INVISCID LIMIT OF COMPRESSIBLE VISCOELASTIC EQUATIONS WITH THE NO-SLIP BOUNDARY CONDITION

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Abstract. The inviscid limit for the two-dimensional compressible viscoelastic equations in the half plane is considered under the no-slip boundary condition. When the initial deformation tensor is a perturbation of the identity matrix and the initial density is near a positive constant, we establish the uniform estimates of solutions to the compressible viscoelastic flows in the conormal Sobolev spaces. It is well-known that for the corresponding inviscid limit of the compressible Navier-Stokes equations with the no-slip boundary condition, one does not expect the uniform energy estimates of solutions due to the appearance of strong boundary layers. However, when the deformation tensor effect is taken into account, our results show that the deformation tensor plays an important role in the vanishing viscosity process and can surprisingly prevent the formation of strong boundary layers. As a result we are able to justify the inviscid limit of solutions for the compressible viscous flows under the no-slip boundary condition governed by the viscoelastic equations, based on the uniform conormal regularity estimates achieved in this paper.

1. Introduction

In this paper we consider the inviscid limit for the two-dimensional compressible viscoelastic equations in the half plane:

$$\begin{cases} \partial_t\rho^\varepsilon + \nabla\cdot(\rho^\varepsilon \mathbf{u}^\varepsilon) = 0, \\ \rho^\varepsilon\partial_t\mathbf{u}^\varepsilon + \rho^\varepsilon\mathbf{u}^\varepsilon\cdot\nabla\mathbf{u}^\varepsilon - \varepsilon\mu\triangle\mathbf{u}^\varepsilon - \varepsilon(\mu+\lambda)\nabla\mathrm{div}\mathbf{u}^\varepsilon + \nabla p(\rho^\varepsilon) = \mathrm{div}(\rho^\varepsilon\mathbf{F}^\varepsilon\mathbf{F}^{\varepsilon\top}), \\ \partial_t\mathbf{F}^\varepsilon + \mathbf{u}^\varepsilon\cdot\nabla\mathbf{F}^\varepsilon = \nabla\mathbf{u}^\varepsilon\cdot\mathbf{F}^\varepsilon, \quad t>0, \quad \mathbf{x}=(x,y)\in\mathbb{R}^2_+ := \mathbb{R}\times\mathbb{R}_+, \end{cases} \tag{1.1}$$

where ρ^ε denotes the density, $\mathbf{u}^\varepsilon = (u^\varepsilon, v^\varepsilon)$ the velocity, and $\mathbf{F}^\varepsilon = (F_1^\varepsilon, F_2^\varepsilon)^\top$ the deformation tensor matrix with $F_1^\varepsilon = (1+f_1^\varepsilon, f_2^\varepsilon), F_2^\varepsilon = (f_3^\varepsilon, 1+f_4^\varepsilon)$; the viscosity coefficients $\mu\varepsilon$ and $\lambda\varepsilon$ satisfy $\mu>0$ and $(\mu+\lambda)>0$ with $\varepsilon\in(0,1)$ being a small parameter, and the pressure $p(\rho)$ is a function of the density ρ that is given by the following formula in the isentropic case:

$$p(\rho) = \rho^\gamma, \qquad \gamma\ge 1, \tag{1.2}$$

where γ is the adiabatic constant. We refer the readers to [7,19,34] for the discussions on the physical background of viscoelasticity. The initial data of (1.1) is given by

$$\rho^\varepsilon(0,x,y) = \rho_0(x,y), \quad \mathbf{u}^\varepsilon(0,x,y) = \mathbf{u_0}(x,y), \quad \mathbf{F}^\varepsilon(0,x,y) = \mathbf{F_0}(x,y), \tag{1.3}$$

and the no-slip boundary condition is imposed on the velocity,

$$\mathbf{u}^\varepsilon(t,x,0) = \mathbf{0}. \tag{1.4}$$

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Since the equations of deformation tensor $\mathbf{F}^\varepsilon$ are a hyperbolic system, one does not need to impose any boundary condition for $\mathbf{F}^\varepsilon$ due to (1.4), and the value of $\mathbf{F}^\varepsilon$ on the boundary is determined by its initial value. In this paper, we consider the case that

$$\mathbf{F_0}(x,0) = \mathbb{I}_{2\times 2}, \tag{1.5}$$

where $\mathbb{I}_{2\times 2}$ is a 2×2 identity matrix. Formally, when $\varepsilon = 0$, the equations in (1.1) are reduced to the following ideal compressible elastodynamic equations:

$$\begin{cases} \partial_t\rho^0 + \nabla\cdot(\rho^0\mathbf{u}^0) = 0, \\ \rho^0\partial_t\mathbf{u}^0 + \rho^0\mathbf{u}^0\cdot\nabla\mathbf{u}^0 + \nabla p(\rho^0) = \mathrm{div}(\rho^0\mathbf{F}^0\mathbf{F}^{0\top}), \\ \partial_t\mathbf{F}^0 + \mathbf{u}^0\cdot\nabla\mathbf{F}^0 = \nabla\mathbf{u}^0\cdot\mathbf{F}^0, \quad t>0, \quad \mathbf{x}=(x,y)\in\mathbb{R}^2_+. \end{cases} \tag{1.6}$$

The aim of this paper is to justify the vanishing viscosity limit from the viscoelastic equations (1.1) to the inviscid elastodynamic equations (1.6) as $\varepsilon \to 0$ under the no-slip boundary condition (1.4) in the half plane.

There have been extensive studies on the existence of solutions to both the incompressible and compressible viscoelastic equations; see [13, 15, 16, 22–24, 28], the survey paper [14] and the references therein. The inviscid limit of solutions for the Cauchy problem was studied in many papers such as [1, 6, 20, 30, 38] for the incompressible Navier-Stokes equations and in [3] for the incompressible viscoelastic equations; see also [5, 8, 12, 17, 21] and their references for other related vanishing viscosity limits of the Cauchy problem for the compressible Navier-Stokes equations. When the inviscid limit problem is considered in a domain with a physical boundary, the vanishing viscosity limit problem is usually more challenging due to the possible presence of boundary layers [10,32,37,39,41]. In particular, if a strong boundary layer appears, the inviscid limit usually becomes extremely difficult because of the uncontrollability of the vorticity of boundary layer corrector. If the no-slip boundary condition (1.4) is replaced by the so-called Navier-slip boundary conditions, the strong boundary layer will disappear, and the inviscid limit has been established in [40,42] for the compressible Navier-Stokes equations. For the corresponding inviscid limit of the incompressible Navier-Stokes equations with the Navier-slip boundary conditions, we refer the readers to [2, 4, 18, 31, 43] and the references therein.

When the no-slip boundary condition is imposed, the inviscid limit problem in a domain with a boundary is more complicated and less developed in analysis. To the best of our knowledge, the inviscid limit of the unsteady incompressible Navier-Stokes equations with the no-slip boundary condition was proved only in the analytic function framework or in the Gevrey settings; see [9, 29, 35, 36] and the references therein. For the incompressible magneto-hydrodynamic (MHD) equations with the no-slip boundary condition, the well-posedness of solutions to the MHD boundary layer equations and the validity of Prandtl boundary layer expansion in the Sobolev spaces were obtained in [25, 26] provided that the tangential component of magnetic field does not degenerate near the physical boundary initially; and it was proved in [27] that there are no strong boundary layers in the inviscid limit for the incompressible non-resistive MHD system when the normal component of magnetic field does not degenerate near the physical boundary initially. However, the inviscid limit of the compressible Navier-Stokes equations with the no-slip boundary condition in the half plane is still open, except for the linearized Navier-Stokes equations [44], even in the analytic function spaces or in the Gevrey class owing to the appearance of strong boundary layers [10, 41]. In this paper, we consider the inviscid limit for the compressible viscoelastic equations in the half plane with the no-slip boundary condition. We find that the deformation tensor in viscoelasticity has a significant effect on the vanishing viscosity process and can prevent the formation of strong boundary layers. For this reason we are able to justify the inviscid limit of solutions for the compressible viscous flows governed by the viscoelastic equations (1.1) under the no-slip boundary condition.

To formulate our main results, let us define the conormal Sobolev spaces that will be used in this paper. Set the conormal derivative operators as the following:

$$Z_0 = \partial_t, \qquad Z_1 = \partial_x, \qquad Z_2 = \varphi(y)\partial_y, \qquad Z^\alpha = Z_0^{\alpha_0} Z_1^{\alpha_1} Z_2^{\alpha_2},$$

with $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ and $|\alpha| = \alpha_0 + \alpha_1 + \alpha_2$. Here the weight function $\varphi(y)$ satisfies $\varphi(0) = 0$, $\varphi'(0) > 0$, $\|\partial_y^i \varphi\|_{L^\infty} \leq C$ ($i = 0, ..., m$ for some integer $m > 0$), and $\varphi(y)$ has uniform lower and upper positive bounds away from the physical boundary, that is $C^{-1} \leq \varphi(y) \leq C$ for some $C > 1$ when $y \geq \delta > 0$ with some constant $\delta > 0$. For example, $\varphi(y) = y/(1+y)$ may be used as a weight function. Define the following two conormal Sobolev spaces:

$$H^m_{co}([0,t] \times \mathbb{R}^2_+) = \{f : Z^\alpha f \in L^2([0,t] \times \mathbb{R}^2_+), |\alpha| \leq m\},$$

and

$$\mathcal{H}^m_{co}([0,t] \times \mathbb{R}^2_+) = \{f : Z^\alpha f \in L^\infty([0,t], L^2(\mathbb{R}^2_+)), |\alpha| \leq m\}.$$

For a given $t > 0$,

$$\|f(t)\|_m^2 = \sum_{|\alpha| \leq m} \|Z^\alpha f(t,\cdot)\|^2_{L^2(\mathbb{R}^2_+)},$$

then

$$\|f\|^2_{H^m_{co}} = \int_0^t \|f(s)\|_m^2 ds, \qquad \|f\|^2_{\mathcal{H}^m_{co}} = \sup_{0 \leq s \leq t} \|f(s)\|^2_m.$$

As usual we use the notation:

$$W^{m,\infty}_{co}([0,t] \times \mathbb{R}^2_+) = \{f : Z^\alpha f \in L^\infty([0,t] \times \mathbb{R}^2_+), |\alpha| \leq m\},$$

and

$$\|f(t)\|_{m,\infty} = \sum_{|\alpha| \leq m} \|Z^\alpha f(t,\cdot)\|_{L^\infty}.$$

Denote the energy by

$$\begin{aligned} N_m(t) =& \|(\rho^\varepsilon - 1, \mathbf{u}^\varepsilon, \mathbf{F}^\varepsilon - \mathbb{I}_{2\times 2})\|^2_{\mathcal{H}^m_{co}} + \varepsilon \left(\|\partial_y(\rho^\varepsilon, f_2^\varepsilon)\|^2_{\mathcal{H}^{m-1}_{co}} + \|\partial_y^2(\rho^\varepsilon, f_2^\varepsilon)\|^2_{\mathcal{H}^{m-2}_{co}} \right) \\ &+ \|\partial_y(\rho^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{F}^\varepsilon)\|^2_{H^{m-1}_{co}} + \|\partial_y^2(\rho^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{F}^\varepsilon)\|^2_{H^{m-2}_{co}} + \varepsilon \|\nabla \mathbf{u}^\varepsilon\|^2_{H^m_{co}} \\ &+ \varepsilon^2 \left(\|\partial_y^2 \mathbf{u}^\varepsilon\|^2_{H^{m-1}_{co}} + \|\partial_y^3 \mathbf{u}^\varepsilon\|^2_{H^{m-2}_{co}} \right). \end{aligned} \tag{1.7}$$

We always take $0 < \varepsilon < 1$ and define

$$\Lambda^m(t) = \{(\rho - 1, \mathbf{u}, \mathbf{F} - \mathbb{I}_{2\times 2}) \in \mathcal{H}^m_{co},\ \partial_y(\rho, \mathbf{u}, \mathbf{F}) \in H^{m-1}_{co},\ \partial_y^2(\rho, \mathbf{u}, \mathbf{F}) \in H^{m-2}_{co}\}.$$

Now we state our main theorem as follows.

Theorem 1.1. *Let $m > 8$ be an integer. Suppose that the initial data $(\rho_0, \mathbf{u_0}, \mathbf{F_0})$ satisfies*

$$\|(\rho_0 - 1, \mathbf{u_0}, \mathbf{F_0} - \mathbb{I}_{2\times 2})\|^2_m + \|\partial_y(\rho_0, \mathbf{u_0}, \mathbf{F_0})\|^2_{m-1} + \|\partial_y(\nabla\rho_0, \nabla\mathbf{u_0}, \nabla\mathbf{F_0})\|^2_{m-2} \leq \sigma_0, \tag{1.8}$$

for some sufficiently small positive constant σ_0, and

$$\rho_0 \det(\mathbf{F_0}) = 1, \qquad \operatorname{div}(\rho_0 \mathbf{F}_0^\top) = 0. \tag{1.9}$$

Then, there exist a time $T>0$ that is independent of ε and a unique solution $U^\varepsilon=(\rho^\varepsilon,\mathbf{u}^\varepsilon,\mathbf{F}^\varepsilon)$ $\in\Lambda^m(T)$ to (1.1)-(1.4), such that
(1) *the following estimate holds for $t\in[0,T]$,*

$$N_m(t)+\|(\rho^\varepsilon-1,\mathbf{u}^\varepsilon,\mathbf{F}^\varepsilon-\mathbb{I}_{2\times 2})(t)\|_{1,\infty}+\|\nabla(\rho^\varepsilon,\mathbf{u}^\varepsilon,\mathbf{F}^\varepsilon)(t)\|_{1,\infty}\le C\sigma_0, \tag{1.10}$$

where $C>0$ is some constant independent of ε;
(2) *there exists a function $U^0=(\rho^0,\mathbf{u^0},\mathbf{F^0})\in\Lambda^m(T)$ satisfying the following limit:*

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]}\left\|(U^\varepsilon-U^0,\partial_y(U^\varepsilon-U^0))(\cdot,t)\right\|_{L^\infty(\mathbb{R}^2_+)}=0, \tag{1.11}$$

and $U^0=(\rho^0,\mathbf{u^0},\mathbf{F^0})$ is a unique solution to the ideal compressible elastodynamic equations (1.6) with the same initial data $(\rho_0,\mathbf{u_0},\mathbf{F_0})$ and the no-slip boundary condition.

Remark 1.1. Since we are considering the solutions to the compressible flows of viscoelasticity in the conormal Sobolev spaces, we need to avoid the appearance of vacuum and degeneracy of deformation tensor matrix, which is guaranteed by the smallness condition (1.8).

Remark 1.2. The time regularity requirements on the initial data can be changed to the spatial regularity requirements through the equations. We believe that the regularity requirements in Theorem 1.1 are not optimal.

Remark 1.3. It is noted that the identity matrix $\mathbb{I}_{2\times 2}$ is not essential in the analysis. In fact, we only need to assume that the component $1+f_4^\varepsilon$ is not zero initially. We choose the initial data of the deformation tensor as a small perturbation of the identity matrix solely for the sake of simplicity of presentation. Moreover, the form of pressure is also not essential, and our results can be extended to more general forms of pressure without causing more difficulties.

Remark 1.4. Based on the uniform conormal energy estimates (1.10) achieved in the first part of Theorem 1.1, the inviscid limit in the second part of Theorem 1.1 can be regarded as a direct consequence of the first part by using some compactness arguments as in [31].

Next we shall explain the main difficulties and the strategy to prove the main theorem. It is well known that when the inviscid limit is considered in a domain with a physical boundary, the uniform estimates of normal derivatives for solutions with respect to the small viscosity parameter are very difficult to obtain. Usually, it is impossible to achieve these uniform estimates due to the presence of strong boundary layers for the solutions to both the incompressible and compressible Navier-Stokes equations with the no-slip boundary condition. Surprisingly, if the deformation tensor in viscoelasticity is taken into account, even though the no-slip boundary condition is imposed on the velocity, the uniform estimates of normal derivatives for solutions to the compressible viscoelastic fluid equations can be achieved, which is the main finding of this paper. In other words, our results in Theorem 1.1 show that the deformation tensor can prevent the strong boundary layers from occurring. These observations are obviously different from both the compressible and incompressible Navier-Stokes equations with the no-slip boundary condition. The effect of the deformation tensor is essentially used in deriving the conormal energy estimates.

We shall present below our strategy to establish the uniform estimates of normal derivatives for all components of $\mathbf{u}^\varepsilon,\mathbf{F}^\varepsilon$ and p^ε in four main steps.

Step I: Estimates of $\partial_y v^\varepsilon$ and $\partial_y u^\varepsilon$. From the second and third equations in (4.2) on the deformation tensor $\mathbf{F}^\varepsilon$, we can write the normal derivatives in terms of the components of $\mathbf{F}^\varepsilon$

as the following:

$$\partial_y v^\varepsilon = \frac{1}{1+f_4^\varepsilon}\left(\partial_t f_4^\varepsilon + u^\varepsilon \partial_x f_4^\varepsilon + v^\varepsilon \partial_y f_4^\varepsilon - f_2^\varepsilon \partial_x v^\varepsilon\right),$$

and

$$\partial_y u^\varepsilon = \frac{1}{1+f_4^\varepsilon}\left(\partial_t f_2^\varepsilon + u^\varepsilon \partial_x f_2^\varepsilon + v^\varepsilon \partial_y f_2^\varepsilon - f_2^\varepsilon \partial_x u^\varepsilon\right).$$

Then, using the estimate

$$\begin{aligned}\|v^\varepsilon \partial_y f_4^\varepsilon\|_{m-1} &= \left\|\frac{v^\varepsilon}{\varphi(y)}\varphi(y)\partial_y f_4^\varepsilon\right\|_{m-1} \lesssim \left\|\frac{v^\varepsilon}{\varphi(y)}\right\|_{m-1}\|Z_2 f_4^\varepsilon\|_{L^\infty} + \left\|\frac{v^\varepsilon}{\varphi(y)}\right\|_{L^\infty}\|f_4^\varepsilon\|_m\\ &\lesssim \|\partial_y v^\varepsilon\|_{m-1}\|Z_2 f_4^\varepsilon\|_{L^\infty} + \left\|\frac{v^\varepsilon}{\varphi(y)}\right\|_{L^\infty}\|f_4^\varepsilon\|_m,\end{aligned}$$

we see that, at least for the case of suitably small $\|Z_2 f_4^\varepsilon\|_{L^\infty}$, we can control $\|\partial_y v^\varepsilon\|_{m-1}$ by the quantity $(1+P(Q(t)))\|(u^\varepsilon, v^\varepsilon, f_2^\varepsilon, f_4^\varepsilon)\|_m$, where $Q(t)$ denotes the $W^{1,\infty}_{co}$-norm of the solution and its first-order derivatives, and P is a generic polynomial that will be frequently used in the estimates of the paper. Using the similar arguments and the estimate of $\|\partial_y v^\varepsilon\|_{m-1}$, one can derive the estimate of $\|\partial_y u^\varepsilon\|_{m-1}$. Here the *a priori* assumption that $1+f_4^\varepsilon$ has a positive lower bound is required, which is guaranteed by the requirement that the initial data of the deformation tensor matrix is a small perturbation of the identity matrix. We remark that the deformation tensor $\mathbf{F}^\varepsilon$ plays an essential role here. It is not clear how to obtain the uniform conormal estimates of the normal derivatives for the tangential velocity u^ε without the viscoelasticity effect under the no-slip boundary condition.

Step II: Estimates of $\partial_y f_2^\varepsilon$. As for the estimate of $\partial_y f_2^\varepsilon$, the equation of u^ε will be used. However, notice that there is also a second-order normal derivative term of $\varepsilon\mu\partial_y^2 u^\varepsilon$ in the equation of u^ε, as a consequence we need to estimate $\rho^\varepsilon(1+f_4^\varepsilon)\partial_y f_2^\varepsilon + \varepsilon\mu\partial_y^2 u^\varepsilon$ instead of $\rho^\varepsilon(1+f_4^\varepsilon)\partial_y f_2^\varepsilon$. Due to the conormal derivatives terms on the right hand side of the equation, taking the L^2-norm on both sides will produce a mixed term of $2\mu\varepsilon\rho^\varepsilon(1+f_4^\varepsilon)\partial_y f_2^\varepsilon\partial_y^2 u^\varepsilon$. To handle the mixed term, we apply the operator ∂_y on the equation of f_2^ε and multiply this equation by $2\mu\varepsilon\partial_y f_2^\varepsilon$, then we produce the same mixed term with the opposite sign. Adding these two estimates together will cancel the mixed terms and achieve the L^2 estimates of $\rho^\varepsilon(1+f_4^\varepsilon)\partial_y f_2^\varepsilon$ and $\varepsilon\mu\partial_y^2 u^\varepsilon$. Similarly, the H^{m-1}_{co} norms also can be done. Here the *a priori* assumption that $1+f_4^\varepsilon$ and ρ^ε have positive lower bounds is required, which is guaranteed by the requirement that the initial data of the deformation tensor matrix is a small perturbation of the identity matrix and the density is a small perturbation of the constant 1.

Step III: Estimates of $\partial_y p^\varepsilon$. By the similar arguments to those in Step II, we use the equations of v^ε and ρ^ε in the following manner:

$$\partial_y p^\varepsilon - (2\mu+\lambda)\varepsilon\partial_y^2 v^\varepsilon = ..., \qquad \partial_t p^\varepsilon + \gamma p^\varepsilon \partial_y v^\varepsilon = ...$$

Moreover, the following relationship will be essentially used:

$$\rho^\varepsilon f_3^\varepsilon \partial_y f_3^\varepsilon = -\frac{(f_3^\varepsilon)^2}{\gamma(\rho^\varepsilon)^{\gamma-1}}\partial_y p^\varepsilon + ...,$$

and

$$\rho^\varepsilon(1+f_4^\varepsilon)\partial_y f_4^\varepsilon = -\frac{(1+f_4^\varepsilon)^2}{\gamma(\rho^\varepsilon)^{\gamma-1}}\partial_y p^\varepsilon + ...$$

due to the equations $\partial_x(\rho^\varepsilon(1+f_1^\varepsilon))+\partial_y(\rho^\varepsilon f_3^\varepsilon)=0$ and $\partial_x(\rho^\varepsilon f_2^\varepsilon)+\partial_y(\rho^\varepsilon(1+f_4^\varepsilon))=0$, which are guaranteed by imposing the same constraint on the initial data (see (1.9) and Proposition 2.1). This relationship is used to change the terms involving $\partial_y f_3^\varepsilon$ and $\partial_y f_4^\varepsilon$ in the equation of v^ε to the form of $\partial_y p^\varepsilon$, then it can be merged into $\partial_y p^\varepsilon$ on the left hand side. In this way, the L^2 estimates of $\left(1+\frac{(f_3^\varepsilon)^2+(1+f_4^\varepsilon)^2}{\gamma(\rho^\varepsilon)^{\gamma-1}}\right)\partial_y p^\varepsilon$ and $(2\mu+\lambda)\varepsilon\partial_y^2 v^\varepsilon$ are established. By the same line, the H_{co}^{m-1} norms will be obtained.

Step IV: Estimates of $\partial_y f_3^\varepsilon, \partial_y f_4^\varepsilon$ and $\partial_y f_1^\varepsilon$. In this paper, the initial data of $\mathbf{F_0}$ and ρ_0 are required to satisfy the natural constraints (1.9), then the smooth solutions also satisfy the same relationship (c.f. [15]). Consequently, it follows that

$$\partial_y f_3^\varepsilon=\frac{1}{\rho^\varepsilon}\left(-\partial_x(\rho^\varepsilon(1+f_1^\varepsilon))-f_3^\varepsilon\partial_y\rho^\varepsilon\right),\quad \partial_y f_4^\varepsilon=\frac{1}{\rho^\varepsilon}\left(-\partial_x(\rho^\varepsilon f_2^\varepsilon)-(1+f_4^\varepsilon)\partial_y\rho^\varepsilon\right).$$

Thus, the estimates of $\partial_y f_3^\varepsilon$ and $\partial_y f_4^\varepsilon$ can be derived. As for the estimate of $\partial_y f_1^\varepsilon$, it can be directly deduced from the following equation:

$$\begin{aligned}\partial_y f_1^\varepsilon=&\frac{1}{(1+f_4^\varepsilon)}\left\{\partial_y\left(\frac{1}{\rho^\varepsilon}\right)+\partial_y(f_2^\varepsilon f_3^\varepsilon)-(1+f_1^\varepsilon)\partial_y f_4^\varepsilon\right\}\\=&\frac{1}{(1+f_4^\varepsilon)}\left\{\partial_y\left(\frac{1}{\rho^\varepsilon}\right)+f_3^\varepsilon\partial_y f_2^\varepsilon-\frac{f_2^\varepsilon}{\rho^\varepsilon}(f_3^\varepsilon\partial_y\rho^\varepsilon+\partial_x(\rho^\varepsilon f_1^\varepsilon))+\frac{f_1^\varepsilon}{\rho^\varepsilon}(f_4^\varepsilon\partial_y\rho^\varepsilon+\partial_x(\rho^\varepsilon f_2^\varepsilon))\right\},\end{aligned}$$

using the property: $\rho^\varepsilon\det(\mathbf{F}^\varepsilon)=1$.

