A Wiener Path Integral Formalism for Treating Nonlinear Systems with Non-Markovian Response Processes

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Abstract: A novel formalism of the Wiener path integral (WPI) technique for determining the stochastic response of diverse dynamical systems is developed. It can be construed as a generalization of earlier efforts to account, in a direct manner, also for systems with non-Markovian response processes. Specifically, first, the probability of a path and the associated transition probability density function (PDF) corresponding to the Wiener excitation process are considered. Next, a functional change of variables is employed, in conjunction with the governing stochastic differential equation, for deriving the system response joint transition PDF as a functional integral over the space of possible paths connecting the initial and final states of the response vector. In comparison to alternative derivations in the literature, which resort to the Chapman-Kolmogorov equation as the starting point, the herein-developed novel formalism circumvents the Markovian assumption for the system response process. Overall, the veracity and mathematical legitimacy of the WPI technique to treat also non-Markovian system response processes are demonstrated. In this regard, nonlinear systems with a history-dependent state, such as hysteretic structures or oscillators endowed with fractional derivative elements, can be accounted for in a direct manner—that is, without resorting to any ad hoc modifications of the WPI technique pertaining, typically, to employing additional auxiliary filter equations and state variables. A Biot hysteretic oscillator with cubic nonlinearities and an oscillator with asymmetric nonlinearities and fractional derivative elements are considered as illustrative numerical examples for demonstrating the reliability of the developed technique. Comparisons with relevant Monte Carlo simulation (MCS) data are included as well. **DOI: 10.1061/JENMDT.EMENG-6873.** © *2022 American Society of Civil Engineers.*

Introduction

Various methodologies have been developed over the last 6 decades in the field of stochastic engineering dynamics for determining response statistics of diverse structural and mechanical systems (e.g., Roberts and Spanos 2003; Li and Chen 2009). Indicatively, relying on the Markovian assumption for the system response process, a wide range of techniques have been developed for solving the Fokker-Planck partial differential equation governing the system response joint transition probability density function (PDF); see Risken (1984) and Soize (1994) for a broad perspective. Also, related research efforts include discrete versions of the Chapman-Kolmogorov equation for propagating the system response joint PDF in short time steps (e.g., Wehner and Wolfer 1983; Kapitaniak 1985; Grigoriu 1990; Naess and Johnsen 1993); see also Di Paola and Alotta (2020) for a recent review paper.

Nevertheless, for a wide range of systems, the convenient Markovian response assumption cannot be reasonably justified. Indicative examples include systems exhibiting hysteresis or subjected to non-white stochastic excitations. Systems endowed with fractional derivative elements can be construed as a special case of hysteretic systems because their state depends on its history (e.g., Rossikhin and Shitikova 2010). It is remarked, in passing, that fractional calculus has been successfully employed recently in theoretical and applied mechanics for developing nonlocal continuum mechanics theories (e.g., Di Paola et al. 2013; Tarasov 2017), and for modeling viscoelastic materials (e.g., Di Paola et al. 2011).

It is readily seen that such systems with non-Markovian response processes cannot be treated in a straightforward manner by stochastic dynamics techniques that inherently rely on the Markovian response assumption, such as solution schemes based on the Chapman-Kolmogorov and Fokker-Planck equations. Typically, this challenge is bypassed in the literature by augmenting the response vector and by considering additional state variables. These relate to auxiliary equations that are utilized for modeling non-white excitation processes as the output of filters to white noise (e.g., Spanos 1986; Alotta et al. 2014), or for describing the dependence of the state of the system on its history (e.g., Ikhouane and Rodellar 2007; Di Paola et al. 2012). In this regard, the original system is recast into an equivalent one with a Markovian response, and thus, standard random vibration theory tools can be applied (e.g., Roberts and Spanos 2003). Nevertheless, this kind of solution treatment relates usually to increased computational cost due to the increased dimensionality of the problem.

Further, stochastic averaging constitutes a versatile approximate technique for treating diverse nonlinear/hysteretic systems, even when subjected to non-white and non-stationary stochastic excitation (e.g., Kougioumtzoglou and Spanos 2013; Spanos et al. 2018). Specifically, stochastic averaging relates to a Markovian approximation of an appropriately chosen amplitude of the system response and to a dimension reduction of the original problem (e.g., Roberts and Spanos 1986; Zhu 1996). Furthermore, it is worth referring to alternative approaches aiming at developing

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Note. This manuscript was submitted on July 19, 2022; approved on September 2, 2022; published online on November 8, 2022. Discussion period open until April 8, 2023; separate discussions must be submitted for individual papers. This paper is part of the ***Journal of Engineering Mechanics***, © ASCE, ISSN 0733-9399.

and solving generalized versions of the Fokker-Planck equation governing the evolution in time of the non-Markovian response process PDF; see, indicatively, Sapsis and Athanassoulis (2008), Mamis et al. (2019), and references therein. Clearly, such approaches exhibit significant theoretical merit. However, in general, they lack versatility and resort to ad hoc approximations, whereas in many cases the computational challenges can be significant.

One of the promising solution techniques in stochastic engineering dynamics relates to the concept of Wiener path integral (WPI) (e.g., Wiener 1921; Chaichian and Demichev 2001). Specifically, a WPI-based technique for stochastic response determination of diverse dynamical systems has recently been developed (e.g., Kougioumtzoglou and Spanos 2012; Psaros et al. 2018; Petromichelakis et al. 2020). In this regard, the system response joint transition PDF is given as a functional integral over the space of all possible paths connecting the initial and the final states of the response vector. Notably, this functional integral is rarely amenable to analytical evaluation. Thus, an approximate calculation is pursued by considering, ordinarily, the contribution only of the path with the maximum probability of occurrence. This is known as the most probable path and corresponds to an extremum of the functional integrand. In this regard, the most probable path is determined by solving a functional minimization problem that takes the form of a deterministic boundary value problem.

Remarkably, the technique exhibits both high accuracy and low computational cost (e.g., Petromichelakis and Kougioumtzoglou 2020; Psaros and Kougioumtzoglou 2020). However, the formulation of the WPI technique has been developed to-date in conjunction with the Markovian assumption for the system response process. In this paper, an alternative novel formalism is developed for the WPI technique for circumventing the Markovian response assumption. Specifically, considering the probability of a path corresponding to the Wiener (excitation) process, and employing a functional change of variables in conjunction with the governing stochastic differential equation, yields the probability of a path corresponding to the response process. This leads to representing the system response joint transition PDF as a functional integral over the space of possible paths connecting the initial and final states of the response vector. This is done without invoking the assumption of a Markovian response process. Overall, the veracity and mathematical legitimacy of the WPI technique to treat also non-Markovian system response processes are demonstrated. In this regard, nonlinear systems with a history-dependent state, such as hysteretic structures or oscillators endowed with fractional derivative elements, can be accounted for in a direct manner—that is, without resorting to any ad hoc modifications of the WPI technique pertaining, typically, to employing additional auxiliary filter equations and state variables.

Preliminaries

Markov Processes and Stochastic Differential Equations

In this section, the basic aspects of Markov processes and the associated Chapman-Kolmogorov equation are presented for completeness; see also, indicatively, Gardiner (1985) and Grigoriu (2002) for a broader perspective.

