb{C})$. There are multiple equivalent definitions that have been given for the Peterson, and we provide one that works in all Lie types. In Definition 11, we provide another definition specific to the case that $G/B = Fl(n; \mathbb{C})$. Let w_0 denote the longest word in the Weyl group W, and $e \in \mathfrak{b}$ a principal nilpotent element in the Lie algebra of B. Define G^e be the centralizer of e. The Peterson variety is defined as the closure in G/B of an orbit of G^e on the point $w_0 B$, as follows:

$$Y := \overline{G^e w_0 B} \hookrightarrow G/B.$$

Kostant elaborated on the connection to integrable systems, showing that the quantum cohomology ring of $Fl(n; \mathbb{C})$ is isomorphic to the coordinate ring of an open dense affine subvariety of the Peterson variety [16]. Rietsch generalizes these results to G/P for any parabolic P, and proved the Peterson variety is paved by these affine varieties as P varies [18]. Her work revealed an explicit relationship among geometric, algebraic and combinatorial descriptions of quantum cohomology, which she subsequently generalized to equivariant quantum cohomology, noting that each stratum may also play the role of a “mirror symmetry phenomenon" for G/P [19].

The Peterson variety Y in $Fl(n; \mathbb{C})$ is invariant under the action of a one-dimensional subgroup S of T (specified in Sect. 3.1). We describe the product structure of the S-equivariant cohomology $H_S^*(Y)$ in a specific linear basis, termed the *Peterson Schubert basis*. In particular, we show that the product is *positive* in an appropriate sense by providing an explicit positive combinatorial formula for the S-equivariant and ordinary structure constants (see Theorems 1, 3, 5, 6, and their corollaries).

The (equivariant) cohomology of the Peterson variety has been formulated and described in several ways. Tymoczko showed the Peterson variety has a paving by affine cells [21], implying its cohomology groups are nonzero only in even degrees. Tymoczko and Insko explore the non-equivariant cohomology through the study of its homology groups [15]. The ring structure has been described both as a quotient ring and as a subring of a sum of polynomial rings in work by Brion and Carrel [6], Harada et al. [13], and Fukukawa et al. [9], and via a connection with hyperplane arrangements [2]. Harada and Tymoczko [14] introduced a Schubert-type basis for the S-equivariant cohomology of the Peterson variety as a module over the S-equivariant cohomology of a point and proved a *manifestly positive* Chevalley–Monk formula for the equivariant cohomology of the Peterson variety of $Fl(n;\mathbb{C})$. Drellich extended the Chevalley–Monk formula proved by Harada and Tymoczko to all Lie types as well as proved Giambelli's formula for Y in all Lie types [8]. After the appearance of this manuscript on the arXiv, Abe, Horiguchi, Kuwata, and Zeng posted a paper that computes the structure constants for the ordinary cohomology of Y [1].

Harada and Tymoczko's insight was to use a natural composition

$$j : H_T^*(Fl(n;\mathbb{C})) \longrightarrow H_S^*(Fl(n;\mathbb{C})) \longrightarrow H_S^*(Y)$$

to obtain a basis of $H_S^*(Y)$ (as a module over H_S^*) as the image of a specific subset of Schubert classes on $Fl(n;\mathbb{C})$ indexed by subsets

$$A \subseteq [n-1] = \{1, \dots, n-1\}.$$

More specifically, let $\alpha_1, \dots \alpha_{n-1}$ denote the simple roots ordered by adjacency in the Dynkin diagram, and $s_1, \dots, s_{n-1}$ the corresponding reflections. For $A = \{a_1, \dots, a_k\}$ listed in increasing order, let

$$v_A = s_{a_1} s_{a_2} \dots s_{a_k}$$

and σ_{v_A} the corresponding Schubert class. The *Peterson Schubert classes* p_A are defined by

$$p_A = j(\sigma_{v_A}).$$

The set $\{p_A\}_{A\subseteq[n-1]}$ forms a module basis of $H_S^*(Y)$. Thus the product of two Peterson Schubert classes is an H_S^*-linear combination of Peterson Schubert classes. For $A, B, C \subseteq \{1, \dots, n-1\}$, define the *structure constant* $b_{A,B}^C \in H_S^*$ by

$$p_A p_B = \sum_{C\subseteq\{1,\dots,n-1\}} b_{A,B}^C \, p_C. \tag{1}$$

Harada and Tymoczko show that $b_{A,B}^C$ is a nonnegative integer multiple of a power of t when $A = \{i\}$ consists of a single element, and provide a positive (counting) formula for the coefficients $b_{\{i\},B}^C$.

Their work raises the enticing question of whether the product structure is positive in the equivariant sense, i.e. whether the structure constants b_{AB}^C are polynomials with nonnegative coefficients for all A, B, C. Our main results are combinatorially positive formulas for these equivariant Peterson Schubert structure coefficients when $G = Gl(n, \mathbb{C})$. The explicit formulas are found in Theorems 1, 3, 5, and 6, which together provide manifestly positive formulas for the equivariant structure constants of $H_S^*(Y)$ in the basis $\{p_A : A \subset \{1, \ldots, n-1\}\}$ of Peterson Schubert classes. As a result, we obtain both the statement that structure constants are nonnegative, as well as simple criteria for when they are positive. The first author explores a geometric proof of positivity in all Lie types in separate work [10].

We call a subset $C_k \subset C \subset \{1, \ldots, n-1\}$ *maximal consecutive* if C_k is consecutive set such that

$$(\min C_k - 1) \notin C \text{ and } (\max C_k + 1) \notin C.$$

Corollary 7, **Theorem** 8 *The equivariant structure constants $b_{A,B}^C$ defined by* (1) *are nonnegative, integral multiples of powers of t. They have positive coefficients if and only if $A \cup B \subseteq C$ and each maximal consecutive subset C_k* of *C satisfies $|C_k| \leq |C_k \cap A| + |C_k \cap B|$.*

One consequence of these theorems is a manifestly positive formula for the structure constants in ordinary Peterson Schubert calculus (Corollary 2).

The proofs in this paper are combinatorial rather than geometric. A crucial step for the proof is an unexpected combinatorial identity (Theorem 9), a generalization of Vandermonde's identity, which we prove using a technique we term *bike lock moves*.

The structure of the paper is as follows. In Sect. 2 we state the main positivity theorems which together provide a full picture of the positivity of the structure constants. In Sect. 3 we define the basics of equivariant cohomology, Peterson varieties, and positivity. We prove the main positivity theorems in Sect. 4, and the crucial combinatorial theorem in Sect. 5.

2 Positivity Theorems

In this section, we describe the main results on the structure constants for the equivariant cohomology $H_S^*(Y)$ of the Peterson variety Y in $Fl(n; \mathbb{C})$ (both defined in Sect. 3), which show directly their positivity. To each subset $A \subseteq \{1, 2, \ldots, n-1\}$, we define an element $p_A \in H_S^*(Y)$ in Sect. 3.3 as the pullback of a specific Schubert class from G/B. We call p_A a *Peterson Schubert class*, The collection $\{p_A : A \subset \{1, \ldots, n-1\}\}$ a free module basis for the equivariant cohomology $H_S^*(Y)$ over $H_S^* := H_S^*(pt)$. Define the *structure constants* $b_{A,B}^C \in H_S^*$ by

$$p_A p_B = \sum_{C \subseteq \{1,2,\ldots,n-1\}} b_{A,B}^C \, p_C. \tag{2}$$

By construction, $p_\emptyset = 1$, and thus the coefficients $b_{A,B}^C$ are easy to calculate when A, B or C is empty: $b_{A,\emptyset}^A = b_{\emptyset,A}^A = 1$ for all $A \subseteq \{1, \ldots, n-1\}$, and all other coefficients vanish.

For A, B, C nonempty, Theorem 1 gives an explicit positive, integral formula for the coefficients $b_{A,B}^C$ when A and B are consecutive. Theorems 3, 5 and 6 describe the constants in the nonconsecutive cases. Nonvanishing conditions for the structure constants are described in Theorem 8. Proofs are relegated to Sect. 4.

We recall notation found in [14]. For $A \subseteq \{1, \dots, n-1\}$ with A nonempty and consecutive, let $\mathcal{T}_A = \min\{a \in A\}$ and $\mathcal{H}_A = \max\{a \in A\}$, called the *tail* and *head* of A, respectively.

Theorem 1. $(A, B, C$ consecutive$)$ *Let $A, B, C \subseteq \{1, \dots, n-1\}$ be nonempty consecutive subsets. If $C \supseteq A \cup B$ and $|C| \le |A| + |B|$, then*

$$b_{A,B}^{C} = d! \binom{\mathcal{H}_A - \mathcal{T}_B + 1}{d,\ \mathcal{T}_A - \mathcal{T}_C,\ \mathcal{H}_C - \mathcal{H}_B} \binom{\mathcal{H}_B - \mathcal{T}_A + 1}{d,\ \mathcal{T}_B - \mathcal{T}_C,\ \mathcal{H}_C - \mathcal{H}_A} t^d \tag{3}$$

for $d := |A| + |B| - |C|$.

Example 1 Let $A = \{1, 2\}$, $B = \{2, 3, 4\}$ and $C = \{1, 2, 3, 4\}$. Then C is consecutive, contains $A \cup B$ and $|C| = 4 \le |A| + |B| = 5$, so that $b_{A,B}^{C}$ is given by (3). Observe

$$\begin{array}{cccc} \mathcal{H}_A = 2 & \mathcal{T}_A = 1 & \mathcal{H}_B = 4 & \mathcal{T}_B = 2 \\ \mathcal{T}_C = 1 & \mathcal{H}_C = 4 & d = 1 & \end{array}$$

so that $b_{A,B}^{C} = 1!\binom{1}{1,\ 0,\ 0}\binom{4}{1,\ 1,\ 2}t^1 = \frac{4!}{2!}t = 12t$.

An immediate consequence of Theorem 1 is a formula for the ordinary cohomology structure constants. For degree reasons, the product $p_A p_B$ in ordinary cohomology requires simply summing over classes p_C such that $|C| = |A| + |B|$.

Corollary 2 *Let $A, B, C \subseteq \{1, \dots, n-1\}$ be nonempty consecutive subsets. Suppose $A \cup B \subseteq C$, and $|C| = |A| + |B|$. Without loss of generality, assume that $\mathcal{T}_A \le \mathcal{T}_B$. Then $b_{A,B}^{C}$ is the product of binomial coefficients:*

$$b_{A,B}^{C} = \binom{\mathcal{H}_A - \mathcal{T}_B + 1}{\mathcal{T}_A - \mathcal{T}_C} \binom{\mathcal{H}_B - \mathcal{T}_A + 1}{\mathcal{T}_B - \mathcal{T}_C}.$$

Proof By the degree assumption, $\mathcal{H}_C - \mathcal{T}_C + 1 = (\mathcal{H}_A - \mathcal{T}_A + 1) + (\mathcal{H}_B - \mathcal{T}_B + 1)$. Thus $\mathcal{H}_A - \mathcal{T}_B + 1 = (\mathcal{T}_A - \mathcal{T}_C) + (\mathcal{H}_C - \mathcal{H}_B)$ and

$$\mathcal{H}_B - \mathcal{T}_A + 1 = (\mathcal{T}_B - \mathcal{T}_C) + (\mathcal{H}_C - \mathcal{H}_A).$$

The corollary follows. □

We successively loosen the restrictive demand of Theorem 1 that A, B and C are each sets with consecutive numbers, as follows:

- Sets $A \cup B$ and C consecutive (Theorem 3),
- The set C is consecutive (Theorem 5), and
- No constraint on A, B, C (Theorem 6).

When A, B or C are not consecutive, there are non-equivariant analogs for ordinary cohomology. We won't list them, however, as each result is identical to the corresponding theorem with an additional hypothesis to ensure the degree is correct: any coefficient $b^G_{E,F}$ occurring in the formula are set to 0 unless $|E| + |F| = |G|$.

Theorem 3 ($A \cup B$, C consecutive) *Let $A, B, C \subseteq \{1, \dots, n-1\}$ be nonempty subsets with $A \cup B$ and C consecutive. Rename the maximal consecutive subsets of A and B by $E_1, \dots, E_v$ ordered with increasing tails i.e. $\mathcal{T}_{E_1} \le \mathcal{T}_{E_2} \le \cdots \le \mathcal{T}_{E_v}$. Then*

$$b^C_{A,B} = \sum_{(C_2,\dots,C_{v-1})} b^{C_2}_{E_1,E_2} b^{C_3}_{C_2,E_3} b^{C_4}_{C_3,E_4} \dots b^{C_{v-1}}_{C_{v-2},E_{v-1}} b^{C}_{C_{v-1},E_v} \tag{4}$$

where the sum is over $v-2$-tuples of consecutive sets C_i.

Note that, for each term in the sum of Theorem 3, the factors $b^{C_2}_{E_1,E_2}$ and $b^{C_{i+1}}_{C_i,E_{i+1}}$ are each calculated using Theorem 1 (as C_i, E_{i+1} and C_{i+1} are all consecutive).

Example 4 Let $A = \{1, 2, 4, 5\}$, $B = \{2, 3, 4\}$ and $C = \{1, 2, 3, 4, 5, 6\}$. We use Theorem 3 to compute $b^C_{A,B}$ noting that $A \cup B$ is consecutive.

By ordering according to the smallest element in each maximal consecutive set, choose $E_1 = \{1, 2\}$, $E_2 = B$, $E_3 = \{4, 5\}$ and note $v = 3$. Thus the sum (4) is

$$b^C_{A,B} = \sum_{\substack{(C_2) \\ C_2 \text{ consecutive}}} b^{C_2}_{E_1,E_2} b^{C}_{C_2,E_3}.$$

By Theorem 1, $b^{C_2}_{E_1,E_2} \neq 0$ implies C_2 contains $E_1 \cup E_2 = \{1, 2, 3, 4\}$ and $|C_2| \le |E_1| + |E_2| = 5$. Since C_2 is consecutive, the two possibilities are $C_2 = \{1, 2, 3, 4\}$ and $C_2 = \{1, 2, 3, 4, 5\}$. Thus by Theorem 3

$$b^C_{A,B} = b^{\{1,2,3,4\}}_{E_1,E_2} b^C_{\{1,2,3,4\},E_3} + b^{\{1,2,3,4,5\}}_{E_1,E_2} b^C_{\{1,2,3,4,5\},E_3}.$$

Each factor of each term can be computed using Theorem 1:

$$\begin{aligned}
b^{\{1,2,3,4\}}_{E_1,E_2} &= 1!\binom{1}{1,\ 0,\ 0}\binom{4}{1,\ 1,\ 2}t^1 = 12t \\
b^C_{\{1,2,3,4\},E_3} &= 0!\binom{1}{0,\ 0,\ 1}\binom{5}{0,\ 3,\ 2}t^0 = 10 \\
b^{\{1,2,3,4,5\}}_{E_1,E_2} &= 0!\binom{1}{0,\ 0,\ 1}\binom{4}{0,\ 1,\ 3}t^0 = 4 \\
b^C_{\{1,2,3,4,5\},E_3} &= 1!\binom{2}{1,\ 0,\ 1}\binom{5}{1,\ 3,\ 1}t^1 = 40t.
\end{aligned}$$

Therefore $b^C_{A,B} = 12t \cdot 10 + 4 \cdot 40t = 280t$.

The following theorem is a complete description of the product when C is consecutive.

Theorem 5 (C consecutive) *Let $A, B, C \subseteq \{1, \ldots, n-1\}$ be nonempty subsets with C consecutive. Let $A \cup B = D_1 \cup \cdots \cup D_u$ be a union of maximal consecutive subsets. Write $A^i = D_i \cap A$ and $B^i = D_i \cap B$, and note that $D_i = A^i \cup B^i$. Then*

$$b^C_{A,B} = \sum_{\substack{(E_1,\ldots,E_u):\ D_i \subseteq E_i, \\ E_i \text{ consecutive}}} \left(\prod_{i=1}^{u} b^{E_i}_{A^i,B^i} \right) b^C_{E_1,\ldots,E_u},$$

where $b^{E_i}_{A^i,B^i}$ is calculated using Theorem 3, and $b^C_{E_1,\ldots,E_u}$ is the coefficient of p_C in the product $\prod_{i=1}^{u} p_{E_i}$.

If $\cup_i E_i$ is consecutive, $b^C_{E_1,\ldots,E_u}$ may be calculated by Theorems 1 and 3. If $\cup_i E_i$ is not consecutive,

$$b^C_{E_1,\ldots,E_u} = \sum_{\substack{(F^{(1)},F^{(2)},\ldots,F^{(u-2)}) \\ \text{consecutive}}} b^{F^{(1)}}_{E^{(1)}_{j_1},E^{(1)}_{k_1}} b^{F^{(2)}}_{E^{(2)}_{j_2},E^{(2)}_{k_2}} \cdots b^{F^{(u-2)}}_{E^{(u-2)}_{j_{u-2}},E^{(u-2)}_{k_{u-2}}} b^C_{E^{(u-1)}_{j_{u-1}},E^{(u-1)}_{k_{u-1}}} \quad (5)$$

where $E^{(1)}_i = E_i$, and the sets $E^{(s)}_i$ for $s = 2, \ldots, u-1$ are defined inductively as follows. $E^{(s)}_{j_s}$ and $E^{(s)}_{k_s}$ are chosen so that their union is consecutive, the sum is over consecutive sets $F^{(s)}$ containing $E^{(s)}_{j_s} \cup E^{(s)}_{k_s}$, and the sets $E^{(s+1)}_i$ are a relabeling of the $u-s$ sets

$$F^{(s)}, E^{(s)}_1, \ldots, \widehat{E}^{(s)}_{j_s}, \widehat{E}^{(s)}_{k_s}, \ldots, E^{(s)}_{(u-s+1)}$$

in which the two sets $E^{(s)}_{j_s}$ and $E^{(s)}_{k_s}$ have been excluded. The sum is independent of choices involved with ordering. Each term $b^{F^{(s)}}_{E^{(s)}_{j_s},E^{(s)}_{k_s}}$ may be calculated using Theorem 1 as $E^{(s)}_i$ is consecutive.

Note that the sum in (5) is not independent of the order of $F^{(1)}, \ldots F^{(u-2)}$. The set of possible $F^{(s)}$ depend on the term $F^{(s-1)}$ in the prior sum, as well as the choice of sets $E^{(s)}_{j_s}$ and $E^{(s)}_{k_s}$ whose union is consecutive. Theorem 5 guarantees that these sets exist for each s when the coefficient is nonzero.

Finally, when C is not consecutive, $b^C_{A,B}$ is a product of coefficients with consecutive superscripts.

Theorem 6 *Let $A, B, C \subseteq \{1, \ldots, n-1\}$ be subsets such that $b^C_{A,B} \neq 0$. Then*

$$b^C_{A,B} = \prod_{k=1}^{m} b^{C_k}_{A \cap C_k, B \cap C_k}.$$

where $C = C_1 \cup \cdots \cup C_m$ is written as a union of maximal consecutive subsequences.

An immediate corollary to these theorems is that the structure constants for multiplication of $\{p_A\}$ in $H_S^*(Y)$, and hence in $H^*(Y)$ are nonnegative.