With the above four steps we obtain the estimates of the first order normal derivatives. Finally, to close the energy estimates, it suffices to control $Q(t)$ by the conormal energy estimates. According to Lemma 2.2, in order to estimate $Q(t)$, we still need to derive the conormal estimates of the second order normal derivatives. We repeat the above four steps for the second order normal derivatives to complete the energy estimate procedure, and then justify the inviscid limit of (1.1) under the no-slip boundary condition.

The rest of the paper is organized as follows. In Section 2, we give some preliminaries and technical lemmas. Section 3 is devoted to deriving the uniform conormal energy estimates of solutions to (1.1)-(1.4). In Section 4, we establish the conormal estimates for the normal derivatives of solutions to (1.1)-(1.4). Based on the uniform estimates established in Sections 3 and 4, we prove the main Theorem 1.1 in Section 5.

2. Preliminary

In this section, we shall present some technical lemmas that will be used frequently in the analysis of the paper later.

We first recall the following generalized Sobolev-Gagliardo-Nirenberg-Moser inequality in the conormal Sobolev spaces (see [11] and the proof):

Lemma 2.1. *For the functions $f,g\in L^\infty([0,t]\times\mathbb{R}_+^2)\cap H_{co}^m([0,t]\times\mathbb{R}_+^2)$, it holds that*

$$\int_0^t\|(Z^\alpha fZ^\beta g)(s)\|^2ds\lesssim\|f\|_{L^\infty_{t,\mathbf{x}}}^2\int_0^t\|g(s)\|_m^2ds+\|g\|_{L^\infty_{t,\mathbf{x}}}^2\int_0^t\|f(s)\|_m^2ds\quad for\ \ |\alpha|+|\beta|\le m.$$

Here we note that the notation $A\lesssim B$ means $A\le CB$ for some generic constant C and $\|\cdot\|:=\|\cdot\|_{L^2(\mathbb{R}_+^2)}$.

Then we recall the following anisotropic Sobolev embedding property in the conormal Sobolev spaces (see [33] and the proof):

Lemma 2.2. *Let $f(t,\mathbf{x}) \in H^3_{co}([0,t]\times\mathbb{R}^2_+)$ and $\partial_y f(t,\mathbf{x}) \in H^2_{co}([0,t]\times\mathbb{R}^2_+)$, then*

$$\|f\|^2_{L^\infty_{t,\mathbf{x}}} \lesssim \|f(0)\|_2^2 + \|\partial_y f(0)\|_1^2 + \int_0^t \left(\|f(s)\|_3^2 + \|\partial_y f(s)\|_2^2\right) ds.$$

To handle the commutators, it is helpful to introduce the following formula (see [33] and the proof):

Lemma 2.3. *There exist two families of bounded smooth functions $\{\phi_{k,m}(y)\}_{0\le k\le m-1}$ and $\{\phi^{k,m}(y)\}_{0\le k\le m-1}$, such that*

$$[Z_2^m, \partial_y] = \sum_{k=0}^{m-1} \phi_{k,m}(y) Z_2^k \partial_y = \sum_{k=0}^{m-1} \phi^{k,m}(y)\partial_y Z_2^k.$$

Based on Lemma 2.3, the following lemma holds true.

Lemma 2.4. *There exists a generic constant $C>1$, such that*

$$C^{-1}\sum_{k=0}^{m} \|\nabla Z^k u\|^2 \le \|\nabla u\|_m^2 \le C\sum_{k=0}^{m}\|\nabla Z^k u\|^2.$$

Proof. Denote $k=(k_0,k_1,k_2)$, then

$$\begin{aligned}\sum_{k=0}^{m}\|\nabla Z^k u\| =&\sum_{k=0}^{m}\|Z^k\nabla u\| + \sum_{k=0}^{m}\|Z_0^{k_0}Z_1^{k_1}[\partial_y, Z_2^{k_2}]u\|,\\ \le&\|\nabla u\|_m + \sum_{j=0}^{k_2-1}\|\phi_{j,k_2}(y)Z_0^{k_0}Z_1^{k_1}Z_2^j\partial_y u\| \le C\|\nabla u\|_m,\end{aligned}$$

where the commutator $[\partial_x, Z^k]=0$ is used. And the other inequality can be proved similarly. □

Lemma 2.5. *There exist two families of bounded smooth functions $\{\phi_{1,k,m}(y),\phi_{2,k,m}(y)\}_{0\le k\le m-1}$ and $\{\phi^{1,k,m}(y),\phi^{2,k,m}(y)\}_{0\le k\le m-1}$, such that*

$$[Z_2^m,\partial_y^2] = \sum_{k=0}^{m-1}\left(\phi_{1,k,m}(y)Z_2^k\partial_y + \phi_{2,k,m}(y)Z_2^k\partial_y^2\right) = \sum_{k=0}^{m-1}\left(\phi^{1,k,m}(y)\partial_y Z_2^k + \phi^{2,k,m}(y)\partial_y^2 Z_2^k\right).$$

Lemma 2.6. *There exists a family of bounded smooth functions $\{\psi_{k,m}(y)\}_{0\le k\le m-1}$, such that*

$$[Z_2^m, 1/\varphi(y)]f = \sum_{k=0}^{m-1}\psi_{k,m}(y)Z_2^k(f/\varphi);$$

and there exists a family of bounded smooth functions $\{\psi^{k,m}(y)\}_{0\le k\le m-1}$, such that

$$[Z_2^m,\varphi(y)]f = \sum_{k=0}^{m-1}\psi^{k,m}(y)Z_2^k(\varphi(y)f).$$

The above two Lemmas 2.5 and 2.6 and the proofs can also be found in [33].

Proposition 2.1. *Assume that $(\rho^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{F}^\varepsilon)$ is a smooth solution to (1.1)-(1.4). Then, the following identities*

$$\rho^\varepsilon \det(\mathbf{F}^\varepsilon) = 1, \tag{2.1}$$

and

$$\operatorname{div}(\rho^\varepsilon \mathbf{F}^{\varepsilon\top}) = 0 \tag{2.2}$$

hold for $t \in [0, T]$, provided that these constraints are satisfied initially.

The proof of Proposition 2.1 can be found in [15].

3. Conormal Energy Estimates

In this section we shall derive the uniform conormal estimates of solutions to (1.1)-(1.4). Firstly, we set

$$\rho^\varepsilon = 1 + \tilde{\rho}^\varepsilon.$$

For simplicity of presentation, we omit the symbols ε and "$\sim$" in the following sections without causing any confusion. It is convenient to rewrite system (1.1) as the following:

$$\begin{cases} \partial_t \rho + \nabla \cdot ((1+\rho)\mathbf{u}) = 0, \\ (1+\rho)\partial_t \mathbf{u} + (1+\rho)\mathbf{u} \cdot \nabla \mathbf{u} - (1+\rho)(\mathbf{G_1} + \mathbf{e_1}) \cdot \nabla \mathbf{G_1} \\ \qquad -(1+\rho)(\mathbf{G_2} + \mathbf{e_2}) \cdot \nabla \mathbf{G_2} + \nabla p = -\varepsilon\mu \nabla \times \omega + \varepsilon(2\mu + \lambda)\nabla \operatorname{div} \mathbf{u}, \\ \partial_t \mathbf{G_1} + \mathbf{u} \cdot \nabla \mathbf{G_1} = (\mathbf{G_1} + \mathbf{e_1}) \cdot \nabla \mathbf{u}, \\ \partial_t \mathbf{G_2} + \mathbf{u} \cdot \nabla \mathbf{G_2} = (\mathbf{G_2} + \mathbf{e_2}) \cdot \nabla \mathbf{u}, \end{cases} \tag{3.1}$$

with

$$\mathbf{u} = (u, v), \ \omega = \partial_y u - \partial_x v, \ \mathbf{G_1} = (f_1, f_3), \ \mathbf{G_2} = (f_2, f_4), \ \mathbf{e_1} = (1, 0), \ \mathbf{e_2} = (0, 1),$$

and $\nabla \times = (-\partial_y, \ \partial_x)$.

The no-slip boundary condition is imposed as the following:

$$\mathbf{u}(t, x, 0) = 0. \tag{3.2}$$

We will establish the following uniform conormal energy estimates in this section.

Proposition 3.1. *Under the assumptions in Theorem 1.1, there exists a sufficiently small $\varepsilon_0 > 0$, such that for any $0 < \varepsilon < \varepsilon_0$, the smooth solutions $(\rho, \mathbf{u}, \mathbf{G_1}, \mathbf{G_2})$ to (3.1)-(3.2) satisfy the following a priori estimates:*

$$\begin{aligned} &\|(p-1, \mathbf{u}, \mathbf{G_1}, \mathbf{G_2})(t)\|_m^2 + \varepsilon \int_0^t \|\nabla \mathbf{u}(\tau)\|_m^2 d\tau \\ \lesssim &\|(p-1, \mathbf{u}, \mathbf{G_1}, \mathbf{G_2})(0)\|_m^2 + \delta \int_0^t \|\nabla p(\tau)\|_{m-1}^2 d\tau + \delta\varepsilon^2 \int_0^t \|\nabla^2 \mathbf{u}(\tau)\|_{m-1}^2 d\tau \\ &+ (1 + P(Q(t))) \int_0^t \left(\|\nabla \mathbf{u}(\tau)\|_{m-1}^2 + \|\nabla \mathbf{G_1}(\tau)\|_{m-1}^2 + \|\nabla \mathbf{G_2}(\tau)\|_{m-1}^2 \right) d\tau \\ &+ (1 + P(Q(t))) \int_0^t (\|(p-1)(\tau)\|_m^2 + \|\mathbf{u}(\tau)\|_m^2 + \|\mathbf{G_1}(\tau)\|_m^2 + \|\mathbf{G_2}(\tau)\|_m^2) d\tau, \end{aligned} \tag{3.3}$$

for some small $\delta > 0$ to be determined later, where

$$Q(t) = \sup_{0 \le \tau \le t} \{\|(p-1, \mathbf{u}, \mathbf{G_1}, \mathbf{G_2})(\tau)\|_{1,\infty} + \|(\nabla p, \nabla \mathbf{u}, \nabla \mathbf{G_1}, \nabla \mathbf{G_2})(\tau)\|_{1,\infty}\},$$

and $P(\cdot)$ is a polynomial.

Proof. Applying the conormal derivative operator Z^α ($|\alpha| \le m$) to the system (3.1) yields the following system:

$$\begin{cases} \partial_t Z^\alpha \rho + Z^\alpha \nabla \cdot ((1+\rho)\mathbf{u}) = 0, \\ (1+\rho)\partial_t Z^\alpha \mathbf{u} + (1+\rho)\mathbf{u}\cdot\nabla Z^\alpha \mathbf{u} - (1+\rho)(\mathbf{G_1}+\mathbf{e_1})\cdot\nabla Z^\alpha \mathbf{G_1} \\ \qquad\qquad -(1+\rho)(\mathbf{G_2}+\mathbf{e_2})\cdot\nabla Z^\alpha \mathbf{G_2} + Z^\alpha \nabla p \\ \qquad = -\varepsilon\mu Z^\alpha \nabla\times\omega + \varepsilon(2\mu+\lambda)Z^\alpha\nabla \mathrm{div}\mathbf{u} + \sum_{i=1}^{3}\mathcal{C}_i^\alpha, \\ \partial_t Z^\alpha \mathbf{G_1} + \mathbf{u}\cdot\nabla Z^\alpha \mathbf{G_1} = (\mathbf{G_1}+\mathbf{e_1})\cdot\nabla Z^\alpha \mathbf{u} + \mathcal{C}_4^\alpha, \\ \partial_t Z^\alpha \mathbf{G_2} + \mathbf{u}\cdot\nabla Z^\alpha \mathbf{G_2} = (\mathbf{G_2}+\mathbf{e_2})\cdot\nabla Z^\alpha \mathbf{u} + \mathcal{C}_5^\alpha, \end{cases} \tag{3.4}$$

with

$$\begin{cases} \mathcal{C}_1^\alpha = -[Z^\alpha, (1+\rho)]\mathbf{u}_t = -\sum_{|\beta|\ge 1, \beta+\kappa=\alpha} C_{\alpha\beta} Z^\beta \rho Z^\kappa \mathbf{u}_t, \\ \mathcal{C}_2^\alpha = -[Z^\alpha, (1+\rho)\mathbf{u}\cdot\nabla]\mathbf{u} \\ \quad = -\sum_{|\beta|\ge 1, \beta+\kappa=\alpha} C_{\alpha\beta} Z^\beta((1+\rho)\mathbf{u}) Z^\kappa \nabla \mathbf{u} - (1+\rho)\mathbf{u}\cdot[Z^\alpha,\nabla]\mathbf{u}, \\ \mathcal{C}_3^\alpha = [Z^\alpha, (1+\rho)(\mathbf{G_1}+\mathbf{e_1})\cdot\nabla]\mathbf{G_1} + [Z^\alpha, (1+\rho)(\mathbf{G_2}+\mathbf{e_2})\cdot\nabla]\mathbf{G_2} \\ \quad = \sum_{|\beta|\ge 1, \beta+\kappa=\alpha} C_{\alpha\beta} Z^\beta((1+\rho)(\mathbf{G_1}+\mathbf{e_1})) Z^\kappa\nabla\mathbf{G_1} + (1+\rho)(\mathbf{G_1}+\mathbf{e_1})\cdot[Z^\alpha,\nabla]\mathbf{G_1} \\ \qquad + \sum_{|\beta|\ge 1, \beta+\kappa=\alpha} C_{\alpha\beta} Z^\beta((1+\rho)(\mathbf{G_2}+\mathbf{e_2})) Z^\kappa\nabla\mathbf{G_2} + (1+\rho)(\mathbf{G_2}+\mathbf{e_2})\cdot[Z^\alpha,\nabla]\mathbf{G_2}, \end{cases}$$

and

$$\begin{cases} \mathcal{C}_4^\alpha = -[Z^\alpha, \mathbf{u}\cdot\nabla]\mathbf{G_1} + [Z^\alpha, (\mathbf{G_1}+\mathbf{e_1})\cdot\nabla]\mathbf{u} \\ \quad = -\sum_{|\beta|\ge 1, \beta+\kappa=\alpha} C_{\alpha\beta} Z^\beta \mathbf{u} Z^\kappa \nabla\mathbf{G_1} - \mathbf{u}\cdot[Z^\alpha,\nabla]\mathbf{G_1} \\ \qquad + \sum_{|\beta|\ge 1, \beta+\kappa=\alpha} C_{\alpha\beta} Z^\beta(\mathbf{G_1}+\mathbf{e_1}) Z^\kappa\nabla\mathbf{u} - (\mathbf{G_1}+\mathbf{e_1})\cdot[Z^\alpha,\nabla]\mathbf{u}, \\ \mathcal{C}_5^\alpha = -[Z^\alpha, \mathbf{u}\cdot\nabla]\mathbf{G_2} + [Z^\alpha, (\mathbf{G_2}+\mathbf{e_2})\cdot\nabla]\mathbf{u} \\ \quad = -\sum_{|\beta|\ge 1, \beta+\kappa=\alpha} C_{\alpha\beta} Z^\beta \mathbf{u} Z^\kappa \nabla\mathbf{G_2} - \mathbf{u}\cdot[Z^\alpha,\nabla]\mathbf{G_2} \\ \qquad + \sum_{|\beta|\ge 1, \beta+\kappa=\alpha} C_{\alpha\beta} Z^\beta(\mathbf{G_2}+\mathbf{e_2}) Z^\kappa\nabla\mathbf{u} - (\mathbf{G_2}+\mathbf{e_2})\cdot[Z^\alpha,\nabla]\mathbf{u}. \end{cases}$$

Multiplying the second equation in (3.4) by $Z^\alpha \mathbf{u}$, the third equation by $(1+\rho)Z^\alpha\mathbf{G_1}$, and the fourth equation by $(1+\rho)Z^\alpha\mathbf{G_2}$, adding the resulting equations together, and integrating them over $\mathbb{R}^2_+$, we have

$$\frac{d}{dt}\int \frac{1}{2}(1+\rho)(|Z^\alpha\mathbf{u}|^2 + |Z^\alpha\mathbf{G_1}|^2 + |Z^\alpha\mathbf{G_2}|^2)d\mathbf{x} + \int Z^\alpha\nabla p\cdot Z^\alpha\mathbf{u}d\mathbf{x}$$

$$
\begin{aligned}
=&-\mu\varepsilon\int Z^\alpha\nabla\times\omega\cdot Z^\alpha\mathbf{u}d\mathbf{x}+(2\mu+\lambda)\varepsilon\int Z^\alpha\nabla\mathrm{div}\mathbf{u}\cdot Z^\alpha\mathbf{u}d\mathbf{x} \\
&+\int(\mathcal{C}_1^\alpha+\mathcal{C}_2^\alpha+\mathcal{C}_3^\alpha)\cdot Z^\alpha\mathbf{u}d\mathbf{x}+\int(1+\rho)\mathcal{C}_4^\alpha\cdot Z^\alpha\mathbf{G_1}d\mathbf{x}+\int(1+\rho)\mathcal{C}_5^\alpha\cdot Z^\alpha\mathbf{G_2}d\mathbf{x},
\end{aligned}\tag{3.5}
$$

where the integration by parts, the boundary conditions (3.2) and the following facts are used:

$$
\partial_t\rho+\nabla\cdot((1+\rho)\mathbf{u})=0,\qquad \mathrm{div}((1+\rho)\mathbf{F}^\top)=0,
$$

due to the first equation in (3.1) and Proposition 2.1.

Notice that

$$
\begin{aligned}
&-\varepsilon\int Z^\alpha\nabla\times\omega\cdot Z^\alpha\mathbf{u}d\mathbf{x} \\
=&-\varepsilon\int\nabla\times Z^\alpha\omega\cdot Z^\alpha\mathbf{u}d\mathbf{x}-\varepsilon\int[Z^\alpha,\nabla\times]\omega\cdot Z^\alpha\mathbf{u}d\mathbf{x} \\
\le&-\varepsilon\int Z^\alpha\omega\nabla\times Z^\alpha\mathbf{u}d\mathbf{x}+C\varepsilon\|\nabla^2\mathbf{u}\|_{m-1}\|\mathbf{u}\|_m \\
=&-\varepsilon\int Z^\alpha\omega Z^\alpha\nabla\times\mathbf{u}d\mathbf{x}-\varepsilon\int Z^\alpha\omega[Z^\alpha,\nabla\times]\mathbf{u}d\mathbf{x}+C\varepsilon\|\nabla^2\mathbf{u}\|_{m-1}\|\mathbf{u}\|_m \\
\le&-\varepsilon\int|Z^\alpha\omega|^2d\mathbf{x}+C\varepsilon\|\nabla\mathbf{u}\|_m\|\nabla\mathbf{u}\|_{m-1}+C\varepsilon\|\nabla^2\mathbf{u}\|_{m-1}\|\mathbf{u}\|_m \\
\le&-\varepsilon\|\nabla\times Z^\alpha\mathbf{u}\|^2+\delta\varepsilon^2\|\nabla^2\mathbf{u}\|_{m-1}^2+\delta\varepsilon\|\nabla\mathbf{u}\|_m^2+C_\delta(\varepsilon\|\nabla\mathbf{u}\|_{m-1}^2+\|\mathbf{u}\|_m^2),
\end{aligned}
$$

for some small $\delta>0$ to be determined later, where for the first and second inequalities Lemmas 2.3 and 2.4 are used. Similarly, one has

$$
\begin{aligned}
&\varepsilon\int Z^\alpha\nabla\mathrm{div}\mathbf{u}\cdot Z^\alpha\mathbf{u}d\mathbf{x} \\
=&\varepsilon\int\nabla Z^\alpha\mathrm{div}\mathbf{u}\cdot Z^\alpha\mathbf{u}d\mathbf{x}+\varepsilon\int[Z^\alpha,\nabla]\mathrm{div}\mathbf{u}\cdot Z^\alpha\mathbf{u}d\mathbf{x} \\
\le&-\varepsilon\int Z^\alpha\mathrm{div}\mathbf{u}\cdot\mathrm{div}Z^\alpha\mathbf{u}d\mathbf{x}+\varepsilon\|\nabla^2\mathbf{u}\|_{m-1}\|\mathbf{u}\|_m \\
\le&-\varepsilon\|\mathrm{div}Z^\alpha\mathbf{u}\|^2+\delta\varepsilon^2\|\nabla^2\mathbf{u}\|_{m-1}^2+\delta\varepsilon\|\nabla\mathbf{u}\|_m^2+C_\delta(\varepsilon\|\nabla\mathbf{u}\|_{m-1}^2+\|\mathbf{u}\|_m^2).
\end{aligned}
$$

Combining (3.5) and the following inequality

$$
2c_1\|\nabla Z^\alpha\mathbf{u}\|^2\lesssim\mu\|\nabla\times Z^\alpha\mathbf{u}\|^2+(2\mu+\lambda)\|\mathrm{div}Z^\alpha\mathbf{u}\|^2,
$$

where c_1 is a generic constant, we obtain

$$
\begin{aligned}
&\int\frac{1}{2}(1+\rho)(|Z^\alpha\mathbf{u}|^2+|Z^\alpha\mathbf{G_1}|^2+|Z^\alpha\mathbf{G_2}|^2)d\mathbf{x}+c_1\varepsilon\int_0^t\|\nabla Z^\alpha\mathbf{u}\|^2d\tau \\
&\quad+\int_0^t\int Z^\alpha\nabla p\cdot Z^\alpha\mathbf{u}d\mathbf{x}d\tau \\
\lesssim&\int\frac{1}{2}(1+\rho_0)(|Z^\alpha\mathbf{u}|^2(0)+|Z^\alpha\mathbf{G_1}|^2(0)+|Z^\alpha\mathbf{G_2}(0)|^2)d\mathbf{x} \qquad (3.6)\\
&\quad+\delta\varepsilon^2\int_0^t\|\nabla^2\mathbf{u}(\tau)\|_{m-1}^2d\tau+\delta\varepsilon\int_0^t\|\nabla\mathbf{u}(\tau)\|_m^2d\tau+C_\delta\varepsilon\int_0^t\|\nabla\mathbf{u}(\tau)\|_{m-1}^2d\tau
\end{aligned}
$$

$$
+ C_\delta \int_0^t \|\mathbf{u}(\tau)\|_m^2 d\tau + \int_0^t (\|\mathcal{C}_1^\alpha\|^2 + \|\mathcal{C}_2^\alpha\|^2 + \|\mathcal{C}_3^\alpha\|^2 + \|\mathcal{C}_4^\alpha\|^2 + \|\mathcal{C}_5^\alpha\|^2) d\tau
+ C \int_0^t (\|Z^\alpha \mathbf{u}\|^2 + \|Z^\alpha \mathbf{G_1}\|^2 + \|Z^\alpha \mathbf{G_2}\|)^2 d\tau,
$$

where the *a priori* assumption of $\|\rho\|_{L^\infty} \le 1/2$ is used.