In this regard, consider a vector stochastic process $\boldsymbol{\alpha}$, where $\boldsymbol{\alpha} = [\alpha_j]_{n\times 1}$ is a n-length vector whose components α_j, $\forall j \in \{1, \ldots, n\}$, are themselves scalar stochastic processes. Next, consider $\boldsymbol{\alpha}$ to be a Markov process. That is, for every l and for $t_0 < \cdots < t_l$, the following relationship for the corresponding transition PDF holds true:

$$p(\boldsymbol{\alpha}_l, t_l|\boldsymbol{\alpha}_{l-1}, \ldots, \boldsymbol{\alpha}_1; t_{l-1}, \ldots, t_0) = p(\boldsymbol{\alpha}_l, t_l|\boldsymbol{\alpha}_{l-1}, t_{l-1}) \quad (1)$$

Considering Eq. (1), it can be readily seen that for a Markov process $\boldsymbol{\alpha}$, each state $\{t_l, \boldsymbol{\alpha}_l\}$ depends only on the previous (in time) state $\{t_{l-1}, \boldsymbol{\alpha}_{l-1}\}$. Also, for any three distinct time instants $t_{l-1} < t_l < t_{l+1}$, the Chapman-Kolmogorov equation is satisfied. That is,

$$p(\boldsymbol{\alpha}_{l+1}, t_{l+1}|\boldsymbol{\alpha}_{l-1}, t_{l-1}) = \int p(\boldsymbol{\alpha}_{l+1}, t_{l+1}|\boldsymbol{\alpha}_l, t_l)\, p(\boldsymbol{\alpha}_l, t_l|\boldsymbol{\alpha}_{l-1}, t_{l-1})\, \mathrm{d}\boldsymbol{\alpha}_l \quad (2)$$

It is remarked that for a wide range of engineering dynamics applications (e.g., Li and Chen 2009) the governing equations of motion take the form of a coupled system of stochastic differential equations, i.e.,

$$\mathrm{d}\boldsymbol{\alpha} = \boldsymbol{A}(\boldsymbol{\alpha}, t)\mathrm{d}t + \boldsymbol{B}(\boldsymbol{\alpha}, t)\mathrm{d}\boldsymbol{W} \quad (3)$$

where $\boldsymbol{A}(\boldsymbol{\alpha}, t)$ and $\boldsymbol{B}(\boldsymbol{\alpha}, t)$ represent the drift vector and diffusion matrix, respectively; and $\boldsymbol{W}$ denotes the Wiener process, which is a Markov process with independent increments; i.e., $\boldsymbol{W}(t_{l+1}) = \boldsymbol{W}(t_l) + \Delta\boldsymbol{W}(t_l, t_{l+1})$, $\forall l \geq 0$, with $\Delta\boldsymbol{W}(t_l, t_{l+1})$ being statistically independent from any other increment corresponding to different time instants. Furthermore, $\boldsymbol{W}$ has continuous, nowhere differentiable, sample paths and is a Gaussian stochastic process. Notwithstanding some loss of mathematical rigor, Eq. (3) is written alternatively as

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{A}(\boldsymbol{\alpha}, t) + \boldsymbol{B}(\boldsymbol{\alpha}, t)\boldsymbol{\eta}(t) \quad (4)$$

where the dot above a variable denotes differentiation with respect to time t; and $\boldsymbol{\eta}$ denotes a zero-mean and delta-correlated Gaussian white-noise stochastic process of intensity one. That is, $\mathbb{E}[\eta_j(t)] = 0$ and $\mathbb{E}[\eta_j(t_l)\eta_k(t_{l+1})] = \delta_{jk}\delta(t_l - t_{l+1})$ for any $j, k \in \{1, \ldots, n\}$, where δ_{jk} is the Kronecker delta, and $\delta(t)$ is the Dirac delta function. Regarding the relation between the Wiener and the whitenoise processes, $\boldsymbol{\eta}(t)$ can be defined as an infinitesimal jump of the Wiener process, i.e., $\boldsymbol{\eta}(t)\mathrm{d}t = \mathrm{d}\boldsymbol{W}$. Thus, it is often, informally, written as the time derivative of the Wiener process in the form $\boldsymbol{\eta}(t) = \mathrm{d}\boldsymbol{W}/\mathrm{d}t$; see also Gardiner (1985) and Øksendal (2003) for a more detailed discussion on the topic.

Note that the Markovian assumption for the process $\boldsymbol{\alpha}$ as the response of a dynamical system governed by Eq. (4) has been adopted, routinely, by various solution methodologies developed in the field of stochastic engineering dynamics (e.g., Li and Chen 2009). Indicatively, based on preliminary work in theoretical physics (Wehner and Wolfer 1983), numerical path integration has been developed and firmly established as a robust solution tool in stochastic engineering dynamics (e.g., Naess and Johnsen 1993; Chen et al. 2018). Specifically, numerical path integration essentially constitutes a discrete version of the Chapman-Kolmogorov Eq. (2), which utilizes an appropriately chosen short-time transition PDF for advancing in time the system response joint PDF. The scheme exhibits excellent accuracy in determining even the tails of the response PDF. However, it becomes computationally prohibitive with increasing dimensionality. This is due to the fact that a multiconvolution integral needs to be computed for each and every time step, whereas the requisite time increment must remain short. Regarding the short-time transition PDF, it was shown by Dekker (1976) that this can be approximated as $\epsilon = t_{l+1} - t_l \to 0$ by

$$p(\boldsymbol{\alpha}_{i+1}, t_{i+1}|\boldsymbol{\alpha}_i, t_i) = \left[\sqrt{(2\pi\varepsilon)^n \det[\tilde{\boldsymbol{B}}(\boldsymbol{\alpha}_i, t_i)]}\right]^{-1} \times \exp\left(-\frac{1}{2}\frac{[\boldsymbol{\alpha}_{i+1} - \boldsymbol{\alpha}_i - \epsilon \boldsymbol{A}(\boldsymbol{\alpha}_i, t_i)]^T [\tilde{\boldsymbol{B}}(\boldsymbol{\alpha}_i, t_i)]^{-1} [\boldsymbol{\alpha}_{i+1} - \boldsymbol{\alpha}_i - \epsilon \boldsymbol{A}(\boldsymbol{\alpha}_i, t_i)]}{\epsilon}\right) \tag{5}$$

where $\tilde{\boldsymbol{B}}(\boldsymbol{\alpha}, t) = \boldsymbol{B}(\boldsymbol{\alpha}, t)\boldsymbol{B}^T(\boldsymbol{\alpha}, t)$ (e.g., Langouche et al. 1979).

Clearly, numerical path integration relies on the Markovian response assumption. Thus, it cannot be used in a direct manner for treating stochastic processes with a non-Markovian behavior, such as the response of systems exhibiting hysteresis or endowed with fractional derivative elements; see also Di Paola and Alotta (2020) for a relevant discussion. Typically, the aforementioned limitation is bypassed in the literature either by considering an augmented response vector or by applying a stochastic averaging treatment.

In the former case, auxiliary equations are utilized for describing the dependence of the state of the system on its history; see, for example, the popular Bouc-Wen hysteretic model (e.g., Ikhouane and Rodellar 2007), or a recent approach by Di Paola and Alotta (2020), where the equation of motion of a single degree-of-freedom (DOF) linear oscillator with a fractional derivative term is recast into a set of coupled linear equations involving integer-order derivatives and additional state variables.

In the latter case, stochastic averaging relates to a Markovian approximation of an appropriately chosen amplitude of the system response, and to a dimension reduction of the original problem (e.g., Spanos et al. 2018). Notably, stochastic averaging has been used in conjunction with numerical path integration for treating arbitrary forms of non-white and non-stationary stochastic excitations, as well as systems with diverse hysteretic behaviors (e.g., Naess and Moe 1996; Kougioumtzoglou and Spanos 2013).

However, modeling the history-dependent response behavior via additional state variables and equations can be a challenging task that also increases the associated computational cost, whereas a stochastic averaging solution treatment introduces considerable approximations.

Wiener Path Integral Formalism Based on Markovian Response Assumption

This section presents in a concise manner the salient elements of a recently developed technique for determining approximately the stochastic response of diverse structural and mechanical systems (e.g., Kougioumtzoglou and Spanos 2012; Kougioumtzoglou 2017; Petromichelakis and Kougioumtzoglou 2020). The technique relies on functional integration concepts for representing the system response joint PDF as a WPI over the space of all possible paths (e.g., Chaichian and Demichev 2001). The WPI formalism has been developed to-date in conjunction with the Markovian assumption for the response process (e.g., Psaros and Kougioumtzoglou 2020).