Corollary 7 *For any $A, B, C \subseteq \{1, \dots, n-1\}$, $b_{A,B}^C$ is a nonnegative, integral multiple of a power of t.*

Proof If A and B are consecutive, then this follows immediately from Theorem 1 as $b_{A,B}^C$ is 0, 1, or described by Eq. (3). If A or B is not consecutive, but $A \cup B$ is consecutive, then Theorem 3 implies that $b_{A,B}^C$ is a sum of products of the terms for consecutive A and B. Finally, Theorems 5 and 6 show that when $A \cup B$ is not consecutive, the terms associated with consecutive pieces are nonnegative and integral, and the terms associated with the product of those terms is also nonnegative and integral. $\square$

Finally, we state a nonvanishing result for arbitrary A, B, C.

Theorem 8 *Let $A, B, C \subseteq \{1, \dots, n-1\}$ be arbitrary subsets. The structure constant $b_{A,B}^C \neq 0$ if and only if*

- *$A \cup B \subseteq C$, and*
- *For each maximal consecutive subset C_k of C, $|C_k| \leq |C_k \cap A| + |C_k \cap B|$.*

Theorem 8 and Corollary 7 imply these structure constants are positive (i.e. are monomials with positive coefficients) when they are non-vanishing.

The proof of Theorem 1 relies heavily on the following combinatorial result, a generalization of Vandermonde's formula.

Theorem 9 *Let $m, n, w, x, y, z \in \mathbb{Z}$ with $w + x = y + z$ and $m, n \geq 0$. Then*

$$\begin{aligned} &\binom{w+m}{w}\binom{y+m}{x}\binom{w+n}{y}\binom{z+n}{z} \\ &= \sum_{\substack{0 \leq i \leq m \\ 0 \leq j \leq n}} \binom{w+i+n}{w+i+j}\binom{w+m+j}{i,\ j,\ m-i,\ x-i-j,\ z-x+j,\ y-x+i}. \end{aligned} \tag{6}$$

We have thusfar not found this result in the literature, and it may stand alone as a worthwhile combinatorial identity, proved in Sect. 5.

3 Background and Notation

3.1 Flag Varieties, Peterson Varieties, and Fixed Points

Let $G = Gl(n; \mathbb{C})$, B upper triangular invertible matrices, B_- lower triangular invertible matrices, and T the set of diagonal matrices in G. Recall G/B is naturally isomorphic to the set of complete flags

$$Fl(n; \mathbb{C}) = \{V_\bullet := (V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq \mathbb{C}^n) |\ V_i \text{ is a subspace of } \mathbb{C}^n, \dim_{\mathbb{C}}(V_i) = i\}.$$

The flag $V_\bullet$ corresponds to a a coset gB, where $g \in Gl(n, \mathbb{C})$ is any matrix whose first k columns form a basis for V_k, for $k = 1, \ldots, n$. Note that right multiplication by an upper triangular matrix (in B) preserves the vector space spanned by the first k columns, for all k. The fixed points $(G/B)^T$ are isolated, and indexed by elements of the Weyl group, $W \cong S_n$. In particular,

$$(G/B)^T = \{wB/B : w \in W\}.$$

Following Tymoczko [21], we describe Hessenberg varieties in $Fl(n; \mathbb{C})$ as a set of flags whose vector spaces satisfy linear conditions imposed by a principal nilpotent operator. The equivalence of this description with the original definition by Kostant is known to experts and proven in [10].

Definition 10 Let $h : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a function satisfying $i \leq h(i)$ for all $i \in \{1, \ldots, n\}$ and let M be any $n \times n$ matrix M. The *Hessenberg variety* $H(h, M)$ corresponding to h and M is the collection of flags $V_\bullet \in Fl(n; \mathbb{C})$ satisfying $MV_i \subseteq V_{h(i)}$ for all $1 \leq i \leq n$.

The Peterson variety Y is a specific Hessenberg variety, with h given by:

$$h(i) = \begin{cases} i+1 & 1 \leq i \leq n-1 \\ n & i = n. \end{cases} \tag{7}$$

Definition 11 The *Peterson variety* in $Fl(n; \mathbb{C})$ is the Hessenberg variety $Y = H(h, M)$ where h is the function defined in Eq. (7) and M is a principal nilpotent operator. Equivalently the Jordan canonical form for M consists of one block and M has eigenvalue 0.

Example 12 Let $n = 3$, $h(1) = 2$, $h(2) = 3$, $h(3) = 3$ and $M = \begin{pmatrix} 0&1&0\\0&0&1\\0&0&0 \end{pmatrix}$. The Peterson variety in $Fl(\mathbb{C}^3)$ consists of flags represented by matrices of the following forms:

$$\begin{pmatrix} a&b&1\\b&1&0\\1&0&0 \end{pmatrix}, \quad \begin{pmatrix} c&1&0\\1&0&0\\0&0&1 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0\\0&d&1\\0&1&0 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix} \tag{8}$$

where $a, b, c, d \in \mathbb{C}$. We verify the condition that $MV_i \subseteq V_{h(i)}$ for the first matrix above. We check that $MV_1 \subseteq V_2$ (clearly $MV_2 \subseteq V_3 = \mathbb{C}^3$):

$$\begin{pmatrix} 0&1&0\\0&0&1\\0&0&0 \end{pmatrix} \begin{pmatrix} a\\b\\1 \end{pmatrix} = \begin{pmatrix} b\\1\\0 \end{pmatrix} \in \operatorname{span}\left\{ \begin{pmatrix} a\\b\\1 \end{pmatrix}, \begin{pmatrix} b\\1\\0 \end{pmatrix} \right\} = V_2.$$

As T consists of diagonal, unitary matrices, we write elements as n-tuples $(a_1, \dots a_n)$ listing the diagonal entires. The variety Y is not T-stable, however it is stable under a subgroup isomorphic to S^1. Let

$$S = \{(z^n, z^{n-1}, \dots, z^2, z) : \; z \in \mathbb{C}^*, ||z||^2 = 1\} \subseteq T.$$

We observe that S preserves Y, as follows. Let $e_i \in \mathbb{C}^n$ be the vector with 1 in the ith coordinate, and 0 elsewhere. For any vector $v \in \mathbb{C}^n$ given by $v = \sum_{i=1}^n a_i e_i$, we have

$$Mv = \sum_{i=1}^{n-1} a_{i+1} e_i.$$

On the other hand, for each element s of S given by a diagonal matrix with entries $(z^n, z^{n-1}, \dots, z)$, we have $s \cdot v = \sum_{i=1}^n z^{n-i+1} a_i e_i$. A quick calculation shows that $s \cdot Mv$ and $M(s \cdot v)$ span the same line:

$$s \cdot Mv = \sum_{i=1}^{n-1} z^{n-i+1} a_{i+1} e_i = z \sum_{i=1}^{n-1} z^{n-i} a_{i+1} e_i = zM(s \cdot v).$$

It follows that $M(s \cdot V_k)$ is in the span of $s \cdot MV_k$. If $V_\bullet \in Y$, then $MV_k \subseteq V_{k+1}$ implies $M(s \cdot V_k) \subseteq s \cdot MV_k \subseteq s \cdot V_{k+1}$, and hence $s \cdot V_\bullet \in Y$.

As S is a regular one-parameter subgroup of T, the S-fixed points of G/B are the same as the T-fixed points. It follows that the fixed point set Y^S may be described as the intersection $Y^S = Y \cap (G/B)^T$.

Explicitly, Y^S consists of flags represented by block diagonal matrices where the diagonal blocks are anti-diagonal with 1's on the anti-diagonal:

$$\begin{pmatrix} 0 \cdots 1 & & \\ \vdots \;\; \iddots \;\; \vdots & & \\ 1 \cdots 0 & & \\ & \ddots & \\ & & 0 \cdots 1 \\ & & \vdots \;\; \iddots \;\; \vdots \\ & & 1 \cdots 0 \end{pmatrix}.$$

For example if $n = 2$ then Y^S consists of flags represented by matrices (8) in the previous example with $a = b = c = d = 0$.

Each simple root α_i corresponds to a simple reflection $s_i := s_{\alpha_i}$ that interchanges i and $i+1$. Recall an element $w \in S_n$ can be written as a product of simple reflections $s_1, \dots, s_{n-1}$, corresponding to the simple roots $\alpha_1, \dots, \alpha_{n-1}$, respectively. When $w = s_{i_1} s_{i_2} \cdots s_{i_{\ell(w)}}$ is written as a product with as few simple reflections as possible, $\ell(w)$ is called the *length* of w. The expression $s_{i_1} s_{i_2} \cdots s_{i_{\ell(w)}}$ is called a *reduced word* *decomposition* for w. To distinguish the product (resulting in w) from a sequence of $\ell(w)$ simple reflections in a reduced word decomposition, we refer to the index sequence $(i_1, i_2, \ldots, i_{\ell(w)})$ as a *reduced word sequence* for w. Recall the Bruhat order for $u, v \in S_n$: we say $u \leq v$ if there exists a substring of a reduced word for v whose corresponding product of reflections is u. There exists a unique element w_0 in S_n with maximal length, and it satisfies $w \leq w_0$ for all $w \in S_n$.

Elements of Y^S are represented by a specific set of permutations:

$$Y^S = \{w_A \in S_n : A \subseteq \{1, \ldots, n-1\}\}, \tag{9}$$

where the permutation w_A associated to a subset A is given as follows. Let $A = A_1 \cup A_2 \cup \cdots \cup A_k$ where each A_i is a maximal consecutive subset of A. For each i, denote by w_{A_i} the long word of the subgroup H_i of S_n generated by reflections s_j for $j \in A_i$, noting that $H_i \cong S_{|A_i|+1}$ is itself a permutation group. Then

$$w_A = w_{A_1} w_{A_2} \cdots w_{A_k}$$

is the long word of the subgroup $H_1 \times H_2 \times \cdots \times H_k \subseteq S_n$. A matrix representing a $w_A B \in Y^S$ has anti-diagonal blocks of size $|A_i| + 1$.

3.2 The Equivariant Cohomology Ring of G/B and Schubert Classes

Define B-invariant Schubert varieties $X^w := \overline{BwB}/B$ in G/B, and let $[X^w]$ denote the corresponding T-equivariant homology class, following [5]. We use Poincaré duality between equivariant homology and equivariant cohomology to define a dual basis $\{\sigma_w : w \in W\}$ of $H_T^*(G/B)$ to the equivariant homology basis $\{[X^w] : w \in W\}$. These bases satisfy the property that $\langle \sigma_w, [X^v] \rangle = \delta_{wv}$, where $\langle \ , \ \rangle$ denotes the equivariant cap product, followed by the pushforward to a point.

Alternatively, σ_w is Poincaré dual to the equivariant homology class of the opposite Schubert variety $X_w := \overline{B_- wB}/B$, which has finite codimension in the mixing space for G/B.

The inclusion $(G/B)^T \hookrightarrow G/B$ induces a map on cohomology

$$H_T^*(G/B) \to H_T^*((G/B)^T) = \bigoplus_{w \in W} H_T^*(wB/B) = \bigoplus_{w \in W} \mathbb{C}[x_1, \ldots, x_n] \tag{10}$$

that is known to be injective [7,11].

Suppose $W = (i_1, \ldots, i_\ell)$ is a reduced word sequence for $w \in W$. If $U = (i_{j_1}, \ldots, i_{j_d})$ with $\{j_1, \ldots, j_d\} \subset \{1, \ldots, \ell\}$ and $j_1 < \cdots < j_d$, we write $U \subseteq W$. It is possible that $U \subseteq W$ in multiple ways, if W has repeated indices. If U is also is a reduced word sequence for $u = s_{i_{j_1}} \cdots s_{i_{j_d}}$, then clearly $u \leq w$; we say that U is a *reduced word for u occurring as* a subword of W.

The image of Schubert class σ_u under the map in Eq. (10) may be computed using the AJS-Billey formula [3,4]:

Theorem 13 ([3,4], AJS-Billey Restriction Formula) *Given a fixed reduced word sequence $V = (i_1, i_2, \dots i_{\ell(v)})$ for v, define*

$$r(k, V) := s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k}).$$

For $U = (i_{j_1}, i_{j_2}, \cdots, i_{j_{\ell(u)}}) \subseteq V$, we write

$$\prod_{k \in U} r(k, V) := r(j_1, V) r(j_2, V) \cdots r(j_{\ell(u)}, V).$$

Then for any $u, v \in S_n$,

$$\sigma_u|_v = \sum_{U \subseteq V} \prod_{k \in U} r(k, V),$$

where the sum is over reduced words U occurring as subwords of V.

An immediate corollary is that $\sigma_u|_v = 0$ unless $u \leq v$.

3.3 The Equivariant Cohomology of the Peterson *Y* and Peterson Schubert Classes

The inclusion $S \hookrightarrow T$ given by $z \mapsto (z^n, z^{n-1}, z^{n-2}, \dots, z)$ for z a complex number with $|z| = 1$, induces a map on Lie algebras, $\mathfrak{s} \to \mathfrak{t}$ given by

$$1 \mapsto (n, n-1, n-2, \dots, 2, 1).$$

Using the dual coordinate basis $\{x_j\}$ of $\mathfrak{t}^*$ introduced above, the dual map $\mathfrak{t}^* \to \mathfrak{s}^*$ induced by the inclusion is given by $x_j \mapsto (n - j + 1)t$ for $j = 1, \dots, n$, where $t \in \mathfrak{s}^*$ is the dual coordinate to $1 \in \mathfrak{s}$. The inclusion $S \hookrightarrow T$

thus induces a map $H_T^* \to H_S^*$ in which

$$\alpha_i \mapsto t$$

for $i = 1, 2, \dots, n-1$. This observation justifies the decision to call $b \in H_S^*$ *positive* if it is a polynomial in t with positive coefficients.

The map on equivariant cohomology in turn induces a map of modules for any T-space X, which we also denote by π:

$$H_T^*(X) \xrightarrow{\pi} H_S^*(X).$$

When $X = G/B$, this is a surjective map of free modules. The S-equivariant inclusion $\iota : Y \hookrightarrow G/B$ of the Peterson variety induces a surjective map:

$$H_S^*(G/B) \xrightarrow{\iota^*} H_S^*(Y),$$

and these maps naturally commute with the restrictions to fixed points. We thus obtain a commutative diagram:

$$\begin{array}{ccccc} H_T^*(G/B) & \xrightarrow{\pi} & H_S^*(G/B) & \xrightarrow{\iota^*} & H_S^*(Y) \\ \downarrow & & \downarrow & & \downarrow \\ H_T^*((G/B)^T) & \longrightarrow & H_S^*((G/B)^S) & \xrightarrow{\iota^*_{fps}} & H_S^*(Y^S) \\ \| & & \| & & \| \\ \bigoplus_{w\in W} H_T^* & \xrightarrow{\bigoplus_{w\in W}\pi} & \bigoplus_{w\in W} H_S^* & \longrightarrow & \bigoplus_{w_A\in S_n} H_S^* \end{array}$$

where ι^* is the map induced by the inclusion $Y \hookrightarrow G/B$, and ι^*_{fps} is the induced map from the inclusion of fixed point sets on Y to those on G/B. The kernel of ι^*_{fps} consists of all copies of $H_S^*(wB/B)$ with wB/B not in Y, i.e. $w \neq w_A$ for any $A \subseteq \{1, \ldots, n-1\}$.

All vertical maps of the commutative diagram are obtained from the inclusion of fixed point sets. As discussed, the first two vertical maps are injective. In [14], the authors prove that the third vertical map is injective, and that $H_S^*(Y)$ is a free module over the equivariant cohomology of a point.

Theorem 14 ([14], Thoerem 3.2) *Let S act on the Peterson variety Y as described above. Then $H_S^*(Y)$ is a free module over H_S^*, and in particular,*

$$H_S^*(Y) \simeq H^*(Y) \otimes_{\mathbb{C}} H_S^*.$$

In addition, the inclusion $Y^S \hookrightarrow Y$ induces an injection

$$H_S^*(Y) \longrightarrow H_S^*(Y^S).$$

The authors also discovered a basis of $H_S^*(Y)$ by mapping a subset of Schubert classes across the vertical arrows of the commuting diagram.

For any subset $A \subseteq \{1, \ldots, n-1\}$, define the *Peterson Schubert class* corresponding to $A \subseteq \{1, \ldots, n-1\}$ by

$$p_A := \iota^* \circ \pi(\sigma_{v_A}) \in H_S^*(Y),$$

where $v_A = s_{a_1} s_{a_2} \cdots s_{a_k}$ with $a_i \in A$ and $a_i < a_j$ whenever $i < j$, and $\sigma_{v_A} \in H_T^*(G/B)$ is the corresponding Schubert class. The degree of p_A is $2\ell(v_A) = 2|A|$.

Theorem 15 ([14], Theorem 4.12) *The collection $\{p_A\}_{A\subseteq\{1,\ldots,n-1\}}$ form an H_S^*-module basis for $H_S^*(Y)$. We call this basis the* Peterson Schubert basis *of $H_S^*(Y)$.*

3.4 Peterson Schubert Classes: Basic Properties

Here we collect together a number of properties of Peterson Schubert classes, their products, and their restrictions.

For $A \subseteq \{1, \ldots, n-1\}$ with $j \in A$, $\mathcal{T}_A(j)$ is the smallest integer in the maximal consecutive subset of A containing j, and similarly, $\mathcal{H}_A(j)$ is the largest integer of the same set. Write $A = A_1 \cup \cdots \cup A_k$ as a union of maximally consecutive sets. Consider the reduced word sequence for the longest word w_{A_i} given by

$$W_{A_i} = (\mathcal{T}_A(j), \mathcal{T}_A(j)+1, \ldots, \mathcal{H}_A(j), \mathcal{T}_A(j), \mathcal{T}_A(j)+1, \ldots \mathcal{H}_A(j)-1, \\ \ldots, \mathcal{T}_A(j), \mathcal{T}_A(j)+1, \mathcal{T}_A(j)). \quad (11)$$

Observe that W_{A_i} is independent of $j \in A_i$ since A_i is consecutive. One reduced word sequence W_A for w_A is given by the concatenation of sequences W_{A_i} for $i = 1, \ldots, k$, i.e. $W_A = W_{A_1} W_{A_2} \cdots W_{A_k}$.

The following restriction formula is a tiny generalization of a formula proved in [14, Proposition 5.9].

Lemma 16 *Let $\sigma_u \in H_T^*(G/B)$ be a Schubert class and let w_A be the S-fixed point of the Peterson variety Y associated to $A \subseteq \{1, \ldots, n-1\}$. Let $A = A_1 \cup \cdots \cup A_k$ be written as a union of maximally consecutive sets, and let W_A be the reduced word sequence for w_A given by the concatenation $W_{A_1} W_{A_2} \cdots W_{A_k}$ of sequences W_{A_i} given in Eq. (11) for $i = 1, \ldots, k$. Then*

$$\iota^* \circ \pi(\sigma_u)|_{w_A} = \sum_U n_{W_A}(U) \left(\prod_{j \in U} (j - \mathcal{T}_A(j) + 1) \right) t^{\ell(u)} \quad (12)$$

where the sum is over distinct reduced words U of u, $n_{W_A}(U)$ is the number times the word U occurs as a subword of W_A.

Since the Peterson Schubert class $p_A = \iota^* \circ \pi(\sigma_{v_A})$, Lemma 16 implies the following Corollary.

Corollary 17 ([14], Theorem 4.12) $p_A|_{w_C} = 0$ *unless* $A \subseteq C$.