Next, we handle the term involving the pressure. First,

$$
\begin{aligned}
&\int_0^t \int Z^\alpha \nabla p \cdot Z^\alpha \mathbf{u} d\mathbf{x} d\tau = \int_0^t \int Z^\alpha \nabla (p-1) \cdot Z^\alpha \mathbf{u} d\mathbf{x} d\tau \\
&= \int_0^t \int \nabla Z^\alpha (p-1) \cdot Z^\alpha \mathbf{u} d\mathbf{x} d\tau + \int_0^t \int [Z^\alpha, \nabla](p-1) \cdot Z^\alpha \mathbf{u} d\mathbf{x} d\tau \\
&\ge - \int_0^t \int Z^\alpha (p-1) \cdot \mathrm{div} Z^\alpha \mathbf{u} d\mathbf{x} d\tau - \int_0^t \|\mathbf{u}\|_m \|\nabla p\|_{m-1} d\tau \\
&\ge - \int_0^t \int Z^\alpha (p-1) \cdot Z^\alpha \mathrm{div} \mathbf{u} d\mathbf{x} d\tau - \delta \int_0^t \|\nabla p\|_{m-1}^2 d\tau \\
&\quad - C_\delta \int_0^t (\|p-1\|_m^2 + \|\mathbf{u}\|_m^2 + \|\nabla \mathbf{u}\|_{m-1}^2) d\tau.
\end{aligned}
$$

Then, it follows from the first equation in (3.1) that

$$
\mathrm{div} \mathbf{u} = -\frac{p_t}{\gamma p} - \frac{\mathbf{u}}{\gamma p} \cdot \nabla p = -\frac{(p-1)_t}{\gamma p} - \frac{\mathbf{u}}{\gamma p} \cdot \nabla (p-1).
$$

Applying the operator $Z^\alpha (|\alpha| \le m)$ on the above equation gives

$$
\begin{aligned}
Z^\alpha \mathrm{div} \mathbf{u} = &- \frac{Z^\alpha (p-1)_t}{\gamma p} - \frac{\mathbf{u}}{\gamma p} \cdot Z^\alpha \nabla (p-1) \\
&- \sum_{|\beta| \ge 1, \beta + \kappa = \alpha} C_{\alpha\beta} Z^\beta \left(\frac{1}{\gamma p} \right) Z^\kappa (p-1)_t \\
&- \sum_{|\beta| \ge 1, \beta + \kappa = \alpha} C_{\alpha\beta} Z^\beta \left(\frac{\mathbf{u}}{\gamma p} \right) Z^\kappa \nabla (p-1).
\end{aligned}
$$

Now we deal with the above right hand side term by term as follows. For the first term, one has,

$$
\begin{aligned}
&\int_0^t \int Z^\alpha (p-1) \cdot \frac{Z^\alpha (p-1)_t}{\gamma p} d\mathbf{x} d\tau \\
&= \int_0^t \int \left(\frac{|Z^\alpha (p-1)|^2}{2\gamma p} \right)_t d\mathbf{x} d\tau - \int_0^t \int |Z^\alpha (p-1)|^2 \left(\frac{1}{2\gamma p} \right)_t d\mathbf{x} d\tau \\
&\ge \int \frac{|Z^\alpha (p-1)(t)|^2}{2\gamma p(t)} d\mathbf{x} - \int \frac{|Z^\alpha (p-1)(0)|^2}{2\gamma p(0)} d\mathbf{x} - C \left\| \frac{p_t}{p^2} \right\|_{L^\infty} \int_0^t \|Z^\alpha (p-1)\|^2 d\tau, \\
&\ge \int \frac{|Z^\alpha (p-1)(t)|^2}{2\gamma p(t)} d\mathbf{x} - \int \frac{|Z^\alpha (p-1)(0)|^2}{2\gamma p(0)} d\mathbf{x} - (1 + P(Q(t))) \int_0^t \|(p-1)(\tau)\|_m^2 d\tau,
\end{aligned}
$$

where and hereafter we use the *a priori* assumption that $\|\rho\|_{L^\infty} \leq 1/2$, which will be justified later by choosing σ_0 in Theorem 1.1 suitably small; for the second term,

$$\begin{aligned}
&\int_0^t \int Z^\alpha(p-1)\frac{\mathbf{u}}{\gamma p}\cdot Z^\alpha \nabla(p-1)d\mathbf{x}d\tau \\
=&\int_0^t \int Z^\alpha(p-1)\nabla Z^\alpha(p-1)\cdot\frac{\mathbf{u}}{\gamma p}d\mathbf{x}d\tau + \int_0^t\int Z^\alpha(p-1)\frac{\mathbf{u}}{\gamma p}[\nabla, Z^\alpha](p-1)d\mathbf{x}d\tau \\
\geq& -\int \nabla\cdot\frac{\mathbf{u}}{2\gamma p}|Z^\alpha(p-1)(t)|^2d\mathbf{x} - \delta\|\nabla p\|_{m-1}^2 d\tau - C_\delta\left\|\frac{\mathbf{u}}{p}\right\|_{L^\infty}^2\int_0^t \|(p-1)(\tau)\|_m^2 d\tau \\
\geq& -\delta\int_0^t \|\nabla p\|_{m-1}^2 d\tau - C_\delta(1+P(Q(t)))\int_0^t \|(p-1)(\tau)\|_m^2 d\tau,
\end{aligned}$$

for the third term, by direct calculations we have

$$\begin{aligned}
&\sum_{|\beta|\geq 1, \beta+\kappa=\alpha} C_{\alpha\beta}\int_0^t\int Z^\alpha(p-1)Z^\beta\left(\frac{1}{\gamma p}\right)Z^\kappa(p-1)_t \\
\geq& -C\|p\|_{1,\infty}\int_0^t \|(p-1)(\tau)\|_m\|Z^\alpha(p-1)(\tau)\|d\tau \\
\geq& -(1+P(Q(t)))\int_0^t \|(p-1)(\tau)\|_m\|Z^\alpha(p-1)(\tau)\|d\tau \\
\geq& -(1+P(Q(t)))\int_0^t \|(p-1)(\tau)\|_m^2 d\tau,
\end{aligned}$$

where in the first inequality Lemma 2.1 is used; and similarly for the fourth term,

$$\begin{aligned}
&\sum_{|\beta|\geq 1, \beta+\kappa=\alpha} C_{\alpha\beta}\int_0^t\int Z^\alpha(p-1)Z^\beta\left(\frac{\mathbf{u}}{\gamma p}\right)Z^\kappa\nabla(p-1) \\
\geq& -C\|\frac{\mathbf{u}}{\gamma p}\|_{1,\infty}\int_0^t \|\nabla p(\tau)\|_{m-1}\|Z^\alpha(p-1)(\tau)\|d\tau \\
&-C\|p\|_{1,\infty}\int_0^t \|\frac{\mathbf{u}}{p}(\tau)\|_m\|Z^\alpha(p-1)(\tau)\|d\tau \\
\geq& -\delta\int_0^t \|\nabla p\|_{m-1}^2 d\tau - C_\delta(1+P(Q(t)))\int_0^t(\|\mathbf{u}(\tau)\|_m^2 + \|(p-1)(\tau)\|_m^2)d\tau.
\end{aligned}$$

Next, we estimate the terms involving $\mathcal{C}_i^\alpha$ $(i=1,...,5)$ in (3.6) as follows. First, we have the following estimates,

$$\begin{aligned}
\int_0^t \|\mathcal{C}_1^\alpha\|^2 d\tau \lesssim& \sum_{|\beta|\geq 1, \beta+\kappa=\alpha}\int_0^t \|Z^\beta\rho Z^\kappa \mathbf{u}_t\|^2 d\tau \\
\lesssim& \|\rho\|_{1,\infty}^2\int_0^t \|\mathbf{u}_t\|_{m-1}^2 d\tau + \|\mathbf{u}_t\|_{L^\infty}^2\int_0^t \|\rho\|_m^2 d\tau \\
\lesssim&(1+P(Q(t)))\int_0^t(\|\rho\|_m^2 + \|\mathbf{u})\|_m^2)d\tau,
\end{aligned}$$

and

$$\int_0^t \|\mathcal{C}_2^\alpha\|^2 d\tau \lesssim \sum_{|\beta|\geq 1, \beta+\kappa=\alpha} \int_0^t \|Z^\beta((1+\rho)\mathbf{u})Z^\kappa \nabla \mathbf{u}\|^2 d\tau + \|(1+\rho)\mathbf{u}\|_{L^\infty}^2 \int_0^t \|\nabla \mathbf{u}\|_{m-1}^2 d\tau$$
$$\lesssim (1+P(Q(t))) \int_0^t (\|\nabla \mathbf{u}(\tau)\|_{m-1}^2 + \|\rho(\tau)\|_m^2 + \|\mathbf{u}(\tau)\|_m^2) d\tau.$$

Similarly, one has,

$$\int_0^t \|\mathcal{C}_3^\alpha\|^2 d\tau \lesssim (1+P(Q(t))) \int_0^t (\|\nabla \mathbf{G_1}(\tau)\|_{m-1}^2 + \|\nabla \mathbf{G_2}(\tau)\|_{m-1}^2) d\tau$$
$$+ (1+P(Q(t))) \int_0^t (\|\rho(\tau)\|_m^2 + \|\mathbf{G_1}(\tau)\|_m^2 + \|\mathbf{G_2}(\tau)\|_m^2) d\tau,$$

and

$$\int_0^t (\|\mathcal{C}_4^\alpha\|^2 + \|\mathcal{C}_5^\alpha\|^2) d\tau$$
$$\lesssim (1+P(Q(t))) \int_0^t (\|\nabla \mathbf{u}(\tau)\|_{m-1}^2 + \|\nabla \mathbf{G_1}(\tau)\|_{m-1}^2 + \|\nabla \mathbf{G_2}(\tau)\|_{m-1}^2) d\tau$$
$$+ (1+P(Q(t))) \int_0^t (\|\mathbf{u}(\tau)\|_m^2 + \|\mathbf{G_1}(\tau)\|_m^2 + \|\mathbf{G_2}(\tau)\|_m^2) d\tau.$$

Substituting all of the above estimates into (3.6), we obtain

$$\begin{aligned}
&\int \frac{1}{2}(1+\rho)(|Z^\alpha \mathbf{u}|^2 + |Z^\alpha \mathbf{G_1}|^2 + |Z^\alpha \mathbf{G_2}|^2) d\mathbf{x} \\
&\qquad + \int \frac{|Z^\alpha (p-1)(t)|^2}{2\gamma p(t)} d\mathbf{x} + c_1 \varepsilon \int_0^t \|\nabla Z^\alpha \mathbf{u}\|^2 d\tau \\
&\lesssim \int \frac{1}{2}(1+\rho_0)(|Z^\alpha \mathbf{u}|^2(0) + |Z^\alpha \mathbf{G_1}|^2(0) + |Z^\alpha \mathbf{G_2}(0)|^2) d\mathbf{x} + \int \frac{|Z^\alpha (p-1)(0)|^2}{2\gamma p(0)} d\mathbf{x} \\
&\quad + \delta \int_0^t \|\nabla p(\tau)\|_{m-1}^2 d\tau + C_\delta (1+P(Q(t))) \int_0^t \|(p-1)(\tau)\|_m^2 d\tau \\
&\quad + \delta \varepsilon^2 \int_0^t \|\nabla^2 \mathbf{u}(\tau)\|_{m-1}^2 d\tau + \delta \varepsilon \int_0^t \|\nabla \mathbf{u}(\tau)\|_m^2 d\tau \\
&\quad + (1+P(Q(t))) \int_0^t (\|\nabla \mathbf{u}(\tau)\|_{m-1}^2 + \|\nabla \mathbf{G_1}(\tau)\|_{m-1}^2 + \|\nabla \mathbf{G_2}(\tau)\|_{m-1}^2 d\tau \\
&\quad + (1+P(Q(t))) \int_0^t (\|\rho(\tau)\|_m^2 + \|\mathbf{u}(\tau)\|_m^2 + \|\mathbf{G_1}(\tau)\|_m^2 + \|\mathbf{G_2}(\tau)\|_m^2) d\tau.
\end{aligned} \tag{3.7}$$

Summing (3.7) over $|\alpha| \leq m$, choosing δ suitably small and using Lemma 2.4, we have

$$\begin{aligned}
&\|(p-1, \mathbf{u}, \mathbf{G_1}, \mathbf{G_2})(t)\|_m^2 + \varepsilon \int_0^t \|\nabla \mathbf{u}(\tau)\|_m^2 d\tau \\
&\lesssim \|(p-1, \mathbf{u}, \mathbf{G_1}, \mathbf{G_2})(0)\|_m^2 + \delta \int_0^t \|\nabla p(\tau)\|_{m-1}^2 d\tau + \delta \varepsilon^2 \int_0^t \|\nabla^2 \mathbf{u}(\tau)\|_{m-1}^2 d\tau \\
&\quad + (1+P(Q(t))) \int_0^t (\|\nabla \mathbf{u}(\tau)\|_{m-1}^2 + \|\nabla \mathbf{G_1}(\tau)\|_{m-1}^2 + \|\nabla \mathbf{G_2}(\tau)\|_{m-1}^2 d\tau
\end{aligned} \tag{3.8}$$

$$+ (1 + P(Q(t))) \int_0^t (\|(p-1)(\tau)\|_m^2 + \|\mathbf{u}(\tau)\|_m^2 + \|\mathbf{G_1}(\tau)\|_m^2 + \|\mathbf{G_2}(\tau)\|_m^2) d\tau,$$

where the following fact of equivalence is used:

$$C^{-1}\|\rho\|_m^2 \leq \|p-1\|_m^2 \leq C\|\rho\|_m^2 \tag{3.9}$$

holds for some generic constant $C > 1$, due to (1.2) and the *a priori* assumption that $\|\rho\|_{L^\infty} \leq 1/2$. Therefore, the proof of Proposition 3.1 is completed. □

To close the energy estimates, it suffices to derive the estimates of $Q(t)$, $\|\nabla(\mathbf{u}, \mathbf{G_1}, \mathbf{G_2})\|_{m-1}$ and $\|\nabla p\|_{m-1}$, which is the main task in the next section.

4. Estimates of Normal Derivatives

To estimate $\|\nabla(\mathbf{u}, \mathbf{G_1}, \mathbf{G_2}, p)\|_{m-1}$ on the right hand side of (3.8), it suffices to estimate $\|\partial_y(\mathbf{u}, \mathbf{G_1}, \mathbf{G_2}, p)\|_{m-1}$, since $\|\partial_x(\mathbf{u}, \mathbf{G_1}, \mathbf{G_2}, p)\|_{m-1} \leq \|(\mathbf{u}, \mathbf{G_1}, \mathbf{G_2}, p-1)\|_m$ as $\partial_x = Z_1$. In this section, we focus on the estimates of the normal derivatives for $(\mathbf{u}, \mathbf{G_1}, \mathbf{G_2})$ and p. We will derive the conormal estimates for both the first and second order normal derivatives of each component for $(\mathbf{u}, \mathbf{G_1}, \mathbf{G_2})$ and p in the subsequent subsections.

Proposition 4.1. *Under the assumptions in Theorem 1.1, there exists a sufficiently small $\varepsilon_0 > 0$, such that for any $0 < \varepsilon < \varepsilon_0$, the smooth solution $(\rho, \mathbf{u}, \mathbf{G_1}, \mathbf{G_2})$ to (3.1)-(3.2) satisfies the following a priori estimate:*

$$\begin{aligned}
&\|(p-1, \mathbf{u}, \mathbf{G_1}, \mathbf{G_2})(t)\|_m^2 + \varepsilon(\|\partial_y f_2(t)\|_{m-1}^2 + \|\partial_y^2 f_2(t)\|_{m-2}^2 + \|\partial_y p(t)\|_{m-1}^2 + \|\partial_y^2 p(t)\|_{m-2}^2)\\
&+ \varepsilon \int_0^t \|\nabla \mathbf{u}(\tau)\|_m^2 d\tau + \int_0^t (\|\partial_y p(\tau)\|_{m-1}^2 + \|\partial_y \mathbf{u}(\tau)\|_{m-1}^2 + \|\partial_y \mathbf{G_1}(\tau)\|_{m-1}^2 + \|\partial_y \mathbf{G_2}(\tau)\|_{m-1}^2) d\tau\\
&+ \int_0^t (\|\partial_y^2 p(\tau)\|_{m-2}^2 + \|\partial_y^2 \mathbf{u}(\tau)\|_{m-2}^2 + \|\partial_y^2 \mathbf{G_1}(\tau)\|_{m-2}^2 + \|\partial_y^2 \mathbf{G_2}(\tau)\|_{m-2}^2) d\tau\\
&+ \varepsilon^2 \int_0^t (\|\partial_y^2 u(\tau)\|_{m-1}^2 + \|\partial_y^3 u(\tau)\|_{m-2}^2 + \|\partial_y^2 v(\tau)\|_{m-1}^2 + \|\partial_y^3 v(\tau)\|_{m-2}^2) d\tau\\
\lesssim &\|(p-1, \mathbf{u}, \mathbf{G_1}, \mathbf{G_2})(0)\|_m^2 + \varepsilon(\|\partial_y f_2(0)\|_{m-1}^2 + \|\partial_y^2 f_2(0)\|_{m-2}^2 + \|\partial_y p(0)\|_{m-1}^2 + \|\partial_y^2 p(0)\|_{m-2}^2)\\
&+ (1 + P(Q(t))) \int_0^t \big(P(\|\rho(\tau)\|_m) + \|\mathbf{u}(\tau)\|_m^2 + P(\|\mathbf{G_1}(\tau)\|_m) + P(\|\mathbf{G_2}(\tau)\|_m)\big) d\tau,
\end{aligned} \tag{4.1}$$

where

$$Q(t) = \sup_{0 \leq \tau \leq t} \{\|(p-1, \mathbf{u}, \mathbf{G_1}, \mathbf{G_2})(\tau)\|_{1,\infty} + \|(\nabla p, \nabla \mathbf{u}, \nabla \mathbf{G_1}, \nabla \mathbf{G_2})(\tau)\|_{1,\infty}\},$$

and $P(\cdot)$ is a polynomial.

4.1. **Estimates of $\partial_y v$ and $\partial_y^2 v$.** To estimate the normal derivatives of each component, it is convenient to rewrite the equations of $(\mathbf{G_1}, \mathbf{G_2})$ in (3.1) as

$$\begin{cases}
\partial_t f_1 + u\partial_x f_1 + v\partial_y f_1 = (1 + f_1)\partial_x u + f_3 \partial_y u,\\
\partial_t f_2 + u\partial_x f_2 + v\partial_y f_2 = f_2 \partial_x u + (1 + f_4)\partial_y u,\\
\partial_t f_3 + u\partial_x f_3 + v\partial_y f_3 = (1 + f_1)\partial_x v + f_3 \partial_y v,\\
\partial_t f_4 + u\partial_x f_4 + v\partial_y f_4 = f_2 \partial_x v + (1 + f_4)\partial_y v.
\end{cases} \tag{4.2}$$

From the fourth equation in (4.2), we have

$$\partial_y v = \frac{1}{1+f_4}(\partial_t f_4 + u\partial_x f_4 + v\partial_y f_4 - f_2\partial_x v). \tag{4.3}$$

Step 1. Applying the operator Z^α ($|\alpha| \le m-1$) on (4.3), we get

$$Z^\alpha \partial_y v = Z^\alpha \left\{ \frac{1}{1+f_4}(\partial_t f_4 + u\partial_x f_4 + v\partial_y f_4 - f_2\partial_x v) \right\}.$$

Notice that

$$\begin{aligned}
&\left\| Z^\alpha \left(\frac{v}{1+f_4}\partial_y f_4 \right) \right\| = \left\| Z^\alpha \left(\frac{1}{1+f_4}\frac{v}{\varphi(y)}\varphi(y)\partial_y f_4 \right) \right\| \\
\lesssim & \left\| \frac{1}{1+f_4}\frac{v}{\varphi(y)} \right\|_{L^\infty} \|f_4\|_m + \|\varphi(y)\partial_y f_4\|_{L^\infty} \left\| \frac{1}{1+f_4}\frac{v}{\varphi(y)} \right\|_{m-1} \\
\lesssim & \|\partial_y v\|_{L^\infty}\|f_4\|_m + \|f_4\|_{1,\infty}(\|\partial_y v\|_{m-1} + \|\partial_y v\|_{L^\infty} P(\|f_4\|_{m-1})) \\
\lesssim & (1+P(Q(t)))P(\|f_4\|_m) + \|f_4\|_{1,\infty}\|\partial_y v\|_{m-1},
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
&\left\| Z^\alpha \left(\frac{\partial_t f_4}{1+f_4} \right) \right\| + \left\| Z^\alpha \left(\frac{u}{1+f_4}\partial_x f_4 \right) \right\| + \left\| Z^\alpha \left(\frac{f_2}{1+f_4}\partial_x v \right) \right\| \\
\lesssim & \left\| \frac{1}{1+f_4} \right\|_{L^\infty} \|f_4\|_m + \|\partial_t f_4\|_{L^\infty} \left\| Z^\alpha \left(\frac{1}{1+f_4} \right) \right\| + \left\| \frac{u}{1+f_4} \right\|_{L^\infty} \|f_4\|_m \\
& + \|\partial_x f_4\|_{L^\infty} \left\| Z^\alpha \left(\frac{u}{1+f_4} \right) \right\| + \left\| \frac{f_2}{1+f_4} \right\|_{L^\infty} \|v\|_m + \|\partial_x v\|_{L^\infty} \left\| Z^\alpha \left(\frac{f_2}{1+f_4} \right) \right\| \\
\lesssim & (1+P(Q(t)))(P(\|f_4\|_m) + \|v\|_m + \|f_2\|_{m-1} + \|u\|_{m-1}),
\end{aligned}$$

where we used the *a priori* assumption of $\|f_4\|_{L^\infty} \le 1/2$. Summing all of above inequalities over $|\alpha| \le m-1$ and using the *a priori* assumption of $\|f_4\|_{1,\infty} \le C_0\sigma_0$ with σ_0 being in Theorem 1.1 and C_0 being a suitably large constant independent of σ_0 and ε to be determined later, we obtain the following estimate by choosing σ_0 sufficiently small once C_0 is fixed,

$$\|\partial_y v\|_{m-1} \lesssim (1+P(Q(t)))(P(\|f_4\|_m) + \|v\|_m + \|f_2\|_{m-1} + \|u\|_{m-1}). \tag{4.5}$$

Step 2. To control $\|\nabla v\|_{1,\infty}$ in $Q(t)$, it is necessary to derive the conormal estimates of $\partial_y^2 v$. Applying the operator $Z^\alpha\partial_y$ ($|\alpha| \le m-2$) on the equation (4.3) gives

$$Z^\alpha \partial_y^2 v = Z^\alpha \partial_y \left\{ \frac{1}{1+f_4}(\partial_t f_4 + u\partial_x f_4 + v\partial_y f_4 - f_2\partial_x v) \right\}.$$

Then

$$\begin{aligned}
\|Z^\alpha \partial_y^2 v\| \le & \left\| Z^\alpha \partial_y \left(\frac{\partial_t f_4}{1+f_4} \right) \right\| + \left\| Z^\alpha \partial_y \left(\frac{u\partial_x f_4}{1+f_4} \right) \right\| \\
& + \left\| Z^\alpha \partial_y \left(\frac{v\partial_y f_4}{1+f_4} \right) \right\| + \left\| Z^\alpha \partial_y \left(\frac{f_2\partial_x v}{1+f_4} \right) \right\|.
\end{aligned}$$

Now, we estimate each of the terms on the right hand side as follows. Firstly,

$$\begin{aligned}
&\left\| Z^\alpha \partial_y \left(\frac{\partial_t f_4}{1+f_4} \right) \right\| \le \left\| Z^\alpha \left(\frac{1}{1+f_4}\partial_y\partial_t f_4 \right) \right\| + \left\| Z^\alpha \left(\partial_t f_4 \partial_y \left(\frac{1}{1+f_4} \right) \right) \right\| \\
\lesssim & \|\partial_y f_4\|_{1,\infty} P(\|f_4\|_{m-2}) + \|\partial_y f_4\|_{m-1}
\end{aligned}$$

$$
\begin{aligned}
&+\|\partial_t f_4\|_{L^\infty}(\|\partial_y f_4\|_{m-2}+\|\partial_y f_4\|_{L^\infty}P(\|f_4\|_{m-2}))+\|\partial_y f_4\|_{L^\infty}\|f_4\|_{m-1}\\
\lesssim&(1+P(Q(t)))(P(\|f_4\|_{m-1})+\|\partial_y f_4\|_{m-1}),
\end{aligned}
$$

where the *a priori* estimate of $\|f_4\|_{L^\infty}\leq 1/2$ is used again; secondly,

$$
\begin{aligned}
&\left\|Z^\alpha\partial_y\left(\frac{u\partial_x f_4}{1+f_4}\right)\right\|\leq\left\|Z^\alpha\left(\frac{u}{1+f_4}\partial_y\partial_x f_4\right)\right\|+\left\|Z^\alpha\left(\partial_x f_4\partial_y\left(\frac{u}{1+f_4}\right)\right)\right\|\\
\lesssim&(1+P(Q(t)))\|\partial_y f_4\|_{1,\infty}(\|u\|_{m-2}+P(\|f_4\|_{m-2}))+\|u\|_{L^\infty}\|\partial_y f_4\|_{m-1}\\
&+(1+P(Q(t)))\|\partial_x f_4\|_{L^\infty}(\|\partial_y u\|_{m-2}+\|\partial_y f_4\|_{m-2})+\|\partial_y(u,f_4)\|_{L^\infty}\|f_4\|_{m-1}\\
\lesssim&(1+P(Q(t)))(\|u\|_{m-2}+P(\|f_4\|_{m-1})+\|\partial_y f_4\|_{m-1}+\|\partial_y u\|_{m-2}).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\left\|Z^\alpha\partial_y\left(\frac{f_2\partial_x v}{1+f_4}\right)\right\|\leq\left\|Z^\alpha\left(\frac{f_2}{1+f_4}\partial_y\partial_x v\right)\right\|+\left\|Z^\alpha\left(\partial_x v\partial_y\left(\frac{f_2}{1+f_4}\right)\right)\right\|\\
\lesssim&(1+P(Q(t)))(P(\|f_4\|_{m-2})+\|f_2\|_{m-2}+\|v\|_{m-1}+\|\partial_y f_4\|_{m-2}+\|\partial_y f_2\|_{m-2}+\|\partial_y v\|_{m-1}).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\|Z^\alpha\partial_y\left(\frac{v\partial_y f_4}{1+f_4}\right)\right\|\leq\left\|Z^\alpha\left(\frac{v}{1+f_4}\partial_y\partial_y f_4\right)\right\|+\left\|Z^\alpha\left(\partial_y f_4\partial_y\left(\frac{v}{1+f_4}\right)\right)\right\|\\
=&\left\|Z^\alpha\left(\frac{1}{(1+f_4)}\frac{v}{\varphi(y)}\varphi(y)\partial_y\partial_y f_4\right)\right\|+\left\|Z^\alpha\left(\partial_y f_4\partial_y\left(\frac{v}{1+f_4}\right)\right)\right\|\\
\lesssim&(1+P(Q(t)))(\|\partial_y v\|_{m-2}+P(\|f_4\|_{m-2})+\|\partial_y f_4\|_{m-1})\\
&+(1+P(Q(t)))(\|\partial_y v\|_{m-2}+\|\partial_y f_4\|_{m-2}+\|v\|_{m-2}+P(\|f_4\|_{m-2}))\\
\lesssim&(1+P(Q(t)))(P(\|f_4\|_{m-2})+\|v\|_{m-2}+\|\partial_y f_4\|_{m-1}+\|\partial_y v\|_{m-2}).
\end{aligned}
$$

Consequently, summing all of above inequalities over $|\alpha|\leq m-2$ yields that

$$
\begin{aligned}
\|\partial_y^2 v\|_{m-2}\lesssim&(1+P(Q(t)))(P(\|f_4\|_{m-1})+\|f_2\|_{m-2}+\|u\|_{m-2}+\|v\|_{m-1})\\
&+(1+P(Q(t)))(\|\partial_y f_4\|_{m-1}+\|\partial_y f_2\|_{m-2}+\|\partial_y v\|_{m-1}+\|\partial_y u\|_{m-2})\\
\lesssim&(1+P(Q(t)))(P(\|f_4\|_m)+\|f_2\|_{m-1}+\|u\|_{m-1}+\|v\|_m)\\
&+(1+P(Q(t)))(\|\partial_y f_4\|_{m-1}+\|\partial_y f_2\|_{m-2}+\|\partial_y u\|_{m-2}),
\end{aligned}
\tag{4.6}
$$

where (4.5) is used in the second inequality.