Specifically, consider the probability of the process $\boldsymbol{\alpha}$ propagating through some infinitesimally thin tube surrounding a path $\boldsymbol{\alpha}(t)$, $\forall t \in [t_i, t_f]$, with fixed initial and final states $\{t_i, \boldsymbol{\alpha}_i\}$ and $\{t_f, \boldsymbol{\alpha}_f\}$, respectively. This can be construed as the probability of the compound event that the path $\boldsymbol{\alpha}(t)$ successively passes through gates corresponding to specific time instants (e.g., Chaichian and Demichev 2001). Next, relying on the Markov properties of $\boldsymbol{\alpha}$, the probability of the compound event is expressed, equivalently, as the product of the probabilities corresponding to the independent events. The independent events are described by Eq. (5), and thus, the product of the probabilities takes the form

$$P[\boldsymbol{\alpha}(t)] = \lim_{\epsilon \to 0}\left\{\left[\prod_{l=0}^{L}\left(\left[\sqrt{(2\pi\epsilon)^n \det[\tilde{\boldsymbol{B}}(\boldsymbol{\alpha}_l, t_l)]]}\right)^{-1}\right)\right]\left[\prod_{l=1}^{L}\prod_{j=1}^{n} \mathrm{d}\alpha_{j,l}\right] \times \exp\left(-\frac{1}{2}\sum_{l=0}^{L}\frac{[\boldsymbol{\alpha}_{l+1} - \boldsymbol{\alpha}_l - \epsilon \boldsymbol{A}(\boldsymbol{\alpha}_l, t_l)]^T [\tilde{\boldsymbol{B}}(\boldsymbol{\alpha}_l, t_l)]^{-1} [\boldsymbol{\alpha}_{l+1} - \boldsymbol{\alpha}_l - \epsilon \boldsymbol{A}(\boldsymbol{\alpha}_l, t_l)]}{\epsilon}\right)\right\} \tag{6}$$

In Eq. (6), the time domain is discretized into $L+2$ points, ϵ apart (with $L \to \infty$ as $\epsilon \to 0$), as $t_i = t_0 < t_1 < \cdots < t_{L+1} = t_f$, and the path $\boldsymbol{\alpha}(t)$ is represented by its values $\boldsymbol{\alpha}_l$ at the discrete time points t_l, for $l \in \{0, \ldots, L+1\}$. Also, $\mathrm{d}\boldsymbol{\alpha}_{jl}$ denotes the (infinite in number) infinitesimal gates through which the path propagates. In the continuous limit, Eq. (6) becomes

$$P[\boldsymbol{\alpha}(t)] = \exp\left(-\int_{t_i}^{t_f} \mathcal{L}[\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}]\right)\mathcal{D}[\boldsymbol{\alpha}(t)] \tag{7}$$

where the Lagrangian functional $\mathcal{L}[\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}]$ is given by

$$\mathcal{L}[\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}] = \frac{1}{2}[\dot{\boldsymbol{\alpha}} - \boldsymbol{A}(\boldsymbol{\alpha}, t)]^T [\tilde{\boldsymbol{B}}(\boldsymbol{\alpha}, t)]^{-1} [\dot{\boldsymbol{\alpha}} - \boldsymbol{A}(\boldsymbol{\alpha}, t)] \tag{8}$$

and $\mathcal{D}[\boldsymbol{\alpha}(t)]$ takes the form

$$\mathcal{D}[\boldsymbol{\alpha}(t)] = \prod_{j=1}^{n}\mathcal{D}[\boldsymbol{\alpha}_j(t)] = \prod_{j=1}^{n}\prod_{t=t_i}^{t_f}\frac{\mathrm{d}\alpha_j(t)}{\sqrt{2\pi(\det[\tilde{\boldsymbol{B}}(\boldsymbol{\alpha}, t)])^{1/n}\mathrm{d}t}} \tag{9}$$

Further, it is rather intuitive to argue that the respective probabilities of each and every path given by Eq. (7) need to be accounted for, and loosely speaking, summed up to evaluate the

total probability of $\boldsymbol{\alpha}$ starting from $\boldsymbol{\alpha}_i$ at time t_i and reaching $\boldsymbol{\alpha}_f$ at time t_f. In this regard, denoting the set of all paths with initial state $\boldsymbol{\alpha}_i$ at time t_i and final state $\boldsymbol{\alpha}_f$ at time t_f by $\mathcal{C}\{\boldsymbol{\alpha}_i, t_i; \boldsymbol{\alpha}_f, t_f\}$, the joint transition PDF $p(\boldsymbol{\alpha}_f, t_f|\boldsymbol{\alpha}_i, t_i)$ takes the form of a functional integral over $\mathcal{C}\{\boldsymbol{\alpha}_i, t_i; \boldsymbol{\alpha}_f, t_f\}$, i.e.,

$$p(\boldsymbol{\alpha}_f, t_f|\boldsymbol{\alpha}_i, t_i) = \int_{\mathcal{C}\{\boldsymbol{\alpha}_i, t_i; \boldsymbol{\alpha}_f, t_f\}} \exp(-\mathcal{S}[\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}])\mathcal{D}[\boldsymbol{\alpha}(t)] \tag{10}$$

where

$$\mathcal{S}[\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}] = \int_{t_i}^{t_f} \mathcal{L}[\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}]\mathrm{d}t \tag{11}$$

It can be readily seen that the WPI formalism presented in this section relies on the assumption that the system response behaves as a Markov process. Thus, similarly to the numerical path integration scheme discussed previously, it appears that the aforementioned WPI formalism cannot treat in a direct manner stochastic processes with a non-Markovian behavior, such as the response of systems exhibiting hysteresis or endowed with fractional derivative elements. Nevertheless, the WPI technique has been capable of determining the stochastic response of such systems with a history-dependent state provided that they are modeled by utilizing additional auxiliary equations and degrees of freedom; see, for instance, Petromichelakis et al. (2020) where a stochastically excited Bouc-Wen nonlinear hysteretic oscillator was considered.

Further, a conceptually different WPI-based approach was followed by Di Matteo et al. (2014) for treating nonlinear systems with fractional derivative elements subject to Gaussian white-noise excitation. Specifically, by employing the Lagrangian functional of a Gaussian white-noise process that is known in closed-form (e.g., Chaichian and Demichev 2001), a Lagrangian functional referring to the system response process was defined. This was done in a rather heuristic manner by simply substituting the left-hand side of the equation of motion into the white-noise Lagrangian expression and by interpreting the resulting Lagrangian functional as the one corresponding to the response process. In this regard, although the excitation (input) process is Markovian, the Markovian assumption for the system response process is not invoked; see also Di Paola and Alotta (2020) for a discussion. Remarkably, the aforementioned WPI-based technique exhibited a high degree of accuracy in determining the system response joint PDF, which further supported the veracity of the expression for the Lagrangian functional proposed by Di Matteo et al. (2014).

In the ensuing analysis, to demonstrate the mathematical legitimacy of the approach proposed by Di Matteo et al. (2014), a rigorous derivation of a novel WPI formalism is developed based on functional change of variables. The novel formulation circumvents the Markovian assumption for the system response process. Thus, it is shown that the WPI technique can also treat, in a straightforward manner, systems whose response exhibits non-Markovian characteristics, such as the response of hysteretic systems or of oscillators with fractional derivative elements.

Mathematical Formulation

Circumventing the Markovian Response Assumption: A Novel Wiener Path Integral Formalism Based on Functional Change of Variables

Consider an m-DOF nonlinear system governed by

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{g}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) = \boldsymbol{w}(t) \tag{12}$$

where $\boldsymbol{x}$ is the displacement vector process ($\boldsymbol{x} = [x_1, \ldots, x_m]^T$); $\boldsymbol{M}$ is the $m \times m$ mass matrix of the system; and $\boldsymbol{g}(\cdot)$ is an arbitrary nonlinear vector function that can account for possible dependence of the state of the system on its history. In this regard, $\boldsymbol{g}(\cdot)$ can include, indicatively, fractional derivatives or other integro-differential operators modeling hysteresis. Thus, in general, the response process cannot be modeled as Markovian. Further, $\boldsymbol{w}(t)$ is a white-noise stochastic vector process with $\mathbb{E}[\boldsymbol{w}(t)] = 0$ and $\mathbb{E}[\boldsymbol{w}(t)\boldsymbol{w}^T(t+\tau)] = \boldsymbol{S}_{\boldsymbol{w}}\delta(\tau)$, where $\boldsymbol{S}_{\boldsymbol{w}} \in \mathbb{R}^{m\times m}$ is a nonsingular symmetric matrix.

In the following, a novel WPI formalism is developed capable of also treating systems whose response exhibits a non-Markovian behavior. In fact, considering the probability of a path and the associated transition PDF corresponding to the Wiener (excitation) process, and employing a functional change of variables in conjunction with the governing stochastic differential equation, yields the joint transition PDF of the system response. Remarkably, this is expressed as a functional integral given by Eq. (10) without invoking the assumption of a Markovian response process.