Observe that in the poset of subsets ordered by inclusion, $C = A$ is the minimal subset for which $p_A|_{w_C}$ may not vanish. See Corollary 21. As a consequence, the structure constants also satisfy support conditions:

Lemma 18 *Let $A, B, C \subseteq \{1, 2, \ldots, n-1\}$. Then $b_{A,B}^C \neq 0$ implies $A \cup B \subseteq C$ and $|C| \leq |A| + |B|$.*

Proof Assume $A \cup B \not\subseteq C$, then either $A \not\subseteq C$ or $B \not\subseteq C$, so the product $p_A p_B|_{w_C}$ vanishes by Corollary 17. Similarly, $p_D|_{w_C} = 0$ unless $D \subseteq C$. Thus

$$p_A p_B|_{w_C} = \sum_{D \subseteq C} b_{A,B}^D p_D|_{w_C} = 0. \quad (13)$$

Note that $D \subseteq C$ implies $A \cup B \nsubseteq C$, else $A \cup B \subseteq C$. If $|C| = 0$, the sum is over a single term $C = \emptyset$, so $b^C_{A,B} p_C|_{w_C} = 0$. However $p_C|_{w_C} \neq 0$ by Lemma 16, so $b^C_{A,B} = 0$. Make the inductive assumption that $A \cup B \nsubseteq C$ implies $b^C_{A,B} = 0$ for $|C| \leq k$. Then for $|C| = k+1$, Eq. (13) may be written

$$p_A p_B|_{w_C} = \sum_{D \subsetneq C} b^D_{A,B} p_D|_{w_C} + b^C_{A,B} p_C|_{w_C} = 0.$$

If D is a proper subset of C, if $|D| \leq k$, and by the inductive assumption, $b^D_{A,B} = 0$. Thus as before, we conclude $b^C_{A,B} p_C|_{w_C} = 0$ and, since $p_C|_{w_C} \neq 0$ that $b^C_{A,B} = 0$. Since $\deg(p_A p_B) = |A|+|B|$ (as a polynomial), each summand $b^C_{A,B} p_C$ in the product $p_A p_B$ has degree $|A| + |B|$, and therefore $b^C_{A,B} \neq 0$ implies that $|C| = \deg(p_C) \leq |A| + |B|$. □

Lemma 16 also implies that the restrictions of Peterson Schubert classes remain constant when nonconsecutive elements are added to a fixed point.

Corollary 19 *Let $A \subseteq C^0$ with C^0 consecutive, and let $C \supset C^0$ be any set so that $C \setminus C^0$ is not consecutive with C^0. Then*

$$p_A|_{w_{C^0}} = p_A|_{w_C}.$$

Proof Let $A = \{a_1, \ldots, a_k\}$ with $a_i < a_j$ for $i < j$. There is only one reduced word decomposition $v_A = s_{a_1} s_{a_1} \ldots s_{a_k}$ and thus one reduced word sequence $V_A = (a_1, \ldots, a_k)$. Neither $\mathcal{T}_{C^0} - 1$ nor $\mathcal{H}_{C^0} + 1$ are in C, so we may choose $W_C = W_{C^0} W_{C \setminus C^0}$ for some choice $W_{C \setminus C^0}$. Lemma 16 therefore implies

$$p_A|_{w_{C^0}} = n_{W_{C^0}}(V_A) \left(\prod_{j \in V_A} (j - \mathcal{T}_{C^0}(j) + 1) \right) t^{|A|}, \text{ and}$$

$$p_A|_{w_C} = n_{W_C}(V_A) \left(\prod_{j \in V_A} (j - \mathcal{T}_C(j) + 1) \right) t^{|A|}.$$

As $A \subseteq C^0$ and $W_C = W_{C^0} W_{C \setminus C^0}$, $n_{W_C}(V_A) = n_{W_{C^0}}(V_A)$. Note that the product over the entries j of V_A consists of a single factor for each $j \in A$. Furthermore, $j \in A$ implies $\mathcal{T}_C(j) = \mathcal{T}_{C^0}(j)$ since C does not contain $\mathcal{T}_{C^0} - 1$. Thus the products have identical factors. □

Lemma 20 *Suppose $A \cup B \subseteq C^0$ (not necessarily consecutive) and $C \supset C^0$ is any set so that $C \setminus C^0$ is nonempty and not consecutive with C^0. Then $b^C_{A,B} = 0$.*

Proof By Corollary 19, $p_A p_B|_{w_C} = p_A p_B|_{w_{C^0}}$. Since the restrictions are the same,

$$\sum_{D \subseteq C} b^D_{A,B} p_D|_{w_C} = \sum_{D \subseteq C^0} b^D_{A,B} p_D|_{w_{C^0}}$$

and in particular also by Corollary 19,

$$\sum_{D:D\subsetneq C^0, D\subseteq C} b^D_{A,B}\, p_D|_{w_C} = 0. \tag{14}$$

We proceed inductively on $|C'|$. If $C' = \{m\}$ consists of one element, the sum is over one set $D = C$, so $b^C_{A,B}\, p_C|_{w_C} = 0$. Since $p_c|_{w_C} \neq 0$, we conclude $b^C_{A,B} = 0$. More generally, the sum (14) is

$$\sum_{D:C^0\subsetneq D\subsetneq C} b^D_{A,B}\, p_D|_{w_C} + b^C_{A,B}\, p_C|_{w_C} = 0$$

where the first sum is 0 by the inductive assumption. Thus $b^C_{A,B} = 0$. □

Lemma 16 also implies an easy formula for the restriction of any Peterson Schubert class p_A to its minimal fixed point w_A.

Corollary 21 *Let A be consecutive. Then*

$$p_A|_{w_A} = |A|!\, t^{|A|}.$$

Proof We calculate directly using the Peterson Schubert restriction formula.

$$p_A|_{w_A} = \iota^*\pi(\sigma_{v_A}|_{w_A}) = n_{W_A}(V_A)\left(\prod_{j\in A}(j - \mathcal{T}_A + 1)\right) t^{|A|}.$$

Then V_A occurs in W_A exactly one time, so the restriction is

$$\prod_{j\in A}(j - \mathcal{T}_A + 1)t = |A|!\, t^{|A|}.$$

□

A fundamental observation is that $p_{A\cup B} = p_A p_B$ when A and B are disjoint strings of consecutive integers separated by at least one number.

Lemma 22 ([14], Lemma 6.7) *Let $A \subseteq \{1, \ldots, n-1\}$ and suppose*

$$A = A_1 \cup A_2 \cup \ldots \cup A_k$$

where each A_i is a nonempty maximal consecutive string of integers and $A_i \neq A_j$ for $i \neq j$. Then

$$p_A = \prod_{1\leq i\leq k} p_{A_i}.$$

4 Proof of Main Theorems and Lemmas

Here we prove Theorems 1, 3, 5, 6, and 8. There are two substantial cases required to prove Theorem 1, recalling that A and B are consecutive by hypothesis. In the first case, either $A \cap B$ is nontrival but neither set contains the other, or the two sets are consecutive to each other. In the second case, one set is contained in the other.

Definition 23 Let A and B be consecutive sequences of $\{1, 2, \ldots, n-1\}$. We say that A and B are *intertwined* if $\mathcal{T}_A \leq \mathcal{T}_B \leq \mathcal{H}_A \leq \mathcal{H}_B$ or $\mathcal{T}_B \leq \mathcal{T}_A \leq \mathcal{H}_B \leq \mathcal{H}_A$.

Lemma 24 *Suppose A, B and $A \cup B$ are consecutive. Then*

$$p_A|_{w_{A\cup B}} = \binom{\mathcal{H}_{A\cup B} - \mathcal{T}_A + 1}{|A|} \frac{(\mathcal{H}_A - \mathcal{T}_{A\cup B} + 1)!}{(\mathcal{T}_A - \mathcal{T}_{A\cup B})!} t^{|A|}.$$

In particular, if A and B are intertwined or if A and B are consecutive to each other and nonintersecting,

$$p_A|_{w_{A\cup B}} = \frac{|A \cup B|!}{|B \setminus A|!} t^{|A|}.$$

Proof According to Lemma 16,

$$p_A|_{w_{A\cup B}} = n_{W_{A\cup B}}(V_A) \left(\prod_{j \in V_A} (j - \mathcal{T}_{A\cup B}(j) + 1) \right) t^{|A|}. \tag{15}$$

We claim that $n_{W_{A\cup B}}(V_A) = \binom{\mathcal{H}_{A\cup B} - \mathcal{T}_A + 1}{|A|}$, where $V_A = (\mathcal{T}_A, \mathcal{T}_A + 1, \cdots, \mathcal{H}_A)$. Choose the reduced decomposition $W_{A\cup B}$ of $w_{A\cup B}$ given by the sequence (read from left to right and top to bottom) in Fig. 1 (left panel), ignoring the grid and path within.

Each increasing consecutive string of $W_{A\cup B}$ is written on its own line, all left-aligned. All rows finishing in numbers $\mathcal{H}_A$ or larger contain the string $\mathcal{T}_A \mathcal{T}_A{+}1 \cdots \mathcal{H}_A$. To count the number of occurrences of V_A in this product, we first draw a grid around all of these strings except for the one appearing in the first row. The grid has $\mathcal{H}_{A\cup B} - \mathcal{H}_A$ rows and $\mathcal{H}_A - \mathcal{T}_A + 1$ columns.

For example, suppose $A = \{2, 3\}$ and $A \cup B = \{1, \ldots, 6\}$. Let $w_{A\cup B}$ be the longest word for the permutations group generated by $\{s_i : i \in A \cup B\}$. Then

$$W_{A\cup B} = (1, 2, 3, 4, 5, 6, 1, 2, 3, 4, 5, 1, 2, 3, 4, 1, 2, 3, 1, 2, 1),$$

and $V_A = (2, 3)$. We have the grid containing the 2 and 3 in the second, third and fourth rows of $W_{A\cup B}$, pictured in Fig. 1, (right panel).

There is a one-to-one correspondence between paths from the top left corner to the bottom right corner of this grid (moving only right and down) and occurrences of V_A inside of W_A. Each instance of V_A inside of $W_{A\cup B}$ is "underlined" by the horizontal components of a path, as indicated with the red path in Fig. 1 (left panel). For example,

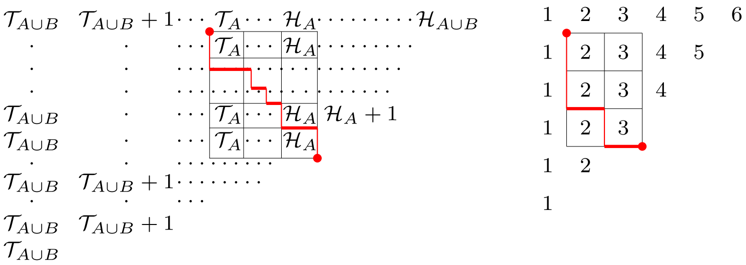


Fig. 1 Finding reduced words V_A occurring in W_A (left panel) and an example (right panel)

V_A is given by the subset of $W_{A\cup B}$ underlined by the path in Fig. 1 (right panel), it selects the subset indicated by boxed elements:

$$(1, 2, 3, 4, 5, 6, 1, 2, 3, 4, 5, 1, \boxed{2}, 3, 4, 1, 2, \boxed{3}, 1, 2, 1).$$

The dimensions of the grid are $(\mathcal{H}_{A\cup B} - \mathcal{H}_A) \times (\mathcal{H}_A - \mathcal{T}_A + 1)$ and hence the number of reduced words for v_A inside of $W_{A\cup B}$ is the count of such paths, known to be the number of “right" (or “down") moves among the total moves given by the sum of the row and column lengths. Therefore,

$$n_{W_{A\cup B}}(V_A) = \binom{\mathcal{H}_{A\cup B} - \mathcal{T}_A + 1}{\mathcal{H}_A - \mathcal{T}_A + 1} = \binom{\mathcal{H}_{A\cup B} - \mathcal{T}_A + 1}{|A|}.$$

We turn our attention to the factor $\left(\prod_{j\in V_A}(j - \mathcal{T}_{A\cup B}(j) + 1)\right) t^{|A|}$ in Eq. (15). Since $A \cup B$ is consecutive and the product is over $|A|$ elements with the highest j occurring at $j = \mathcal{H}_A$, but only descending $|A|$ terms:

$$\begin{aligned}\left(\prod_{j\in V_A}(j - \mathcal{T}_{A\cup B} + 1)\right) t^{|A|} &= \frac{(\mathcal{H}_A - \mathcal{T}_{A\cup B} + 1)!}{(\mathcal{H}_A - \mathcal{T}_{A\cup B} + 1 - |A|)!} t^{|A|} \\ &= \frac{(\mathcal{H}_A - \mathcal{T}_{A\cup B} + 1)!}{(\mathcal{T}_A - \mathcal{T}_{A\cup B})!} t^{|A|}.\end{aligned}$$

We put the two terms together to get the formula.

If A and B are intertwined, then if $\mathcal{T}_A = \mathcal{T}_{A\cup B}$ and $\mathcal{H}_B = \mathcal{H}_{A\cup B}$,

$$\binom{\mathcal{H}_{A\cup B} - \mathcal{T}_A + 1}{|A|} = \binom{|A\cup B|}{|A|}, \qquad \frac{(\mathcal{H}_A - \mathcal{T}_{A\cup B} + 1)!}{(\mathcal{T}_A - \mathcal{T}_{A\cup B})!} = |A|!$$

so the product is $\frac{|A\cup B|!}{(|A\cup B|-|A|)!} = \frac{|A\cup B|!}{|B\setminus A|!}$. If $\mathcal{T}_B = \mathcal{T}_{A\cup B}$ and $\mathcal{H}_A = \mathcal{H}_{A\cup B}$,

$$\binom{\mathcal{H}_{A\cup B} - \mathcal{T}_A + 1}{|A|} = \binom{|A|}{|A|} = 1, \qquad \frac{(\mathcal{H}_A - \mathcal{T}_{A\cup B} + 1)!}{(\mathcal{T}_A - \mathcal{T}_{A\cup B})!} = \frac{|A\cup B|!}{|B\setminus A|!},$$

resulting in the same product. □

The following Lemma serves as the base case for an inductive argument in the proof of Theorem 1.

Lemma 25 *Suppose A, B are consecutive. When A and B are intertwined, or when A and B are consecutive to each other and nonintersecting,*

$$b_{A,B}^{A\cup B} = \frac{|A\cup B|!}{|B\setminus A|!|A\setminus B|!} t^{|A\cap B|}.$$

Proof Restrict the product

$$(p_A p_B)|_{w_{A\cup B}} = \sum_{C:A\cup B\subseteq C} b_{A,B}^C p_C|_{w_{A\cup B}} = b_{A,B}^{A\cup B} p_{A\cup B}|_{w_{A\cup B}},$$

since $p_C|_{w_{A\cup B}} = 0$ unless $C \subseteq A\cup B$. By Lemma 24,

$$p_A|_{w_{A\cup B}} p_B|_{w_{A\cup B}} = \frac{|A\cup B|!}{|B\setminus A|!} \frac{|A\cup B|!}{|A\setminus B|!} t^{|A|+|B|}.$$

By Corollary 21, $p_{A\cup B}|_{w_{A\cup B}} = |A\cup B|! t^{|A\cup B|}$. We then solve:

$$b_{A,B}^{A\cup B} = \frac{1}{|A\cup B|! t^{|A\cup B|}} \frac{|A\cup B|!}{|B\setminus A|!} \frac{|A\cup B|!}{|A\setminus B|!} t^{|A|+|B|} = \frac{|A\cup B|!}{|B\setminus A|!|A\setminus B|!} t^{|A\cap B|}.$$

□

When $B\subseteq A$, the structure constant $b_{A,B}^C$ can be recast in terms of another structure constant with intertwined sets.

Lemma 26 *Suppose A, B are consecutive and C any set with $B\subseteq A\subseteq C$. Then*

$$|A|!\,|B|!\,b_{A,B}^C = |A'|!\,|B'|!\,b_{A',B'}^C,$$

where $A' = \{a\in A : a\leq \mathcal{H}_B\}$ and $B' = \{b\in A : b\geq \mathcal{T}_B\}$.

Proof We show that

$$|A|!\,|B|!\,p_A p_B = |A'|!\,|B'|!\,\ p_{A'} p_{B'}, \tag{16}$$

which implies that the coefficients have the desired relationship since $\{p_C\}$ forms a basis of $H_S^*(Y)$.

By Lemma 24 if C is consecutive, and Lemma 19 otherwise,

$$\begin{aligned} p_A|_{w_C} &= \binom{\mathcal{H}_C - \mathcal{T}_A + 1}{|A|} \frac{(\mathcal{H}_A - \mathcal{T}_C + 1)!}{(\mathcal{T}_A - \mathcal{T}_C)!} t^{|A|} \\ &= \frac{(\mathcal{H}_C - \mathcal{T}_A + 1)!}{|A|!\,(H_C - H_A)!} \frac{(\mathcal{H}_A - \mathcal{T}_C + 1)!}{(\mathcal{T}_A - \mathcal{T}_C)!} t^{|A|} \end{aligned}$$

with a similar formula for $p_B|_{w_C}$. Using the relationships

$$\mathcal{T}_{A'} = \mathcal{T}_A, \quad \mathcal{H}_{A'} = \mathcal{H}_B, \quad \mathcal{T}_{B'} = \mathcal{T}_B, \quad \mathcal{H}_{B'} = \mathcal{H}_A \tag{17}$$

and simplifying as above,

$$p_{A'}|_{w_C} = \frac{(\mathcal{H}_C - \mathcal{T}_A + 1)!}{|A'|!\,(\mathcal{H}_C - \mathcal{H}_B)!} \frac{(\mathcal{H}_B - \mathcal{T}_C + 1)!}{(\mathcal{T}_A - \mathcal{T}_C)!} t^{|A'|}$$

with a similar formula for $p_{B'}|_{w_C}$. Since $|A| + |B| = |A'| + |B'|$, we conclude

$$|A|!\,|B|!\; p_A|_{w_C} p_B|_{w_C} = |A'|!\,|B'|!\;\; p_{A'}|_{w_C} p_{B'}|_{w_C} \tag{18}$$

for all sets C containing $A \supseteq B$. By Theorem 14, the equality at every fixed point implies Eq. (16) holds. □

Finally, we state the crucial lemma for the proof of Theorem 1.

Definition 27 Let A, B, and $A \cup B$ be consecutive. Define

$${}_iD_j := \{\mathcal{T}_{A\cup B} - i, \mathcal{T}_{A\cup B} - i + 1, \ldots, \mathcal{H}_{A\cup B} + j - 1, \mathcal{H}_{A\cup B} + j\}.$$

For convenience, denote $D = {}_0D_0 = A \cup B$.

Lemma 28 *Let A, B and ${}_mD_n$ be consecutive for $m = 0, 1, \ldots, \ell$, $n = 0, 1 \ldots, r$, with $D = {}_0D_0 = A \cup B$, and $|A \cap B| = \ell + r$. If A and B are intertwined or if A and B are consecutive to each other and disjoint,*

$$b_{A,B}^{{}_mD_n} = \frac{|A \cup B|!|A \cap B|!}{(|A \cap B| - m - n)!\, m!\, n!\, (|A \setminus B| + m)!\, (|B \setminus A| + n)!} t^{|A\cap B|-m-n}.$$

Proof We prove this by induction on $m + n$. When $m = n = 0$, this formula is the statement of Lemma 25.