4.2. **Estimates of $\partial_y u$ and $\partial_y^2 u$.** From the second equation in (4.2), we have

$$
\partial_y u=\frac{1}{1+f_4}\left(\partial_t f_2+u\partial_x f_2+v\partial_y f_2-f_2\partial_x u\right).
\tag{4.7}
$$

A similar argument to (4.5) yields that

$$
\|\partial_y u\|_{m-1}\lesssim(1+P(Q(t)))(\|(u,f_2)\|_m+P(\|f_4\|_{m-1})+\|\partial_y v\|_{m-1}).
$$

Then, by using (4.5), we have

$$
\|\partial_y u\|_{m-1}\lesssim(1+P(Q(t)))(\|(u,v,f_2)\|_m+P(\|f_4\|_m)).
\tag{4.8}
$$

Applying the operator $Z^\alpha\partial_y$ ($|\alpha|\leq m-2$) on (4.7) gives

$$
Z^\alpha\partial_y^2 u=Z^\alpha\partial_y\left\{\frac{1}{1+f_4}(\partial_t f_2+u\partial_x f_2+v\partial_y f_2-f_2\partial_x u)\right\}.
\tag{4.9}
$$

Similar arguments to (4.6) give

$$\begin{aligned}
&\|\partial_y^2 u\|_{m-2}\\
\lesssim&(1+P(Q(t)))\big(\|v\|_{m-2}+P(\|f_4\|_{m-2})+\|(u,f_2)\|_{m-1}\\
&\qquad\qquad\qquad+\|\partial_y(v,f_4)\|_{m-2}+\|\partial_y(u,f_2)\|_{m-1}\big)\\
\lesssim&(1+P(Q(t)))\big(\|(u,v,f_2)\|_m+P(\|f_4\|_m)+\|\partial_y f_2\|_{m-1}+\|\partial_y f_4\|_{m-2}\big),
\end{aligned}\tag{4.10}$$

where in the second inequality both (4.5) and (4.8) are used.

4.3. **Estimate of $\partial_y f_2$.** It is convenient to rewrite the momentum equations in (3.1) as the following:

$$\begin{cases}
(1+\rho)\partial_t u+(1+\rho)u\partial_x u+(1+\rho)v\partial_y u-(1+\rho)(1+f_1)\partial_x f_1-(1+\rho)f_3\partial_y f_1\\
\quad-(1+\rho)f_2\partial_x f_2-(1+\rho)(1+f_4)\partial_y f_2-\mu\varepsilon\partial_y^2 u-\mu\varepsilon\partial_x^2 u\\
\quad-(\mu+\lambda)\varepsilon\partial_x(u_x+v_y)+\partial_x p=0,\\
(1+\rho)\partial_t v+(1+\rho)u\partial_x v+(1+\rho)v\partial_y v-(1+\rho)(1+f_1)\partial_x f_3-(1+\rho)f_3\partial_y f_3\\
\quad-(1+\rho)f_2\partial_x f_4-(1+\rho)(1+f_4)\partial_y f_4-\mu\varepsilon\partial_y^2 v-\mu\varepsilon\partial_x^2 v\\
\quad-(\mu+\lambda)\varepsilon\partial_y(u_x+v_y)+\partial_y p=0.
\end{cases}\tag{4.11}$$

Step 1. According to the first equation in (4.11), we have

$$\begin{aligned}
&(1+\rho)(1+f_4)\partial_y f_2+\mu\varepsilon\partial_y^2 u\\
=&(1+\rho)\partial_t u+(1+\rho)u\partial_x u+(1+\rho)v\partial_y u-(1+\rho)(1+f_1)\partial_x f_1-(1+\rho)f_3\partial_y f_1\\
&-(1+\rho)f_2\partial_x f_2-\mu\varepsilon\partial_x^2 u-(\mu+\lambda)\varepsilon\partial_x(u_x+v_y)+\partial_x p.
\end{aligned}\tag{4.12}$$

Applying the operator Z^α ($|\alpha|\le m-1$) on the both sides of (4.12), one has,

$$\begin{aligned}
&(1+\rho)(1+f_4)Z^\alpha\partial_y f_2+\varepsilon\mu Z^\alpha\partial_y^2 u\\
=&Z^\alpha\{(1+\rho)\partial_t u+(1+\rho)u\partial_x u+(1+\rho)v\partial_y u\}\\
&+Z^\alpha\{-(1+\rho)(1+f_1)\partial_x f_1-(1+\rho)f_3\partial_y f_1-(1+\rho)f_2\partial_x f_2\}\\
&+Z^\alpha\{-\mu\varepsilon\partial_x^2 u-(\mu+\lambda)\varepsilon\partial_x(u_x+v_y)+\partial_x p\}-[Z^\alpha,(1+\rho)(1+f_4)]\partial_y f_2.
\end{aligned}\tag{4.13}$$

Taking the L^2 inner product over $\mathbb{R}_+^2$ on the both sides of the above equality yields that

$$\begin{aligned}
&\|(1+\rho)(1+f_4)Z^\alpha\partial_y f_2\|^2+\mu^2\varepsilon^2\|Z^\alpha\partial_y^2 u\|^2+2\mu\varepsilon\int(1+\rho)(1+f_4)Z^\alpha\partial_y f_2\cdot Z^\alpha\partial_y^2 u d\mathbf{x}\\
\lesssim&\|(1+\rho)\partial_t u\|_{m-1}^2+\|(1+\rho)u\partial_x u\|_{m-1}^2+\|(1+\rho)v\partial_y u\|_{m-1}^2\\
&+\|(1+\rho)(1+f_1)\partial_x f_1\|_{m-1}^2+\|(1+\rho)f_3\partial_y f_1\|_{m-1}^2+\|(1+\rho)f_2\partial_x f_2\|_{m-1}^2\\
&+\varepsilon^2\|\partial_x u\|_m^2+\varepsilon^2\|\partial_y v\|_m^2+\|\partial_x p\|_{m-1}^2\\
&+\|Z((1+\rho)(1+f_4))\|_{L^\infty}^2\|\partial_y f_2\|_{m-2}^2+\|\partial_y f_2\|_{L^\infty}^2\|Z((1+\rho)(1+f_4))\|_{m-2}^2\\
\lesssim&(1+P(Q(t)))\left(\|(u,f_1,f_2)\|_m^2+\|(\rho,v,f_3,f_4)\|_{m-1}^2+\|\partial_y u\|_{m-1}^2+\|\partial_y f_2\|_{m-2}^2\right)\\
&+\|\partial_x p\|_{m-1}^2+\varepsilon^2\|(\partial_x u,\partial_y v)\|_m^2+\|f_3\|_{L^\infty}^2\|\partial_y f_1\|_{m-1}^2\\
\lesssim&(1+P(Q(t)))\left(\|(u,v,f_1,f_2)\|_m^2+\|(\rho,f_3)\|_{m-1}^2+P(\|f_4\|_m)+\|\partial_y f_2\|_{m-2}^2\right)\\
&+\|\partial_x p\|_{m-1}^2+\varepsilon^2\|(\partial_x u,\partial_y v)\|_m^2+\|f_3\|_{L^\infty}^2\|\partial_y f_1\|_{m-1}^2,
\end{aligned}\tag{4.14}$$

where (4.8) is used in the last inequality.

Step 2. It remains to handle the mixed term $2\mu\varepsilon\int(1+\rho)(1+f_4)Z^\alpha\partial_y f_2\cdot Z^\alpha\partial_y^2 u d\mathbf{x}$ on the left hand side of (4.14). From the second equation in (4.2), we have

$$\frac{1}{(1+f_4)}\partial_t f_2-\partial_y u=\frac{1}{(1+f_4)}(f_2\partial_x u-u\partial_x f_2-v\partial_y f_2). \tag{4.15}$$

Applying the operator $Z^\alpha\partial_y$ ($|\alpha|\le m-1$) on the equation (4.15) leads to

$$\begin{aligned}&\frac{1}{(1+f_4)}\partial_t Z^\alpha\partial_y f_2-Z^\alpha\partial_y^2 u\\ =&Z^\alpha\partial_y\left(\frac{f_2}{1+f_4}\partial_x u\right)-\frac{u}{(1+f_4)}\partial_x Z^\alpha\partial_y f_2-\frac{v}{(1+f_4)}\partial_y Z^\alpha\partial_y f_2+\mathcal{C}_6^\alpha,\end{aligned} \tag{4.16}$$

where

$$\mathcal{C}_6^\alpha=-\left[Z^\alpha\partial_y,\frac{1}{(1+f_4)}\partial_t\right]f_2-\left[Z^\alpha\partial_y,\frac{u}{(1+f_4)}\partial_x\right]f_2-\left[Z^\alpha\partial_y,\frac{v}{(1+f_4)}\partial_y\right]f_2. \tag{4.17}$$

Multiplying (4.16) by $2\mu\varepsilon(1+\rho)(1+f_4)Z^\alpha\partial_y f_2$ and integrating the resulting equation over $\mathbb{R}^2_+$ give that

$$\begin{aligned}&\mu\varepsilon\frac{d}{dt}\|\sqrt{(1+\rho)}Z^\alpha\partial_y f_2\|^2-2\mu\varepsilon\int(1+\rho)(1+f_4)Z^\alpha\partial_y f_2 Z^\alpha\partial_y^2 u d\mathbf{x}\\ =&2\mu\varepsilon\int(1+\rho)(1+f_4)Z^\alpha\partial_y f_2 Z^\alpha\partial_y\left(\frac{f_2}{1+f_4}\partial_x u\right)d\mathbf{x}+2\mu\varepsilon\int(1+\rho)(1+f_4)Z^\alpha\partial_y f_2\mathcal{C}_6^\alpha d\mathbf{x},\end{aligned} \tag{4.18}$$

where the equation of $\partial_t\rho+\partial_x((1+\rho)u)+\partial_y((1+\rho)v)=0$ is used. For the terms on the right hand side of (4.18), by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\left|2\mu\varepsilon\int(1+\rho)(1+f_4)Z^\alpha\partial_y f_2 Z^\alpha\partial_y\left(\frac{f_2}{1+f_4}\partial_x u\right)d\mathbf{x}\right|\\ &+\left|2\mu\varepsilon\int(1+\rho)(1+f_4)Z^\alpha\partial_y f_2\mathcal{C}_6^\alpha d\mathbf{x}\right|\\ =&\left|2\mu\varepsilon\int(1+\rho)(1+f_4)Z^\alpha\partial_y f_2 Z^\alpha\left(\partial_y\left(\frac{f_2}{1+f_4}\right)\partial_x u+\frac{f_2}{1+f_4}\partial_x\partial_y u\right)d\mathbf{x}\right|\\ &+\left|2\mu\varepsilon\int(1+\rho)(1+f_4)Z^\alpha\partial_y f_2\mathcal{C}_6^\alpha d\mathbf{x}\right|\\ \le&\delta\|(1+\rho)(1+f_4)Z^\alpha\partial_y f_2\|^2\\ &+C_\delta\varepsilon^2\times\left(\left\|Z^\alpha\left(\partial_y\left(\frac{f_2}{1+f_4}\right)\partial_x u+\frac{f_2}{1+f_4}\partial_x\partial_y u\right)\right\|^2+\|\mathcal{C}_6^\alpha\|^2\right),\end{aligned}$$

for some small constant $\delta>0$ to be determined later. Note that

$$\begin{aligned}&\left\|Z^\alpha\left(\partial_y\left(\frac{f_2}{1+f_4}\right)\partial_x u+\frac{f_2}{1+f_4}\partial_x\partial_y u\right)\right\|^2\\ \lesssim&\left\|\partial_y\left(\frac{f_2}{1+f_4}\right)\right\|_{L^\infty}^2\|u\|_m^2+\|\partial_x u\|_{L^\infty}^2\left\|\partial_y\left(\frac{f_2}{1+f_4}\right)\right\|_{m-1}^2\\ &+\left\|\frac{f_2}{1+f_4}\right\|_{L^\infty}^2\|\partial_y u\|_m^2+\|\partial_y u\|_{1,\infty}^2\left\|\frac{f_2}{1+f_4}\right\|_{m-1}^2\end{aligned}$$

$$\lesssim (1+P(Q(t)))(\|u\|_m^2+\|\partial_y u\|_m^2+\|\partial_y(f_2,f_4)\|_{m-1}^2+\|f_2\|_{m-1}^2+P(\|f_4\|_{m-1})).$$

Step 3. For the second term on the right hand side of (4.18), we need to estimate the commutator $\mathcal{C}_6^\alpha$ defined in (4.17). First, we have

$$\begin{aligned}
&\left\|\left[Z^\alpha\partial_y,\frac{1}{(1+f_4)}\partial_t\right]f_2\right\|^2\\
=&\left\|Z^\alpha\left(\partial_y\left(\frac{1}{1+f_4}\right)\partial_t f_2\right)+\left[Z^\alpha,\frac{1}{1+f_4}\right]\partial_t\partial_y f_2\right\|^2\\
\lesssim&\left\|\partial_y\left(\frac{1}{1+f_4}\right)\right\|_{L^\infty}^2\|f_2\|_m^2+\|\partial_t f_2\|_{L^\infty}^2\left\|\partial_y\left(\frac{1}{1+f_4}\right)\right\|_{m-1}^2\\
&+\left\|Z\left(\frac{1}{1+f_4}\right)\right\|_{L^\infty}^2\|\partial_y f_2\|_{m-1}^2+\|\partial_y f_2\|_{1,\infty}^2\left\|Z\left(\frac{1}{1+f_4}\right)\right\|_{m-2}^2\\
\lesssim&(1+P(Q(t)))(\|f_2\|_m^2+\|\partial_y(f_2,f_4)\|_{m-1}^2+P(\|f_4\|_{m-1})),
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\left\|\left[Z^\alpha\partial_y,\frac{u}{(1+f_4)}\partial_x\right]f_2\right\|^2\\
=&\left\|Z^\alpha\left(\partial_y\left(\frac{u}{1+f_4}\right)\partial_x f_2\right)+\left[Z^\alpha,\frac{u}{(1+f_4)}\right]\partial_x\partial_y f_2\right\|^2\\
\lesssim&\left\|\partial_y\left(\frac{u}{1+f_4}\right)\right\|_{L^\infty}^2\|f_2\|_m^2+\|\partial_x f_2\|_{L^\infty}^2\left\|\partial_y\left(\frac{u}{1+f_4}\right)\right\|_{m-1}^2\\
&+\left\|Z\left(\frac{u}{1+f_4}\right)\right\|_{L^\infty}^2\|\partial_y f_2\|_{m-1}^2+\|\partial_y f_2\|_{1,\infty}^2\left\|Z\left(\frac{u}{1+f_4}\right)\right\|_{m-2}^2\\
\lesssim&(1+P(Q(t)))(\|f_2\|_m^2+\|u\|_{m-1}^2+P(\|f_4\|_{m-1})+\|\partial_y(u,f_2,f_4)\|_{m-1}^2).
\end{aligned}$$

Next, we notice that

$$\begin{aligned}
&\left[Z^\alpha\partial_y,\frac{v}{(1+f_4)}\partial_y\right]f_2\\
=&Z^\alpha\left\{\partial_y\left(\frac{v}{1+f_4}\right)\partial_y f_2\right\}+\left[Z^\alpha,\frac{v}{(1+f_4)}\right]\partial_y\partial_y f_2+\frac{v}{1+f_4}[Z^\alpha,\partial_y]\partial_y f_2
\end{aligned}\tag{4.19}$$

with

$$\left[Z^\alpha,\frac{v}{(1+f_4)}\right]\partial_y\partial_y f_2=\sum_{|\beta|\geq1,\beta+\kappa=\alpha}C_{\alpha\beta}Z^\beta\left(\frac{v}{1+f_4}\right)Z^\kappa\partial_y\partial_y f_2.$$

The first term in (4.19) can be estimated as the following:

$$\begin{aligned}
&\left\|Z^\alpha\left\{\partial_y\left(\frac{v}{1+f_4}\right)\partial_y f_2\right\}\right\|^2\\
\lesssim&(1+P(Q(t)))(\|\partial_y v\|_{m-1}^2+\|\partial_y f_2\|_{m-1}^2+\|\partial_y f_4\|_{m-1}^2+P(\|f_4\|_{m-1})+\|v\|_{m-1}^2).
\end{aligned}$$

Step 4. We now estimate the second and the third terms, that is, the two commutators in (4.19). For the first commutator, we have the following computation,

$$
\begin{aligned}
&Z^{\beta}\left(\frac{v}{1+f_4}\right)Z^{\kappa}\partial_y\partial_y f_2\\
=&\left(Z^{\beta}\left(\frac{v}{(1+f_4)\varphi(y)}\right)+\left[Z^{\beta},\frac{1}{\varphi(y)}\right]\frac{v}{1+f_4}\right)\left(Z^{\kappa}\varphi(y)\partial_y\partial_y f_2+[Z^{\kappa},\varphi(y)]\partial_y\partial_y f_2\right),
\end{aligned}
$$

where

$$
\left[Z^{\beta},\frac{1}{\varphi(y)}\right]\frac{v}{1+f_4}=\sum_{\eta=0}^{\beta-1}\psi_{\eta,\beta}(y)Z^{\eta}\left(\frac{v}{\varphi(y)(1+f_4)}\right)
$$

for some bounded smooth functions $\psi_{\eta,\beta}(y)$ due to Lemma 2.6, and similarly,

$$
[Z^{\kappa},\varphi(y)]\partial_y\partial_y f_2=\sum_{\theta=0}^{\kappa-1}\psi_{\theta,\kappa}(y)Z^{\theta}(\varphi(y)\partial_y\partial_y f_2)
$$

for some bounded smooth functions $\psi_{\theta,\kappa}(y)$. Then, according to Lemma 2.1, we have the following estimate:

$$
\begin{aligned}
&\left\|\left[Z^{\alpha},\frac{v}{(1+f_4)}\right]\partial_y\partial_y f_2\right\|^2\\
\lesssim&\sum_{|\beta|\geq 1,\beta+\kappa=\alpha}\left\|Z^{\beta}\left(\frac{v}{1+f_4}\right)Z^{\kappa}\partial_y\partial_y f_2\right\|^2\\
\lesssim&(1+P(Q(t)))(\|\partial_y v\|_{m-1}^2+\|\partial_y f_2\|_{m-1}^2+P(\|f_4\|_{m-1})).
\end{aligned}
$$

For the second commutator in (4.19), we write

$$
[Z^{\alpha},\partial_y]\partial_y f_2=\sum_{\theta=0}^{m-2}\phi_{\theta,\alpha}(y)\partial_y Z^{\theta}\partial_y f_2
$$

with $\phi_{\theta,\alpha}(y)$ being bounded smooth functions due to Lemma 2.3. Then,

$$
\begin{aligned}
&\left\|\frac{v}{1+f_4}[Z^{\alpha},\partial_y]\partial_y f_2\right\|^2\\
=&\left\|\frac{v}{(1+f_4)\varphi(y)}\varphi(y)[Z^{\alpha},\partial_y]\partial_y f_2\right\|^2\\
\lesssim&(1+P(Q(t)))\|\partial_y f_2\|_{m-1}^2.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\left\|\left[Z^{\alpha}\partial_y,\frac{v}{(1+f_4)}\partial_y\right]f_2\right\|^2\\
\lesssim&(1+P(Q(t)))\big(\|\partial_y v\|_{m-1}^2+\|\partial_y f_2\|_{m-1}^2+\|\partial_y f_4\|_{m-1}^2+P(\|f_4\|_{m-1})+\|v\|_{m-1}^2\big).
\end{aligned}
$$

Substituting all of the above estimates into (4.18) yields the following estimate:

$$
\begin{aligned}
&\mu\varepsilon\frac{d}{dt}\|\sqrt{(1+\rho)}Z^{\alpha}\partial_y f_2\|^2-2\mu\varepsilon\int(1+\rho)(1+f_4)Z^{\alpha}\partial_y f_2Z^{\alpha}\partial_y^2 u d\mathbf{x}\\
\leq&\delta\|(1+\rho)(1+f_4)Z^{\alpha}\partial_y f_2\|^2 \qquad (4.20)
\end{aligned}
$$

$$+ C_\delta \varepsilon^2 (1+P(Q(t)))\Big(\|u\|_m^2 + \|f_2\|_m^2 + P(\|f_4\|_{m-1}) + \|v\|_{m-1}^2 + \|\partial_y u\|_m^2 + \|\partial_y v\|_{m-1}^2 + \|\partial_y f_2\|_{m-1}^2 + \|\partial_y f_4\|_{m-1}^2\Big).$$

Step 5. Combining (4.14) and (4.20) together and choosing δ suitably small, we obtain

$$\begin{aligned}
&\mu\varepsilon\frac{d}{dt}\|\sqrt{(1+\rho)}Z^\alpha\partial_y f_2\|^2 + \|(1+\rho)(1+f_4)Z^\alpha\partial_y f_2\|^2 + \mu^2\varepsilon^2\|Z^\alpha\partial_y^2 u\|^2\\
\lesssim&(1+P(Q(t)))\left(\|(u,v,f_1,f_2)\|_m^2 + \|(\rho,f_3)\|_{m-1}^2 + P(\|f_4\|_m) + \|\partial_y f_2\|_{m-2}^2\right)\\
&+\|\partial_x p\|_{m-1}^2 + \varepsilon^2\|(\partial_x u,\partial_y v)\|_m^2 + \|f_3\|_{L^\infty}^2\|\partial_y f_1\|_{m-1}^2\\
&+\varepsilon^2(1+P(Q(t)))\left(\|\partial_y u\|_m^2 + \|\partial_y v\|_{m-1}^2 + \|\partial_y f_2\|_{m-1}^2 + \|\partial_y f_4\|_{m-1}^2\right).
\end{aligned}\tag{4.21}$$

Choosing ε_0 to be sufficiently small and for $0<\varepsilon<\varepsilon_0$ summing the above inequalities over $|\alpha|\le m-1$ lead to

$$\begin{aligned}
&\mu\varepsilon\frac{d}{dt}\|\partial_y f_2\|_{m-1}^2 + \|\partial_y f_2\|_{m-1}^2 + \varepsilon^2\|\partial_y^2 u\|_{m-1}^2\\
\lesssim&(1+P(Q(t)))\left(\|(u,v,f_1,f_2)\|_m^2 + \|(\rho,f_3)\|_{m-1}^2 + P(\|f_4\|_m)\right)\\
&+\|\partial_x p\|_{m-1}^2 + \varepsilon^2\|(\partial_x u,\partial_y v)\|_m^2 + \|f_3\|_{L^\infty}^2\|\partial_y f_1\|_{m-1}^2\\
&+\varepsilon^2(1+P(Q(t)))\left(\|\partial_y u\|_m^2 + \|\partial_y f_4\|_{m-1}^2\right),
\end{aligned}\tag{4.22}$$

where the mathematical induction arguments and the following *a priori* estimates are used:

$$\|\rho\|_{L^\infty}\le 1/2, \qquad \|f_4\|_{L^\infty}\le 1/2, \quad Q(t)\le C.$$

More precisely, notice that the order of conormal derivatives is up to $m-1$ on the left hand side of (4.21), and there exist terms of $\|\partial_y f_2\|_{m-2}^2$ and $\varepsilon^2(1+P(Q(t)))\|\partial_y f_2\|_{m-1}^2$ on the right hand side of (4.21), then the first term is absorbed by using the mathematical induction arguments, and the second term is absorbed by choosing ε sufficiently small and the *a priori* assumption of $Q(t)\le C$. And (4.5) is also used in deriving (4.22).