Specifically, by employing a state variable formulation, Eq. (12) is recast in the form of Eq. (4) with $n = 2m$, where

$$\boldsymbol{\alpha} = \begin{bmatrix} \boldsymbol{x} \\ \dot{\boldsymbol{x}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \end{bmatrix} \tag{13}$$

$$\boldsymbol{A}(\boldsymbol{\alpha}, t) = \begin{bmatrix} \boldsymbol{v} \\ -\boldsymbol{M}^{-1}\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{v}, t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_1 \\ \boldsymbol{A}_2 \end{bmatrix} \tag{14}$$

and

$$\boldsymbol{B}(\boldsymbol{\alpha}, t) = \begin{bmatrix} \boldsymbol{0}_{m\times m} & \boldsymbol{0}_{m\times m} \\ \boldsymbol{0}_{m\times m} & \boldsymbol{M}^{-1}\sqrt{\boldsymbol{S}_{\boldsymbol{w}}} \end{bmatrix} \tag{15}$$

where the square root of matrix $\boldsymbol{S}_{\boldsymbol{w}}$ is given by $\sqrt{\boldsymbol{S}_{\boldsymbol{w}}}\sqrt{\boldsymbol{S}_{\boldsymbol{w}}}^T = \boldsymbol{S}_{\boldsymbol{w}}$.

Next, considering a discretization of the time domain into $L+2$ points ϵ apart as in Eq. (6), i.e., $t_i = t_0 < \cdots < t_{L+1} = t_f$, it has been shown (e.g., Chaichian and Demichev 2001) that the probability of a path corresponding to the n-dimensional Wiener vector process of Eq. (3) with $\boldsymbol{W}(t_0) = \boldsymbol{W}_0$, $\boldsymbol{W}(t_f) = \boldsymbol{W}_f$ and $\Delta\boldsymbol{W}_l = \boldsymbol{W}_{l+1} - \boldsymbol{W}_l$ is given by

$$\mathcal{P}[\boldsymbol{W}(t)] = \lim_{\epsilon\to 0}\left\{\exp\left(-\frac{1}{2\epsilon}\sum_{l=0}^{L}\Delta\boldsymbol{W}_l^T\Delta\boldsymbol{W}_l\right) \times \prod_{l=0}^{L}\left[\sqrt{(2\pi\epsilon)^n}\right]^{-1}\prod_{j=1}^{n}\left[\prod_{l=1}^{L+1}\mathrm{d}W_{j,l}\right]\right\} \tag{16}$$

Alternatively, Eq. (16) can be derived by setting $\boldsymbol{A}(\boldsymbol{\alpha}, t) = \boldsymbol{0}$ and $\boldsymbol{B}(\boldsymbol{\alpha}, t) = \boldsymbol{I}$ in Eq. (3). In this regard, $\boldsymbol{\alpha} = \boldsymbol{W}$ and Eq. (6) degenerates to Eq. (16). Further, considering Eq. (16) and accounting for the probabilities of all possible paths that the Wiener process $\boldsymbol{W}$ can follow, the corresponding transition PDF is given as the limit of an L-dimensional integral (with $L \to \infty$, or equivalently $\epsilon \to 0$) in the form

$$p(\boldsymbol{W}_f, t_f|\boldsymbol{W}_i, t_i) = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(\boldsymbol{W}_0, \ldots, \boldsymbol{W}_{L+1}) \times \prod_{l=0}^{L}\left[\sqrt{(2\pi\epsilon)^n}\right]^{-1}\prod_{j=1}^{n}\left[\prod_{l=1}^{L+1}\mathrm{d}W_{j,l}\right] \tag{17}$$

where

$$f(\boldsymbol{W}_0, \ldots, \boldsymbol{W}_{L+1}) = \exp\left(-\frac{1}{2\epsilon}\sum_{l=0}^{L}\Delta\boldsymbol{W}_l^T\Delta\boldsymbol{W}_l\right) \tag{18}$$

Furthermore, the transition PDF of a stochastic process $\boldsymbol{\alpha}$ that is related to $\boldsymbol{W}$ at an arbitrary time instant t_l via $\boldsymbol{W}_l = h_l(\boldsymbol{\alpha}_0, \ldots, \boldsymbol{\alpha}_l)$, where $h_l(\cdot)$ is a differentiable function, can be evaluated by employing a functional change of variables in Eq. (17). That is,

$$\begin{aligned} &p(\boldsymbol{\alpha}_f, t_f|\boldsymbol{\alpha}_i, t_i) \\ &= \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(h_0(\boldsymbol{\alpha}_0), \ldots, h_l(\boldsymbol{\alpha}_0, \ldots, \boldsymbol{\alpha}_l), \ldots, \\ &\quad h_{L+1}(\boldsymbol{\alpha}_0, \ldots, \boldsymbol{\alpha}_{L+1}))\times\det(\boldsymbol{J})\prod_{l=0}^{L}\left[\sqrt{(2\pi\epsilon)^n}\right]^{-1}\prod_{j=1}^{n}\left[\prod_{l=1}^{L+1}\mathrm{d}\alpha_{j,l}\right] \end{aligned} \tag{19}$$

where $\boldsymbol{J}$ is the Jacobian of the transformation (e.g., Chaichian and Demichev 2001).

Clearly, in the herein considered problem, $h_l(\cdot)$ corresponds to the relationship between $\boldsymbol{W}$ and $\boldsymbol{\alpha}$ as given by Eq. (3). In fact, note that $h_l(\cdot)$ in Eq. (19) is defined so that at t_l it depends not only on the variable $\boldsymbol{\alpha}_l$, but also on the previous in time variables $\boldsymbol{\alpha}_0, \ldots, \boldsymbol{\alpha}_{l-1}$. In this manner, $h_l(\cdot)$ can account also for cases where Eq. (3), or equivalently Eq. (12), incorporates hysteretic elements. In this regard, employing the Itô discretization rule (e.g., Grigoriu 2002; Øksendal 2003), the discretized in time Eq. (3) takes the form

$$\Delta\boldsymbol{\alpha}_l = \epsilon\boldsymbol{A}_l + \boldsymbol{B}_l\Delta\boldsymbol{W}_l \tag{20}$$

where $\Delta\boldsymbol{\alpha}_l = \boldsymbol{\alpha}_{l+1} - \boldsymbol{\alpha}_l$. Further, to bypass the singularity of matrix $\boldsymbol{B}_l$ in Eq. (20) given by Eq. (15), an auxiliary variable $\beta \to 0$ is employed, and Eq. (15) is cast in the form

$$\bar{\boldsymbol{B}}(\boldsymbol{\alpha}, t) = \begin{bmatrix} \sqrt{\beta}\boldsymbol{I}_{m\times m} & \boldsymbol{0}_{m\times m} \\ \boldsymbol{0}_{m\times m} & \hat{\boldsymbol{B}} \end{bmatrix} \tag{21}$$

with $\hat{\boldsymbol{B}} = \boldsymbol{M}^{-1}\sqrt{\boldsymbol{S}_w}$. Next, Eq. (21) is inverted and Eq. (20) is solved for $\Delta\boldsymbol{W}_l$ yielding

$$\Delta\boldsymbol{W}_l = \begin{bmatrix} \frac{1}{\sqrt{\beta}}[\Delta\boldsymbol{x}_l - \epsilon\boldsymbol{v}_l] \\ \hat{\boldsymbol{B}}_l^{-1}[\Delta\boldsymbol{v}_l - \epsilon\boldsymbol{A}_{2,l}] \end{bmatrix} \tag{22}$$

where Eqs. (13) and (14) have been taken into account. Thus, considering Eq. (22), the product $\Delta\boldsymbol{W}_l^T\Delta\boldsymbol{W}_l$ becomes

$$\begin{aligned} \Delta\boldsymbol{W}_l^T\Delta\boldsymbol{W}_l = &\frac{1}{\beta}[\Delta\boldsymbol{x}_l - \epsilon\boldsymbol{v}_l]^T[\Delta\boldsymbol{x}_l - \epsilon\boldsymbol{v}_l] \\ &+ [\Delta\boldsymbol{v}_l - \epsilon\boldsymbol{A}_{2,l}]^T\tilde{\boldsymbol{B}}_l^{-1}[\Delta\boldsymbol{v}_l - \epsilon\boldsymbol{A}_{2,l}] \end{aligned} \tag{23}$$

where

$$\tilde{\boldsymbol{B}}_l^{-1} = (\hat{\boldsymbol{B}}_l\hat{\boldsymbol{B}}_l^T)^{-1} = (\hat{\boldsymbol{B}}_l^{-1})^T\hat{\boldsymbol{B}}_l^{-1} \tag{24}$$