For ease of notation, let K denote ${}_mD_n$. Restrict $p_A p_B = \sum_C b_{A,B}^C p_C$ to w_K.

$$b_{A,B}^K p_K|_{w_K} = p_A|_{w_K} p_B|_{w_K} - \sum_{\substack{0\le i\le m, 0\le j\le n\\ i+j<m+n}} b_{A,B}^{{}_iD_j} p_{{}_iD_j}|_{w_K}. \tag{19}$$

For all $0 \le i \le m$, $0 \le j \le n$ and $i + j < m + n$ assume

$$b_{A,B}^{{}_iD_j} = \frac{|A \cup B|!|A \cap B|!}{(|A \cap B| - i - j)!\, i!\, j!\, (|A \setminus B| + i)!\, (|B \setminus A| + j)!} t^{|A\cap B|-i-j}.$$

Assume without loss of generality that $\mathcal{T}_A \le \mathcal{T}_B$. Then if A or B are intertwined or disjoint and consecutive to each other, $|A \cap B| = \mathcal{H}_A - \mathcal{T}_B + 1$ and $|A \cup B| = \mathcal{H}_B - \mathcal{T}_A + 1$.

Using Lemma 24 for each restriction, we obtain:

$$p_A|_{w_K} = \binom{\mathcal{H}_K - \mathcal{T}_A + 1}{|A|} \frac{(\mathcal{H}_A - \mathcal{T}_K + 1)!}{(\mathcal{T}_A - \mathcal{T}_K)!} t^{|A|} = \binom{|A \cup B| + n}{|A|} \frac{(|A| + m)!}{m!} t^{|A|}.$$

By the inductive assumption,

$$\begin{aligned} b^K_{A,B} p_K|_{w_K} =& p_A|_{w_K}\ p_B|_{w_K} - \sum_{\substack{0 \le i \le m \\ 0 \le j \le n \\ i+j<m+n}} b^{{}_iD_j}_{A,B} p_{{}_iD_j}|_{w_K} \\ =& \binom{|A \cup B| + n}{|A|} \frac{(|A| + m)!}{m!} \binom{|B| + n}{|B|} \frac{(|A \cup B| + m)!}{(|A \setminus B| + m)!} t^{|A|+|B|} \\ &- \sum_{\substack{0 \le i \le m \\ 0 \le j \le n \\ i+j<m+n}} \left[\frac{|A \cup B|!|A \cap B|!}{i!j!(|A \setminus B| + i)!(|B \setminus A| + j)!(|A \cap B| - i - j)!} t^{|A \cap B| - i - j} \right. \\ &\left. \binom{|A \cup B| + i + n}{|A \cup B| + i + j} \frac{(|A \cup B| + m + j)!}{(m - i)!} t^{|A \cup B| + i + j} \right]. \end{aligned}$$

Multiply both sides of Eq. (19) by $\frac{1}{|A \cup B|!\ |A \cap B|!}$. Since $p_K|_{w_K} = |K|!t^{|K|}$, the left-hand side of Eq. (19) becomes

$$\frac{1}{|A \cup B|!\ |A \cap B|!} b^K_{A,B} p_K|_{w_K} = \frac{(|A \cup B| + m + n)!}{|A \cup B|!\ |A \cap B|!} t^{|A \cup B| + m + n}\ b^K_{A,B}.$$

Now by rearranging terms while noting that $t^{|A \cap B| - i - j} t^{|A \cup B| + i + j} = t^{|A|+|B|}$, $|A \setminus B| = \mathcal{T}_B - \mathcal{T}_A$, and $|B \setminus A| = \mathcal{H}_B - \mathcal{H}_A$, the right hand side of Eq. (19) becomes

$$\begin{aligned} &\left[\binom{|A \cup B| + m}{|A \cup B|} \binom{|A| + m}{|A \cap B|} \binom{|B| + n}{|B|} \binom{|A \cup B| + n}{|A|} \right. \\ &- \sum_{\substack{0 \le i \le m \\ 0 \le j \le n \\ i+j<m+n}} \binom{|A \cup B| + i + n}{|A \cup B| + i + j} \\ &\left. \binom{|A \cup B| + m + j}{i,\ j,\ |A \setminus B| + i,\ |B \setminus A| + j,\ |A \cap B| - i - j,\ m - i} \right] t^{|A|+|B|}. \end{aligned}$$

Let $x = |A \cap B|$, $w = |A \cup B|$, $y = |A|$ and $z = |B|$ to rewrite the expression as

$$\begin{aligned} &\left[\binom{w + m}{w} \binom{y + m}{x} \binom{z + n}{z} \binom{w + n}{y} \right. \\ &\left. - \sum_{\substack{0 \le i \le m \\ 0 \le j \le n \\ i+j<m+n}} \binom{w + i + n}{w + i + j} \binom{w + m + j}{i,\ j,\ (y - x) + i,\ (z - x) + j,\ x - i - j,\ m - i} \right] t^{y+z}. \end{aligned}$$

We recognize the coefficient as the term with $i = m$, and $j = n$ of the sum on the right hand side of Theorem 9. Using the same variables, we then simplify the equation:

$$\frac{(w+m+n)!}{w!\,x!} t^{|A\cup B|+m+n}\, b^K_{A,B} = \binom{w+m+n}{m,\ n,\ x-m-n,\ z-x+n,\ y-x+m} t^{|A|+|B|}.$$

Finally, we solve for $b^K_{A,B}$ and substitute back for x, y, w, z to obtain

$$b^K_{A,B} = \frac{|A\cup B|!\,|A\cap B|!}{m!\,n!\,(|A\setminus B|+m)!(|B\setminus A|+n)!(|A\cap B|-m-n)!} t^{|A\cap B|-m-n}.$$

□

Proof of Theorem 1 Assume A, B, and C are consecutive and that $A\cup B\subseteq C$ with $|C|\le |A|+|B|$. Without loss of generality, assume also that $\mathcal{T}_A\le\mathcal{T}_B$.

If A and B are disjoint, then C consecutive and $|C|\le|A|+|B|$ forces $C = A\cup B$ and thus A and B are adjacent. If either A and B are intertwined, or if A and B are adjacent and disjoint,

$$|A\cup B| = \mathcal{H}_B - \mathcal{T}_A + 1 \qquad |A\cap B| = \mathcal{H}_A - \mathcal{T}_B + 1$$
$$|A\setminus B| = \mathcal{T}_B - \mathcal{T}_A \qquad |B\setminus A| = \mathcal{H}_B - \mathcal{H}_A.$$

As C is consecutive, $C = {}_mD_n$ where $m = \mathcal{T}_A - \mathcal{T}_C$ and $n = \mathcal{H}_C - \mathcal{H}_B$. It follows that $|A\setminus B| + m = \mathcal{T}_B - \mathcal{T}_C$ and $|B\setminus A| + n = \mathcal{H}_C - \mathcal{H}_A$. Then by Lemma 28 with $d := |A| + |B| - |C| = |A\cap B| - m - n$,

$$b^C_{A,B} = \frac{(\mathcal{H}_A-\mathcal{T}_B+1)!(\mathcal{H}_B-\mathcal{T}_A+1)!}{d!(\mathcal{T}_A-\mathcal{T}_C)!(\mathcal{H}_C-\mathcal{H}_B)!(\mathcal{T}_B-\mathcal{T}_C)!(\mathcal{H}_C-\mathcal{H}_A)!} t^d.$$

To prove the case when $B\subseteq A$ we construct two intertwined sets from A and B and apply Lemma 26. Let

$$A' := \{a\in A:\ a\le\mathcal{H}_B\} \text{ and } B' := \{b\in A:\ b\ge\mathcal{T}_B\}.$$

Then A' and B' are intertwined and also satisfy the relationships in (17) with

$$d = |A'| + |B'| - |C| = |A| + |B| - |C|.$$

Furthermore, $\mathcal{T}_{A'} = \mathcal{T}_A$, $\mathcal{T}_{B'} = \mathcal{T}_B$, $\mathcal{H}_{B'} = \mathcal{H}_A$, and $\mathcal{H}_{A'} = \mathcal{H}_B$. Thus by the formula above for the intertwined case,

$$\begin{aligned} b^C_{A',B'} &= \frac{(\mathcal{H}_{A'} - \mathcal{T}_{B'} + 1)!(\mathcal{H}_{B'} - \mathcal{T}_{A'} + 1)!}{d!(\mathcal{T}_{A'} - \mathcal{T}_C)!(\mathcal{H}_C - \mathcal{H}_{B'})!(\mathcal{T}_{B'} - \mathcal{T}_C)!(\mathcal{H}_C - \mathcal{H}_{A'})!} t^d \\ &= \frac{(\mathcal{H}_B - \mathcal{T}_B + 1)!(\mathcal{H}_A - \mathcal{T}_A + 1)!}{d!(\mathcal{T}_A - \mathcal{T}_C)!(\mathcal{H}_C - \mathcal{H}_A)!(\mathcal{T}_B - \mathcal{T}_C)!(\mathcal{H}_C - \mathcal{H}_B)!} t^d. \end{aligned}$$

Applying Lemma 26.

$$\begin{aligned} b^C_{A,B} =& \frac{|A'|!\,|B'|!}{|A|!\,|B|!} b^C_{A',B'} \\ =& \frac{(\mathcal{H}_B - \mathcal{T}_A + 1)!(\mathcal{H}_A - \mathcal{T}_B + 1)!}{(\mathcal{H}_A - \mathcal{T}_A + 1)!(\mathcal{H}_B - \mathcal{T}_B + 1)!} \\ & \cdot \frac{(\mathcal{H}_B - \mathcal{T}_B + 1)!(\mathcal{H}_A - \mathcal{T}_A + 1)!}{d!(\mathcal{T}_A - \mathcal{T}_C)!(\mathcal{H}_C - \mathcal{H}_A)!(\mathcal{T}_B - \mathcal{T}_C)!(\mathcal{H}_C - \mathcal{H}_B)!} t^d \\ =& \frac{(\mathcal{H}_A - \mathcal{T}_B + 1)!(\mathcal{H}_B - \mathcal{T}_A + 1)!}{d!(\mathcal{T}_A - \mathcal{T}_C)!(\mathcal{H}_C - \mathcal{H}_A)!(\mathcal{T}_B - \mathcal{T}_C)!(\mathcal{H}_C - \mathcal{H}_B)!} t^d. \end{aligned}$$

To make the formula obviously integral, multiply by $\frac{d!}{d!}$ to obtain

$$b^C_{A,B} = d! \binom{\mathcal{H}_A - \mathcal{T}_B + 1}{d,\ \mathcal{T}_A - \mathcal{T}_C,\ \mathcal{H}_C - \mathcal{H}_B} \binom{\mathcal{H}_B - \mathcal{T}_A + 1}{d,\ \mathcal{T}_B - \mathcal{T}_C,\ \mathcal{H}_C - \mathcal{H}_A} t^d.$$

□

Proof of Theorem 3 Let $A = A_1 \cup \cdots A_k$ and $B = B_1 \cup \cdots \cup B_\ell$ be written as a union of disjoint maximal consecutive subsets. Now rename the sets $\{A_1, \ldots, A_k, B_1, \ldots, B_\ell\}$ by $E_1, \ldots, E_v$ where $v = k + \ell$ so that $\mathcal{T}_{E_i} \leq \mathcal{T}_{E_{i+1}}$ for all i. By assumption, $A \cup B = \cup_j E_j$ is consecutive. Since each E_i is consecutive, the reordering implies $E_i \cup E_{i+1}$ is consecutive. Then by Lemma 22 and expanding the product,

$$\begin{aligned} p_A p_B = \prod_{j=1}^{v} p_{E_v} = p_{E_1} p_{E_2} \prod_{j=3}^{v} p_{E_v} &= \sum_{C_2} b^{C_2}_{E_1,E_2} p_{C_2} \prod_{j=3}^{v} p_{E_v} \\ &= \sum_{(C_2, C_3, C_4, \ldots, C_{v-1}, C)} b^{C_2}_{E_1,E_2} b^{C_3}_{C_2,E_3} \cdots b^{C}_{C_{v-1},E_v} p_C. \end{aligned}$$

By Lemma 18, $b^{C_2}_{E_1,E_2} \neq 0$ implies $E_1 \cup E_2 \subseteq C_2$. If C_2 weren't consecutive, there exists a maximal consecutive subset $C^0 \subset C_2$ with $E_1 \cup E_2 \subseteq C^0$, since $E_1 \cup E_2$ is consecutive. Thus $b^{C_2}_{E_1,E_2} = 0$ by Lemma 20, contrary to assumption. Thus C_2 is consecutive.

Similarly, as C_2 is consecutive and the tails of E_i are increasing with $\cup_j E_j$ consecutive, $C_2 \cup E_3$ is consecutive. Thus $b^{C_3}_{C_2,E_3} \neq 0$ implies C_3 is consecutive and $C_2 \cup E_3 \subseteq C_3$. Inductively it follows that the sum may be taken over sequences in which all C_i are consecutive, and that each coefficient $b^{C_2}_{E_1,E_2}$ and $b^{C_{i+1}}_{C_i,E_{i+1}}$ may be calculated by Theorem 1 as the corresponding sets are consecutive. Therefore,

$$b^C_{A,B} = \sum_{\substack{(C_2,C_3,C_4,\dots,C_{v-1})\\ C_i \text{ consecutive}}} b^{C_2}_{E_1,E_2} b^{C_3}_{C_2,E_3} \cdots b^{C}_{C_{v-1},E_v} \tag{20}$$

is the coefficient of p_C, as stated in Theorem 3. □

Furthermore, all factors of any term in the sum (20) are nonnegative by Theorem 1. Corollary 29 claims that, if a consecutive set C contains $A \cup B$ and $|C| \leq |A| + |B|$, the sum is actually positive.

Corollary 29 *If $A \cup B$ and C are consecutive, $A \cup B \subseteq C$ and $|C| \leq |A| + |B|$, then* $b^C_{A,B} \neq 0$.

Proof of Corollary 29 We need only find a single sequence

$$(C_2, C_3, C_4, \dots, C_{v-1})$$

for which the corresponding summand in (20) is nonzero. As in the proof of Theorem 3, let $E_1, \dots, E_v$ be a reordering of the maximally consecutive subsets of A and of B, as for the proof of Theorem 3. Note that

$$|A| + |B| = \sum_{i=1}^{v} |E_j|. \tag{21}$$

Since $A \cup B$ is consecutive, $E_{j-1} \cup E_j$ is consecutive for each $j = 2, \dots, v$. Let $C_1 = E_1$. We find a set C_j for $j = 2, \dots, v-1$ inductively. Choose $C_j \subset C$ of maximal size such that

(1) C_j is consecutive
(2) $C_{j-1} \cup E_j \subset C_j$, and
(3) $|C_j| \leq |C_{j-1}| + |E_j|$.

If $|C| > |C_{j-1}| + |E_j|$, there exists C_j satisfying (1)-(2) with $|C_j| = |C_{j-1}| + |E_j|$, the maximal allowable size of property (3). If $|C| \leq |C_k| + |E_k|$, for some k, set $C_j = C$ for all $j \geq k+1$ and note that it necessarily satisfies conditions (1)-(3). The sets C_{j-1}, E_j, and C_j are consecutive, and satisfy the degree condition of Theorem 1, ensuring $b^{C_j}_{C_{j-1},E_j} \neq 0$.

We have only to show that the last term in the product is nonzero, i.e. $b^C_{C_{v-1},E_v} \neq 0$. If $C_{v-1} = C$, then the sets C_{v-1}, E_v and C satisfy the conditions of Theorem 1 so the statement holds. If $C \neq C_{v-1}$, then $|C_j| = |C_{j-1}| + |E_j|$ for all $j = 2, 3, \dots, v-1$. Then by Eq. (21),

$$|C_{v-1}| = \sum_{j=1}^{v-1} |E_j| = |A| + |B| - |E_v|.$$

Then

$$|C| \leq |A| + |B| = |C_{v-1}| + |E_v|,$$

which is the degree requirement of Theorem 1. Since C_{v-1}, E_v and C are also consecutive, Theorem 1 implies $b^C_{C_{v-1},E_v} \neq 0$. □

Proof of Theorem 5 Let $A \cup B = D_1 \cup \cdots \cup D_u$ be a union of maximal consecutive components of $A \cup B$. Note that each A_j and each B_j occurs in exactly one D_i. Thus

$$\begin{aligned}
p_A p_B &= p_{A_1} \cdots p_{A_s} p_{B_1} \cdots p_{B_t} \\
&= \prod_{i=1}^{u} p_{A^i} p_{B^i}, \quad \text{where } A^i = A \cap D_i,\, B^i = B \cap D_i \\
&= \prod_{i=1}^{u} \sum_{E} b^E_{A^i,B^i} p_E \\
&= \sum_{E_1,\ldots,E_u} \prod_{i=1}^{u} b^{E_i}_{A^i,B^i} p_{E_i} \\
&= \sum_{E_1,\ldots,E_u} \left(\prod_{i=1}^{u} b^{E_i}_{A^i,B^i} \right) \left(\prod_{i=1}^{u} p_{E_i} \right)
\end{aligned}$$

where the sum is over sequences of consecutive E_i by Lemma 22, each containing $D_i = A^i \cup B^i$ by Lemma 18.

Therefore, the coefficient of p_C in this product is

$$b^C_{A,B} = \sum_{E_1,\ldots,E_u} \left(\prod_{i=1}^{u} b^{E_i}_{A^i,B^i} \right) b^C_{E_1,\ldots,E_u},$$

as stated by Theorem 5. Each factor $b^{E_i}_{A^i,B^i}$ is calculated by Theorem 3 since $A^i \cup B^i$ and E_i are consecutive.

We now take to calculating $\prod_{i=1}^{u} p_{E_i}$ to find the coefficient $b^C_{E_1,\ldots,E_u}$ of p_C, noting that E_i is consecutive for each i.

If $\cup_i E_i$ is consecutive, then as before we order $E_1, \ldots, E_u$ so that their tails are increasing. Then $E_1 \cup E_2$ must be consecutive, and so we apply Theorem 3 to find

$$p_{E_1} p_{E_2} = \sum_{\substack{C \text{ consecutive} \\ C \supset E_1 \cup E_2}} b^C_{E_1,E_2} p_C$$

with $b^C_{E_1,E_2}$ determined by the formula in Theorem 1. Since each C contains E_1 and E_2, the union $C \cup E_3$ is consecutive for all C. Therefore

$$p_{E_1}p_{E_2}p_{E_3} = \sum_{\substack{C \text{ consecutive} \\ C \supset E_1 \cup E_2}} b^C_{E_1,E_2} p_C p_{E_3} = \sum_{\substack{(C_1,C_2) \text{ both consecutive} \\ C_1 \supset E_1 \cup E_2, C_2 \supset C_1 \cup E_3}} b^{C_1}_{E_1,E_2} b^{C_2}_{C_1,E_3} p_{C_2}.$$

Continuing inductively, we arrive at the equation

$$\prod_{i=1}^{u} p_{E_i} = \sum_{(C_1,C_2,\ldots,C_u)} b^{C_1}_{E_1,E_2} b^{C_2}_{C_1,E_3} \cdots b^{C_u}_{C_{u-1},E_u} p_{C_u}$$

where the sum is over consecutive C_s with $C_s \supset C_{s-1} \cup E_{s+1}$. We thus conclude

$$b^C_{E_1,\ldots,E_u} = \sum_{(C_1,C_2,\ldots,C_{u-1})} b^{C_1}_{E_1,E_2} b^{C_2}_{C_1,E_3} \cdots b^{C}_{C_{u-1},E_u},$$

where $C \supseteq \cup_i E_i \supseteq A \cup B$.