4.4. **Estimate of $\partial_y^2 f_2$.** Next, we will derive the conormal energy estimates of $\partial_y^2 f_2$.

Step 1. Applying the operator $Z^\alpha\partial_y$ ($|\alpha|\le m-2$) on the both sides of (4.12) yields

$$\begin{aligned}
&(1+\rho)(1+f_4)Z^\alpha\partial_y^2 f_2 + \varepsilon\mu Z^\alpha\partial_y^3 u\\
&\quad= Z^\alpha\partial_y\{(1+\rho)\partial_t u + (1+\rho)u\partial_x u + (1+\rho)v\partial_y u\}\\
&\qquad+ Z^\alpha\partial_y\{-(1+\rho)(1+f_1)\partial_x f_1 - (1+\rho)f_3\partial_y f_1 - (1+\rho)f_2\partial_x f_2\}\\
&\qquad+ Z^\alpha\partial_y\{-\mu\varepsilon\partial_x^2 u - (\mu+\lambda)\varepsilon\partial_x(u_x+v_y) + \partial_x p\} - [Z^\alpha\partial_y,(1+\rho)(1+f_4)]\partial_y f_2.
\end{aligned}\tag{4.23}$$

Taking the L^2 inner product on the both sides of the above equality, we obtain

$$\begin{aligned}
&\|(1+\rho)(1+f_4)Z^\alpha\partial_y^2 f_2\|^2 + \mu^2\varepsilon^2\|Z^\alpha\partial_y^3 u\|^2 + 2\mu\varepsilon\int(1+\rho)(1+f_4)Z^\alpha\partial_y^2 f_2\cdot Z^\alpha\partial_y^3 u\,d\mathbf{x}\\
\lesssim&\|Z^\alpha\partial_y((1+\rho)\partial_t u)\|^2 + \|Z^\alpha\partial_y((1+\rho)u\partial_x u)\|^2 + \|Z^\alpha\partial_y((1+\rho)v\partial_y u)\|^2\\
&+\|Z^\alpha\partial_y((1+\rho)(1+f_1)\partial_x f_1)\|^2 + \|Z^\alpha\partial_y((1+\rho)f_3\partial_y f_1)\|^2\\
&+\|Z^\alpha\partial_y((1+\rho)f_2\partial_x f_2)\|^2 + \varepsilon^2\|Z^\alpha\partial_y\partial_x^2 u\|^2 + \varepsilon^2\|Z^\alpha\partial_y\partial_{xy}^2 v\|^2\\
&+\|Z^\alpha\partial_y\partial_x p\|^2 + \|[Z^\alpha\partial_y,(1+\rho)(1+f_4)]\partial_y f_2\|^2.
\end{aligned}\tag{4.24}$$

Here we only need to deal several typical terms in (4.24) since other terms can be handled similarly. First, we have

$$\begin{aligned}
&\|Z^\alpha\partial_y((1+\rho)\partial_t u)\|^2\\
\le&\|Z^\alpha(\partial_y\rho\partial_t u)\|^2+\|Z^\alpha((1+\rho)\partial_t\partial_y u)\|^2\\
\le&\|\partial_y\rho\|_{L^\infty}^2\|u\|_{m-1}^2+\|\partial_t u\|_{L^\infty}^2\|\partial_y\rho\|_{m-2}^2+(1+\|\rho\|_{L^\infty}^2)\|\partial_y u\|_{m-1}^2+\|\partial_y u\|_{1,\infty}^2\|\rho\|_{m-2}^2\\
\lesssim&(1+P(Q(t)))(\|u\|_{m-1}^2+\|\rho\|_{m-2}^2+\|\partial_y\rho\|_{m-2}^2+\|\partial_y u\|_{m-1}^2)\\
\lesssim&(1+P(Q(t)))(\|u,v,f_2\|_m^2+\|\rho\|_{m-2}^2+P(\|f_4\|_m)+\|\partial_y\rho\|_{m-2}^2),
\end{aligned}$$

where the estimate (4.8) is used in the last inequality. By the same argument, we get

$$\begin{aligned}
&\|Z^\alpha\partial_y((1+\rho)u\partial_x u)\|^2+\|Z^\alpha\partial_y((1+\rho)(1+f_1)\partial_x f_1)\|^2+\|Z^\alpha\partial_y((1+\rho)f_2\partial_x f_2)\|^2\\
\lesssim&(1+P(Q(t)))(\|(u,f_1,f_2)\|_{m-1}^2+\|\rho\|_{m-2}^2+\|\partial_y\rho\|_{m-2}^2+\|\partial_y(u,f_1,f_2)\|_{m-1}^2)\\
\lesssim&(1+P(Q(t)))\big(\|(u,v,f_2)\|_m^2+\|f_1\|_{m-1}^2+\|\rho\|_{m-2}^2+P(\|f_4\|_m)+\|\partial_y\rho\|_{m-2}^2+\|\partial_y(f_1,f_2)\|_{m-1}^2\big).
\end{aligned}$$

Next,

$$Z^\alpha\partial_y((1+\rho)v\partial_y u)=Z^\alpha(\partial_y\rho v\partial_y u)+Z^\alpha((1+\rho)\partial_y v\partial_y u)+Z^\alpha((1+\rho)v\partial_y\partial_y u),$$

where

$$\begin{aligned}
&\|Z^\alpha(\partial_y\rho v\partial_y u)\|^2+\|Z^\alpha((1+\rho)\partial_y v\partial_y u)\|^2\\
\lesssim&(1+P(Q(t)))(\|(\rho,v)\|_{m-2}^2+\|\partial_y(\rho,u,v)\|_{m-2}^2\\
\lesssim&(1+P(Q(t)))(\|(u,v,f_2)\|_m^2+\|\rho\|_{m-2}^2+P(\|f_4\|_m)+\|\partial_y\rho\|_{m-2}^2,
\end{aligned}$$

and

$$\begin{aligned}
&\|Z^\alpha((1+\rho)v\partial_y^2 u)\|^2=\left\|Z^\alpha\left((1+\rho)\frac{v}{\varphi(y)}\varphi(y)\partial_y\partial_y u\right)\right\|^2\\
\lesssim&(1+P(Q(t)))(\|\rho\|_{m-2}^2+\|\partial_y v\|_{m-2}^2+\|\partial_y u\|_{m-1}^2)\\
\lesssim&(1+P(Q(t)))(\|\rho\|_{m-2}^2+\|(u,v,f_2)\|_m^2+P(\|f_4\|_m)).
\end{aligned}$$

Similarly,

$$Z^\alpha\partial_y((1+\rho)f_3\partial_y f_1)=Z^\alpha(\partial_y\rho f_3\partial_y f_1)+Z^\alpha((1+\rho)\partial_y f_3\partial_y f_1)+Z^\alpha((1+\rho)f_3\partial_y\partial_y f_1),$$

where

$$\begin{aligned}
&\|Z^\alpha(\partial_y\rho f_3\partial_y f_1)\|^2+\|Z^\alpha((1+\rho)\partial_y f_3\partial_y f_1)\|^2\\
\lesssim&(1+P(Q(t)))(\|\rho\|_{m-2}^2+\|f_3\|_{m-2}^2+\|\partial_y\rho\|_{m-2}^2+\|\partial_y f_1\|_{m-2}^2+\|\partial_y f_3\|_{m-2}^2),
\end{aligned}$$

and

$$\begin{aligned}
&\|Z^\alpha((1+\rho)f_3\partial_y\partial_y f_1)\|^2\\
\lesssim&\sum_{\beta+\kappa=\alpha}\|Z^\beta((1+\rho)f_3)Z^\kappa(\partial_y^2 f_1)\|^2\\
=&\sum_{|\beta|\le|\alpha|/2,\beta+\kappa=\alpha}\|Z^\beta((1+\rho)f_3)Z^\kappa(\partial_y^2 f_1)\|^2+\sum_{|\beta|>|\alpha|/2,\beta+\kappa=\alpha}\|Z^\beta((1+\rho)f_3)Z^\kappa(\partial_y^2 f_1)\|^2\\
\lesssim&\sum_{|\beta|\le|\alpha|/2,\beta+\kappa=\alpha}\|Z^\beta((1+\rho)f_3)\|_{L^\infty_{x,y}}^2\|Z^\kappa(\partial_y^2 f_1)\|_{L^2_{\mathbf{x}}}^2
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{|\beta|>|\alpha|/2,\beta+\kappa=\alpha}\|Z^\beta((1+\rho)f_3)\|^2_{L^2_x(L^\infty_y)}\|Z^\kappa(\partial_y^2 f_1)\|^2_{L^\infty_x(L^2_y)}\\
&\lesssim\sum_{|\beta|\le|\alpha|/2,\beta+\kappa=\alpha}\|Z^\beta((1+\rho)f_3)\|_{L^\infty_x(L^2_y)}\|\partial_y Z^\beta((1+\rho)f_3)\|_{L^\infty_x(L^2_y)}\|Z^\kappa(\partial_y^2 f_1)\|^2_{L^2_{\mathbf{x}}}\\
&+\sum_{|\beta|>|\alpha|/2,\beta+\kappa=\alpha}\|Z^\beta((1+\rho)f_3)\|_{L^2_{\mathbf{x}}}\|\partial_y Z^\beta((1+\rho)f_3)\|_{L^2_{\mathbf{x}}}\|Z^\kappa(\partial_y^2 f_1)\|^2_{L^\infty_x(L^2_y)}\\
&\lesssim(1+P(Q(t)))(\|\rho\|^2_{m-2}+\|f_3\|^2_{m-2}+\|\partial_y\rho\|^2_{m-2}+\|\partial_y f_3\|^2_{m-2})\|\partial_y^2 f_1\|^2_{m-3},
\end{aligned}
$$

provided that $m>8$, where Lemma 2.4 is used in the last inequality. Now we deal with the last term of commutator in (4.24). Note that

$$
[Z^\alpha\partial_y,(1+\rho)(1+f_4)]\partial_y f_2=Z^\alpha(\partial_y((1+\rho)(1+f_4))\partial_y f_2)+[Z^\alpha,(1+\rho)(1+f_4)]\partial_y^2 f_2,
$$

where

$$
\|Z^\alpha(\partial_y((1+\rho)(1+f_4))\partial_y f_2)\|^2\lesssim(1+P(Q(t)))(\|(\rho,f_4)\|^2_{m-2}+\|(\partial_y\rho,\partial_y f_2,\partial_y f_4)\|^2_{m-2}),
$$

and

$$
\begin{aligned}
&\|[Z^\alpha,(1+\rho)(1+f_4)]\partial_y^2 f_2\|^2\\
\lesssim&\sum_{|\beta|\ge1,\beta+\kappa=\alpha}\|Z^\beta((1+\rho)(1+f_4))Z^\kappa(\partial_y^2 f_2)\|^2\\
=&\sum_{1\le|\beta|\le|\alpha|/2,\beta+\kappa=\alpha}\|Z^\beta((1+\rho)(1+f_4))Z^\kappa(\partial_y^2 f_2)\|^2\\
&+\sum_{|\beta|>|\alpha|/2,\beta+\kappa=\alpha}\|Z^\beta((1+\rho)(1+f_4))Z^\kappa(\partial_y^2 f_2)\|^2\\
\lesssim&\sum_{1\le|\beta|\le|\alpha|/2,\beta+\kappa=\alpha}\|Z^\beta((1+\rho)(1+f_4))\|^2_{L^\infty_{x,y}}\|Z^\kappa(\partial_y^2 f_2)\|^2_{L^2_{\mathbf{x}}}\\
&+\sum_{|\beta|>|\alpha|/2,\beta+\kappa=\alpha}\|Z^\beta((1+\rho)(1+f_4))\|^2_{L^2_x(L^\infty_y)}\|Z^\kappa(\partial_y^2 f_2)\|^2_{L^\infty_x(L^2_y)}\\
\lesssim&\sum_{1\le|\beta|\le|\alpha|/2,\beta+\kappa=\alpha}\|Z^\beta((1+\rho)(1+f_4))\|_{L^\infty_x(L^2_y)}\|\partial_y Z^\beta((1+\rho)(1+f_4))\|_{L^\infty_x(L^2_y)}\|Z^\kappa(\partial_y^2 f_2)\|^2_{L^2_{\mathbf{x}}}\\
&+\sum_{|\beta|>|\alpha|/2,\beta+\kappa=\alpha}\|Z^\beta((1+\rho)(1+f_4))\|_{L^2_{\mathbf{x}}}\|\partial_y Z^\beta((1+\rho)(1+f_4))\|_{L^2_{\mathbf{x}}}\|Z^\kappa(\partial_y^2 f_2)\|^2_{L^\infty_x(L^2_y)}\\
\lesssim&(1+P(Q(t)))(\|\rho\|^2_{m-2}+\|f_4\|^2_{m-2}+\|\partial_y\rho\|^2_{m-2}+\|\partial_y f_4\|^2_{m-2})\|\partial_y^2 f_2\|^2_{m-3},
\end{aligned}
$$

provided that $m>8$, where Lemma 2.4 is used in the last inequality.

Consequently, from the above estimates and (4.24) we arrive at

$$
\begin{aligned}
&\|(1+\rho)(1+f_4)Z^\alpha\partial_y^2 f_2\|^2+\mu^2\varepsilon^2\|Z^\alpha\partial_y^3 u\|^2+2\mu\varepsilon\int(1+\rho)(1+f_4)Z^\alpha\partial_y^2 f_2\cdot Z^\alpha\partial_y^3 u\,d\mathbf{x}\\
\lesssim&(1+P(Q(t)))(\|(u,v,f_2)\|^2_m+\|f_1\|^2_{m-1}+P(\|f_4\|_m)+\|(\rho,f_3)\|^2_{m-2})\\
&+(1+P(Q(t)))(\|\partial_y(f_1,f_2)\|^2_{m-1}+\|\partial_y(\rho,f_3,f_4)\|^2_{m-2})\\
&+\|\partial_y p\|^2_{m-1}+\varepsilon^2\|\partial_y u\|^2_m+\varepsilon^2\|\partial_y^2 v\|^2_{m-1} \qquad (4.25)\\
&+(1+P(Q(t)))(\|\rho\|^2_{m-2}+\|f_3\|^2_{m-2}+\|\partial_y\rho\|^2_{m-2}+\|\partial_y f_3\|^2_{m-2})\|\partial_y^2 f_1\|^2_{m-3}
\end{aligned}
$$

$$+ (1+P(Q(t)))(\|\rho\|_{m-2}^2+\|f_4\|_{m-2}^2+\|\partial_y\rho\|_{m-2}^2+\|\partial_y f_4\|_{m-2}^2)\|\partial_y^2 f_2\|_{m-3}^2.$$

Step 2. Applying the operator $Z^\alpha\partial_y^2$ ($|\alpha|\le m-2$) on the equation $(1+f_4)\times$(4.15) gives that

$$\partial_t Z^\alpha\partial_y^2 f_2-(1+f_4)Z^\alpha\partial_y^3 u = Z^\alpha\partial_y^2(f_2\partial_x u)-u\partial_x Z^\alpha\partial_y^2 f_2 - v\partial_y Z^\alpha\partial_y^2 f_2+\mathcal{C}_7^\alpha, \tag{4.26}$$

where

$$\mathcal{C}_7^\alpha=[Z^\alpha\partial_y^2,(1+f_4)]\partial_y u-[Z^\alpha\partial_y^2,u\partial_x]f_2-[Z^\alpha\partial_y^2,v\partial_y]f_2. \tag{4.27}$$

Multiplying (4.26) by $2\mu\varepsilon(1+\rho)Z^\alpha\partial_y^2 f_2$ and integrating the resulting equality over $\mathbb{R}_+^2$ give

$$\begin{aligned}&\mu\varepsilon\frac{d}{dt}\|\sqrt{(1+\rho)}Z^\alpha\partial_y^2 f_2\|^2-2\mu\varepsilon\int(1+\rho)(1+f_4)Z^\alpha\partial_y^2 f_2 Z^\alpha\partial_y^3 u d\mathbf{x}\\ =&2\mu\varepsilon\int(1+\rho)Z^\alpha\partial_y^2 f_2 Z^\alpha\partial_y^2(f_2\partial_x u)d\mathbf{x}+2\mu\varepsilon\int(1+\rho)Z^\alpha\partial_y^2 f_2\mathcal{C}_7^\alpha d\mathbf{x}.\end{aligned} \tag{4.28}$$

Since

$$\partial_y^2\left(f_2\partial_x u\right)=\partial_y^2 f_2\partial_x u+2\partial_y f_2\partial_x\partial_y u+f_2\partial_x\partial_y^2 u,$$

by the Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\left|2\mu\varepsilon\int(1+\rho)Z^\alpha\partial_y^2 f_2 Z^\alpha\partial_y^2\left(f_2\partial_x u\right)d\mathbf{x}+2\mu\varepsilon\int(1+\rho)Z^\alpha\partial_y^2 f_2\mathcal{C}_7^\alpha d\mathbf{x}\right|\\ \le&\delta\|(1+\rho)(1+f_4)Z^\alpha\partial_y^2 f_2\|^2+C_\delta\varepsilon^2\|\mathcal{C}_7^\alpha\|^2\\ &+C_\delta\varepsilon^2\left\|Z^\alpha\left(\partial_y^2 f_2\partial_x u+2\partial_y f_2\partial_x\partial_y u+f_2\partial_x\partial_y^2 u\right)\right\|^2.\end{aligned}$$

Notice that

$$\begin{aligned}&\left\|Z^\alpha\left(\partial_y^2 f_2\partial_x u\right)\right\|^2+\|Z^\alpha\left(\partial_y f_2\partial_x\partial_y u\right)\|^2\\ =&\left\|Z^\alpha\left(\frac{\partial_x u}{\varphi(y)}\varphi(y)\partial_y\partial_y f_2\right)\right\|^2+\|Z^\alpha\left(\partial_y f_2\partial_x\partial_y u\right)\|^2\\ \lesssim&\|\partial_y u\|_{1,\infty}^2\|\partial_y f_2\|_{m-1}^2+\|\partial_y f_2\|_{1,\infty}^2\|\partial_y u\|_{m-1}^2\\ \lesssim&(1+P(Q(t)))(\|\partial_y u\|_{m-1}^2+\|\partial_y f_2\|_{m-1}^2),\end{aligned}$$

and

$$\begin{aligned}&\left\|Z^\alpha(f_2\partial_x\partial_y^2 u)\right\|^2\\ \lesssim&\sum_{|\beta|\le|\alpha|/2,\beta+\kappa=\alpha}\left\|Z^\beta f_2 Z^\kappa\partial_x\partial_y^2 u\right\|^2+\sum_{|\beta|>|\alpha|/2,\beta+\kappa=\alpha}\left\|Z^\beta f_2 Z^\kappa\partial_x\partial_y^2 u\right\|^2\\ \lesssim&\sum_{|\beta|\le|\alpha|/2,\beta+\kappa=\alpha}\left\|Z^\beta f_2\right\|_{L^\infty_{x,y}}^2\|Z^\kappa\partial_x\partial_y^2 u\|_{L^2_{\mathbf{x}}}^2\\ &+\sum_{|\beta|>|\alpha|/2,\beta+\kappa=\alpha}\left\|Z^\beta f_2\right\|_{L^2_x(L^\infty_y)}^2\|Z^\kappa\partial_x\partial_y^2 u\|_{L^\infty_x(L^2_y)}^2\\ \lesssim&\sum_{|\beta|\le|\alpha|/2,\beta+\kappa=\alpha}\left\|Z^\beta f_2\right\|_{L^\infty_x(L^2_y)}\left\|\partial_y Z^\beta f_2\right\|_{L^\infty_x(L^2_y)}\|Z^\kappa\partial_x\partial_y^2 u\|_{L^2_{\mathbf{x}}}^2\\ &+\sum_{|\beta|>|\alpha|/2,\beta+\kappa=\alpha}\left\|Z^\beta f_2\right\|_{L^2_{\mathbf{x}}}\left\|\partial_y Z^\beta f_2\right\|_{L^2_{\mathbf{x}}}\|Z^\kappa\partial_x\partial_y^2 u\|_{L^\infty_x(L^2_y)}^2\end{aligned}$$

$$\lesssim (1+P(Q(t)))(\|f_2\|_{m-2}^2+\|\partial_y f_2\|_{m-2}^2)\|\partial_y^2 u\|_{m-1}^2,$$

provided that $m>4$.

Step 3. Next, we deal with the estimates for commutator of $\mathcal{C}_7^\alpha$ defined in (4.27). First,

$$\begin{aligned}&\left\|\left[Z^\alpha\partial_y^2,(1+f_4)\right]\partial_y u\right\|^2\\ \lesssim&\left\|Z^\alpha\left(\partial_y^2(1+f_4)\partial_y u\right)\right\|^2+\|Z^\alpha\left(\partial_y(1+f_4)\partial_y\partial_y u\right)\|^2+\left\|\left[Z^\alpha,(1+f_4)\right]\partial_y^2\partial_y u\right\|^2.\end{aligned}$$

Since

$$\partial_y f_4=\frac{1}{1+\rho}\{-\partial_x((1+\rho)f_2)-(1+f_4)\partial_y\rho\}$$

due to $\partial_x((1+\rho)f_2)+\partial_y((1+\rho)(1+f_4))=0$, then

$$\begin{aligned}&\left\|Z^\alpha\left(\partial_y^2(1+f_4)\partial_y u\right)\right\|^2\\
=&\left\|Z^\alpha\left(\partial_y\left(\frac{1}{1+\rho}\{-\partial_x((1+\rho)f_2)-(1+f_4)\partial_y\rho\}\right)\partial_y u\right)\right\|^2\\
\lesssim&\left\|Z^\alpha\left(\partial_y\left(\frac{\partial_x((1+\rho)f_2)}{1+\rho}\right)\partial_y u\right)\right\|^2+\left\|Z^\alpha\left(\partial_y\left(\frac{(1+f_4)\partial_y\rho}{1+\rho}\right)\partial_y u\right)\right\|^2\\
\lesssim&\left\|Z^\alpha\left(\partial_y\left(\frac{\partial_x((1+\rho)f_2)}{1+\rho}\right)\partial_y u\right)\right\|^2+\left\|Z^\alpha\left(\partial_y\left(\frac{1+f_4}{1+\rho}\right)\partial_y\rho\partial_y u\right)\right\|^2\\
&+\left\|Z^\alpha\left(\frac{(1+f_4)\partial_y u}{1+\rho}\partial_y^2\rho\right)\right\|^2\\
\lesssim&(1+P(Q(t)))\Big(\|\rho\|_{m-2}^2+\|\partial_y\rho\|_{m-2}^2+\|\partial_y u\|_{m-2}^2+\|\partial_y^2 u\|_{m-2}^2\\
&\qquad+\|f_4\|_{m-2}^2+\|\partial_y f_4\|_{m-2}^2\Big)\|\partial_y^2\rho\|_{m-2}^2\\
&+(1+P(Q(t)))\Big(\|\partial_y(\rho,f_2)\|_{m-1}^2+\|(\rho,f_2)\|_{m-1}^2+\|f_4\|_{m-2}^2+\|\partial_y(u,f_4)\|_{m-2}^2\Big),\end{aligned}$$

where the following estimates are used in the last inequality:

$$\begin{aligned}&\left\|Z^\alpha\left(\partial_y\left(\frac{\partial_x((1+\rho)f_2)}{1+\rho}\right)\partial_y u\right)\right\|^2+\left\|Z^\alpha\left(\partial_y\left(\frac{1+f_4}{1+\rho}\right)\partial_y\rho\partial_y u\right)\right\|^2\\
\lesssim&(1+P(Q(t)))\Big(\|\partial_y(\rho,f_2)\|_{m-1}^2+\|(\rho,f_2)\|_{m-1}^2+\|f_4\|_{m-2}^2+\|\partial_y(u,f_4)\|_{m-2}^2\Big),\end{aligned}$$

and

$$\begin{aligned}&\left\|Z^\alpha\left(\frac{(1+f_4)\partial_y u}{1+\rho}\partial_y^2\rho\right)\right\|^2\\
\lesssim&\sum_{|\beta|\le|\alpha|/2,\beta+\kappa=\alpha}\left\|Z^\beta\left(\frac{(1+f_4)\partial_y u}{1+\rho}\right)Z^\kappa\partial_y^2\rho\right\|^2\\
&+\sum_{|\beta|>|\alpha|/2,\beta+\kappa=\alpha}\left\|Z^\beta\left(\frac{(1+f_4)\partial_y u}{1+\rho}\right)Z^\kappa\partial_y^2\rho\right\|^2\\
\lesssim&\sum_{|\beta|\le|\alpha|/2,\beta+\kappa=\alpha}\left\|Z^\beta\left(\frac{(1+f_4)\partial_y u}{1+\rho}\right)\right\|_{L^\infty_{x,y}}\left\|Z^\kappa\partial_y^2\rho\right\|_{L^2_{\mathbf{x}}}^2\end{aligned}$$

$$
\begin{aligned}
&+\sum_{|\beta|>|\alpha|/2,\beta+\kappa=\alpha}\left\|Z^\beta\left(\frac{(1+f_4)\partial_y u}{1+\rho}\right)\right\|_{L^2_x(L^\infty_y)}\|Z^\kappa\partial_y^2\rho\|^2_{L^\infty_x(L^2_y)}\\
\lesssim&\sum_{|\beta|\le|\alpha|/2,\beta+\kappa=\alpha}\left\|Z^\beta\left(\frac{(1+f_4)\partial_y u}{1+\rho}\right)\right\|_{L^\infty_x(L^2_y)}\left\|\partial_y Z^\beta\left(\frac{(1+f_4)\partial_y u}{1+\rho}\right)\right\|_{L^\infty_x(L^2_y)}\left\|Z^\kappa\partial_y^2\rho\right\|^2_{L^2_{\mathbf{x}}}\\
&+\sum_{|\beta|>|\alpha|/2,\beta+\kappa=\alpha}\left\|Z^\beta\left(\frac{(1+f_4)\partial_y u}{1+\rho}\right)\right\|_{L^2_{\mathbf{x}}}\left\|\partial_y Z^\beta\left(\frac{(1+f_4)\partial_y u}{1+\rho}\right)\right\|_{L^2_{\mathbf{x}}}\|Z^\kappa\partial_y^2\rho\|^2_{L^\infty_x(L^2_y)}\\
\lesssim&(1+P(Q(t)))\Big(\|\rho\|^2_{m-2}+\|\partial_y\rho\|^2_{m-2}+\|\partial_y u\|^2_{m-2}+\|\partial_y^2 u\|^2_{m-2}\\
&\qquad+\|f_4\|^2_{m-2}+\|\partial_y f_4\|^2_{m-2}\Big)\|\partial_y^2\rho\|^2_{m-2},
\end{aligned}
$$

provided that $m>6$.