Furthermore, manipulating Eq. (22) yields

$$\boldsymbol{W}_l = \begin{bmatrix} \frac{1}{\sqrt{\beta}}\boldsymbol{x}_l + \boldsymbol{W}_{1,l-1} - \frac{1}{\sqrt{\beta}}[\boldsymbol{x}_{l-1} + \epsilon\boldsymbol{v}_{l-1}] \\ \hat{\boldsymbol{B}}_{l-1}^{-1}\boldsymbol{v}_l + \boldsymbol{W}_{2,l-1} - \hat{\boldsymbol{B}}_{l-1}^{-1}[\boldsymbol{v}_{l-1} + \epsilon\boldsymbol{A}_{2,l-1}] \end{bmatrix} \tag{25}$$

where $\boldsymbol{W} = [\boldsymbol{W}_1, \boldsymbol{W}_2]^T$. Next, employing sequentially $l-1$ times Eq. (25) for eliminating its dependence on the history of $\boldsymbol{W}$, and also considering the Wiener process property $\boldsymbol{W}_0 = 0$ (e.g., Grigoriu 2002), yields

$$\boldsymbol{W}_l = \begin{bmatrix} \frac{1}{\sqrt{\beta}}(\boldsymbol{x}_l - \boldsymbol{x}_0) - \frac{\epsilon}{\sqrt{\beta}}\sum_{k=1}^{l}\boldsymbol{v}_{k-1} \\ \sum_{k=1}^{l}\hat{\boldsymbol{B}}_{k-1}^{-1}(\boldsymbol{v}_k - \boldsymbol{v}_{k-1} - \epsilon\boldsymbol{A}_{2,k-1}) \end{bmatrix} \tag{26}$$

Obviously, comparing the relationship $\boldsymbol{W}_l = h_l(\boldsymbol{\alpha}_0, \ldots, \boldsymbol{\alpha}_l)$ with Eq. (26), it is readily seen that

$$h_l(\boldsymbol{\alpha}_0, \ldots, \boldsymbol{\alpha}_l) = \begin{bmatrix} \frac{1}{\sqrt{\beta}}(\boldsymbol{x}_l - \boldsymbol{x}_0) - \frac{\epsilon}{\sqrt{\beta}}\sum_{k=1}^{l}\boldsymbol{v}_{k-1} \\ \sum_{k=1}^{l}\hat{\boldsymbol{B}}_{k-1}^{-1}(\boldsymbol{v}_k - \boldsymbol{v}_{k-1} - \epsilon\boldsymbol{A}_{2,k-1}) \end{bmatrix} \tag{27}$$

In this regard, the Jacobian matrix $\boldsymbol{J}$ in Eq. (19) takes the form

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{K}_{1,1} & \boldsymbol{K}_{1,2} & & \boldsymbol{K}_{1,L+1} \\ \boldsymbol{K}_{2,1} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \boldsymbol{K}_{L+1,1} & & & \boldsymbol{K}_{L+1,L+1} \end{bmatrix} \tag{28}$$

where

$$\boldsymbol{K}_{l_1,l_2} = \left[\frac{\partial h_{j_1,l_1}}{\partial\alpha_{j_2,l_2}}\right]_{2m\times 2m}, j_1, j_2 \in \{1, \cdots, 2m\}; l_1, l_2 \in \{1, \ldots, L+1\} \tag{29}$$

Note that based on the definition of $h_l(\boldsymbol{\alpha}_0, \ldots, \boldsymbol{\alpha}_l)$, which corresponds to causal systems, h_{j,l_1} does not depend on $\boldsymbol{\alpha}_{l_2}$ for all $j \in \{1, \cdots, 2m\}$ and $l_1 < l_2 \in \{1, \ldots, L+1\}$. Thus,

$$\boldsymbol{K}_{l_1,l_2} = \boldsymbol{0}_{2m\times 2m} \quad \text{for } l_1 < l_2 \tag{30}$$

Consequently, based on Eq. (30), $\boldsymbol{J}$ is a lower-triangular block matrix. Considering Eq. (29) in conjunction with Eq. (27), the diagonal elements $\boldsymbol{K}_{l,l}$ become

$$\boldsymbol{K}_{l,l} = \begin{bmatrix} \frac{1}{\sqrt{\beta}}\boldsymbol{I}_{m\times m} & \boldsymbol{0}_{m\times m} \\ \boldsymbol{0}_{m\times m} & \hat{\boldsymbol{B}}_{l-1}^{-1} \end{bmatrix}, \quad \text{for } l \in \{1, \ldots, L+1\} \tag{31}$$

Thus, the determinant of $\boldsymbol{J}$ is given by (Strang 2016)

$$\begin{aligned} |\boldsymbol{J}| &= \prod_{l=1}^{L+1}|\boldsymbol{K}_{l,l}| = \left[\prod_{l=1}^{L+1}\left(\frac{1}{\sqrt{\beta}}\right)^m\right]\left[\prod_{l=1}^{L+1}|\hat{\boldsymbol{B}}_{l-1}|^{-1}\right] \\ &= \left[\prod_{l=0}^{L}\left(\frac{1}{\sqrt{\beta}}\right)^m\right]\left[\prod_{l=0}^{L}|\tilde{\boldsymbol{B}}_l|^{\frac{-1}{2}}\right] \end{aligned} \tag{32}$$

Next, substituting Eq. (23) and Eq. (32) into Eq. (19) yields

$$p(\boldsymbol{\alpha}_f, t_f|\boldsymbol{\alpha}_i, t_i) = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\prod_{l=0}^{L}\left(\left[\sqrt{(2\pi\epsilon)^n|\tilde{\boldsymbol{B}}_l|}\right]^{-1}\right)\right]\left[\prod_{l=1}^{L}\prod_{j=1}^{n}\mathrm{d}\alpha_{j,l}\right]$$
$$\times\exp\left(-\frac{1}{2\epsilon}\sum_{l=0}^{L}\frac{1}{\beta}[\Delta\boldsymbol{x}_l-\epsilon\boldsymbol{v}_l]^T[\Delta\boldsymbol{x}_l-\epsilon\boldsymbol{v}_l]+[\Delta\boldsymbol{v}_l-\epsilon\boldsymbol{A}_{2,l}]^T\tilde{\boldsymbol{B}}_l^{-1}[\Delta\boldsymbol{v}_l-\epsilon\boldsymbol{A}_{2,l}]\right) \tag{33}$$

or, equivalently

$$p(\boldsymbol{\alpha}_f, t_f|\boldsymbol{\alpha}_i, t_i) = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\prod_{l=0}^{L}\prod_{j=1}^{m}\left[\sqrt{2\pi\epsilon|\tilde{\boldsymbol{B}}_l|^{\frac{1}{m}}}\right]^{-1}\right]\left[\prod_{l=1}^{L}\prod_{j=1}^{m}\mathrm{d}x_{j,l}\right]$$
$$\times\left[\prod_{l=1}^{L}\prod_{j=1}^{m}\mathrm{d}v_{j,l}\right]\exp\left(-\frac{1}{2\epsilon}\sum_{l=0}^{L}[\Delta\boldsymbol{v}_l-\epsilon\boldsymbol{A}_{2,l}]^T\tilde{\boldsymbol{B}}_l^{-1}[\Delta\boldsymbol{v}_l-\epsilon\boldsymbol{A}_{2,l}]\right)\times\left[\prod_{l=0}^{L}\prod_{j=1}^{m}\frac{1}{\sqrt{\beta}}\exp\left(-\frac{1}{2\beta\epsilon}[\Delta x_{jl}-\epsilon v_{jl}]^2\right)\right] \tag{34}$$

Further, taking the limit as $\beta\to 0$ and employing the delta function properties yields (e.g., Demidov 2001)

$$\lim_{\beta\to 0}\frac{1}{\sqrt{\beta}}\exp\left(-\frac{1}{2\beta\epsilon}[\Delta x_{jl}-\epsilon v_{jl}]^2\right)=\epsilon\delta(\Delta x_{jl}-\epsilon v_{jl}) \tag{35}$$