Now suppose $\cup_i E_i$ is not consecutive. If none of the E_i are adjacent or overlapping, then $\prod_{i=1}^{u} p_{E_i} = p_{\cup_i E_i}$ has no p_C term, as C is consecutive. Otherwise, there exist two sets E_{j_1} and E_{k_1} whose union is consecutive. Then

$$\prod_{i=1}^{u} p_{E_i} = p_{E_{j_1}} p_{E_{k_1}} \prod_{i \neq j_1,k_1} p_{E_i} = \sum_{\substack{F_1 \supset E_{j_1} \cup E_{k_1}, \\ \text{consecutive}}} b^{F_1}_{E_{j_1},E_{k_1}} p_{F_1} \prod_{i \neq j_1,k_1} p_{E_i}.$$

For each such F_1, expand the product $p_{F_1} \prod\limits_{i \neq j_1,k_1} p_{E_i}$ with one fewer factor. Relabel the sets $F_1, E_1, \ldots, \widehat{E}_{j_1}, \widehat{E}_{k_1}, \ldots E_u$, and continue inductively. At each step, if the union of the sets is not consecutive, and if no two sets are adjacent, the coefficient of p_C vanishes. If there are any two sets whose union is consecutive, we may expand their product using Theorem 3.

Explicitly, for each F_1, we relabel the sets $F_1, E_1, \ldots, \widehat{E}_{j_1}, \widehat{E}_{k_1}, \ldots E_u$ by $E^{(2)}_1, \ldots, E^{(2)}_{u-1}$. Choose j_2, k_2 such that $E^{(2)}_{j_2} \cup E^{(2)}_{k_2}$ is consecutive. Then

$$\begin{aligned}\prod_{i=1}^{u} p_{E_i} &= \sum_{F_1} b^{F_1}_{E_{j_1},E_{k_1}} \prod_{i=2}^{u} p_{E^{(2)}_j} = \sum_{F_1} b^{F_1}_{E_{j_1},E_{k_1}} \left(p_{E^{(2)}_{j_2}} p_{E^{(2)}_{k_2}} \right) \prod_{i \neq j_2,k_2}^{u} p_{E^{(2)}_j} \\ &= \sum_{F_1} b^{F_1}_{E_{k_1},E_{j_1}} \left(\sum_{F_2} b^{F_2}_{E^{(2)}_{j_2},E^{(2)}_{k_2}} p_{F_2} \right) \prod_{i \neq j_2,k_2} p_{E^{(2)}_j} \\ &= \sum_{F_1,F_2} b^{F_1}_{E_{j_1},E_{k_2}} b^{F_2}_{E^{(2)}_{j_2},E^{(2)}_{k_2}} p_{F_2} \prod_{i \neq j_2,k_2} p_{E^{(2)}_j},\end{aligned}$$

where the sum is over consecutive F_1 and F_2 such that $E_{j_1} \cup E_{k_1} \subseteq F_1$ and $E^{(2)}_{j_2} \cup E^{(2)}_{k_2} \subseteq F_2$. Note that the choice of F_2 over which we sum, and indeed the sets $E^{(2)}_j$ depend on each F_1. We continue inductively. For each sequence $F_1, \ldots, F_s$ with $s < u$, there exist two sets $E^{(s)}_{j_s}, E^{(s)}_{k_s}$ among $F_s, E^{(s)}_1, \ldots, E^{(s)}_{u-s+1}$ whose union is consecutive. Label the sets $F_s, E^{(s)}_1, \ldots, \widehat{E}^{(s)}_{j_s}, \widehat{E}^{(s)}_{k_s}, \ldots, E^{(s)}_{u-s+1}$ by $E^{(s+1)}_1, \ldots, E^{(s+1)}_{u-s}$ for $s = 1, \ldots, u-2$, so that there is one set $E^{(u-1)}_1$ when the super index is $u-1$. We have found:

$$\prod_{i=1}^{u} p_{E_i} = \sum_{(F_1,F_2,\ldots,F_s)} b^{F_1}_{E_{j_1},E_{k_1}} b^{F_2}_{E^{(2)}_{j_2},E^{(2)}_{k_2}} \cdots b^{F_s}_{E^{(s)}_{j_s},E^{(s)}_{k_s}} p_{F_s} \prod_{i \neq j_s,k_s} p_{E^{(s)}_j}$$
$$= \sum_{(F_1,F_2,\ldots,F_{u-2})} b^{F_1}_{E_{j_1},E_{k_1}} b^{F_2}_{E^{(2)}_{j_2},E^{(2)}_{k_2}} \cdots b^{F_{u-2}}_{E^{(u-2)}_{j_{u-2}},E^{(u-2)}_{k_{u-2}}} p_{F_{u-2}} \prod_{i \neq j_{u-2},k_{u-2}} p_{E^{(u-2)}_i},$$

which, by relabeling F_{u-2} and the single $E^{(u-2)}_i$ in the product,

$$= \sum_{(F_1,F_2,\ldots,F_{u-2})} b^{F_1}_{E_{j_1},E_{k_1}} b^{F_2}_{E^{(2)}_{j_2},E^{(2)}_{k_2}} \cdots b^{F_{u-2}}_{E^{(u-2)}_{j_{u-2}},E^{(u-2)}_{k_{u-2}}} p_{E^{(u-1)}_1} p_{E^{(u-1)}_2}$$
$$= \sum_{(F_1,F_2,\ldots,F_{u-2})} b^{F_1}_{E_{j_1},E_{k_1}} b^{F_2}_{E^{(2)}_{j_2},E^{(2)}_{k_2}} \cdots b^{F_{u-2}}_{E^{(u-2)}_{j_{u-2}},E^{(u-2)}_{k_{u-2}}} \left(\sum_C b^C_{E^{(u-1)}_1,E^{(u-1)}_2} p_C \right),$$

and thus

$$b^C_{E_1,\ldots,E_u} = \sum_{(F_1,F_2,\ldots,F_{u-2})} b^{F_1}_{E_{j_1},E_{k_1}} b^{F_2}_{E^{(2)}_{j_2},E^{(2)}_{k_2}} \cdots b^{F_{u-2}}_{E^{(u-2)}_{j_{u-2}},E^{(u-2)}_{k_{u-2}}} b^C_{E^{(u-1)}_1,E^{(u-1)}_2}.$$

Finally, to obtain the statement of Eq. (5) in Theorem 5 we note that j_{u-1} and k_{u-1} must be the two indices 1, 2 as the union of the two sets $E^{(u-1)}_1$ and $E^{(u-1)}_2$ are necessarily consecutive. □

Proof of Theorem 6 We want to show that $b^C_{A,B} = \prod_k b^{C_k}_{A\cap C_k, B\cap C_k}$ where $C = C_1 \cup \cdots \cup C_m$ is a union of (nonempty) maximal consecutive subsets of C. As $C_1, \ldots, C_m$ are maximal consecutive subsets, $A = \cup_k (A \cap C_k)$ is nonconsecutive (though for an individual k, $A \cap C_k$ may be consecutive). Similarly $B = \cup_k (B \cap C_k)$ is nonconsecutive. By Lemma 22,

$$p_A = \prod_k p_{A\cap C_k} \quad \text{and} \quad p_B = \prod_k p_{B\cap C_k},$$

which implies

$$p_A p_B = \prod_k p_{A\cap C_k} p_{B\cap C_k} = \prod_k \sum_E b^E_{A\cap C_k, B\cap C_k} p_E.$$

Note that $b^E_{A\cap C_k, B\cap C_k} = 0$ unless E contains $(A \cap C_k) \cup (B \cap C_k)$ by Lemma 18. We first argue that the only terms $b^E_{A\cap C_k, B\cap C_k} \neq 0$ that contribute to the coefficient p_C are those with $E \subseteq C_k$.

Clearly, if E contains elements not in C, the corresponding terms p_E do not contribute to the coefficient p_C, since for any F, $b^C_{E,F} \neq 0$ implies C contains E. Thus we may suppose $E = E^0 \cup E'$, where E' is not consecutive with, nor intersects, C_k, and $E^0 \subseteq C_k$. Then by Lemma 20, $b^E_{A\cap C_k, B\cap C_k} = 0$.

It follows that the coefficient of p_C in $p_A p_B$ is the coefficient of p_C in

$$\prod_k \sum_{E_k \subseteq C_k} b^{E_k}_{A\cap C_k, B\cap C_k} p_{E_k}.$$

On the other hand, if $E_k \neq C_k$, then $\prod_k p_{E_k} = p_{\cup_k E_k} \neq p_C$, where the first equality follows because $\cup_k E_k$ is a nonconsecutive union (Lemma 22). Therefore

$$\prod_k b^{C_k}_{A\cap C_k, B\cap C_k} p_{C_k} = \left(\prod_k b^{C_k}_{A\cap C_k, B\cap C_k} \right) p_C,$$

as $p_C = p_{C_1} p_{C_2} \dots p_{C_m}$ (Lemma 22 again). □

A slight generalization shows that the non-vanishing of the structure constant holds also when A and B are not consecutive. To prove the general case, we need the following lemma.

Lemma 30 *Let A and B be arbitrary subsets of $\{1, \dots, n-1\}$, and C consecutive. Then $b^C_{A,B} \neq 0$ if and only if C contains $A \cup B$ and $|C| \leq |A| + |B|$.*

Proof of Lemma 30 If $b^C_{A,B} \neq 0$, then $A \cup B \subseteq C$ and $|C| \leq |A| + |B|$ by Lemma 18.

To prove the converse, let $A \cup B = D_1 \cup \dots \cup D_u$ where each D_i is a maximal consecutive subset of $A \cup B$ and let $A^i = D_i \cap A$ and $B^i = D_i \cap B$. By Theorem 5, we have the equality

$$b^C_{A,B} = \sum_{\substack{(E_1,\dots,E_u):\ D_i \subseteq E_i,\\ E_i \text{ consecutive}}} \left(\prod_{i=1}^u b^{E_i}_{A^i, B^i} \right) b^C_{E_1,\dots,E_u}$$

where $b^C_{E_1,\dots,E_u}$ is the coefficient of p_C in the product $\prod_{i=1}^u p_{E_i}$. We prove there exists a sequence of sets $(E_1, \dots, E_u)$ in the index set of the sum such that $b^{E_i}_{A^i,B^i} \neq 0$ for all i, and $b^C_{E_1,\dots,E_u} \neq 0$. Indeed, consider any sequence $(E_1, \dots, E_u)$ with E_i consecutive and containing D_i, with the additional properties that $E_i \subseteq C$ and $|E_i| = \min(|A^i| + |B^i|, |C|)$. Since $D_i = A^i \cup B^i$ is consecutive and $|E_i| \leq |A^i| + |B^i|$, by Corollary 29, $b^{E_i}_{A^i,B^i} \neq 0$. It remains to show that $b^C_{E_1,\dots,E_u} \neq 0$.

If $\cup_i E_i$ consecutive, then by Lemma 29 $b^F_{E_1,\ldots,E_u} \neq 0$ for all consecutive F such that $|F| \leq \sum_i |E_i|$ and F contains $\cup_i E_i$. Since

$$|C| \leq |A| + |B| = \sum_i |A_i| + |B_i| = \sum_i |E_i|,$$

and $\cup_i E_i \subseteq C$, the coefficient $b^C_{E_1,\ldots,E_u} \neq 0$.

If $\cup_i E_i$ is not consecutive, then $|C| \leq |A| + |B| = \sum_i |E_i|$ and C consecutive containing $\cup_i E_i$ implies there are at least two sets E_{j_1}, E_{k_1} whose union is consecutive. Thus

$$\prod_{i=1}^{u} p_{E_i} = p_{E_{j_1}} p_{E_{k_1}} \prod_{i \neq j_1,k_1} p_{E_i} = \left(\sum_{\substack{F \supset E_{j_1} \cup E_{k_1}, \\ \text{consecutive}}} b^F_{E_{j_1},E_{k_1}} p_F \right) \prod_{i \neq j_1,k_1} p_{E_i}$$

and the terms $b^F_{E_{j_1},E_{k_1}}$ are nonzero whenever F satisfies the degree condition $|F| \leq |E_{j_1}| + |E_{k_2}|$. In particular, let $F_1 \subset C$ be a consecutive set containing $E_{j_1} \cup E_{k_1}$ with $|F_1| = \min(|E_{j_1}| + |E_{k_1}|, |C|)$. Then

$$\prod_{i=1}^{u} p_{E_i} = b^{F_1}_{E_{j_1},E_{k_1}} p_{F_1} \prod_{i \neq j_1,k_1} p_{E_i} + \text{nonnegative terms}$$

with $b^{F_1}_{E_{j_1},E_{k_1}} \neq 0$. As in the proof of Theorem 5, we relabel the sets

$$F_1, E_1, \ldots, \widehat{E}_{j_1}, \widehat{E}_{k_1}, \ldots, E_u \text{ by } E^{(2)}_1, \ldots E^{(2)}_{u-1},$$

in which we omit sets with a $\widehat{\ }$. By construction of F_1, $|C| \leq \sum_i |E^{(2)}_i|$ and $\cup_i E^{(2)}_i \subseteq C$. Thus there is a pair of sets $E^{(2)}_{j_1}$ and $E^{(2)}_{k_1}$ whose union is consecutive. We continue inductively, obtaining a sequence of consecutive sets $F_1, \ldots, F_{u-2} \subseteq C$ such that $b^{F_s}_{E^{(s)}_{j_s},E^{(s)}_{k_s}} \neq 0$ and $|F_s| = \min(|E_{j_s}| + |E_{k_s}|, |C|)$ for all s. By picking out the coefficient of p_C in the product, we obtain:

$$b^C_{E_1,\ldots,E_u} = b^{F_1}_{E_{j_1},E_{k_1}} b^{F_2}_{E^{(2)}_{j_2},E^{(2)}_{k_2}} \cdots b^{F_{u-2}}_{E^{(u-2)}_{j_{u-2}},E^{(u-2)}_{k_{u-2}}} b^C_{E^{(u-1)}_1,E^{(u-1)}_2} + \text{nonnegative terms}$$

where the nonnegative terms in the sum are similarly products of coefficients. The first term is nonzero, as its factors are all nonzero by construction. Thus $b^C_{E_1,\ldots,E_u} \neq 0$. □

Proof of Theorem 8. Suppose $b^C_{A,B} = at^d$ with $a > 0$. By Lemma 18, $A \cup B \subseteq C$. Let $C = C_1 \cup \cdots \cup C_m$ be a union of maximal consecutive subsets C_k. Then by Theorem 6,

$$b^C_{A,B} = \prod_{k=1}^{m} b^{C_k}_{A\cap C_k, B\cap C_k}.$$

The hypothesis implies $b^{C_k}_{A\cap C_k, B\cap C_k} \neq 0$, and thus by degree considerations (or Lemma 18), $|C_k| \leq |A \cap C_k| + |B \cap C_k|$.

Now suppose the converse. For $k = 1, \ldots, m$, let $A^k = C_k \cap A$ and $B^k = C_k \cap B$. Note that $A^k \cup B^k \subseteq C_k$ by construction and $|C_k| \leq |A^k| + |B^k|$ by assumption. Then the coefficient $b^{C_k}_{A^k,B^k} \neq 0$ by Corollary 30 as C_k is consecutive.

We show that $b^C_{A,B} \neq 0$. By Lemma 22,

$$p_A p_B = \prod_k p_{A^k} \prod_k p_{B^k} = \prod_k (p_{A^k} p_{B^k})$$

since $A = \cup A^k$ and $B = \cup_k B^k$ are disjoint unions. Each product $p_{A^k} \cdot p_{B^k}$ has at least one nonzero summand in its expansion, since $b^{C_k}_{A^k,B^k} \neq 0$. It follows that the expansion of the product $p_A p_B$ has a nonzero term

$$\prod_k \left(b^{C_k}_{A^k,B^k} p_{C_k} \right) = \prod_k b^{C_k}_{A^k,B^k} \prod_k p_{C_k} = \prod_k b^{C_k}_{A^k,B^k} p_C,$$

where the last equality follows from Lemma 22 as C_k are all disjoint. It is possible that additional terms in the product contribute to the coefficient of p_C, however any additional terms contribute a nonnegative multiple of t^d, where $d = |A| + |B| - |C|$ by Corollary 7. As a result, the coefficient $b^C_{A,B}$ has at least one strictly positive contribution, and thus $b^C_{A,B} = at^d$ with $a > 0$. □

5 Proof of Theorem 9

Fix $m, n, w, x, y, z \in \mathbb{Z}$ with $x, y, z, w, m, n \geq 0$ and $w + x = y + z$. Note that Theorem 9 holds trivially whenever x, y, z or w is less than 0.

We construct an explicit bijection between two sets of sizes given by the right hand and left sides of (6) in Theorem 9. We carry this out as follows: we define two sets $\mathcal{S}$ and $\mathcal{V}$ whose sizes obviously correspond to the left and right hand sides of the identity in Theorem 9. We construct bijections

$$BL^- : \mathcal{S} \to \widetilde{\mathcal{S}} \quad \text{and} \quad BL^\star : \mathcal{V} \to \widetilde{\mathcal{V}},$$

for sets $\widetilde{S}$ and $\widetilde{V}$ that will be rather clearly in one-to-one correspondence with one another. The bijections BL^- and $BL^\star$ are compositions of *bike lock moves*, which we introduce in Sect. 5.2.

Table 1 The number of each letter in $\binom{F}{G} \in \mathcal{S}$, when $i = |P|$ and $j = |T|$

	O	P	Q	R	S	T	U	C	$-$
F	$m-i$	i	$y-x+i$	$x-i-j$	$z-x+j$	j			$n-j$
G							$n-j$	$w+i+j$	$m-i$

5.1 Two Sets with the Right Size

We begin by describing a set $\mathcal{S}$ that indexes the right hand side of Theorem 9. Let $\mathcal{S}$ be the set of $2 \times (w+m+n)$ matrices $\binom{F}{G}$ where the row F rows is a sequence consisting of six letters and a placeholder, denoted by O, P, Q, R, S, T and $-$, respectively, while row G is a sequence consisting of only two letters and a placeholder, U, C, and $-$. We refer to the number of each letter or symbol in the matrix using the absolute value, e.g. $|P|$ refers to the number of Ps occurring in $\binom{F}{G}$.

We insist that the following relationships hold among the numbers of each letter:

- $|O| + |P| = m$
- $|T| + |U| = n$
- $|Q| + |R| + |S| = w$
- $|Q| - |P| = y - x$
- $|S| - |T| = z - x$
- $|C| + |O| + |U| = w + n + m$
- Letters are left-aligned in both sequences, so that any placeholders $-$ occur to the right of all the letters, ensuring each sequence has length $w + m + n$.

For a given pair $\binom{F}{G}$, let $i := |P|$ and $j := |T|$, then $|O| = m - i$ and $|U| = n - j$. It follows that $|Q| = y - x + i$, and $|S| = z - x + j$, so the number of letters in F is $|O| + |P| + |Q| + |R| + |S| + |T| = m + w + j$, and these letters are followed by $n - j$ placeholders. Similarly, the number of letters in $|G|$ is $|U| + |C| = n + w + i$, and the letters are followed by $m - i$ placeholders. We tabulate the counts of each letter in Table 1 for $\binom{F}{G}$.