Moreover,

$$
\begin{aligned}
&\|Z^\alpha\left(\partial_y(1+f_4)\partial_y\partial_y u\right)\|^2\\
\lesssim&(1+P(Q(t)))\|\partial_y f_4\|^2_{m-2}(\|\partial_y^2 u\|^2_{m-2}+\|\partial_y^3 u\|^2_{m-2}),
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\|\left[Z^\alpha,(1+f_4)\right]\partial_y^2\partial_y u\right\|^2\\
\lesssim&\sum_{1\le|\beta|\le|\alpha|/2,\beta+\kappa=\alpha}\left\|Z^\beta\left(1+f_4\right)Z^\kappa\partial_y^3 u\right\|^2+\sum_{|\beta|>|\alpha|/2,\beta+\kappa=\alpha}\left\|Z^\beta\left(1+f_4\right)Z^\kappa\partial_y^3 u\right\|^2\\
\lesssim&\sum_{1\le|\beta|\le|\alpha|/2,\beta+\kappa=\alpha}\left\|Z^\beta\left(1+f_4\right)\right\|_{L^\infty_x(L^2_y)}\left\|\partial_y Z^\beta\left(1+f_4\right)\right\|_{L^\infty_x(L^2_y)}\left\|Z^\kappa\partial_y^3 u\right\|^2_{L^2_{\mathbf{x}}}\\
&+\sum_{|\beta|>|\alpha|/2,\beta+\kappa=\alpha}\left\|Z^\beta\left(1+f_4\right)\right\|_{L^2_{\mathbf{x}}}\left\|\partial_y Z^\beta\left(1+f_4\right)\right\|_{L^2_{\mathbf{x}}}\|Z^\kappa\partial_y^3 u\|^2_{L^\infty_x(L^2_y)}\\
\lesssim&(1+P(Q(t)))(\|f_4\|^2_{m-2}+\|\partial_y f_4\|^2_{m-2})\|\partial_y^3 u\|^2_{m-3},
\end{aligned}
\tag{4.29}
$$

provided that $m>6$.

Similarly,

$$
\begin{aligned}
&\left\|\left[Z^\alpha\partial_y^2,u\partial_x\right]f_2\right\|^2\\
\lesssim&\left\|Z^\alpha(\partial_y^2 u\partial_x f_2)\right\|^2+\|Z^\alpha(\partial_y u\partial_x\partial_y f_2)\|^2+\left\|[Z^\alpha,u/\varphi(y)]\partial_x\varphi(y)\partial_y^2 f_2)\right\|^2\\
\lesssim&(\|\partial_y^2 u\|^2_{m-2}+\|\partial_y^3 u\|^2_{m-2})\|f_2\|^2_{m-1}+(\|\partial_y u\|^2_{m-2}+\|\partial_y^2 u\|^2_{m-2})\|\partial_y f_2\|^2_{m-1}
\end{aligned}
\tag{4.30}
$$

and

$$
\begin{aligned}
&\left\|\left[Z^\alpha\partial_y^2,v\partial_y\right]f_2\right\|^2\\
\lesssim&\left\|Z^\alpha(\partial_y^2 v\partial_y f_2)\right\|^2+\|Z^\alpha(\partial_y v\partial_y\partial_y f_2)\|^2+\left\|[Z^\alpha,v/\varphi(y)]\varphi(y)\partial_y\partial_y^2 f_2)\right\|^2\\
\lesssim&(\|\partial_y^2 v\|^2_{m-2}+\|\partial_y^3 v\|^2_{m-2})\|\partial_y f_2\|^2_{m-2}+(\|\partial_y v\|^2_{m-2}+\|\partial_y^2 v\|^2_{m-2})\|\partial_y^2 f_2\|^2_{m-2}.
\end{aligned}
\tag{4.31}
$$

Step 4. Consequently, from (4.28) we have

$$
\begin{aligned}
&\mu\varepsilon\frac{d}{dt}\|\sqrt{(1+\rho)}Z^\alpha\partial_y^2 f_2\|^2-2\mu\varepsilon\int(1+\rho)(1+f_4)Z^\alpha\partial_y^2 f_2 Z^\alpha\partial_y^3 u d\mathbf{x}\\
\lesssim&\delta\|(1+\rho)(1+f_4)Z^\alpha\partial_y^2 f_2\|^2+\varepsilon^2(\|\partial_y u\|^2_{m-2}+\|\partial_y^2 u\|^2_{m-2})\|\partial_y f_2\|^2_{m-1}
\end{aligned}
$$

$$
\begin{aligned}
&+\varepsilon^2\{(\|\partial_y^2 v\|_{m-2}^2+\|\partial_y^3 v\|_{m-2}^2)\|\partial_y f_2\|_{m-2}^2+(\|\partial_y v\|_{m-2}^2+\|\partial_y^2 v\|_{m-2}^2)\|\partial_y^2 f_2\|_{m-2}^2\}\\
&+\varepsilon^2(1+P(Q(t)))(\|\partial_y(\rho,u,f_2))\|_{m-1}^2+\|(\rho,f_2)\|_{m-1}^2+\|f_4\|_{m-2}^2+\|\partial_y f_4\|_{m-2}^2)\\
&+\varepsilon^2(1+P(Q(t)))(\|f_2\|_{m-2}^2+\|\partial_y f_2\|_{m-2}^2)\|\partial_y^2 u\|_{m-1}^2 \qquad (4.32)\\
&+\varepsilon^2(1+P(Q(t)))(\|\rho\|_{m-2}^2+\|\partial_y\rho\|_{m-2}^2+\|\partial_y u\|_{m-2}^2+\|\partial_y^2 u\|_{m-2}^2\\
&+\|f_4\|_{m-2}^2+\|\partial_y f_4\|_{m-2}^2)\times\|\partial_y^2\rho\|_{m-2}^2\\
&+\varepsilon^2(\|f_4\|_{m-2}^2+\|f_2\|_{m-1}^2+\|\partial_y f_4\|_{m-2}^2)(\|\partial_y^2 u\|_{m-2}^2+\|\partial_y^3 u\|_{m-2}^2).
\end{aligned}
$$

Combining (4.25) and (4.32) and choosing δ suitably small, we have

$$
\begin{aligned}
&\mu\varepsilon\frac{d}{dt}\|\sqrt{(1+\rho)}Z^\alpha\partial_y^2 f_2\|^2+\|(1+\rho)(1+f_4)Z^\alpha\partial_y^2 f_2\|^2+\mu^2\varepsilon^2\|Z^\alpha\partial_y^3 u\|^2\\
\lesssim&(1+P(Q(t)))(\|(u,v,f_2)\|_m^2+\|f_1\|_{m-1}^2+P(\|f_4\|_m)+\|(\rho,f_3)\|_{m-2}^2)\\
&+(1+P(Q(t)))(\|\partial_y(f_1,f_2)\|_{m-1}^2+\|\partial_y(\rho,f_3,f_4)\|_{m-2}^2)\\
&+\|\partial_y p\|_{m-1}^2+\varepsilon^2\|\partial_y u\|_m^2+\varepsilon^2\|\partial_y^2 v\|_{m-1}^2+\varepsilon^2(\|\partial_y u\|_{m-2}^2+\|\partial_y^2 u\|_{m-2}^2)\|\partial_y f_2\|_{m-1}^2\\
&+(1+P(Q(t)))(\|\rho\|_{m-2}^2+\|f_3\|_{m-2}^2+\|\partial_y\rho\|_{m-2}^2+\|\partial_y f_3\|_{m-2}^2)\|\partial_y^2 f_1\|_{m-3}^2\\
&+(1+P(Q(t)))(\|\rho\|_{m-2}^2+\|f_4\|_{m-2}^2+\|\partial_y\rho\|_{m-2}^2+\|\partial_y f_4\|_{m-2}^2)\|\partial_y^2 f_2\|_{m-3}^2.\\
&+\varepsilon^2\{(\|\partial_y^2 v\|_{m-2}^2+\|\partial_y^3 v\|_{m-2}^2)\|\partial_y f_2\|_{m-2}^2+(\|\partial_y v\|_{m-2}^2+\|\partial_y^2 v\|_{m-2}^2)\|\partial_y^2 f_2\|_{m-2}^2\}\\
&+\varepsilon^2(1+P(Q(t)))(\|\partial_y(\rho,u,f_2))\|_{m-1}^2+\|(\rho,f_2)\|_{m-1}^2+\|f_4\|_{m-2}^2+\|\partial_y f_4\|_{m-2}^2)\\
&+\varepsilon^2(1+P(Q(t)))(\|f_2\|_{m-2}^2+\|\partial_y f_2\|_{m-2}^2)\|\partial_y^2 u\|_{m-1}^2 \qquad (4.33)\\
&+\varepsilon^2(1+P(Q(t)))(\|\rho\|_{m-2}^2+\|\partial_y\rho\|_{m-2}^2+\|\partial_y u\|_{m-2}^2+\|\partial_y^2 u\|_{m-2}^2\\
&+\|f_4\|_{m-2}^2+\|\partial_y f_4\|_{m-2}^2)\times\|\partial_y^2\rho\|_{m-2}^2\\
&+\varepsilon^2(\|f_4\|_{m-2}^2+\|f_2\|_{m-1}^2+\|\partial_y f_4\|_{m-2}^2)(\|\partial_y^2 u\|_{m-2}^2+\|\partial_y^3 u\|_{m-2}^2).
\end{aligned}
$$

4.5. **Estimate of $\partial_y p$.** From the second equation in (4.11), we have

$$
\begin{aligned}
\partial_y p-(2\mu+\lambda)\varepsilon\partial_y^2 v=&-(1+\rho)\partial_t v-(1+\rho)u\partial_x v-(1+\rho)v\partial_y v+(1+\rho)(1+f_1)\partial_x f_3\\
&+(1+\rho)f_3\partial_y f_3+(1+\rho)f_2\partial_x f_4+(1+\rho)(1+f_4)\partial_y f_4+\mu\varepsilon\partial_x^2 v+(\mu+\lambda)\varepsilon\partial_y u_x.
\end{aligned}
$$

Since $\partial_x((1+\rho)f_2)+\partial_y((1+\rho)(1+f_4))=0$, then

$$
\begin{aligned}
(1+\rho)(1+f_4)\partial_y f_4=&(1+f_4)\{-\partial_x((1+\rho)f_2)-(1+f_4)\partial_y\rho\}\\
=&-(1+f_4)\partial_x((1+\rho)f_2)-(1+f_4)^2\partial_y\rho\\
=&-(1+f_4)\partial_x((1+\rho)f_2)-\frac{(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\partial_y p,
\end{aligned}
$$

similarly,

$$
(1+\rho)f_3\partial_y f_3=-f_3\partial_x((1+\rho)(1+f_1))-\frac{f_3^2}{\gamma(1+\rho)^{\gamma-1}}\partial_y p,
$$

due to $\partial_x((1+\rho)(1+f_1))+\partial_y((1+\rho)f_3)=0$. Consequently,

$$
\begin{aligned}
&\left(1+\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)\partial_y p-(2\mu+\lambda)\varepsilon\partial_y^2 v\\
=&-(1+\rho)\partial_t v-(1+\rho)u\partial_x v-(1+\rho)v\partial_y v \qquad (4.34)
\end{aligned}
$$

$$
\begin{aligned}
&+(1+\rho)(1+f_1)\partial_x f_3 - f_3\partial_x((1+\rho)(1+f_1))\\
&+(1+\rho)f_2\partial_x f_4-(1+f_4)\partial_x((1+\rho)f_2)+\mu\varepsilon\partial_x^2 v+(\mu+\lambda)\varepsilon\partial_y u_x.
\end{aligned}
$$

Step 1. Applying the operator Z^α ($|\alpha|\le m-1$) on (4.34) leads to

$$
\begin{aligned}
&\left(1+\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)Z^\alpha\partial_y p-(2\mu+\lambda)\varepsilon Z^\alpha\partial_y^2 v\\
=&Z^\alpha\big\{-(1+\rho)\partial_t v-(1+\rho)u\partial_x v-(1+\rho)v\partial_y v\\
&\qquad+(1+\rho)(1+f_1)\partial_x f_3-f_3\partial_x((1+\rho)(1+f_1))\big\}\\
&+Z^\alpha\left\{(1+\rho)f_2\partial_x f_4-(1+f_4)\partial_x((1+\rho)f_2)+\mu\varepsilon\partial_x^2 v+(\mu+\lambda)\varepsilon\partial_y u_x\right\}+\mathcal{C}_8^\alpha, \qquad (4.35)
\end{aligned}
$$

where

$$
\mathcal{C}_8^\alpha=\left[Z^\alpha,\left(1+\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)\right]\partial_y p.
$$

Taking L^2 inner product on the both sides of (4.35) over $\mathbb{R}_+^2$ gives

$$
\begin{aligned}
&\left\|\left(1+\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)Z^\alpha\partial_y p\right\|^2+(2\mu+\lambda)^2\varepsilon^2\|Z^\alpha\partial_y^2 v\|^2\\
&\quad-2(2\mu+\lambda)\varepsilon\int\left(1+\frac{(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)Z^\alpha\partial_y pZ^\alpha\partial_y^2 v d\mathbf{x}\\
\le&\|Z^\alpha\big\{-(1+\rho)\partial_t v-(1+\rho)u\partial_x v-(1+\rho)v\partial_y v\\
&\qquad+(1+\rho)(1+f_1)\partial_x f_3-f_3\partial_x((1+\rho)(1+f_1))\big\}\|^2\\
&+\|Z^\alpha\{(1+\rho)f_2\partial_x f_4-(1+f_4)\partial_x((1+\rho)f_2)+\mu\varepsilon\partial_x^2 v+(\mu+\lambda)\varepsilon\partial_y u_x\}\|^2+\|\mathcal{C}_8^\alpha\|^2\\
\le&\left\|Z^\alpha\left\{-(1+\rho)\partial_t v-(1+\rho)u\partial_x v-(1+\rho)\frac{v}{\varphi(y)}\varphi(y)\partial_y v+(1+\rho)(1+f_1)\partial_x f_3\right\}\right\|^2\\
&+\|Z^\alpha\left\{-f_3\partial_x((1+\rho)(1+f_1))\right\}\|^2 \qquad (4.36)\\
&+\|Z^\alpha\{(1+\rho)f_2\partial_x f_4-(1+f_4)\partial_x((1+\rho)f_2)+\mu\varepsilon\partial_x^2 v+(\mu+\lambda)\varepsilon\partial_y u_x\}\|^2+\|\mathcal{C}_8^\alpha\|^2\\
\lesssim&(1+P(Q(t)))(\|(\rho,v,f_1,f_2,f_3,f_4)\|_m^2+\|u\|_{m-1}^2+\|\partial_y v\|_{m-1}^2)\\
&+\varepsilon^2\|(\partial_x v,\partial_y u)\|_m^2+\|\mathcal{C}_8^\alpha\|^2\\
\lesssim&(1+P(Q(t)))(\|(\rho,v,f_1,f_2,f_3,f_4)\|_m^2+\|f_1\|_{m-1}^2)\\
&+(1+P(Q(t)))\|\partial_y p\|_{m-2}^2+\varepsilon^2\|(\partial_x v,\partial_y u)\|_m^2,
\end{aligned}
$$

where we have used (4.5) and the following estimates in the last inequality:

$$
\begin{aligned}
\|\mathcal{C}_8^\alpha\|^2=&\left\|\left[Z^\alpha,\left(1+\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)\right]\partial_y p\right\|^2\\
\lesssim&\left\|Z\left(1+\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)\right\|_{L^\infty}^2\|\partial_y p\|_{m-2}^2+\|\partial_y p\|_{L^\infty}^2\left\|Z\left(1+\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)\right\|_{m-2}^2\\
\lesssim&(1+P(Q(t)))(\|\partial_y p\|_{m-2}^2+P(\|\rho\|_{m-1})+\|(f_3,f_4)\|_{m-1}^2).
\end{aligned}
$$

Step 2. From the equation of conservation of mass, we have

$$\frac{1}{\gamma p}\partial_t p+\partial_y v=-\frac{1}{\gamma p}(u\partial_x p+v\partial_y p)-\partial_x u. \tag{4.37}$$

Applying the operator $Z^\alpha\partial_y$ ($|\alpha|\le m-1$) on (4.37) yields

$$\frac{1}{\gamma p}\partial_t Z^\alpha\partial_y p+Z^\alpha\partial_y^2 v=-\frac{1}{\gamma p}(u\partial_x Z^\alpha\partial_y p+v\partial_y Z^\alpha\partial_y p)-Z^\alpha\partial_y\partial_x u+\mathcal{C}_9^\alpha, \tag{4.38}$$

with

$$\mathcal{C}_9^\alpha=-\left[Z^\alpha\partial_y,\frac{1}{\gamma p}\partial_t\right]p-\left[Z^\alpha\partial_y,\frac{u}{\gamma p}\partial_x\right]p-\left[Z^\alpha\partial_y,\frac{v}{\gamma p}\partial_y\right]p.$$

Multiplying (4.38) by

$$2(2\mu+\lambda)\varepsilon\left(1+\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)Z^\alpha\partial_y p\triangleq 2a(\rho,f_3,f_4)\varepsilon Z^\alpha\partial_y p$$

and integrating the resulting equality over $\mathbb{R}_+^2$, we obtain

$$\begin{aligned}&\frac{d}{dt}\varepsilon\left\|\sqrt{\frac{a(\rho,f_3,f_4)}{\gamma p}}Z^\alpha\partial_y p\right\|^2+2(2\mu+\lambda)\varepsilon\int\left(1+\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)Z^\alpha\partial_y pZ^\alpha\partial_y^2 vd\mathbf{x}\\ =&\varepsilon\int\left(\left(\frac{a(\rho,f_3,f_4)}{\gamma p}\right)_t+\left(\frac{ua(\rho,f_3,f_4)}{\gamma p}\right)_x+\left(\frac{va(\rho,f_3,f_4)}{\gamma p}\right)_y\right)(Z^\alpha\partial_y p)^2d\mathbf{x}\\ &-\varepsilon\int 2a(\rho,f_3,f_4)Z^\alpha\partial_y pZ^\alpha\partial_y\partial_x ud\mathbf{x}+\varepsilon\int 2a(\rho,f_3,f_4)Z^\alpha\partial_y p\mathcal{C}_9^\alpha d\mathbf{x}.\end{aligned} \tag{4.39}$$

First,

$$\begin{aligned}&\varepsilon\left|\int\left(\left(\frac{a(\rho,f_3f_4)}{\gamma p}\right)_t+\left(\frac{ua(\rho,f_3,f_4)}{\gamma p}\right)_x+\left(\frac{va(\rho,f_3,f_4)}{\gamma p}\right)_y\right)(Z^\alpha\partial_y p)^2d\mathbf{x}\right|\\ \lesssim&\varepsilon(1+P(Q(t)))\|Z^\alpha\partial_y p\|^2,\end{aligned}$$

and

$$\begin{aligned}&\left|\varepsilon\int a(\rho,f_3,f_4)Z^\alpha\partial_y pZ^\alpha\partial_y\partial_x ud\mathbf{x}\right|\\ \lesssim&\varepsilon(1+P(Q(t)))\|Z^\alpha\partial_y p\|\|Z^\alpha\partial_y\partial_x u\|\\ \lesssim&\delta\|Z^\alpha\partial_y p\|^2+C_\delta\varepsilon^2(1+P(Q(t)))\|Z^\alpha\partial_y\partial_x u\|^2.\end{aligned}$$

Step 3. The commutator $\mathcal{C}_9^\alpha$ is estimated as follows. We note that

$$\varepsilon\left|\int a(\rho,f_3,f_4)Z^\alpha\partial_y p\mathcal{C}_9^\alpha d\mathbf{x}\right|\le\delta\|Z^\alpha\partial_y p\|^2+C_\delta\varepsilon^2\|a(\rho,f_3,f_4)\|_{L^\infty}^2\|\mathcal{C}_9^\alpha\|^2.$$