Note that in the continuous limit, Eq. (35) simply enforces the compatibility condition $\dot{\boldsymbol{x}}=\boldsymbol{v}$ on Eq. (34). Lastly, combining Eqs. (34) and (35) and taking the continuous limit, as $\epsilon\to 0$, the transition PDF of the process $\boldsymbol{\alpha}$ converges to the exact same form as in Eq. (10), where the Lagrangian functional is given by Eq. (8). Specifically, employing Eq. (13) and expressing $\boldsymbol{\alpha}$ in terms of $(\boldsymbol{x},\dot{\boldsymbol{x}})$, the response transition PDF of the system of Eq. (12) is given by

$$p(\boldsymbol{x}_f,\dot{\boldsymbol{x}}_f,t_f|\boldsymbol{x}_i,\dot{\boldsymbol{x}}_i,t_i)=\int_{\mathcal{C}\{\boldsymbol{x}_f,\dot{\boldsymbol{x}}_f,t_f;\boldsymbol{x}_i,\dot{\boldsymbol{x}}_i,t_i\}}\exp(-\mathcal{S}[\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}}])\mathcal{D}[\boldsymbol{x}(t)] \tag{36}$$

where

$$\mathcal{S}[\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}}]=\int_{t_i}^{t_f}\mathcal{L}[\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}}]\mathrm{d}t \tag{37}$$

The Lagrangian functional $\mathcal{L}[\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}}]$ in Eq. (37) takes the form

$$\mathcal{L}[\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}}]=\frac{1}{2}[\boldsymbol{M}\ddot{\boldsymbol{x}}+\boldsymbol{g}(\boldsymbol{x},\dot{\boldsymbol{x}},t)]^T\ \boldsymbol{S}_{\boldsymbol{w}}^{-1}[\boldsymbol{M}\ddot{\boldsymbol{x}}+\boldsymbol{g}(\boldsymbol{x},\dot{\boldsymbol{x}},t)] \tag{38}$$

and

$$\mathcal{D}[\boldsymbol{x}(t)]=\prod_{j=1}^{m}\mathcal{D}[\boldsymbol{x}_j(t)]=\prod_{j=1}^{m}\prod_{t=t_i}^{t_f}\frac{\mathrm{d}x_j(t)}{\sqrt{2\pi(\det[\tilde{\boldsymbol{B}}(\boldsymbol{x},t)])^{\frac{1}{m}}\mathrm{d}t}} \tag{39}$$

Overall, an alternative novel WPI formalism has been developed in this section based on functional change of variables. In comparison with previous efforts, which resort to the Chapman-Kolmogorov Eq. (2) as the starting point (e.g., Petromichelakis and Kougioumtzoglou 2020; Psaros and Kougioumtzoglou 2020), the Markovian assumption for the system response process has been circumvented. In fact, the mathematical legitimacy of using the general form of the Lagrangian functional given by Eq. (8), or equivalently Eq. (38), also for systems whose response exhibits non-Markovian characteristics, has been demonstrated. In this regard, systems with a history-dependent state, such as hysteretic structures or oscillators endowed with fractional derivative elements, can be treated in a direct manners. That is, without resorting to ad hoc modifications of the technique pertaining, typically, to considering additional auxiliary equations and state variables.

Most Probable Path Approximation

In this section the basic aspects of the numerical implementation of the WPI technique are presented for completeness. The interested reader is also directed to Di Matteo et al. (2014), to Petromichelakis et al. (2020) and to Petromichelakis et al. (2021a) for more details and a broader perspective.

In this regard, note that the analytical calculation of the WPI of Eq. (36) for determining the transition PDF of the process $\boldsymbol{x}$ is, in general, impossible. Thus, alternative approaches are typically pursued in the literature for evaluating approximately Eq. (36), such as the most probable path approach (e.g., Chaichian and Demichev 2001). It is remarked that the most probable path approximation has exhibited a quite high degree of accuracy in various diverse engineering mechanics applications (e.g., Kougioumtzoglou 2017; Petromichelakis et al. 2018, 2021b). In fact, as proved by Psaros et al. (2020), for the case of linear systems, the most probable path approximation yields the exact joint response PDF.

Specifically, the largest contribution to the functional integral of Eq. (36) relates to the trajectory $\boldsymbol{x}_c(t)$ for which the stochastic action of Eq. (37) becomes as small as possible. This leads to the variational (functional minimization) problem

$$\underset{\mathcal{C}\{\boldsymbol{x}_i,\dot{\boldsymbol{x}}_i,t_i;\boldsymbol{x}_f,\dot{\boldsymbol{x}}_f,t_f\}}{\text{minimize}}\ \mathcal{S}[\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}}] \tag{40}$$

The deterministic problem of Eq. (40) can be readily solved by standard numerical approaches such as Rayleigh-Ritz type schemes (e.g., Zienkiewicz and Morgan 1983; Di Matteo et al. 2014; Petromichelakis et al. 2020). Alternatively, relying on computational algebraic geometry concepts and tools, Petromichelakis et al. (2021a) developed recently a technique based on Groebner basis for solving Eq. (40) and for determining the most probable path $\boldsymbol{x}_c(t)$. Next, following solution of Eq. (40) and determination of $\boldsymbol{x}_c(t)$, the functional integral of Eq. (36) is evaluated approximately as

$$p(\boldsymbol{x}_f,\dot{\boldsymbol{x}}_f,t_f|\boldsymbol{x}_i,\dot{\boldsymbol{x}}_i,t_i)\approx C\exp(-\mathcal{S}[\boldsymbol{x}_c,\dot{\boldsymbol{x}}_c,\ddot{\boldsymbol{x}}_c]) \tag{41}$$

where C is a constant to be determined by the normalization condition

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}p(\boldsymbol{x}_f,\dot{\boldsymbol{x}}_f,t_f|\boldsymbol{x}_i,\dot{\boldsymbol{x}}_i,t_i)\mathrm{d}\boldsymbol{x}_f\mathrm{d}\dot{\boldsymbol{x}}_f=1 \tag{42}$$

Mechanization of the Technique

Concisely stated, the mechanization of the technique comprises the following steps:

1. For a given time instant t_f, consider an effective domain of final states $\{\boldsymbol{x}_f, \dot{\boldsymbol{x}}_f\}$ and discretize it into N^{2m} points, where N is the number of points in each dimension and $2m$ is the number of stochastic dimensions.
2. For each final state $\{\boldsymbol{x}_f, \dot{\boldsymbol{x}}_f\}$, determine the most probable path $\boldsymbol{x}_c(t)$ by solving numerically Eq. (40).
3. Obtain the system response transition PDF by utilizing Eqs. (41) and (42).

Clearly, solution of the functional minimization problem of Eq. (40) yields $\boldsymbol{x}_c(t)$ to be substituted into Eq. (41) for determining a specific point of the response PDF. In this regard, it is readily seen that a brute-force approach for computing the complete $2m$-dimensional transition PDF at time instant t_f dictates, first, the discretization of the $(\boldsymbol{x}_f, \dot{\boldsymbol{x}}_f)$ domain into N^{2m} points, where N is the number of points in each dimension, and second, the numerical solution of N^{2m} problems of the form of Eq. (40). Evidently, the computational cost increases exponentially with increasing number of dimensions. Thus, alternative, more efficient formulations have been developed recently (e.g., Psaros et al. 2018; Zhang et al. 2022) by relying on sparse PDF expansions in conjunction with compressive sampling concepts and tools (e.g., Kougioumtzoglou et al. 2020). Notably, the aforementioned approaches can be coupled with a WPI variational formulation with mixed fixed/free boundaries for further reduction of the computational cost. Specifically, Petromichelakis and Kougioumtzoglou (2020) showed that the system response joint PDF can be marginalized in an a priori manner, and thus, the associated computational cost becomes independent of the total number of stochastic dimensions.

Numerical Examples

Biot Hysteretic Oscillator with Cubic Nonlinearities

Consider a single-DOF oscillator with cubic nonlinearities and a Biot hysteretic element (e.g., Biot 1958; Caughey 1962). In this regard, Eq. (12) becomes

$$m\ddot{x} + g(x, \dot{x}, t) = w(t) \tag{43}$$

where

$$g(x, \dot{x}, t) = k\left(\frac{2\eta}{\pi}\int_{t_i}^{t} I[\lambda(t-\tau)]\dot{x}(\tau)\mathrm{d}\tau\right) + k(x + \epsilon x^3) \tag{44}$$

and

$$I(y) = \int_{y}^{\infty} \frac{e^{-u}}{u}\mathrm{d}u \tag{45}$$

where m and k denote the mass and stiffness coefficients; ϵ controls the nonlinearity magnitude; η is the loss factor of the Biot model; and λ is a constant. Further, the scalar Gaussian white-noise excitation process $w(t)$ in Eq. (43) corresponds to a constant power spectrum S_0. The interested reader is also directed to Spanos and Tsavachidis (2001) and references therein for a detailed presentation and discussion of the Biot hysteretic model.