By allowing $i = |P|$ and $j = |T|$ to vary from 0 to m and n, respectively, we obtain a count of the number of matrices $\binom{F}{G}$ satisfying these conditions. Among the $w + m + j$ letters in F, we choose where to place i entries in of P, j entries of T, $y - x + i$ entries of Q, $z - x + j$ entries of S, and $m - i$ entries for O. The remaining non-letter entries of F are placeholders and have no part in the count as they must be placed at the end of the sequence. Similarly, among the $w + n + i$ letters in G, we choose where to place the $n - j$ copies of U. The remaining letters are all Cs, and the entries of G that aren't letters are placeholders at the end of the sequence. We have shown:

$$|\mathcal{S}| = \sum_{i,j} \frac{(w+m+j)!}{i!(y-x+i)!(x-i-j)!(z-x+j)!j!(m-i)!} \cdot \frac{(w+n+i)!}{(n-j)!(w+i+j)!}.$$

Table 2 Counts of 0s, 1s, and $\star$s in each of $v_1, \dots, v_4$, where $w + x = y + z$

	1	0	$\star$
v_1	w	m	n
v_2	x	$y - x + m$	$z - x + n$
v_3	y	$z - x + n$	m
v_4	z	n	$y - x + m$

This expression is the right hand side of the equation in Theorem 9.

Now we define a set $\mathcal{V}$ that indexes the left-hand side of Theorem 9. Let $\mathcal{V}$ be the set of 4-tuples of sequences $V = (v_1, v_2, v_3, v_4)$ with each v_i a sequence of 1s, 0s, and $\star$s, with any $\star$s occurring to the right of all numbers. We additionally require that

- v_1 consists of w 1s, m 0s, and n $\star$s
- v_2 consists of x 1s, $y - x + m$ 0s, and $z - x + n$ $\star$s
- v_3 consists of y 1s, $z - x + n = w - y + n$ 0s and m $\star$s
- v_4 consists of z 1s, n 0s, and $y - x + m = w - z + m$ $\star$s
- Numbers are left-aligned in all 4 sequences, so any placeholders $\star$ occur to the right of all the numbers, ensuring each sequence has length $w + m + n$.

One quickly observes that

$$|\mathcal{V}| = \binom{w+m}{w}\binom{y+m}{x}\binom{w+n}{y}\binom{z+n}{z},$$

since the $\star$ entries are all placed to in the final spots for each sequence. Observe this is the left-hand side of the equality in Theorem 9.

For future use, we tabulate these values in Table 2.

5.2 Bike Lock Moves

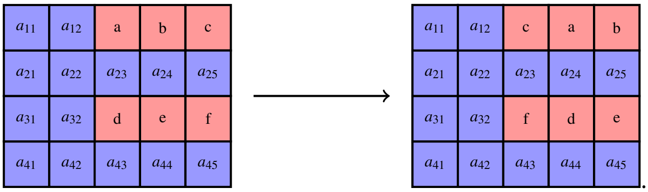
The bijections we construct depend on a series of *bike lock moves* on $r \times c$ matrices. Each move is indexed by a column k, and specifies a set of set of rows on which it will operate (which generally depends on the matrix itself). Each affected row is will rotate its entries from k to c cyclically, by sending the entry in column i to $i + 1$, while the entry in column c will move to column k.

Definition 31 For each k with $1 \le k \le c$, a *bike lock move* BL_k on a set of matrices $\mathcal{M}_c$ with $c > 0$ columns is a map $\mathcal{M}_c \to \mathcal{M}_c$ such that, for all $M \in \mathcal{M}_c$,

1. $BL_k(M)$ is identical to M except in a specified subset of rows $R_{BL_k(M)}$.
2. $BL_k(M)$ cyclically permutes the entries in row $\ell \in R_{BL_k(M)}$ as follows:
   - An entry in column $m < k$ is fixed.
   - An entry in column m with $k \le m < c$ of M sent to column $m + 1$ in the same row.
   - If $m = c$, the entry is sent to the kth column of the same row.

Observe that each bike lock move is determined by its row set.

Example 32 Consider the 4×5 matrix M on the left below. A bike lock move BL_3 on a 4×5 matrix with $R_{BL_3}(M) = \{1, 3\}$ can be seen as follows. Impacted entries are highlighted in red.



Remark 33 Bike lock moves rotate elements starting in a specified column; they do not change the set of entries on each row, nor the number of any repeated entries.

We capture an immediate but more subtle version of this critical property of bike lock moves in the following lemma. Let $(M)_k$ indicate the kth column of the matrix M.

Lemma 34 *Let M be an $r \times c$ matrix, and BL_k a bike lock move with $k \leq c$. Then M and $BL_k(M)$ satisfy the following properties:*

1. *The set of entries in the ℓth row of M is the same as the set of entries in the ℓth row of $BL_k(M)$.*
2. *$(M)_\ell = (BL_k(M))_\ell$ for $\ell = 1, \ldots, k-1$.*
3. *If $\ell \notin R_{BL_k(M)}$, then the ℓth row of $BL_k(M)$ is identical to the ℓth row of M.*
4. *If $\ell \in R_{BL_k(M)}$, each entry in the ℓth row and jth column of M appears in the ℓth row and $j+1$st column of $BL_k(M)$, for $j = k, \ldots, c-1$. In particular, these entries occur in the same (column) order.*

5.3 Bike Lock Moves on $\mathcal{S}$

We define a specific type of bike lock move, and apply a composition of them to elements of $\mathcal{S}$. The idea of the composition of bike lock moves is intuitive but the execution is rather technical. Applied to a 2×9 matrix $\binom{F}{G} \in \mathcal{S}$,

$$\begin{pmatrix} R & Q & O & S & P & R & T & R & - \\ C & C & U & C & C & C & C & C & - \end{pmatrix},$$

for example, the sequence of bike lock moves "shuffle" in the $-$s at the right of the matrix in order to line up the consonants P, Q, R, S and T in the top row with Cs in the bottom row, and line up the vowels O and U with the $-$s:

$$\begin{pmatrix} R & Q & O & - & S & P & R & T & R \\ C & C & - & U & C & C & C & C & C \end{pmatrix}.$$

Details for this example are carried out in Example 36.

Definition 35 The $-$ bike lock move BL_k^- is defined on the set of $2 \times c$ matrices whose entries in the first row are in the set $\{O, P, Q, R, S, T, -\}$, and whose entries in the second row are in $\{C, U, -\}$. Let m_{ij} refer to the (i, j)-entry of M. Define:

$$R_{BL_k^-(M)} = \begin{cases} \{2\} & \text{if } m_{1k} = O, \\ \{1\} & \text{if } m_{2k} = U \text{ and } m_{1k} \neq O, \\ \emptyset & \text{else.} \end{cases} \tag{22}$$

By definition, BL_k^- cyclicly rotates the entries in $R_{BL_k^-(M)}$ in columns $k, k+1, \ldots, c$ one column to the right, with the entry in the last column sent to column k.

Let BL^- be the composition

$$BL^- := BL^-_{w+m+n} \circ BL^-_{w+m+n-1} \circ \cdots \circ BL_2^- \circ BL_1^-.$$

We restrict the domain to $\mathcal{S}$, and let

$$\widetilde{\mathcal{S}} := \{BL^-(S) : S \in \mathcal{S}\}.$$

Example 36 Let $S = \begin{pmatrix} R & Q & O & S & P & R & T & R & - \\ C & C & U & C & C & C & C & C & - \end{pmatrix}$. We find the result of a series of bike lock moves

$$BL^-(S) = BL_9^- \circ BL_8^- \circ \cdots \circ BL_2^- \circ BL_1^-(S).$$

The bike lock moves $BL_2^- \circ BL_1^-$ do not change S, since in the first two columns there is no O in the first row or U in the second. When applying BL_3^-, the third column $\binom{O}{U}$ indicates by (22) that we must shift the second row to the right:

R	Q	O	S	P	R	T	R	-
C	C	U	C	C	C	C	C	-

$\xrightarrow{BL_3^-}$

R	Q	O	S	P	R	T	R	-
C	C	-	U	C	C	C	C	C

where we have indicated the shifted row in red. When applying BL_4^- to the result, the fourth column is $\binom{S}{U}$ so we shift the first row.

R	Q	O	S	P	R	T	R	-
C	C	-	U	C	C	C	C	C

$\xrightarrow{BL_4^-}$

R	Q	O	-	S	P	R	T	R
C	C	-	U	C	C	C	C	C

.

The remaining columns have no Us or Os, so this matrix is left unchanged the bike lock moves $BL_9^- \circ \cdots \circ BL_5^-$. Thus $\widetilde{S} = \begin{pmatrix} R & Q & O & - & S & P & R & T & R \\ C & C & - & U & C & C & C & C & C \end{pmatrix}$.

We now prove a basic property of BL^- applied to elements of $\mathcal{S}$. For any $\binom{F}{G} \in \mathcal{S}$, define

$$N_k^- = BL_k^- \circ \cdots \circ BL_1^- \begin{pmatrix} F \\ G \end{pmatrix}.$$

By convention $N_0^- = \binom{F}{G}$.

Lemma 37 *For $k = 1, \ldots, w+m+n$, the bike lock move BL_k^- applied to N_{k-1}^- either leaves it unchanged, or inserts $-$ into the kth column.*

Proof Let $\binom{F}{G} \in \mathcal{S}$. There are $m-i$ Os in the first row F (see Table 1), and therefore the row specification of (22), results indicates there are $m-i$ bike lock moves in the composition BL^- impacting the second row, G. There are $m-i$ placeholders $-$ in G, so each of these bike lock moves will shift a $-$ from the end of G to some earlier part of the sequence.

Similarly, there are $n-j$ Us in G, and thus by (22) at most $n-j$ individual bike lock moves that impact the row F. We argue that *exactly* $n-j$ bike lock moves in the composition BL^- cycle F by showing that each U results in a cycle of the first row.

Referencing (22), the first row is cycled to the right by BL_k^- whenever $(N_{k-1}^-)_k = \binom{*}{U}$ and $* \neq O$. If $(N_{k-1}^-)_k = \binom{O}{U}$, then BL_k^- cycles the second row of N_{k-1}^-, with the result that $(N_k^-) = \binom{O}{-}$ and $(N_k^-)_{k+1} = \binom{*}{U}$; in particular, the same number of Us occur in columns $k+1, \ldots, w+m+n$ of N_k^- as occur in columns $k, \ldots, w+m+n$ of N_{k-1}^-.

For some $\ell \geq k$, $(N_\ell^-)_{\ell+1} = \binom{*}{U}$ with $* \neq O$, as the existence of U in the second row guarantees some non-O entries on the first row (see Table 1). Thus the second row will be cycled by BL_ℓ^-. We have shown that, for each U occurring in G, there is a shift to the right of the original sequence F. Since there are $n-j$ placeholders $-$ at the end of F, each move results in the insertion of $-$ into the column associated with the bike lock move. □

Corollary 38 *The letters of N_k^- are in the same order as the letters of $\binom{F}{G}$ for all $k = 0, \ldots, w+m+n$.*

Proof of Corollary 38 Suppose BL_ℓ^- acts nontrivially on $N_{\ell-1}^-$ for some $\ell \leq k$. By Lemma 37, BL_ℓ^- inserts a $-$ into the ℓth column. Lemma 34 implies that all letters in columns $\ell+1, \ldots, w+m+n-1$ in the impacted row are shifted to the right one column. Thus all letters remain in the same order after each subsequent bike lock move. □

Corollary 39 *The composition BL^- is bijective map from $\mathcal{S}$ to $\widetilde{\mathcal{S}}$.*

Proof of Corollary 39 Let $\binom{F}{G} \in \mathcal{S}$. By Corollary 38, the order of the letters of $BL^-\binom{F}{G}$ in each row are the same as the order of the letters in $\binom{F}{G}$. Observe the letters of $\binom{F}{G}$ are left-aligned. If $BL^-\binom{F'}{G'}$ for some $\binom{F'}{G'} \in \mathcal{S}$, then the letters of $\binom{F'}{G'}$ are also left-aligned, and occur in the same order as $\binom{F}{G}$, so that $\binom{F'}{G'} = \binom{F}{G}$. Therefore, BL^- is injective. Recall $\widetilde{\mathcal{S}}$ is the image of BL^-. □

We now characterize $\widetilde{\mathcal{S}}$.

Proposition 40 *Elements of $\widetilde{\mathcal{S}}$ are exactly $2 \times (w+m+n)$ matrices M satisfying the following:*

1. *The columns of M consist only of 7 types:*

$$\binom{-}{U}, \binom{O}{-}, \binom{P}{C}, \binom{Q}{C}, \binom{R}{C}, \binom{S}{C}, \binom{T}{C}.$$

2. *There are no pairs of adjacent columns in M of the form* $\begin{pmatrix} - & O \\ U & - \end{pmatrix}$.
3. *The number of times each letter or placeholder appears in each row of M is given in Table 1 for some $0 \le i \le m$ and $0 \le j \le n$.*

We prove Proposition 40 in a series of lemmas.

Lemma 41 *Elements of $\widetilde{S}$ satisfy the three conditions of Proposition 40.*

Proof of Lemma 41 By Lemma 37, each nontrivial bike lock move inserts a $-$ into the corresponding column. $BL_k^-(N_{k-1}^-)$ has a nontrivial row set exactly when there is an O or a U in the kth column of N_{k-1}^-. Thus all columns in $BL^-\binom{F}{G}$ with an O or a U are of the form $\binom{O}{-}$ or $\binom{-}{U}$. All other columns are possible, and listed in the proposition, proving Property 1.

Observe that $-$s occur before letters in N_k only in columns $1, \ldots, k$. Thus $(N_k)_{k+1}$ is not $\binom{-}{U}$ for any k, unless no letters follow on the first row, in which case the column $\binom{-}{U}$ cannot be followed by $\binom{O}{-}$. On the other hand, a column of the form $\binom{O}{U}$ results in a shift on the second row. As a result, the column $\binom{-}{U}$ is never followed by $\binom{O}{-}$. This establishes Property 2.

Finally, Lemma 34 ensures that the counts of $BL^-\binom{F}{G}$ are the same as those of $\binom{F}{G}$. These counts are given in Table 1, establishing Property 3. □

We now show that any matrix M satisfying these conditions is $BL^-\binom{F}{G}$ for some $\binom{F}{G} \in S$. Consider any matrix M satisfying the conditions of Proposition 40 for some i, j. Observe that the first row of M consists of entries in $\{O, P, Q, R, S, T, -\}$ and the second row consists of entries in $\{U, C, -\}$. In each row of M, remove all placeholders, left align all letters and place the placeholders to the right of the last letter. Note that this operation does not change the number of individual letters listed in each row. The resulting matrix is of the form $\binom{F}{G}$, with the number of letters of each type given in Table 1. Therefore $\binom{F}{G}$ satisfies the bulleted listed in §5.1, implying $\binom{F}{G} \in \mathcal{S}$.

We verify that

$$M = BL_{w+m+n}^- \circ \cdots \circ BL_2^- \circ BL_1^- \binom{F}{G}$$

using an inductive argument on the columns of each matrix. We begin with some properties of the series of applications of bike lock moves on $\binom{F}{G}$.

Lemma 42 *All letters of N_k^- in columns $k+1, \ldots, w+m+n$ are left aligned, meaning that all letters occur before any $-$ in these columns.*

Proof of Lemma 42 Observe that all letters of $\binom{F}{G}$ occur to the left of all copies of $-$. The application of BL_1^- to $\binom{F}{G}$ results in either no change, or a cyclic shift to the right by one row, resulting in an entry of the first column $\binom{F}{G}$ moving to the second column and all other entries moving to the right, with the last entry of the row cycling to the first column. If all entries of F or G are $-$, then a rotation of that row will has entries that are vacuously left-aligned from the second column. If either begins with a letter, then a cycling of that row will move that letter to the right one unit, possibly inserting a $-$ in the first column. The resulting matrix remains left-aligned from column 2.

Similarly, suppose the letters of N_{k-1}^- are left-aligned among columns $k, k+1, \ldots, w+m+n$ with $-$ occurring at the the end of the matrix and/or possibly in the first $k-1$ columns in N_{k-1}^-. The application of BL_k^- to N_{k-1}^- has either no effect, or it rotates one row in columns $k, k+1, \ldots, w+m+n$ by one unit to the right with the entry in column $w+m+n$ moving to column k. If there is no effect, then clearly $N_k^- = BL_k^-(N_{k-1}^-)$ is left-aligned in columns $k+1, \ldots, w+m+n$. If a rotated row of N_{k-1}^- has a letter in column k, then that letter is moved to the $k+1$st column and thus N_k^- is left-aligned from column $k+1$. If the entry of a rotated row of N_{k-1}^- is $-$, then N_{k-1}^- has only $-$ in rows $k, k+1, \ldots w+m+n$, since it is left-aligned from column k. Therefore N_k^- has only $-$ in that row in columns $k+1, \ldots, w+m+n$, so its letters in these columns are vacuously left-aligned. ◻

Lemma 43 *Let $\binom{F}{G}$ have counts of letters in Table 1. For $k = 0, 1, \ldots, w+n+m$,*

- *The number of Os in the first row of N_k^- in columns $k+1, \ldots, w+m+n$ is the same as the number of $-$s in the second row of N_k^- in columns $k+1, \ldots, w+m+n$, and*
- *The number of Us in the second row of N_k^- in columns $k+1, \ldots, w+m+n$ is the same as the number of $-$s in the first row of N_k^- in columns $k+1, \ldots, w+m+n$.*

Proof of Lemma 43 Observe that these two properties hold for $\binom{F}{G}$ by a quick check on Table 1. If $\binom{F}{G}_1 = \binom{O}{-}$, then all entries of G are $-$ which implies by Table 1 that all entries of F are O. Similarly if $\binom{F}{G}_1 = \binom{-}{U}$, all entries of F are $-$ since the letters of F are left-aligned, and thus by Table 1, all entries of G are U. In both cases, $N_k^- = \binom{F}{G}$ for all k, so the statement holds.

If $\binom{F}{G}$ consists of two consonants, then $\left(BL_1^-\binom{F}{G}\right)_1 = \binom{F}{G}$ and hence the number of O, Us, and $-$ in each row and in columns $2, \ldots, w+m+n$, is the same for N_1^- and $\binom{F}{G}$.

If $\binom{F}{G}_1 = \binom{O}{*}$ for any $* \neq -$, then $\left(BL_1^-\binom{F}{G}\right)_1 = \binom{O}{-}$ since BL_1^- applied to $\binom{F}{G}$ rotates of the second row, and Table 1 ensures there is a $-$ at the end of G (since there is an O in F). Therefore the first row of $BL_1^-\binom{F}{G}$ has one fewer O in columns $2, \ldots, w+n+m$ than $\binom{F}{G}$, and one fewer $-$ in the second row in those columns. Since $1 \notin R_{BL_1^-\binom{F}{G}}$, Lemma 34(3) implies The number of $-$ occurring in the first row of $\binom{F}{G}$ is the same as the number in $BL_1^-\binom{F}{G}$. By Lemma 34(4), since $2 \in R_{BL_1^-\binom{F}{G}}$, the number of U in columns $2, \ldots, w+m+n$ in $BL_1^-\binom{F}{G}$ is also unchanged.