For the first term in $\mathcal{C}_9^\alpha$, one has

$$\left\|\left[Z^\alpha\partial_y,\frac{1}{\gamma p}\partial_t\right]p\right\|\le\left\|Z^\alpha\left(\partial_y\left(\frac{1}{\gamma p}\right)\partial_t p\right)\right\|+\left\|\left[Z^\alpha,\frac{1}{\gamma p}\right]\partial_t\partial_y p\right\|,$$

where

$$\left\|Z^\alpha\left(\partial_y\left(\frac{1}{\gamma p}\right)\partial_t p\right)\right\|=\left\|Z^\alpha\left(\left(\frac{\partial_y p}{\gamma p^2}\right)\partial_t p\right)\right\|$$

$$
\begin{aligned}
&\lesssim \|\partial_y p\|_{L^\infty}\|p_t\|_{m-1}+\|p_t\|_{L^\infty}(1+P(Q(t)))(\|\partial_y p\|_{m-1}+P(\|p-1\|_{m-1}))\\
&\lesssim (1+P(Q(t)))(\|\partial_y p\|_{m-1}+P(\|p-1\|_m)),
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\|\left[Z^\alpha,\frac{1}{\gamma p}\right]\partial_t\partial_y p\right\|\\
\lesssim &\sum_{|\beta|\geq 1,\beta+\kappa=\alpha}\left\|Z^\beta\left(\frac{1}{\gamma p}\right)Z^\kappa\partial_t\partial_y p\right\|\\
\lesssim &\left\|Z\left(\frac{1}{\gamma p}\right)\right\|_{L^\infty}\|\partial_y p\|_{m-1}+\|\partial_y p\|_{1,\infty}P(\|p-1\|_{m-1})\\
\lesssim &(1+P(Q(t)))(P(\|p-1\|_{m-1})+\|\partial_y p\|_{m-1}).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\left\|\left[Z^\alpha\partial_y,\frac{u}{\gamma p}\partial_x\right]p\right\|=\left\|Z^\alpha\left(\partial_y\left(\frac{u}{\gamma p}\right)\partial_x p\right)+\left[Z^\alpha,\frac{u}{\gamma p}\right]\partial_x\partial_y p\right\|\\
\lesssim&\left\|\partial_y\left(\frac{u}{\gamma p}\right)\right\|_{L^\infty}\|\partial_x p\|_{m-1}+\left\|\partial_y\left(\frac{u}{\gamma p}\right)\right\|_{m-1}\|\partial_x p\|_{L^\infty}+\sum_{|\beta|\geq 1,\beta+\kappa=\alpha}\left\|Z^\beta\left(\frac{u}{\gamma p}\right)Z^\kappa\partial_x\partial_y p\right\|\\
\lesssim&\left\|\partial_y\left(\frac{u}{\gamma p}\right)\right\|_{L^\infty}\|\partial_x p\|_{m-1}+\left\|\partial_y\left(\frac{u}{\gamma p}\right)\right\|_{m-1}\|\partial_x p\|_{L^\infty}\\
&+\sum_{|\beta|\geq 1,\beta+\kappa=\alpha}\left\|Z\left(\frac{u}{\gamma p}\right)\right\|_{L^\infty}\|\partial_y p\|_{m-1}+\|\partial_y p\|_{1,\infty}\left\|\frac{u}{\gamma p}\right\|_{m-1}\\
\lesssim&(1+P(Q(t)))(P(\|p-1\|_m)+\|u\|_{m-1}+\|\partial_y p\|_{m-1}+\|\partial_y u\|_{m-1}),
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\|\left[Z^\alpha\partial_y,\frac{v}{\gamma p}\right]\partial_y p\right\|\\
=&\left\|Z^\alpha\left(\partial_y\left(\frac{v}{\gamma p}\right)\partial_y p\right)+\left[Z^\alpha,\frac{v}{\gamma p}\right]\partial_y\partial_y p\right\|\\
=&\left\|Z^\alpha\left(\partial_y\left(\frac{v}{\gamma p}\right)\partial_y p\right)+\left[Z^\alpha,\frac{1}{\gamma p}\frac{v}{\varphi(y)}\right]\varphi(y)\partial_y\partial_y p\right\|\\
\lesssim&\left\|Z^\alpha\left(\partial_y\left(\frac{v}{\gamma p}\right)\partial_y p\right)\right\|+\sum_{|\beta|\geq 1,\beta+\kappa=\alpha}\left\|Z^\beta\left(\frac{1}{\gamma p}\frac{v}{\varphi(y)}\right)Z^\kappa\left(\varphi(y)\partial_y\partial_y p\right)\right\|\\
\lesssim&(1+P(Q(t)))(\|\partial_y p\|_{m-1}+\|\partial_y v\|_{m-1}+P(\|p-1\|_{m-1})+\|v\|_{m-1}).
\end{aligned}
$$

Substituting all of the above inequalities into (4.39) gives

$$
\begin{aligned}
&\frac{d}{dt}\varepsilon\left\|\sqrt{\frac{a(\rho,f_3,f_4)}{\gamma p}}Z^\alpha\partial_y p\right\|^2+2(2\mu+\lambda)\varepsilon\int\left(1+\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)Z^\alpha\partial_y pZ^\alpha\partial_y^2 v d\mathbf{x}\\
\lesssim&\ \varepsilon^2(1+P(Q(t)))\big(\|\partial_y p\|_{m-1}^2+\|\partial_y u\|_{m-1}^2+\|\partial_y v\|_{m-1}^2+\|u\|_{m-1}^2+\|v\|_{m-1}+P(\|p-1\|_m)\big)\\
&+\delta\|Z^\alpha\partial_y p\|^2+\varepsilon(1+P(Q(t)))\|Z^\alpha\partial_y p\|^2+\varepsilon^2(1+P(Q(t)))\|\partial_y u\|_m^2. \qquad (4.40)
\end{aligned}
$$

Combining (4.36) and (4.40) and choosing δ and ε suitably small, we have

$$\begin{aligned}&\frac{d}{dt}\varepsilon\left\|\sqrt{\frac{a(\rho,f_3,f_4)}{\gamma p}}Z^\alpha\partial_y p\right\|^2+\left\|\left(1+\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)Z^\alpha\partial_y p\right\|^2+(2\mu+\lambda)^2\varepsilon^2\|Z^\alpha\partial_y^2 v\|^2\\ \lesssim&(1+P(Q(t)))(\|(v,f_1,f_2,f_3,f_4)\|_m^2+\|u\|_{m-1}^2+P(\|p-1\|_m)+\|\partial_y p\|_{m-2}^2)\\ &+\varepsilon^2(1+P(Q(t)))\|(\partial_x v,\partial_y u)\|_m^2+\varepsilon^2(1+P(Q(t)))(\|\partial_y p\|_{m-1}^2+\|\partial_y u\|_{m-1}^2+\|\partial_y v\|_{m-1}^2).\end{aligned}\tag{4.41}$$

We remark that in order to derive (4.41) we have used the equivalence between $\|\rho\|_m$ and $\|p-1\|_m$, and the *a priori* assumption that $\|\rho\|_{L^\infty}\le 1/2$, $\|f_4\|_{L^\infty}\le 1/2$ and $Q(t)\le C$; moreover, the smallness of ε is required, which is used to absorb the term of $\varepsilon(1+P(Q(t)))\|Z^\alpha\partial_y p\|$ on the right hand side of (4.40).

Summing (4.41) over $|\alpha|\le m-1$, choosing ε suitably small and using the mathematical induction argument yield that

$$\begin{aligned}&\frac{d}{dt}\varepsilon\left\|\sqrt{\frac{a(\rho,f_3,f_4)}{\gamma p}}\partial_y p\right\|_{m-1}^2+\left\|\left(1+\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)\partial_y p\right\|_{m-1}^2+(2\mu+\lambda)^2\varepsilon^2\|\partial_y^2 v\|_{m-1}^2\\ \lesssim&(1+P(Q(t)))(\|(u,v,f_1,f_2,f_3,f_4)\|_m^2+P(\|p-1\|_m))+\varepsilon^2(1+P(Q(t)))\|(\partial_x v,\partial_y u)\|_m^2.\end{aligned}\tag{4.42}$$

where (4.5) and (4.8) are used.

4.6. **Estimate of $\partial_y^2 p$.** We now derive the estimates on $\partial_y^2 p$.

Step 1. Applying the operator $Z^\alpha\partial_y$ ($|\alpha|\le m-2$) on (4.34) leads to

$$\begin{aligned}&\left(1+\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)Z^\alpha\partial_y^2 p-(2\mu+\lambda)\varepsilon Z^\alpha\partial_y^3 v\\ =&Z^\alpha\partial_y\big\{-(1+\rho)\partial_t v-(1+\rho)u\partial_x v-(1+\rho)v\partial_y v\\ &\qquad+(1+\rho)(1+f_1)\partial_x f_3-f_3\partial_x((1+\rho)(1+f_1))\big\}\\ &+Z^\alpha\partial_y\{(1+\rho)f_2\partial_x f_4-(1+f_4)\partial_x((1+\rho)f_2)+\mu\varepsilon\partial_x^2 v+(\mu+\lambda)\varepsilon\partial_y u_x\}+\mathcal{C}_{10}^\alpha\end{aligned}\tag{4.43}$$

with

$$\mathcal{C}_{10}^\alpha=\left[Z^\alpha\partial_y,\ 1+\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right]\partial_y p.$$

Taking L^2 inner product on both sides of (4.43) over $\mathbb{R}_+^2$, we arrive at

$$\begin{aligned}&\left\|\left(1+\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)Z^\alpha\partial_y^2 p\right\|^2+(2\mu+\lambda)^2\varepsilon^2\|Z^\alpha\partial_y^3 v\|^2\\ &-2(2\mu+\lambda)\varepsilon\int\left(1+\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)Z^\alpha\partial_y^2 pZ^\alpha\partial_y^3 v d\mathbf{x}\\ \le&\|Z^\alpha\partial_y\{-(1+\rho)\partial_t v-(1+\rho)u\partial_x v-(1+\rho)v\partial_y v\\ &\qquad+(1+\rho)(1+f_1)\partial_x f_3-f_3\partial_x((1+\rho)(1+f_1))\}\|^2\\ &+\|Z^\alpha\partial_y\{(1+\rho)f_2\partial_x f_4-(1+f_4)\partial_x((1+\rho)f_2)+\mu\varepsilon\partial_x^2 v+(\mu+\lambda)\varepsilon\partial_y u_x\}\|^2+\|\mathcal{C}_{10}^\alpha\|^2\\ \lesssim&(1+P(Q(t)))(\|(\rho,v,f_1,f_2,f_3,f_4)\|_{m-1}^2+\|u\|_{m-2}^2)\end{aligned}$$

$$
\begin{aligned}
&+(1+P(Q(t)))(\|\partial_y(\rho,v,f_2,f_4)\|_{m-1}^2+\|\partial_y(u,f_1)\|_{m-2}^2)+\varepsilon^2\|\partial_y v\|_m^2 \qquad (4.44)\\
&+\varepsilon^2\|\partial_y^2 u\|_{m-1}^2+\|\mathcal{C}_{10}^\alpha\|^2\\
\lesssim&(1+P(Q(t)))(\|(\rho,f_1,f_2,f_3)\|_{m-1}^2+\|u\|_{m-2}^2+P(\|f_4\|_m)+\|v\|_m^2)\\
&+(1+P(Q(t)))(\|\partial_y(\rho,f_2,f_4)\|_{m-1}^2+\|\partial_y f_1\|_{m-2}^2)+\varepsilon^2\|\partial_y v\|_m^2+\varepsilon^2\|\partial_y^2 u\|_{m-1}^2+\|\mathcal{C}_{10}^\alpha\|^2,
\end{aligned}
$$

where (4.5) and (4.8) are used. in the last inequality. It remains to estimate the term of commutator $\mathcal{C}_{10}^\alpha$. First we note that

$$
\begin{aligned}
\|\mathcal{C}_{10}^\alpha\|^2=&\left\|\left[Z^\alpha\partial_y,\left(1+\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)\right]\partial_y p\right\|^2\\
=&\left\|Z^\alpha\left(\partial_y\left(\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)\partial_y p\right)\right\|^2+\left\|\left[Z^\alpha,\left(1+\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)\right]\partial_y^2 p\right\|^2\\
\lesssim&(1+P(Q(t)))(P(\|f_4\|_{m-2})+P(\|\rho\|_{m-2})+\|\partial_y(\rho,f_4)\|_{m-2}^2)\\
&+\sum_{|\beta|\ge1,\beta+\kappa=\alpha}\left\|Z^\beta\left(\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)Z^\kappa\partial_y^2 p\right\|^2.
\end{aligned}
$$

By the similar arguments to (4.29), we have

$$
\begin{aligned}
&\sum_{|\beta|\ge1,\beta+\kappa=\alpha}\left\|Z^\beta\left(\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)Z^\kappa\partial_y^2 p\right\|^2\\
\lesssim&\sum_{1\le|\beta|\le|\alpha|/2,\beta+\kappa=\alpha}\|Z^\kappa\partial_y^2 p\|_{L^2_{\mathbf{x}}}^2\left\|Z^\beta\left(\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)\right\|_{L^\infty_{\mathbf{x}}}^2\\
&+\sum_{|\beta|>|\alpha|/2,\beta+\kappa=\alpha}\|Z^\kappa\partial_y^2 p\|_{L^\infty_x(L^2_y)}^2\left\|Z^\beta\left(\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)\right\|_{L^2_x(L^\infty_y)}^2\\
\lesssim&\sum_{1\le|\beta|\le|\alpha|/2,\beta+\kappa=\alpha}\|Z^\kappa\partial_y^2 p\|_{L^2_{\mathbf{x}}}^2\left\|Z^\beta\left(\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)\right\|_{L^\infty_x(L^2_y)}\left\|\partial_y Z^\beta\left(\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)\right\|_{L^\infty_x(L^2_y)}\\
&+\sum_{|\beta|>|\alpha|/2,\beta+\kappa=\alpha}\|Z^\kappa\partial_y^2 p\|_{L^\infty_x(L^2_y)}^2\left\|Z^\beta\left(\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)\right\|_{L^2_{\mathbf{x}}}\left\|\partial_y Z^\beta\left(\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)\right\|_{L^2_{\mathbf{x}}}\\
\lesssim&(1+P(Q(t)))(P(\|f_3\|_{m-2})+P(\|f_4\|_{m-2})+P(\|\rho\|_{m-2})+\|\partial_y f_4\|_{m-2}^2+\|\partial_y\rho\|_{m-2}^2)\|\partial_y^2 p\|_{m-3}^2,
\end{aligned}
$$

provided that $m>8$. Consequently,

$$
\begin{aligned}
\|\mathcal{C}_{10}^\alpha\|^2\lesssim&(1+P(Q(t)))\big(P(\|f_3\|_{m-2})+P(\|f_4\|_{m-2})+P(\|\rho\|_{m-2})\\
&+\|\partial_y f_4\|_{m-2}^2+\|\partial_y\rho\|_{m-2}^2\big)\|\partial_y^2 p\|_{m-3}^2\\
&+(1+P(Q(t)))(P(\|f_4\|_{m-2})+P(\|\rho\|_{m-2})+\|\partial_y(\rho,f_4)\|_{m-2}^2).
\end{aligned}
$$

Then, we have from (4.44),

$$
\begin{aligned}
&\left\|\left(1+\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)Z^\alpha\partial_y^2 p\right\|^2+(2\mu+\lambda)^2\varepsilon^2\|Z^\alpha\partial_y^3 v\|^2\\
&-2(2\mu+\lambda)\varepsilon\int\left(1+\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)Z^\alpha\partial_y^2 pZ^\alpha\partial_y^3 v d\mathbf{x}
\end{aligned}
$$

$$\begin{aligned}\lesssim&(1+P(Q(t)))(\|(\rho,f_2,f_3)\|_{m-1}^2+\|(u,f_1)\|_{m-2}^2+P(\|f_4\|_m)+\|v\|_m^2)\\&+(1+P(Q(t)))(\|\partial_y(\rho,f_2,f_3,f_4)\|_{m-1}^2+\|\partial_y f_1\|_{m-2}^2)+\varepsilon^2\|\partial_y v\|_m^2+\varepsilon^2\|\partial_y^2 u\|_{m-1}^2\\&+(1+P(Q(t)))\big(P(\|f_3\|_{m-2})+P(\|f_4\|_{m-2})+P(\|\rho\|_{m-2})\\&\qquad+\|\partial_y f_4\|_{m-2}^2+\|\partial_y\rho\|_{m-2}^2\big)\|\partial_y^2 p\|_{m-3}^2.\end{aligned}\tag{4.45}$$

Step 2. Again, from the equation of conservation of mass, we have

$$\partial_t p+\gamma p\partial_y v=-(u\partial_x p+v\partial_y p)-\gamma p\partial_x u.\tag{4.46}$$

Applying the operator $Z^\alpha\partial_y^2$ ($|\alpha|\le m-2$) on (4.46) yields

$$\partial_t Z^\alpha\partial_y^2 p+\gamma pZ^\alpha\partial_y^3 v=-u\partial_x Z^\alpha\partial_y^2 p-v\partial_y Z^\alpha\partial_y^2 p-Z^\alpha\partial_y^2(\gamma p\partial_x u)+\mathcal{C}_{11}^\alpha,\tag{4.47}$$

with

$$\mathcal{C}_{11}^\alpha=-\left[Z^\alpha\partial_y^2,\gamma p\right]\partial_y v-\left[Z^\alpha\partial_y^2,u\partial_x\right]p-\left[Z^\alpha\partial_y^2,v\partial_y\right]p.$$

Multiplying (4.47) by

$$2(2\mu+\lambda)\varepsilon\frac{1}{\gamma p}\left(1+\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)Z^\alpha\partial_y^2 p\triangleq 2\frac{a(\rho,f_3,f_4)}{\gamma p}\varepsilon Z^\alpha\partial_y^2 p$$

and integrating the resulting equality over $\mathbb{R}_+^2$, we get

$$\begin{aligned}&\frac{d}{dt}\varepsilon\left\|\sqrt{\frac{a(\rho,f_3,f_4)}{\gamma p}}Z^\alpha\partial_y^2 p\right\|^2+2(2\mu+\lambda)\varepsilon\int\left(1+\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)Z^\alpha\partial_y^2 pZ^\alpha\partial_y^3 v d\mathbf{x}\\=&\varepsilon\int\left(\left(\frac{a(\rho,f_3,f_4)}{\gamma p}\right)_t+\left(\frac{ua(\rho,f_3,f_4)}{\gamma p}\right)_x+\left(\frac{va(\rho,f_3,f_4)}{\gamma p}\right)_y\right)(Z^\alpha\partial_y^2 p)^2 d\mathbf{x}\\&-\varepsilon\int 2\frac{a(\rho,f_3,f_4)}{\gamma p}Z^\alpha\partial_y^2 pZ^\alpha\partial_y^2(\gamma p\partial_x u)d\mathbf{x}+\int 2\frac{a(\rho,f_3,f_4)}{\gamma p}\varepsilon Z^\alpha\partial_y^2 p\mathcal{C}_{11}^\alpha d\mathbf{x}.\end{aligned}\tag{4.48}$$

First,

$$\begin{aligned}&\varepsilon\left|\int\left(\left(\frac{a(\rho,f_3,f_4)}{\gamma p}\right)_t+\left(\frac{ua(\rho,f_3,f_4)}{\gamma p}\right)_x+\left(\frac{va(\rho,f_3,f_4)}{\gamma p}\right)_y\right)(Z^\alpha\partial_y^2 p)^2 d\mathbf{x}\right|\\&\lesssim\varepsilon(1+P(Q(t)))\|Z^\alpha\partial_y^2 p\|^2,\end{aligned}$$

and

$$\begin{aligned}&|\varepsilon\int\frac{a(\rho,f_3,f_4)}{\gamma p}Z^\alpha\partial_y^2 pZ^\alpha\partial_y^2(\gamma p\partial_x u)d\mathbf{x}|\\\lesssim&\delta\|Z^\alpha\partial_y^2 p\|^2+C_\delta\varepsilon^2(1+P(Q(t)))\|Z^\alpha\partial_y^2(\gamma p\partial_x u)\|^2\\\lesssim&\delta\|Z^\alpha\partial_y^2 p\|^2+C_\delta\varepsilon^2(1+P(Q(t)))(\|Z^\alpha(\partial_y^2 p\partial_x u)\|^2+\|Z^\alpha(\partial_y p\partial_x\partial_y u)\|^2+\|Z^\alpha(p\partial_x\partial_y^2 u)\|^2).\end{aligned}$$

Note that

$$\begin{aligned}&\left\|Z^\alpha\left(\partial_y^2 p\partial_x u\right)\right\|^2+\|Z^\alpha\left(\partial_y p\partial_x\partial_y u\right)\|^2\\=&\left\|Z^\alpha\left(\frac{\partial_x u}{\varphi(y)}\varphi(y)\partial_y\partial_y p\right)\right\|^2+\|Z^\alpha\left(\partial_y p\partial_x\partial_y u\right)\|^2\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|\partial_y u\|_{1,\infty}^2 \|\partial_y p\|_{m-1}^2 + \|\partial_y p\|_{1,\infty}^2 \|\partial_y u\|_{m-1}^2 \\
&\lesssim (1+P(Q(t)))(\|\partial_y u\|_{m-1}^2 + \|\partial_y p\|_{m-1}^2),
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\|Z^\alpha(p\partial_x\partial_y^2 u)\right\|^2 \\
\lesssim &\sum_{|\beta|\le|\alpha|/2,\beta+\kappa=\alpha} \left\|Z^\beta p Z^\kappa \partial_x \partial_y^2 u\right\|^2 + \sum_{|\beta|>|\alpha|/2,\beta+\kappa=\alpha} \left\|Z^\beta p Z^\kappa \partial_x \partial_y^2 u\right\|^2 \\
\lesssim &\sum_{|\beta|\le|\alpha|/2,\beta+\kappa=\alpha} \left\|Z^\beta p\right\|_{L^\infty_{x,y}}^2 \left\|Z^\kappa \partial_x \partial_y^2 u\right\|_{L^2_{\mathbf{x}}}^2 \\
&+ \sum_{|\beta|>|\alpha|/2,\beta+\kappa=\alpha} \left\|Z^\beta p\right\|_{L^2_x(L^\infty_y)}^2 \left\|Z^\kappa \partial_x \partial_y^2 u\right\|_{L^\infty_x(L^2_y)}^2 \\
\lesssim &\sum_{|\beta|\le|\alpha|/2,\beta+\kappa=\alpha} \left\|Z^\beta p\right\|_{L^\infty_x(L^2_y)} \left\|\partial_y Z^\beta p\right\|_{L^\infty_x(L^2_y)} \left\|Z^\kappa \partial_x \partial_y^2 u\right\|_{L^2_{\mathbf{x}}}^2 \\
&+ \sum_{|\beta|>|\alpha|/2,\beta+\kappa=\alpha} \left\|Z^\beta p\right\|_{L^2_{\mathbf{x}}} \left\|\partial_y Z^\beta p\right\|_{L^2_{\mathbf{x}}} \left\|Z^\kappa \partial_x \partial_y^2 u\right\|_{L^\infty_x(L^2_y)}^2 \\
\lesssim &(1+P(Q(t)))(1+\|p-1\|_{m-2}^2 + \|\partial_y p\|_{m-2}^2)\|\partial_y^2 u\|_{m-1}^2,
\end{aligned}
$$

provided that $m > 4$.