Obviously, the response of the Biot hysteretic oscillator depends on its history based on Eq. (44). Thus, a solution treatment relying on the Markovian response assumption cannot be employed. In contrast, the herein developed alternative formulation of the WPI technique can treat in a direct manner the oscillator of Eq. (43). In this regard, utilizing the parameter values $m = 1$, $k = 1$, $\epsilon = 2.2$, $\eta = 1.2$, $\lambda = 0.1$, $S_0 = 0.125$, $t_i = 0$, and $N = 85$, the functional minimization problem of Eq. (40) is solved based on a Rayleigh-Ritz scheme for determining the most probable path $x_c(t)$. This is done in conjunction with the Lagrangian functional of Eq. (38) that takes the form

$$\mathcal{L}[x, \dot{x}, \ddot{x}] = \frac{1}{4\pi S_0}\left[m\ddot{x} + k\left(\frac{2\eta}{\pi}\int_{t_i}^{t} I[\lambda(t-\tau)]\dot{x}(\tau)\mathrm{d}\tau\right) + k(x + \epsilon x^3)\right]^2 \tag{46}$$

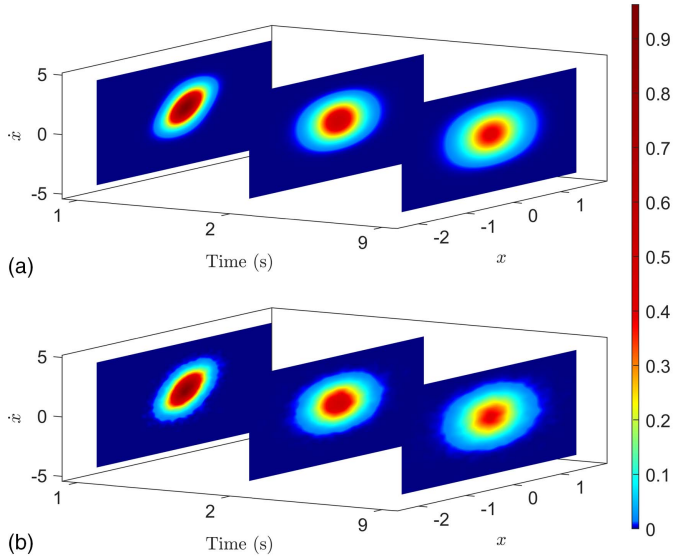

Fig. 1. Response joint PDFs at indicative time instants $t_f = 1$, 2, and 9 s corresponding to a stochastically excited Biot hysteretic oscillator with cubic nonlinearities ($m = 1$, $k = 1$, $\epsilon = 2.2$, $\eta = 1.2$, $\lambda = 0.1$, and $S_0 = 0.125$): (a) results obtained by the WPI technique; and (b) comparison with MCS data (50,000 realizations).

Next, the joint response PDF is evaluated based on Eq. (41) and plotted in Fig. 1(a) for various indicative time instants. Further, the corresponding marginal response displacement and velocity PDFs are shown in Fig. 2. Comparisons with Monte Carlo simulation (MCS)-based estimates (50,000 realizations) corresponding to the joint response PDF in Fig. 1(b), and to the marginal PDFs in Fig. 2 demonstrate a quite high degree of accuracy. It is noted that the MCS data were obtained by resorting to the recursive algorithm developed by Spanos and Tsavachidis (2001) for integrating numerically the equation of motion given by Eqs. (43) and (44)

Nonlinear Oscillator with Asymmetric Response PDF and Fractional Derivative Elements

Consider next a single-DOF oscillator with asymmetric nonlinearities and fractional derivative elements. Its equation of motion takes the form of Eq. (43) with the nonlinear function $g(x, \dot{x}, t)$ given by

$$g(x, \dot{x}, t) = c\,{}_{t_i}^{C}D_t^a x + k(x + \epsilon x^2) \tag{47}$$

where m, c, and k are constant coefficients; the parameter ϵ controls the nonlinearity magnitude; and ${}_i^{C}D_t^a x$ denotes the left Caputo fractional derivative of order a, defined as

$${}_{t_i}^{C}D_t^a x(t) = \frac{1}{\Gamma(1-a)}\int_{t_i}^{t} \frac{\dot{x}(\tau)}{(t-\tau)^a}\mathrm{d}\tau \tag{48}$$

where $\Gamma(\cdot)$ = gamma function; and $0 < a < 1$.

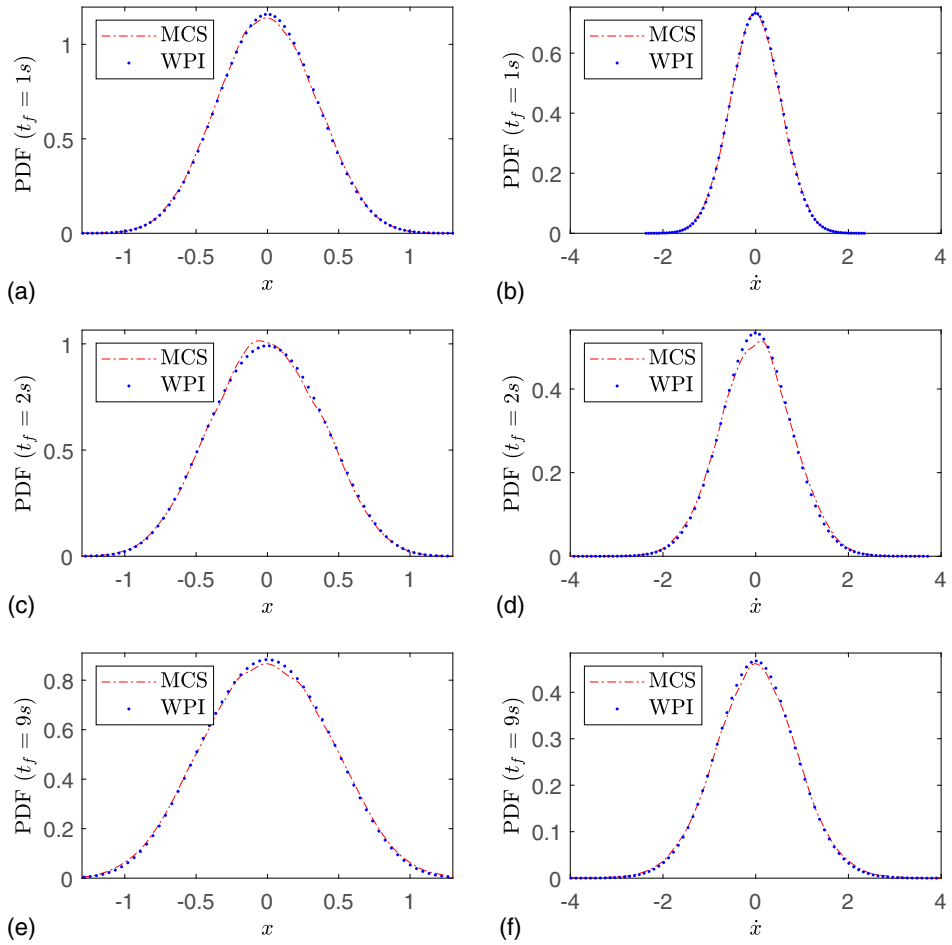

Fig. 2. (a–d) Nonstationary; and (e and f) stationary response displacement and velocity PDFs at various time instants corresponding to a stochastically excited Biot hysteretic oscillator with cubic nonlinearities ($m = 1$, $k = 1$, $\epsilon = 2.2$, $\eta = 1.2$, $\lambda = 0.1$, and $S_0 = 0.125$), and comparisons with MCS data (50,000 realizations).