If $\binom{F}{G}_1 = \binom{*}{U}$ for $* \neq -$ and $* \neq O$, then $\left(BL_1^-\binom{F}{G}\right)_1 = \binom{-}{U}$ since Table 1 ensures there is a $-$ at the end of F. Therefore the first row of $BL_1^-\binom{F}{G}$ has one fewer $-$ in columns $2, \dots, w+n+m$ than $\binom{F}{G}$, and one fewer U in the second row in those columns. Since $* \neq O$, $1 \in R_{BL_k^-\binom{F}{G}}$. Only $-$ are rotated into the first column. By Lemma 34 the count of O in the first row and the count of $-$ in the second row in $\binom{F}{G}$, columns $2, \dots, w+n+m$, are the same as those in $BL_1^-\binom{F}{G}$.

Now suppose that the equalities hold for N_{k-1}^-. If $\left(N_{k-1}^-\right)_k = \binom{O}{-}$ (or $\binom{-}{U}$), then all entries of N_{k-1}^- in columns $k, k+1, \dots w+m+n$ in the second row (or first row) are $-$, since the letters of N_{k-1}^- are left-aligned (see Lemma 42). By the inductive assumption, all entries of N_{k-1}^- in columns $k, k+1, \dots w+m+n$ in the first row (or second row) are O (or U). Then $N_k^- = N_{k-1}^-$ and there are both one fewer $-$ and one fewer O (or U) in subsequent columns, preserving the equality of the counts.

If $(N_{k-1}^-)_k = \binom{O}{*}$ for any $* \neq -$, then BL_k^- requires the rotation of the second row. The inductive assumption ensures that there is a $-$ at the end of the second row of N_{k-1}^-. Therefore the first row of N_k^- has one fewer O and the second row has one fewer $-$ in columns $k+1, \dots, w+n+m$ than N_{k-1}^- has in columns $k, \dots, w+n+m$.

To check the other equality, if $(N_{k-1}^-)_k = \binom{O}{*}$, then BL_k^- rotates the second row, so that the count of Us in N_k^- in columns $k+1, \dots, w+m+n$ equals the count of Us in N_{k-1}^- in columns $k, \dots, w+n+m$. It follows that the lemma holds for N_k^- when $(N_{k-1}^-)_k = \binom{O}{*}$.

Similarly, if $(N_{k-1}^-)_k = \binom{*}{U}$ for $* \neq -$ and $* \neq O$, it must be the case that $\left(BL_k^-(N_{k-1}^-)\right)_1 = \binom{-}{U}$ since Table 1 ensures there is a $-$ at the end of the first row of N_{k-1}^-. Thus there is one fewer $-$ in the first row and one fewer U in the second row of N_k^- in columns $k+1, \dots, w+m+n$, compared to the counts of the same in N_{k-1}^- in columns $k, k+1, \dots, w+m+n$. As both values are reduced by 1, they remain equal. The counts of Os in the first row and $-$ in the second row do not change as they are not present in the kth column of N_{k-1}^- or N_k^-.

It follows that the lemma holds for N_k^- in all cases. □

We use an inductive argument to show that $(M)_k = (N_k^-)_k$ for all k. The base case is established in the following lemma.

Lemma 44 *Let $(M)_1$ denote the first column of M. Then $(M)_1 = (N_1^-)_1$.*

Proof of Lemma 44 If the entries of column $(M)_1$ are consonants, then BL_1^- does not change $\binom{F}{G}$. Therefore, $(N_1^-)_1 = (BL_1^-\binom{F}{G})_1 = (M)_1$ in this case.

Suppose $(M)_1 = \binom{O}{-}$. It follow that the first column of $\binom{F}{G}$ is $\binom{O}{U}$, $\binom{O}{C}$, or $\binom{O}{-}$ In all cases, the bike lock move BL_1^- rotates the second row of $\binom{F}{G}$ (see (22)). Table 1 guarantees a $-$ at the end of the second row of $\binom{F}{G}$ because there exists an O in the first row. It follows that $\left(N_1^-\right)_1 = \binom{O}{-} = (M)_1$.

Suppose $(M)_1 = \binom{-}{U}$. The first column of $\binom{F}{G}$ is thus $\binom{*}{U}$, where $*$ is an element of $\{O, P, Q, R, S, T, -\}$. If $* = O$, then the first non-placeholder in the first row of M

would be O. If this occurs in column ℓ, then $(M)_\ell = \binom{O}{-}$, as this is the only permitted column with an O in the first row. For the same reason, $(M)_{\ell-1} = \binom{-}{U}$, and so M contains a disallowed pair $\begin{pmatrix} - & O \\ U & - \end{pmatrix}$.

On the other hand, if $*$ is one of $\{P, Q, R, S, T, -\}$, then by (22) BL_1^- cycles the first row starting in column 1, and introduces the last element in F to column 1. Table 1 ensures this symbol is $-$ since the U in the second row ensures a $-$ at the end of F. Thus $\left(BL_1^-\binom{F}{G}\right)_1 = \binom{-}{U} = (M)_1$. □

Having established the base case, we assume that $(N_{k-1}^-)_\ell = (M)_\ell$ for $\ell \leq k-1$ and show that

$$\left(N_k^-\right)_\ell = \left(BL_k^- \circ BL_{k-1}^- \circ \cdots \circ BL_1^- \begin{pmatrix} F \\ G \end{pmatrix}\right)_\ell = (M)_\ell\,, \quad \text{for } \ell \leq k$$

in each three cases of the possible columns of M in Proposition 40: when $(M)_k$ consists of consonants, when $(M)_k = \binom{O}{-}$ and when $(M)_k = \binom{-}{U}$.

Observe that $(N_{k-1}^-)_\ell = (M)_\ell$ for all $\ell \leq k-1$ implies that $(N_k^-)_\ell = (M)_\ell$ for all $\ell \leq k-1$ as BL_k^- does not change any of the first $k-1$ columns (see Lemma 34, Property 2). Thus we need only show that $(N_k^-)_k = (M)_k$.

Lemma 45 *Suppose $(M)_k$ consists of consonants, and $(N_{k-1}^-)_\ell = (M)_\ell$ for all $\ell \leq k-1$. Then $(N_k^-)_k = (M)_k$.*

Proof of Lemma 45 Suppose $(M)_k = \binom{m_{1k}}{m_{2k}}$. By Remark 33, the number of each symbol that occur in M and in N_{k-1}^- are the same. The two matrices agree on the first $k-1$ columns., so m_{1k} and m_{2k} appear in the first and second rows of N_{k-1}^- in columns ℓ_1 and ℓ_2, respectively, where $\ell_1, \ell_2 \geq k$. By Corollary 38, the order of the letters are the same in M and in N_{k-1}^-, so m_{1k} and m_{2k} are the the first letters to appear in N_{k-1}^- in column k or later, in their respective rows. By Lemma 42, the letters in columns $k, k+1, \ldots, w+m+n$ of N_{k-1}^- are left-aligned, and thus $\ell_1 = \ell_2 = k$. It follows that $(N_{k-1}^-)_k = (M)_k$, and thus $(N_{k-1}^-)_k$ consists of consonants. By Definition 35, $N_k^- = N_{k-1}^-$, and thus $(N_k^-)_k = (N_{k-1}^-)_k = (M)_k$, as desired. □

Lemma 46 *Suppose $(M)_k$ consists of $\binom{O}{-}$ and $(N_{k-1}^-)_\ell = (M)_\ell$ for all $\ell \leq k-1$. Then $(N_k^-)_k = (M)_k$.*

Proof of Lemma 46 By Corollary 38, the letters of F are in the same order as the letters of M. Thus the first row entry of $(N_k^-)_k$ is either O or $-$.

Case 1 Suppose $(N_k^-)_k = \binom{O}{*}$ for $*$ either C or U. Then either $(N_{k-1}^-)_k = \binom{O}{*}$, in which case BL_k^- does not change the first entry, or $(N_{k-1}^-)_k = \binom{*}{U}$ for $*$ some symbol not O. Observe that in the latter case, the bike lock move BL_k^- whose row shifts are specified in (22) results in a shift of the first row. Following Lemma 43, there are exactly as many copies of U in the second row, columns $k, \ldots, w+m+n$ as there are

$-$ in the first row in these columns of N_{k-1}^-. By Lemma 42, there is a $-$ at the end of the first row of N_{k-1}^-. As a consequence, $\left(N_k^-\right)_k = \left(BL_k^-(N_{k-1}^-)\right)_k = \binom{-}{U}$, contrary to assumption. Thus we may assume that $(N_{k-1}^-)_k = \binom{O}{*}$. Using Lemma 43 again, there are as many $-$ in the second row in columns $k, \ldots, w+m+n$ of N_{k-1}^- as there are Os in the first row, so that

$$\left(N_k^-\right)_k = \left(BL_k^-(N_{k-1}^-)\right)_k = \begin{pmatrix} O \\ - \end{pmatrix} = (M)_k.$$

This establishes that $(N_k^-)_k = \binom{O}{*}$ implies $(N_k^-)_k = (M)_k$.

Case 2 Suppose $\left(N_k^-\right)_k = \binom{-}{*}$ for some symbol $*$. Since $N_k^- = BL_k^-(N_{k-1}^-)$, the specification of row shifts in (22) of BL_k^- implies $(N_{k-1}^-)_k = \binom{-}{*}$ for some symbol $*$, or $(N_{k-1}^-)_k = \binom{*}{U}$ for a symbol $*$ that is not O.

If $(N_{k-1}^-)_k = \binom{*}{U}$ with $* \neq O$, then by Lemmas 43 and 42, there is a $-$ at the end of N_{k-1}^- in the first row. As a result, $(N_k^-)_k = (BL_k^-(N_{k-1}^-)_k = \binom{-}{U}$. On the other hand, if $(N_{k-1}^-)_k = \binom{-}{*}$, but $* \neq U$, then $(N_k^-)_k = \binom{-}{*}$ as BL_k^- has no effect. In either case, by Corollary 38, N_k^- must have an O in the first row of some column $\ell > k$ and $-$ in rows $k, k+1, \ldots, \ell-1$, since M has an O in the first row in column k. But then the letters of N_k^- are not left-aligned from from column $k+1$, contrary to Lemma 42.

These two cases establish that $(M)_k = \binom{O}{-}$ implies $(N_k^-)_k = (M)_k$. ◻

Lemma 47 *Suppose* $(M)_k = \binom{-}{U}$ *and* $(N_{k-1}^-)_\ell = (M)_\ell$ *for all* $\ell \leq k-1$. *Then* $(N_k^-)_k = (M)_k$.

Proof of Lemma 47 The letters of the second row of N_{k-1}^- are in the same order as those of M by Corollary 38. Thus the second entry of $(N_k^-)_k$ is either U or $-$.

Case 1 Suppose $(N_k^-)_k = \binom{*}{U}$. By Definition 35, $(N_{k-1}^-)_k$ has either an O in the first row or a U in the second row. If $(N_{k-1}^-)_k = \binom{O}{*}$, then BL_k^- rotates the second row of N_{k-1}^- starting in column k. By Lemmas 43 and 42, there is a $-$ at the end of the second row of N_{k-1}^- Therefore, using (22), $(N_k^-)_k = \binom{O}{-}$, contrary to assumption.

Alternatively $(N_{k-1}^-)_k = \binom{*}{U}$ with $* \neq O$. Then Lemmas 43 and 42 imply that there is a $-$ at the end of the first row of N_{k-1}^-. It follows that

$$\left(N_k^-\right)_k = \begin{pmatrix} - \\ U \end{pmatrix} = (M)_k\,.$$

Case 2 Suppose $(N_k^-)_k = \binom{*}{-}$. If $(N_{k-1}^-)_k$ also has a $-$ in the kth column second row, then all the entries in the second row of N_{k-1}^- in columns $k, k+1, \ldots w+m+n$ are $-$ as letters are left-aligned (see Lemma 42). However, this contradicts the fact that $(M)_k$ has a U in the second row, as the letters must be the same as those in M in columns $k, k+1, \ldots w+m+n$ (by Lemma 43).

We may therefore assume that the second row of $(N_{k-1}^-)_k$ is not $-$. In this case BL_k^- moves the second row to ensure that $(N_k^-)_k = \binom{*}{-}$. However BL_k^- moves the second row if and only if the first entry is O, so $(N^-_{k-1})_k = \binom{O}{*}$. It follows that $(N^-_k)_k = \binom{O}{-}$. Then by Corollary 38, the first letter occurring in the first row in columns $k+1, \ldots w+m+n$ of M must be O. If this occurs in column ℓ with $\ell > k$, then $(M)_\ell = \binom{O}{-}$, since this is the only permitted column with an O. For the same reason, $(M)_{\ell-1} = \binom{-}{U}$, and so M contains a disallowed pair $\begin{pmatrix} - & O \\ U & - \end{pmatrix}$. We conclude that Case 2 cannot occur. □

We finally have the ingredients to prove Proposition 40.

Proof of Proposition 40 If $S \in \widetilde{S}$, then S satisfies the three properties of the proposition, by Lemma 41. On the other hand, if M satisfies these three properties, then construct $\binom{F}{G}$ by removing all $-$ from each row, left aligning all letters, and placing all $-$ at the end of the corresponding row, as done earlier.

Observe that $(BL^-\binom{F}{G})_\ell = (N^-_k)_\ell$ whenever $\ell \le k$ (see Lemma 34). By Lemma 44, the first columns of M and N^-_1 agree. Therefore, the first columns of M and $BL^-\binom{F}{G}$ agree.

By way of induction we assume that the first $k-1$ columns of M and $BL^-\binom{F}{G}$ agree. Then by Lemma 34, the first $k-1$ columns of M and of N^-_{k-1} agree. Lemmas 45, 46 and 47 imply that the kth columns of M and N^-_k agree, and hence that the kth columns of M and of $BL^-\binom{F}{G}$ agree. We conclude that all columns of M agree with all columns of $BL^-\binom{F}{G}$, i.e. $M = BL^-\binom{F}{G}$. □

5.4 Bike Lock Moves on Elements of $\mathcal{V}$

Definition 48 We define a bike lock move $BL^\star_k$ on the set of $4 \times c$ matrices M with entries in $\{0, 1, \star\}$, with row shifts listed in Table 3. By definition, $BL^\star_k$ cyclicly rotates the entries in each row of $R_{BL^\star_k(M)}$ and columns $k, k+1, \ldots, c$ one column to the right, with the entry in the last column sent to column k.

The rows $R_{BL^\star_k(M)}$ that $BL^\star_k$ shifts depend on the columns of M, as indicated in Table 3. Let $(M)_k$ denote the kth column of M.

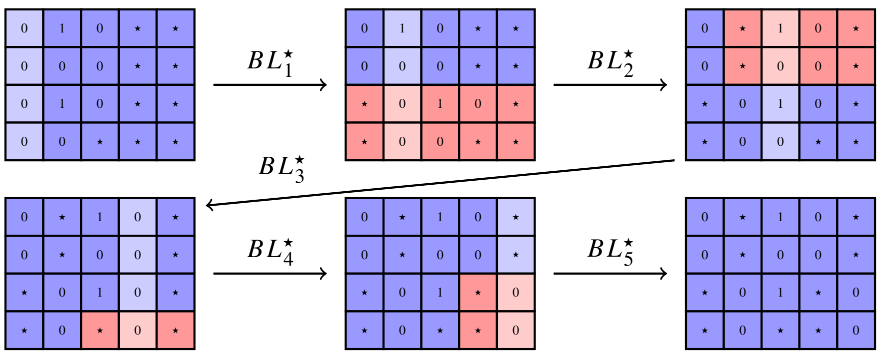
Example 49 We show that

$$BL^\star_5 \circ BL^\star_4 \circ BL^\star_3 \circ BL^\star_2 \circ BL^\star_1 \begin{pmatrix} 0\,1\,0\,\star\,\star \\ 0\,0\,0\,\star\,\star \\ 0\,1\,0\,\star\,\star \\ 0\,0\,\star\,\star\,\star \end{pmatrix} = \begin{pmatrix} 0\,\star\,1\,0\,\star \\ 0\,\star\,0\,0\,\star \\ \star\,0\,1\,\star\,0 \\ \star\,0\,\star\,\star\,0 \end{pmatrix}.$$

Apply each bike lock move referring to $R_{BL^\star_k(M)}$ in (3) for the appropriate rows to shift. In each case we highlight the column that determines the row shift, and color the impacted cells that have changed with each bike lock move.

Table 3 Rows moved by $BL_k^\star$, depending on the kth column

$(M)_k$				$R_{BL_k^\star(M)}$
$\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\\star\\0\\0\end{pmatrix}$		$\{1\}$
$\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\\1\end{pmatrix}$	$\begin{pmatrix}\star\\1\\0\\0\end{pmatrix}$		$\{2\}$
$\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\\1\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\\star\end{pmatrix}$		$\{3\}$
$\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\\1\end{pmatrix}$	$\begin{pmatrix}0\\0\\\star\\1\end{pmatrix}$		$\{4\}$
$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$		$\{1, 2\}$
$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$	$\{3, 4\}$
	else			$\emptyset$



We prove a series of properties of $BL_k^\star$ that will allow us to completely describe $\widetilde{\mathcal{V}} := \{BL^\star(V) :\ V \in \mathcal{V}\}$. Let

$$N_k^\star := BL_k^\star \circ \cdots \circ BL_1^\star(V)$$

with the convention $N_0^\star = V$.

Lemma 50 *All* 0*s and* 1*s of* $N_k^\star$ *in columns* $k + 1, \ldots, w + m + n$ *are left-aligned, meaning that all numbers occur before any* $\star$ *in these columns.*

Proof The 0s and 1s in V are all left-aligned by definition. If a row of V consists of all $\star$s it is vacuously left-aligned. Assume inductively that for all $\ell \leq k$ that the 0s and 1s in $N_\ell^\star$ in columns $k+1, \ldots, w+m+n$ are left aligned. We consider $BL_{k+1}^\star(N_k^\star)$. By hypothesis the numbers in the rows of columns $k+1$ through $w+m+n$ are left aligned; if $BL_k^\star(N_k^\star)$ is trivial they remain left aligned and in particular the numbers in columns $k+2$ to $w+m+n$ remain left-aligned.

If $BL_{k+1}^\star(N_k^\star)$ rotates one of the rows of $N_k^\star$, then in inserts the last entry in the rotated row into $(N_k^\star)_{k+1}$ and shifts the remaining entries to the right by one. Thus the numbers in the affected row remain left-aligned in columns $k+2$ and any unchanged rows also preserve the property. □

Corollary 51 *All the $\star$s of $N_k^\star$ are right aligned, meaning that if the ℓth row of $N_k^\star$ is a $\star$, then so is every entry of row ℓ in columns $k+2$ to $w+m+n$.*

Proof Since the numbers in rows $k+1$ to $w+m+n$ are left-aligned by Lemma 50, the entries to the right of all of the numbers in a row in columns $k+2$ to $w+m+n$ must all be $\star$s. □

Lemma 52 *Suppose $V \in \mathcal{V}$, or V has counts of 0s, 1s, and $\star$s given in Table 2. For $k = 0, 1, \ldots, w+n+m$,*

- *The number of $\star$s in the first row of $N_k^\star$ in columns $k+1, \ldots, w+m+n$ is the same as the number of 0s in the fourth row of $N_k^\star$ in columns $k+1, \ldots, w+m+n$;*
- *The number of $\star$s in the second row of $N_k^\star$ in columns $k+1, \ldots, w+m+n$ is the same as the number of 0s in the third row of $N_k^\star$ in columns $k+1, \ldots, w+m+n$;*
- *The number of $\star$s in the third row of $N_k^\star$ in columns $k+1, \ldots, w+m+n$ is the same as the number of 0s in the first row of $N_k^\star$; and*
- *The number of $\star$s in the fourth row of $N_k^\star$ in columns $k+1, \ldots, w+m+n$ is the same as the number of 0s in the second row of $N_k^\star$.*

Proof By referencing Table 2 the above statement is true for $k = 0$. Assume by induction that the statements in the lemma hold for all $0 \leq \ell < k$ and consider $BL_k^\star(N_{k-1}^\star)$. We consider each row separately.