The commutator of $\mathcal{C}_{11}^\alpha$ is estimated as follows. For the first term in $\mathcal{C}_{11}^\alpha$, it is noted that

$$
\left\|\left[Z^\alpha \partial_y^2, \gamma p\right] \partial_y v\right\| \le \left\|\gamma Z^\alpha\left(\partial_y^2 p \partial_y v\right)\right\| + \left\|2\gamma Z^\alpha\left(\partial_y p \partial_y^2 v\right)\right\| + \left\|[Z^\alpha, \gamma p]\partial_y^3 v\right\|.
$$

For the above three terms on the right hand side, we have the following estimates:

$$
\begin{aligned}
&\left\|Z^\alpha\left(\partial_y^2 p \partial_y v\right)\right\|^2 \\
= &\int \left(\sum_{|\beta|\le|\alpha|/2,\beta+\kappa=\alpha} Z^\beta(\partial_y^2 p) Z^\kappa \partial_y v + \sum_{|\beta|>|\alpha|/2,\beta+\kappa=\alpha} Z^\beta(\partial_y^2 p) Z^\kappa \partial_y v \right)^2 d\mathbf{x} \\
\lesssim &\int \left(\sum_{|\beta|\le|\alpha|/2,\beta+\kappa=\alpha} Z^\beta\left(\partial_y^2 p\right) Z^\kappa \partial_y v \right)^2 d\mathbf{x} \\
&+ \int \left(\sum_{|\beta|>|\alpha|/2,\beta+\kappa=\alpha} Z^\beta\left(\partial_y^2 p\right) Z^\kappa \partial_y v \right)^2 d\mathbf{x} \\
\lesssim &\sum_{|\beta|\le|\alpha|/2,\beta+\kappa=\alpha} \|Z^\kappa \partial_y v\|_{L^2_x(L^\infty_y)}^2 \left\|Z^\beta \partial_y^2 p\right\|_{L^\infty_x(L^2_y)}^2 \\
&+ \sum_{|\beta|>|\alpha|/2,\beta+\kappa=\alpha} \|Z^\kappa \partial_y v\|_{L^\infty_x(L^\infty_y)}^2 \left\|Z^\beta \partial_y^2 p\right\|_{L^2_x(L^2_y)}^2 \\
\lesssim &\sum_{|\beta|\le|\alpha|/2,\beta+\kappa=\alpha} \|Z^\kappa \partial_y v\|_{L^2_x(L^2_y)} \|\partial_y Z^\kappa \partial_y v\|_{L^2_x(L^2_y)} \left\|Z^\beta \partial_y^2 p\right\|_{L^2_x(L^2_y)} \left\|Z^{\beta+1} \partial_y^2 p\right\|_{L^\infty_x(L^2_y)}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{|\beta|>|\alpha|/2,\beta+\kappa=\alpha}\|Z^\kappa\partial_y v\|_{L_x^2(L_y^2)}^{1/2}\|\partial_y Z^\kappa\partial_y v\|_{L_x^2(L_y^2)}^{1/2}\|Z^{\kappa+1}\partial_y v\|_{L_x^2(L_y^2)}^{1/2}\\
&\qquad\times\|\partial_y Z^{\kappa+1}\partial_y v\|_{L_x^2(L_y^2)}^{1/2}\left\|Z^\beta\partial_y^2 p\right\|_{L_x^2(L_y^2)}^2\\
&\lesssim(1+P(Q(t)))(\|\partial_y v\|_{m-2}^2+\|\partial_y^2 v\|_{m-2}^2)\|\partial_y^2 p\|_{m-2}^2,
\end{aligned}
$$

and similarly,

$$
\left\|[Z^\alpha,\gamma p]\,\partial_y\partial_y^2 v\right\|^2\lesssim(1+P(Q(t)))(\|p-1\|_{m-2}^2+\|\partial_y p\|_{m-2}^2)\|\partial_y^3 v\|_{m-3}^2,
$$

as well as

$$
\|2Z^\alpha\left(\partial_y(\gamma p)\partial_y\partial_y v\right)\|^2\lesssim(1+P(Q(t)))\|\partial_y p\|_{m-2}^2(\|\partial_y^2 v\|_{m-2}^2+\|\partial_y^3 v\|_{m-2}^2).
$$

Consequently, for the first term in $\mathcal{C}_{11}^\alpha$, one has

$$
\begin{aligned}
&\left\|\left[Z^\alpha\partial_y^2,\gamma p\right]\partial_y v\right\|\\
\lesssim&(1+P(Q(t)))\big\{\|\partial_y v\|_{m-2}^2+\|\partial_y^2 v\|_{m-2}^2)\|\partial_y^2 p\|_{m-2}^2\big\}\\
&+(1+P(Q(t)))\big\{(\|p-1\|_{m-2}^2+\|\partial_y p\|_{m-2}^2)\|\partial_y^3 v\|_{m-3}^2+\|\partial_y p\|_{m-2}^2(\|\partial_y^2 v\|_{m-2}^2+\|\partial_y^3 v\|_{m-2}^2)\big\}.
\end{aligned}\tag{4.49}
$$

The second term in $\mathcal{C}_{11}^\alpha$ can be dealt with similarly as the following:

$$
\begin{aligned}
&\|\left[Z^\alpha\partial_y^2,u\partial_x\right]p\|^2\\
\lesssim&(1+P(Q(t)))\big\{\|\partial_y u\|_{m-2}^2+\|\partial_y p\|_{m-1}^2\big\}\\
&+(1+P(Q(t)))\big\{(\|p-1\|_{m-1}^2+\|\partial_y p\|_{m-1}^2)\|\partial_y^2 u\|_{m-2}^2+\|\partial_y^2 p\|_{m-2}^2(\|u\|_{m-2}^2+\|\partial_y u\|_{m-2}^2)\big\}.
\end{aligned}\tag{4.50}
$$

For the third term in $\mathcal{C}_{11}^\alpha$, we notice that

$$
\begin{aligned}
\left[Z^\alpha\partial_y^2,v\partial_y\right]p=&Z^\alpha\left(\partial_y^2 v\partial_y p\right)+Z^\alpha\left(2\partial_y v\partial_y^2 p\right)+[Z^\alpha,v]\,\partial_y^3 p\\
=&Z^\alpha\left(\partial_y^2 v\partial_y p\right)+2Z^\alpha(\partial_y v\partial_y^2 p)+\left[Z^\alpha,\frac{v}{\varphi(y)}\right]\varphi(y)\partial_y\partial_y^2 p,
\end{aligned}
$$

which can also be handled similarly as the following:

$$
\begin{aligned}
&\left\|\left[Z^\alpha\partial_y^2,v\partial_y\right]p\right\|\\
\lesssim&(1+P(Q(t)))\big\{\|\partial_y p\|_{m-2}^2(\|\partial_y^2 v\|_{m-2}^2+\|\partial_y^3 v\|_{m-2}^2)+\|\partial_y^2 p\|_{m-2}^2(\|\partial_y v\|_{m-2}^2+\|\partial_y^2 v\|_{m-2}^2)\big\}.
\end{aligned}\tag{4.51}
$$

Consequently, we have from (4.48)-(4.51),

$$
\begin{aligned}
&\frac{d}{dt}\varepsilon\left\|\sqrt{\frac{a(\rho,f_3,f_4)}{\gamma p}}Z^\alpha\partial_y^2 p\right\|^2+2(2\mu+\lambda)\varepsilon\int\left(1+\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)Z^\alpha\partial_y^2 pZ^\alpha\partial_y^3 v d\mathbf{x}\\
&\lesssim(\delta+(1+P(Q(t)))\varepsilon)\|Z^\alpha\partial_y^2 p\|^2+\varepsilon^2(1+P(Q(t)))(\|\partial_y u\|_{m-1}^2+\|\partial_y p\|_{m-1}^2)\\
&+\varepsilon^2(1+P(Q(t)))(1+\|p-1\|_{m-2}^2+\|\partial_y p\|_{m-2}^2)\|\partial_y^2 u\|_{m-1}^2\\
&+\varepsilon^2(1+P(Q(t)))\big\{(\|\partial_y v\|_{m-2}^2+\|\partial_y^2 v\|_{m-2}^2)\|\partial_y^2 p\|_{m-2}^2\big\}\\
&+\varepsilon^2(1+P(Q(t)))\big\{(\|p-1\|_{m-2}^2+\|\partial_y p\|_{m-2}^2)\|\partial_y^3 v\|_{m-3}^2+\|\partial_y p\|_{m-2}^2(\|\partial_y^2 v\|_{m-2}^2+\|\partial_y^3 v\|_{m-2}^2)\big\}\\
&+\varepsilon^2(1+P(Q(t)))\big\{(\|p-1\|_{m-1}^2+\|\partial_y p\|_{m-1}^2)\|\partial_y^2 u\|_{m-2}^2+\|\partial_y^2 p\|_{m-2}^2(\|u\|_{m-2}^2+\|\partial_y u\|_{m-2}^2)\big\}
\end{aligned}\tag{4.52}
$$

Step 3. Combining (4.45) and (4.52) together and choosing δ and ε suitably small lead to

$$
\begin{aligned}
&\frac{d}{dt}\varepsilon\left\|\sqrt{\frac{a(\rho,f_3,f_4)}{\gamma p}}Z^\alpha\partial_y^2p\right\|^2+\left\|\left(1+\frac{f_3^2+(1+f_4)^2}{\gamma(1+\rho)^{\gamma-1}}\right)Z^\alpha\partial_y^2p\right\|^2+(2\mu+\lambda)^2\varepsilon^2\|Z^\alpha\partial_y^3v\|^2\\
&\lesssim(1+P(Q(t)))(\|(\rho,f_2,f_3)\|_{m-1}^2+\|(u,f_1)\|_{m-2}^2+P(\|f_4\|_m)+\|v\|_m^2)\\
&+(1+P(Q(t)))(\|\partial_y(\rho,f_2,f_3,f_4)\|_{m-1}^2+\|\partial_yf_1\|_{m-2}^2)+\varepsilon^2\|\partial_yv\|_m^2+\varepsilon^2\|\partial_y^2u\|_{m-1}^2\\
&+(1+P(Q(t)))\big(P(\|f_3\|_{m-2})+P(\|f_4\|_{m-2})+P(\|\rho\|_{m-2})\\
&\qquad\qquad+\|\partial_yf_4\|_{m-2}^2+\|\partial_y\rho\|_{m-2}^2\big)\|\partial_y^2p\|_{m-3}^2\\
&+\varepsilon^2(1+P(Q(t)))(\|\partial_yu\|_{m-1}^2+\|\partial_yp\|_{m-1}^2)\\
&+\varepsilon^2(1+P(Q(t)))(1+\|p-1\|_{m-2}^2+\|\partial_yp\|_{m-2}^2)\|\partial_y^2u\|_{m-1}^2\\
&+\varepsilon^2(1+P(Q(t)))\big\{(\|\partial_yv\|_{m-2}^2+\|\partial_y^2v\|_{m-2}^2)\|\partial_y^2p\|_{m-2}^2\big\}\\
&+\varepsilon^2(1+P(Q(t)))\big\{(\|p-1\|_{m-2}^2+\|\partial_yp\|_{m-2}^2)\|\partial_y^3v\|_{m-3}^2+\|\partial_yp\|_{m-2}^2(\|\partial_y^2v\|_{m-2}^2+\|\partial_y^3v\|_{m-2}^2)\big\}\\
&+\varepsilon^2(1+P(Q(t)))\big\{(\|p-1\|_{m-1}^2+\|\partial_yp\|_{m-1}^2)\|\partial_y^2u\|_{m-2}^2+\|\partial_y^2p\|_{m-2}^2(\|u\|_{m-2}^2+\|\partial_yu\|_{m-2}^2)\big\}
\end{aligned}\tag{4.53}
$$

4.7. **Estimates of ∂_yf_1 and $\partial_y^2f_1$.** As for the normal derivatives of f_1, we use the following formulation

$$(1+f_1)(1+f_4)-f_2f_3=\frac{1}{1+\rho}$$

due to $(1+\rho)\det\mathbf{F}=1$. Then

$$
\begin{aligned}
\partial_yf_1=&\frac{1}{(1+f_4)}\left\{\partial_y\left(\frac{1}{1+\rho}\right)+\partial_y(f_2f_3)-(1+f_1)\partial_yf_4\right\}\\
=&\frac{1}{(1+f_4)}\left\{\partial_y\left(\frac{1}{1+\rho}\right)+f_3\partial_yf_2-\frac{f_2}{1+\rho}(f_3\partial_y\rho+\partial_x((1+\rho)(1+f_1)))\right\}\\
&+\frac{1}{(1+f_4)}\left\{\frac{1+f_1}{1+\rho}((1+f_4)\partial_y\rho+\partial_x((1+\rho)f_2))\right\}.
\end{aligned}
$$

due to

$$\partial_x((1+\rho)(1+f_1))+\partial_y((1+\rho)f_3)=0,\qquad\partial_x((1+\rho)f_2)+\partial_y((1+\rho)(1+f_4))=0.$$

Then, the following two inequalities hold true:

$$
\begin{aligned}
&\|\partial_yf_1\|_{m-1}^2\lesssim\left\|\frac{f_3}{1+f_4}\right\|_{L^\infty}^2\|\partial_yf_2\|_{m-1}^2\\
&\quad+(1+P(Q(t)))(\|\partial_y\rho\|_{m-1}^2+\|f_3\|_{m-1}^2+P(\|\rho\|_m)+\|f_1,f_2\|_m^2+P(\|f_4\|_{m-1}))
\end{aligned}\tag{4.54}
$$

and

$$
\begin{aligned}
&\|\partial_y^2f_1\|_{m-2}^2\\
\lesssim&(1+P(Q(t)))(\|f_1,f_2,f_3\|_m^2+P(\|\rho\|_m)+P(\|f_4\|_{m-1}))\\
&+(1+P(Q(t)))\big(\|\partial_y^2\rho\|_{m-2}^2+\|\partial_y\rho\|_{m-1}^2+\|\partial_y^2f_2\|_{m-2}^2\\
&\qquad+\|\partial_yf_1\|_{m-1}^2+\|\partial_yf_3\|_{m-2}^2+\|\partial_yf_4\|_{m-2}^2\big),
\end{aligned}\tag{4.55}
$$

where (4.54) is used.

4.8. **Estimates of $\partial_y^i f_3$ and $\partial_y^i f_4$ $(i=1,2)$.** By the divergence free conditions, we have

$$\partial_y f_3 = \frac{1}{1+\rho}\{-\partial_x((1+\rho)(1+f_1)) - f_3\partial_y\rho\},$$

and

$$\partial_y f_4 = \frac{1}{1+\rho}\{-\partial_x((1+\rho)f_2) - (1+f_4)\partial_y\rho\}.$$

Consequently,

$$\|\partial_y f_3\|_{m-1}^2 \lesssim (1+P(Q(t)))(\|\rho\|_m^2 + \|f_1, f_3\|_m^2) + \|f_3\|_{L^\infty}^2\|\partial_y\rho\|_{m-1}^2, \tag{4.56}$$

By similar arguments, it follows that

$$\|\partial_y f_4\|_{m-1}^2 \lesssim (1+P(Q(t)))(\|\rho\|_m^2 + \|f_2\|_m^2 + \|\partial_y\rho\|_{m-1}^2 + \|f_4\|_{m-1}^2); \tag{4.57}$$

moreover,

$$\begin{aligned}&\|\partial_y^2 f_3\|_{m-2}^2\\ \lesssim&(1+P(Q(t)))(\|\rho\|_{m-1}^2 + \|(f_1,f_3)\|_{m-1}^2 + \|\partial_y(\rho,f_1)\|_{m-1}^2 + \|\partial_y f_3\|_{m-2}^2) + \|f_3\|_{L^\infty}^2\|\partial_y^2\rho\|_{m-2}^2,\end{aligned} \tag{4.58}$$

and

$$\begin{aligned}&\|\partial_y^2 f_4\|_{m-2}^2\\ \lesssim&(1+P(Q(t)))(\|(\rho,f_2,f_4)\|_{m-1}^2 + \|\partial_y(\rho,f_2)\|_{m-1}^2 + \|\partial_y f_4\|_{m-2}^2 + (1+\|f_4\|_m^2)\|\partial_y^2\rho\|_{m-2}^2).\end{aligned} \tag{4.59}$$

Finally, by combining the estimates (3.8), (4.5), (4.6), (4.8), (4.10), (4.22), (4.33), (4.42), (4.53), (4.54)-(4.59), we shall be able to complete the proof of Proposition 4.1. We remark that for this purpose we may apply the multiplications: (4.22) $\times M_0$ and (4.42) $\times M_1$ with M_0 and M_1 being suitably large to cancel the terms $\varepsilon^2\|\partial_y^2 v\|_{m-1}^2$ in (4.33) and $\varepsilon^2\|\partial_y^2 u\|_{m-1}^2$ in (4.53), moreover, it can also cancel $\|\partial_y\rho\|_{m-1}^2$ in the right hand sides of (4.54) and (4.57) due to *a priori* assumption $Q(t)\le C$, and δ in (3.8) is chosen to be suitably small. And the *a priori* assumption of $\|f_3\|_{L^\infty}\le C_0\sigma_0$ is also used. To derive the $L^2_{t,\mathbf{x}}$-norms of second order normal derivatives, we also need the following facts:

$$\sup_{0\le s\le t}\|\partial_y(p,f_2,f_4)(s)\|_{m-2}^2 \lesssim \|\partial_y(p,f_2,f_4)(0)\|_{m-2}^2 + \int_0^t \|\partial_y(p,f_2,f_4)(s)\|_{m-1}^2 ds \le C_0\sigma_0,$$

and

$$\sup_{0\le s\le t}\varepsilon^2\|\partial_y^i(u,v)(s)\|_{m-2}^2 \lesssim \varepsilon^2\|\partial_y^i(u,v)(0)\|_{m-2}^2 + \int_0^t \varepsilon^2\|\partial_y^i(u,v)(s)\|_{m-1}^2 ds \le C_0\sigma_0,\ (i=1,2),$$

due to the condition (1.8) and the *a priori* assumptions:

$$\int_0^t \|\partial_y(p,f_2,f_4)(s)\|_{m-1}^2 ds \le (C_0-1)\sigma_0, \quad \int_0^t \varepsilon^2\|\partial_y^i(u,v)(s)\|_{m-1}^2 ds \le (C_0-1)\sigma_0,\ (i=1,2),$$

where the constant C_0 is suitably large and σ_0 is sufficiently small in Theorem 1.1. Then, the $L^2_{t,\mathbf{x}}$-norms of the second order normal derivatives appearing on the right hand sides of (4.33) and (4.53) can be absorbed by related terms on the left hand sides due to the smallness of σ_0 and ε.

5. Proof of Theorem 1.1

We are ready to prove the estimate (1.10) in Theorem 1.1. Using the estimates in Propositions 4.1, we have

$$
\begin{aligned}
&\|(p-1,\mathbf{u},\mathbf{G_1},\mathbf{G_2})(t)\|_m^2+\varepsilon\|\partial_y f_2(t)\|_{m-1}^2+\varepsilon\left\|\partial_y p\right\|_{m-1}^2+\varepsilon\|\partial_y^2 f_2(t)\|_{m-2}^2+\varepsilon\left\|\partial_y^2 p\right\|_{m-2}^2\\
&+\int_0^t(\varepsilon\|\nabla\mathbf{u}(\tau)\|_m^2+\varepsilon^2(\|\partial_y^2 u\|_{m-1}^2+\|\partial_y^3 u\|_{m-2}^2)+\varepsilon^2(\|\partial_y^2 v\|_{m-1}^2+\|\partial_y^3 v\|_{m-2}^2))d\tau\\
&+\int_0^t(\|\partial_y(\mathbf{u},\mathbf{G_1},\mathbf{G_2})(\tau)\|_{m-1}^2+\|\partial_y^2(\mathbf{u},\mathbf{G_1},\mathbf{G_2})(\tau)\|_{m-2}^2)d\tau\\
&+\int_0^t\left(\|\partial_y p\|_{m-1}^2+\left\|\partial_y^2 p\right\|_{m-2}^2\right)d\tau\\
\lesssim&\|(p-1,\mathbf{u},\mathbf{G_1},\mathbf{G_2})(0)\|_m^2+\varepsilon\|\partial_y f_2(0)\|_{m-1}^2+\varepsilon\left\|\partial_y p(0)\right\|_{m-1}^2+\varepsilon\|\partial_y^2 f_2(0)\|_{m-2}^2\\
&+\varepsilon\left\|\partial_y^2 p(0)\right\|_{m-2}^2+(1+P(Q(t)))\int_0^t\|(\rho,\mathbf{u},\mathbf{G_1},\mathbf{G_2})(\tau)\|_m^2 d\tau. \qquad (5.1)
\end{aligned}
$$

Set

$$
\begin{aligned}
W(t)=&\|(p-1,\mathbf{u},\mathbf{G_1},\mathbf{G_2})(t)\|_m^2+\varepsilon\|\partial_y f_2(t)\|_{m-1}^2+\varepsilon\left\|\partial_y p\right\|_{m-1}^2+\varepsilon\|\partial_y^2 f_2(t)\|_{m-2}^2+\varepsilon\left\|\partial_y^2 p\right\|_{m-2}^2\\
&+\int_0^t(\varepsilon\|\nabla\mathbf{u}(\tau)\|_m^2+\varepsilon^2(\|\partial_y^2 u\|_{m-1}^2+\|\partial_y^3 u\|_{m-2}^2)+\varepsilon^2(\|\partial_y^2 v\|_{m-1}^2+\|\partial_y^3 v\|_{m-2}^2))d\tau\\
&+\int_0^t(\|\partial_y(\mathbf{u},\mathbf{G_1},\mathbf{G_2})(\tau)\|_{m-1}^2+\|\partial_y^2(\mathbf{u},\mathbf{G_1},\mathbf{G_2})(\tau)\|_{m-2}^2)d\tau\\
&+\int_0^t\left(\|\partial_y p\|_{m-1}^2+\left\|\partial_y^2 p\right\|_{m-2}^2\right)d\tau.
\end{aligned}
$$

By Lemma 2.2, we have

$$
\begin{aligned}
&\|(p-1,\mathbf{u},\mathbf{G_1},\mathbf{G_2})(t)\|_{1,\infty}^2\\
\lesssim&\|(p-1,\mathbf{u},\mathbf{G_1},\mathbf{G_2})(0)\|_3^2+\|\partial_y(p,\mathbf{u},\mathbf{G_1},\mathbf{G_2})(0)\|_3^2\\
&+\int_0^t(\|(p-1,\mathbf{u},\mathbf{G_1},\mathbf{G_2})(\tau)\|_4^2+\|\partial_y(p,\mathbf{u},\mathbf{G_1},\mathbf{G_2})(\tau)\|_3^2)d\tau\lesssim W(t)(1+t)+\sigma_0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\|(\nabla p,\nabla\mathbf{u},\nabla\mathbf{G_1},\nabla\mathbf{G_2})(t)\|_{1,\infty}\\
\lesssim&\|(\nabla p,\nabla\mathbf{u},\nabla\mathbf{G_1},\nabla\mathbf{G_2})(0)\|_3^2+\|\partial_y(\nabla p,\nabla\mathbf{u},\nabla\mathbf{G_1},\nabla\mathbf{G_2})(0)\|_3^2\\
&+\int_0^t(\|(\nabla p,\nabla\mathbf{u},\nabla\mathbf{G_1},\nabla\mathbf{G_2})(\tau)\|_4^2+\|\partial_y(\nabla p,\nabla\mathbf{u},\nabla\mathbf{G_1},\nabla\mathbf{G_2})(\tau)\|_3^2)d\tau\\
\lesssim&W(t)(1+t)+\sigma_0,
\end{aligned}
$$

provided that $m>5$. Then one has

$$
\begin{aligned}
W(t)\lesssim&\|(p-1,\mathbf{u},\mathbf{G_1},\mathbf{G_2})(0)\|_m^2+\varepsilon\|\partial_y f_2(0)\|_{m-1}^2+\varepsilon\left\|\partial_y p(0)\right\|_{m-1}^2\\
&+\varepsilon\|\partial_y^2 f_2(0)\|_{m-2}^2+\varepsilon\left\|\partial_y^2 p(0)\right\|_{m-2}^2+(1+P(W(t)(1+t)+\sigma_0))W(t)t. \qquad (5.2)
\end{aligned}
$$

Let the time t and σ_0 be suitably small, it follows that

$$
\begin{aligned}
W(t) =& \|(p-1,\mathbf{u},\mathbf{G_1},\mathbf{G_2})(t)\|_m^2+\varepsilon\|\partial_y f_2(t)\|_{m-1}^2+\varepsilon\left\|\partial_y p\right\|_{m-1}^2+\varepsilon\|\partial_y^2 f_2(t)\|_{m-2}^2+\varepsilon\left\|\partial_y^2 p\right\|_{m-2}^2\\
&+\int_0^t(\varepsilon\|\nabla\mathbf{u}(\tau)\|_m^2+\varepsilon^2(\|\partial_y^2 u\|_{m-1}^2+\|\partial_y^3 u\|_{m-2}^2)+\varepsilon^2(\|\partial_y^2 v\|_{m-1}^2+\|\partial_y^3 v\|_{m-2}^2))d\tau\\
&+\int_0^t(\|\partial_y(\mathbf{u},\mathbf{G_1},\mathbf{G_2})(\tau)\|_{m-1}^2+\|\partial_y^2(\mathbf{u},\mathbf{G_1},\mathbf{G_2})(\tau)\|_{m-2}^2)d\tau\\
&+\int_0^t\left(\|\partial_y p\|_{m-1}^2+\left\|\partial_y^2 p\right\|_{m-2}^2\right)d\tau\\
\lesssim& \|(p-1,\mathbf{u},\mathbf{G_1},\mathbf{G_2})(0)\|_m^2+\varepsilon\|\partial_y f_2(0)\|_{m-1}^2+\varepsilon\left\|\partial_y p(0)\right\|_{m-1}^2\\
&+\varepsilon\|\partial_y^2 f_2(0)\|_{m-2}^2+\varepsilon\left\|\partial_y^2 p(0)\right\|_{m-2}^2,
\end{aligned}\tag{5.3}
$$

and

$$
\begin{aligned}
&\|(p-1,\mathbf{u},\mathbf{G_1},\mathbf{G_2})(t)\|_{1,\infty}^2+\|(\nabla p,\nabla\mathbf{u},\nabla\mathbf{G_1},\nabla\mathbf{G_2})(t)\|_{1,\infty}\\
\lesssim& \|(p-1,\mathbf{u},\mathbf{G_1},\mathbf{G_2})(0)\|_m^2+\varepsilon\|\partial_y f_2(0)\|_{m-1}^2+\varepsilon\left\|\partial_y p(0)\right\|_{m-1}^2\\
&+\varepsilon\|\partial_y^2 f_2(0)\|_{m-2}^2+\varepsilon\left\|\partial_y^2 p(0)\right\|_{m-2}^2+\|\partial_y(p,\mathbf{u},\mathbf{G_1},\mathbf{G_2})(0)\|_3^2\\
&+\|\partial_y(\nabla p,\nabla\mathbf{u},\nabla\mathbf{G_1},\nabla\mathbf{G_2})(0)\|_3^2.
\end{aligned}\tag{5.4}
$$

Consequently, the following *a priori* assumptions hold true:

$$\|\rho\|_{L^\infty}<1/2,\qquad \|f_4\|_{L^\infty}<1/2$$

by letting σ_0 in Theorem 1.1 be suitably small. In fact, the following estimates hold true:

$$\|f_3\|_{L^\infty}\le\frac{C_0}{2}\sigma_0,\qquad \|\rho\|_{1,\infty}\le\frac{C_0}{2}\sigma_0,\qquad \|f_4\|_{1,\infty}\le\frac{C_0}{2}\sigma_0,$$

and

$$\int_0^t\|\partial_y(p,f_2,f_4)(s)\|_{m-1}^2ds\le\frac{C_0}{2}\sigma_0,\qquad \int_0^t\varepsilon^2\|\partial_y^i(u,v)(s)\|_{m-1}^2ds\le\frac{C_0}{2}\sigma_0,\ (i=1,2),$$

where C_0 is a suitably large constant. Based on the uniform *a priori* estimates established above, we can achieve the estimate (1.10) and further verify the inviscid limit in Theorem 1.1 by the similar arguments to those in [31]. We omit the details here. The proof of Theorem 1.1 is completed.

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