Clearly, the oscillator exhibits a hysteretic response behavior due to the presence of the fractional derivative term. Indeed, based on the definition of Eq. (48), it is seen that the fractional derivative operator depends on the oscillator response history from the initial time t_i to the current time t_f. Thus, a solution treatment based on Markovian response modeling cannot be reasonably justified. In fact, as noted by Di Paola and Alotta (2020), numerical path integration schemes cannot be utilized in a straightforward manner for treating systems with a history-dependent response, such as oscillators with fractional derivative elements. This is due to the fact that such schemes constitute essentially discrete versions of the Chapman-Kolmogorov Eq. (2), which inherently assumes a Markovian response process.

In contrast, the herein-developed novel formulation of the WPI technique can treat in a direct manner the oscillator of Eqs. (43) and (47). In this regard, the Lagrangian functional of Eq. (38) becomes

$$\mathcal{L}[x, \dot{x}, \ddot{x}] = \frac{1}{4\pi S_0}[m\ddot{x} + c\,{}_{t_i}^{C}D_t^a x + k(x + \epsilon x^2)]^2 \qquad (49)$$

and a Rayleigh-Ritz scheme is employed next for solving the functional minimization problem of Eq. (40), and for determining the oscillator response joint PDF by Eq. (41); see also Di Matteo et al. (2014) for more details on the Rayleigh-Ritz solution scheme. Further, strictly speaking, the response of the oscillator in Eq. (43) cannot reach stationarity. This is due to the fact that escape from the corresponding potential energy well is possible if the displacement

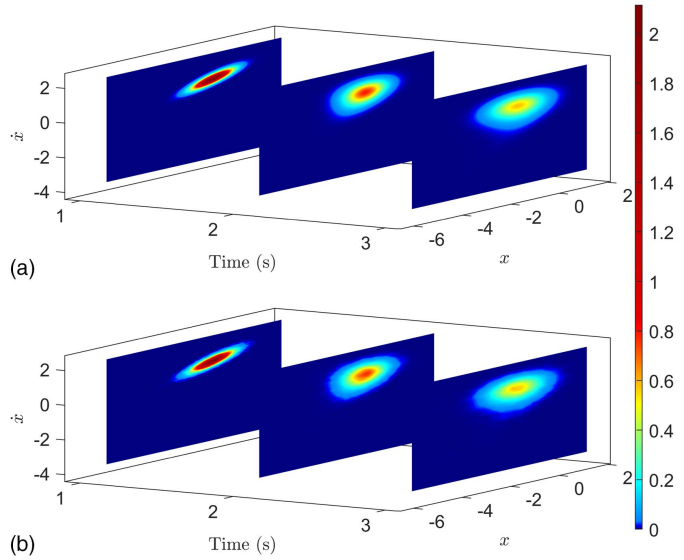

Fig. 3. Nonstationary response joint PDF at indicative time instants $t_f = 1$, 2, and 3 s corresponding to a stochastically excited oscillator with asymmetric nonlinearities and fractional derivative elements ($m = 1$, $c = 0.6$, $k = 1$, $\epsilon = 0.5$, $a = 0.5$, and $S_0 = 0.05$): (a) results obtained by the WPI technique; and (b) comparison with MCS data (50,000 realizations).

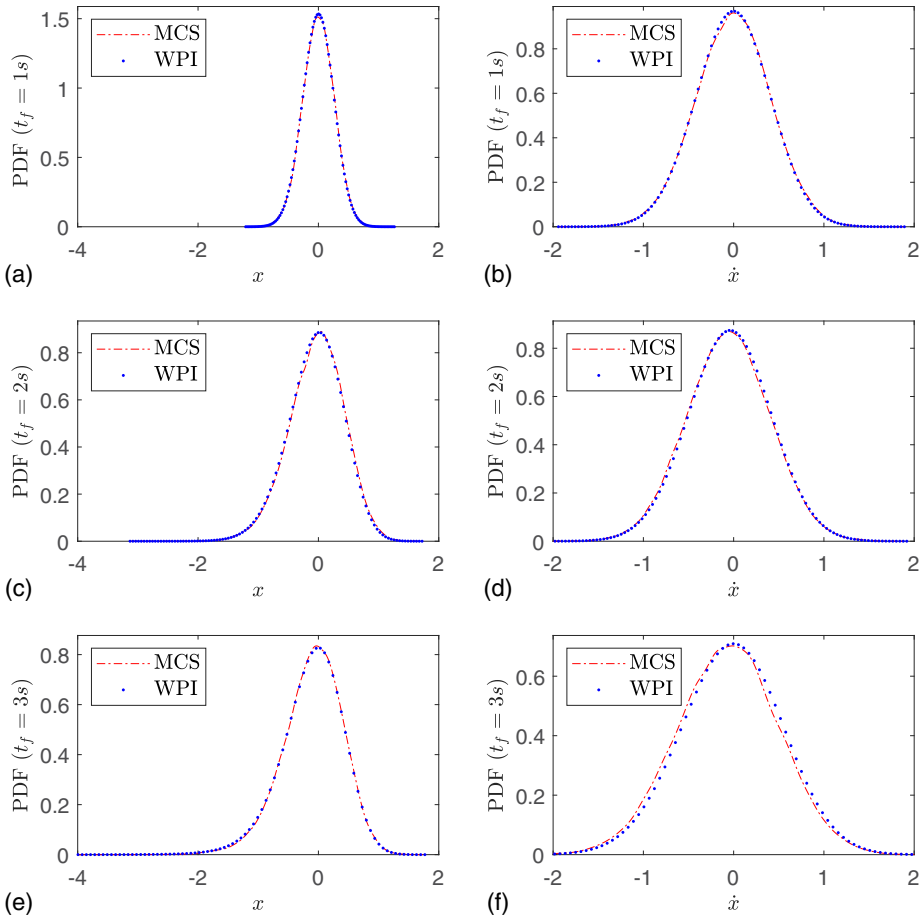

Fig. 4. Nonstationary response displacement and velocity PDFs at various time instants corresponding to a stochastically excited oscillator with asymmetric nonlinearities and fractional derivative elements ($m = 1$, $c = 0.6$, $k = 1$, $\epsilon = 0.5$, $a = 0.5$, and $S_0 = 0.05$), and comparisons with MCS data (50,000 realizations).

exceeds a critical level. Such an escape is followed, typically, by an unbounded response behavior. In the following numerical results, such an escape event has practically zero probability of occurrence for the selected parameters values and final time instants [Section 5. 3.6 in the book by Roberts and Spanos (2003) offers a relevant discussion].

Next, utilizing the values $m = 1$, $c = 0.6$, $k = 1$, $\epsilon = 0.5$, $a = 0.5$, $S_0 = 0.05$, $t_i = 0$, and $N = 105$, the WPI-based estimates are plotted in Fig. 3(a) corresponding to the joint response PDF $p(x, \dot{x})$ at various time instants. Comparisons with MCS-based estimates shown in Fig. 3(b) (50,000 realizations) demonstrate a satisfactory degree of accuracy. For the MCS analyses, the $L1$ algorithm (e.g., Koh and Kelly 1990) has been used for integrating numerically Eq. (43) and for determining response realizations. Further, the quite high accuracy exhibited by the WPI technique is also shown in Fig. 4, where the marginal response displacement and velocity PDFs at various time instants are compared with pertinent MCS data.

Concluding Remarks

In this paper, an alternative novel formalism of the WPI technique has been developed that circumvents the Markovian assumption for the system response process. Specifically, the transition PDF of the response process has been derived as a functional integral over the space of possible paths connecting the initial and final states of the response vector. This has been accomplished by considering the probability of a path corresponding to the Wiener excitation process and by applying a functional change of variables in conjunction with the system equation of motion. In this regard, it has been demonstrated that the WPI technique can treat in a direct manner also cases where the Markovian response assumption cannot be reasonably justified. These pertain, indicatively, to nonlinear systems with a history-dependent state, such as hysteretic structures or oscillators endowed with fractional derivative elements. A Biot hysteretic oscillator with cubic nonlinearities, and an oscillator with asymmetric nonlinearities and fractional derivative elements have been considered as illustrative numerical examples. Comparisons with pertinent MCS data have corroborated further the veracity of the developed formalism.

Data Availability Statement

All data, models, or code that support the findings of this study are available from the corresponding author upon reasonable request.

Acknowledgments

I. A. Kougioumtzoglou gratefully acknowledges the support through his CAREER award by the CMMI Division of the National Science Foundation (Award No. 1748537).

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