By Lemma 50, if there is a $\star$ in the first row of $(N_{k-1}^\star)_k$ then all the remaining entries of the first row must also be $\star$s and therefore by the inductive assumption all the remaining entries in the fourth row of $N_{k-1}^\star$ must be 0.

Note that 1 or 2 is in the row set of $BL_k^\star$ applied to $N_{k-1}^\star$ implies neither 3 nor 4 is in the row set (see Table 3). In particular, in these cases, the third and fourth rows of $N_k^\star$ and $N_{k-1}^\star$ are identical.

If $1 \in R_{BL_k^\star(N_{k-1}^\star)}$ then by referring to Table 3 we see that there must be a 0 in the fourth row of $(N_{k-1}^\star)_k$. By the inductive hypothesis there must be a $\star$ in the first row in columns k through $w+m+n$, by Corollary 51 a $\star$ occur in the last entry of the first row. Thus $BL_k^\star$ rotates a $\star$ into the first row of $(N_{k-1}^\star)_k$. Since the fourth row of $N_k^\star$ is identical to that of $N_{k-1}^\star$, both the number of $\star$s in the first row and 0s in the fourth row of columns $k+1$ to $w+m+n$ of $N_k^\star$ decrease by one.

Similarly, if $2 \in R_{BL_k^\star(N_{k-1}^\star)}$, then Table 3 implies there is a 0 in the third row of $(N_{k-1}^\star)_k$, and also that $3 \notin R_{BL_k^\star}$. By the inductive assumption there must be a $\star$

in the second row among columns $k, \ldots, w+m+n$ of $N^\star_{k-1}$ of $(N^\star_{k-1})_k$, and by Corollary 51, such a $\star$ is found at the end of the second row. Furthermore, since the third row is not cycled by $BL^\star_k$, the matrix $N^\star_k$ has one fewer 0 in row 3, and one fewer $\star$ in row 2, in columns $k+1, \ldots, w+m+n$, compared to the number of each in columns $k, \ldots, w+m+n$ of $N^\star_{k-1}$. Thus the properties of the Lemma hold for $N^\star_k$.

A similar argument applies to prove the case when 3 or 4 is in the row set of $BL^\star_k$ applied to $N^\star_{k-1}$. □

Corollary 53 *For $k = 1, \ldots, w+m+n$, the bike lock move $BL^\star_k$ applied to $N^\star_{k-1}$ either leaves it unchanged, or inserts $\star$ into the kth column.*

Proof If $BL^\star_k$ shifts the first row of $N^\star_{k-1}$ then there is a 0 in the fourth row of $(N^\star_{k-1})_k$. By Lemma 52 and Corollary 51 there is a $\star$ in the first row of $N^\star_{k-1}$ in the $w+m+n$ column. Thus $BL^\star_k$ rotates a $\star$ into the first row of $(N^\star_{k-1})_k$.

Similar arguments apply to prove the cases when $BL^\star_k$ shifts the second, third, and fourth rows of $N^\star_{k-1}$. Thus if $BL^\star_k$ shifts the ℓth of $N^\star_{k-1}$, it cycles a $\star$ into the ℓth row of $(N^\star_{k-1})_k$. □

Corollary 54 *The numbers of $N^\star_k$ are in the same order as the numbers of V for all $k = 0, \ldots, w+m+n$.*

Proof By Corollary 53 if $BL^\star_k$ shifts row ℓ of $N^\star_{k-1}$ then it cycles the entries of row ℓ in columns k to $w+m+n$ to the right by 1. By Corollary 53 $BL^\star_k$ always shifts a $\star$ into the kth column of $N^\star_{k-1}$ and so the original order of the numbers is preserved. □

Lemma 55 *The composition*

$$BL^\star := BL^\star_{w+m+n} \circ \cdots \circ BL^\star_2 \circ BL^\star_1$$

is a bijective map from $\mathcal{V}$ to $\widetilde{\mathcal{V}} := \{BL^\star(V) : \; V \in \mathcal{V}\}$.

Proof of Lemma 55 We verify that $BL^\star$ is injective. Suppose that $BL^\star(V) = BL^\star(V')$ for some $V, V' \in \mathcal{V}$. By Corollary 54 the order of the 0s and 1s is preserved from V to $BL^\star(V)$ and V' to $BL^\star(V')$, implying that the sequence of 0s and 1s in each row of V and V' are the same since $BL^\star(V) = BL^\star(V')$. Since the 0s and 1s in V and V' are left-aligned, it must be the case that $V = V'$. Hence $BL^\star$ injects onto its image. □

We now characterize the elements of $\widetilde{\mathcal{V}}$ by a careful accounting of what each bike lock move $BL^\star_k$ does to columns of $N^\star_k := BL^\star_k \circ \ldots BL^\star_1(V)$ for $V \in \mathcal{V}$.

Proposition 56 *Elements of $\widetilde{\mathcal{V}}$ are exactly $4 \times (w+m+n)$ matrices M satisfying the following:*

1. *The columns consist only of 7 types:*

$$\begin{pmatrix}\star\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\\star\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\\star\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\\\star\end{pmatrix}, \begin{pmatrix}\star\\\star\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\\star\\\star\end{pmatrix}, \; or \; \begin{pmatrix}1\\1\\1\\1\end{pmatrix}.$$

2. *There are no pairs of adjacent columns of the form* $\begin{pmatrix} \star & 0 \\ \star & 0 \\ 0 & \star \\ 0 & \star \end{pmatrix}$.
3. *The number of times each* 1, 0*, or* $\star$ *appears in each row of* M *is given in Table* 2.

We prove Proposition 56 via a series of lemmas.

Lemma 57 *Elements of* $\widetilde{\mathcal{V}}$ *satisfy the three conditions of Proposition* 56.

Proof of Lemma 57 We first show that elements $BL^{\star}(V) \in \widetilde{\mathcal{V}}$ satisfy Property 1, noting that $(BL^{\star}(V))_k = (N_k^{\star})_k$.

If the kth column of $N_{k-1}^{\star}$ consists of only 0s and 1s, then by referencing Table 3 and by Corollary 53 the possibilities for the kth column of $N_k^{\star}$ are

$$(BL^{\star}(V))_k = (N_k^{\star})_k = \begin{pmatrix} \star \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ \star \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \star \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ \star \end{pmatrix}, \begin{pmatrix} \star \\ \star \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \star \\ \star \end{pmatrix}, \text{ or } \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Suppose the kth column of $N_{k-1}^{\star}$ contains a $\star$. We consider each row separately. If $\star$ is the entry of the first row of $(N_{k-1}^{\star})_k$, then every subsequent entry in the first row is $\star$ since the 0s and 1s of $N_{k-1}^{\star}$ are left aligned (see Corollary 51). Hence by Table 2 and Lemma 52, every entry in the fourth row, columns $k, \ldots, w+m+n$, must be a 0. Since there are no $\star$s in the fourth row in columns $k, \ldots, w+m+n$, no entry in the second row of $N_{k-1}^{\star}$ in these columns can be a 0. Therefore the second entry of $(N_{k-1}^{\star})_k$ is 1 or $\star$.

If the entry in the second row of $(N_{k-1}^{\star})_k$ is a $\star$, by a similar argument the entry in the third row of $(N_{k-1}^{\star})_k$ must be a 0 and so $(N_{k-1}^{\star})_k = (\star, \star, 0, 0)^T$.

Suppose $(N_{k-1}^{\star})_k$ has 1 in its second row. Since there are no 0s in the first row of $N_{k-1}^{\star}$, columns $k, \ldots, w+m+n$, there are no $\star$s in the third row in these columns. Thus the only possibilities for $(N_{k-1}^{\star})_k$ are $(N_{k-1}^{\star})_k = (\star, 1, 1, 0)^T$, which is one of the 7 types listed in the first property, or $(N_{k-1}^{\star})_k = (\star, 1, 0, 0)^T$. If $(N_{k-1}^{\star})_k = (\star, 1, 0, 0)^T$, then $BL_k^{\star}$ rotates the second row of $N_{k-1}^{\star}$ and Corollary 53 ensures that $(BL^{\star}(V))_k = (N_k^{\star})_k = (\star, \star, 0, 0)^T$, also one of the types listed in Property 1.

Similar arguments show that if the second, third, or fourth rows of $(N_{k-1}^{\star})_k$ are $\star$ then either $(N_{k-1}^{\star})_k$ is already one of the 7 types or $BL_k^{\star}$ shifts a $\star$ into an appropriate row so that $(BL^{\star}(V))_k = (N_k^{\star})_k$ is one of the types listed in Property 1.

To prove the second property we check that if the kth column of $M \in \widetilde{\mathcal{V}}$ is $(\star, \star, 0, 0)^T$ then the $k+1$st column cannot be $(0, 0, \star, \star)^T$. There are three possiblities for $(N_{k-1}^{\star})_k$ that could lead to $(M)_k = (N_k^{\star})_k = (\star, \star, 0, 0)^T$.

First suppose that $R_{BL_k^{\star}(N_{k-1}^{\star})} = \emptyset$, in which case $(N_{k-1}^{\star})_k = (\star, \star, 0, 0)^T$. By Corollary 51 and Lemma 52 $(N_{k-1}^{\star})_{\ell} = (\star, \star, 0, 0)^T$ for all $k \le \ell \le w+m+n$; in particular $(N_{k+1}^{\star})_{k+1} \neq (0, 0, \star, \star)^T$.

The second possibility is that $R_{BL_k^\star(N_{k-1}^\star)} = \{1\}$. By Table 3 the only choice for $(N_{k-1}^\star)_k$ is $(1, \star, 0, 0)^T$. By Lemma 51 every subsequent entry in the second row of $N_{k-1}^\star$ must be $\star$, which excludes $(N_{k+1}^\star)_{k+1} = (0, 0, \star, \star)^T$.

The last possibility is that $R_{BL_k^\star(N_{k-1}^\star)} = \{2\}$. Again by referencing Table 3 it must be the case that $(N_{k-1}^\star)_k = (\star, 1, 0, 0)^T$. Thus once again by Lemma 51 every entry in the second row of $N_{k-1}^\star$ to the right of the kth column must also be a $\star$ and so $(M)_{k+1} = (N_{k+1}^\star)_{k+1} \neq (0, 0, \star, \star)^T$.

Finally, Property 3 is immediately satisfied by Corollary 54. □

We now verify that any M satisfying the properties of Proposition 56 is $BL^\star(V)$ for some $V \in \mathcal{V}$, and hence $M \in \widetilde{\mathcal{V}}$.

Lemma 58 *Let M be any matrix satisfying Properties 1, 2, and 3 in Proposition 56, and let V be the matrix obtained by right justifying the $\star$s in M. Then $BL^\star(V) = M$.*

Proof Observe that $V \in \mathcal{V}$ due to Property 3. Suppose that the first $k - 1$ columns of $BL^\star(V)$ and M agree, and consider the kth column (k could be 1). If the kth column of $BL^\star(V)$ is any of $(\star, 1, 1, 0)^T$, $(1, \star, 0, 1)^T$, $(0, 1, \star, 1)^T$, $(1, 0, 1, \star)^T$, or $(1, 1, 1, 1)^T$, so too must be the kth column of M, to have preserved the order of the 0s and 1s when removing $\star$s from M to form V.

Suppose the kth column of $BL^\star(V)$ is $(\star, \star, 0, 0)^T$. An appearance of a 1 in the 3rd or 4th row of the kth column of M would disrupt the order of 0s and 1s. Thus the kth column of M must contain only $\star$ or 0s in the 3rd and 4th rows. Only $(\star, \star, 0, 0)^T$ and $(0, 0, \star, \star)^T$ satisfy this condition.

If M has $(\star, \star, 0, 0)^T$ in the kth column, we're done. If not, then it must have $(0, 0, \star, \star)^T$ in the kth column. Then the appearance of a 1 in the first or second row of the $(k + 1)$st column of $BL^\star(V)$ would disrupt the order of the 0s and 1s. Thus the $(k + 1)$st column of $BL^\star(V)$ must contain only $\star$ or 0s in the 1st or 2nd rows. Of the seven possibilities, only $(\star, \star, 0, 0)^T$ and $(0, 0, \star, \star)^T$ satisfy this condition. By the same reasoning, $BL^\star(V)$ has $(\star, \star, 0, 0)^T$ in all subsequent columns, until the first occurrence of $(0, 0, \star, \star)^T$, guaranteed to occur by a simple count. It follows that $BL^\star(V)$ has two adjacent columns of the form disallowed by Property 2.

A similar argument works if the kth column of $BL^\star(V)$ is $(0, 0, \star, \star)^T$. If M has $(0, 0, \star, \star)^T$ in the kth column, we're done. If not, then it must have $(\star, \star, 0, 0)^T$ in the kth column. By the same reasoning, M has $(\star, \star, 0, 0)^T$ in all subsequent columns, until the first occurrence of $(0, 0, \star, \star)^T$, guaranteed to occur by a simple count. Then M has two adjacent columns of the form disallowed by Property 2. □

Proof of Proposition 56 By Lemma 57 any $M \in \widetilde{\mathcal{V}}$ satisfies the conditions listed in Proposition 56; by Lemma 58 any matrix M satisfying the properties is $BL^\star(V)$ for some $V \in \mathcal{V}$ and hence $M \in \widetilde{\mathcal{V}}$. □

5.5 Bijection Between $\mathcal{V}$ and $\mathcal{S}$

Finally, we complete the proof of Theorem 9 by establishing the bijection between sets of the right size.

Proposition 59 *There is a bijection between sets $\mathcal{V}$ and $\mathcal{S}$.*

Proof We establish a bijection between the sets $\widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{S}}$, by mapping the seven vectors $\begin{pmatrix}\star\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\\star\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\\star\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\\\star\end{pmatrix}, \begin{pmatrix}\star\\\star\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\\star\\\star\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ listed in Lemma 56 for elements of $\widetilde{\mathcal{V}}$ to the seven 2-vectors

$$\begin{pmatrix}T\\C\end{pmatrix}, \begin{pmatrix}S\\C\end{pmatrix}, \begin{pmatrix}P\\C\end{pmatrix}, \begin{pmatrix}Q\\C\end{pmatrix}, \begin{pmatrix}-\\U\end{pmatrix}, \begin{pmatrix}O\\-\end{pmatrix}, \begin{pmatrix}R\\C\end{pmatrix}$$

of elements of $\widetilde{\mathcal{S}}$, respectively. Note that the excluded configurations of $\widetilde{\mathcal{V}}$ correspond exactly to the excluded configurations of $\widetilde{\mathcal{S}}$

Lemmas 39 and 55 establish bijections from $\mathcal{S}$ to $\widetilde{\mathcal{S}}$ and from $\mathcal{V}$ to $\widetilde{\mathcal{V}}$, respectively. Thus there is a bijection $\mathcal{S} \to \mathcal{V}$. □

It follows that $|\mathcal{S}| = |\mathcal{V}|$. Since $|\mathcal{V}|$ is given by the left-hand side of Eq. (6) and $|\mathcal{S}|$ is given by the right side of Eq. (6), we have concluded the proof of Theorem 9.

As an immediate corollary, we obtain Verdermonde's Identity. Let $n = 0$ in Theorem 9, and substitute $a = x$, $b = y - x + m$, $s = m$, and $r = i$.

Corollary 60 (Vandermonde) *Let $a, b \in \mathbb{Z}$. Then*

$$\binom{a+b}{s} = \sum_r \binom{a}{r}\binom{b}{s-r}.$$

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References

1. Abe, H., Horiguchi, T., Kuwata, H., Zeng, H.: Geometry of Peterson Schubert calculus in type A and left-right diagrams (2021). ArXiv:2104.02914 [math.AG]
2. Abe, T., Horiguchi, T., Masuda, M., Murai, S., Sato, T.: Hessenberg varieties and hyperplane arrangements. J. Reine Angew. Math. **2020**(764), 241–286 (2020)
3. Andersen, H.H., Jantzen, J.C., Soergel, W.: Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: independence of p. Astérisque **220**, 321 (1994)
4. Billey, S.C.: Kostant polynomials and the cohomology ring for G/B. Duke Math. J. **96**(1), 205–224 (1999)
5. Brion, M.: Poincaré duality and equivariant (co)homology. Michigan Math. J. **48**, 77–92 (2000)
6. Brion, M., Carrell, J.B.: The equivariant cohomology ring of regular varieties. Michigan Math. J. **52**, 189–203 (2002)

7. Chang, T., Skjelbred, T.: The topological Schur lemma and related results. Ann. Math. **100**(2), 307–321 (1974)
8. Drellich, E.: Monk's rule and Giambelli's formula for Peterson varieties of all Lie types. J. Algebr. Comb. **41**(2), 539–575 (2015)
9. Fukukawa, Y., Harada, M., Masuda, M.: The equivariant cohomology rings of Peterson varieties. J. Math. Soc. Jpn. **67**(3), 1147–1159 (2015)
10. Goldin, R., Mihalcea, L., Singh, R.: Positivity of Peterson Schubert calculus (2021). ArXiv:2106.10372 [math.AG]
11. Goresky, M., Kottwitz, R., MacPherson, R.: Equivariant cohomology, Koszul duality, and the localization theorem. Invent. Math. **131**(1), 25–83 (1997)
12. Graham, W.: Positivity in equivariant Schubert calculus. Duke Math. J. **109**(3), 599–614 (2001)
13. Harada, M., Horiguchi, T., Masuda, M.: The equivariant cohomology rings of Peterson varieties in all Lie types. Can. Math. Bull. **58**, 80–90 (2014)
14. Harada, M., Tymoczko, J.S.: A positive Monk formula in the S^1-equivariant cohomology of type A Peterson varieties. Proc. Lond. Math. Soc. **103**(1), 40–72 (2011)
15. Insko, E., Tymoczko, J.: Intersection theory of the Peterson variety and certain singularities of Schubert varieties. Geom. Dedicata **180**, 95–116 (2016)
16. Kostant, B.: Flag manifold quantum cohomology, the Toda lattice, and the representation with highest weight ρ. Sel. Math. New Ser. **2**(1), 43–91 (1996)
17. Peterson, D.: Quantum cohomology of G/P (1997). Lecture Notes, M.I.T
18. Rietsch, K.: Totally positive Toeplitz matrices and quantum cohomology of partial flag varieties. J. Am. Math. Soc. **16**(2), 363–392 (2003)
19. Rietsch, K.: A mirror symmetric construction of $qH_T^*(G/P)_{(q)}$. Adv. Math. **217**, 2401–2442 (2008)
20. Székely, L.A.: Common origin of cubic binomial identities; a generalization of Surányi's proof on Le Jen Shoo's formula. J. Comb. Theory Ser. A. **40**(1), 171–174 (1985)
21. Tymoczko, J.S.: Paving Hessenberg varieties by affines. Sel. Math. New Ser. **13**(2), 353–367 (2007)