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Localization, monoid sets and K -theory

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ABSTRACT

We develop the K -theory of sets with an action of a pointed monoid (or monoid scheme), analogous to the K -theory of modules over a ring (or scheme). In order to form localization sequences, we construct the quotient category of a nice regular category by a Serre subcategory.

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Over any base field k , a (pointed) monoid A realizes to Spec of the monoid algebra $k[A]$, and an A -set realizes to a $k[A]$ -module. Similarly, a monoid scheme S realizes to a scheme over k , and a sheaf of monoid sets on S realizes to a sheaf of modules. In particular, every toric variety is the realization of a unique monoid scheme S , with sheaves of monoid sets on S realizing to sheaves of modules on the toric variety. Thus monoid schemes and monoid-sets are a combinatorial model for toric varieties and many of their cousins.

When A is a noetherian pointed monoid A , there are several interesting categories of A -sets. One is the category of all finitely generated A -sets; it is studied in [8], [5] and [9]. Another is the category of all *partially cancellative* A -sets, which is studied in [11]; see Definition 1.1 below. These A -sets are well-behaved, include free A -sets, and turn

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out to be useful in toric geometry; see [7]. There are similar categories over a noetherian monoid scheme, including categories of A -sets with a restriction on the codimension of support.

This paper is primarily concerned with the K -theory of these categories. One of our themes is that they closely track Quillen’s K' -theory of noetherian rings and schemes. In particular, we show that their K -theory has localization sequences and coniveau spectral sequences. In particular, we prove the monoid-scheme analogue of “Bloch’s formula” [16, V.9.8.1]; see Theorem 6.2: if S is a 0-smooth monoid scheme, and \mathcal{K}'_* is the sheaf associated to the K -theory of partially cancellative \mathcal{A}_S -sets, then:

$$E_2^{p,q} = H^p(S, \mathcal{K}'_{-q}).$$

The fundamental new tool we develop and use is the construction of the localization \mathcal{M}/\mathcal{C} of a nice regular category \mathcal{M} (such as the above categories of A -sets) by a “Serre subcategory” \mathcal{C} . This construction generalizes Gabriel’s localization of abelian categories, and is of independent interest. For example, if U is open in a monoid scheme X , and \mathcal{M} is the category of monoid sets on X , the subcategory \mathcal{C} of monoid sets vanishing on U is a Serre subcategory, and \mathcal{M}/\mathcal{C} is the category of monoid sets on U ; see Example 4.3. For expositional reasons, the lengthy construction of \mathcal{M}/\mathcal{C} is postponed until Section 7; in Section 8, we compare our construction to the double category construction of [3], and obtain the K -theory localization sequence

$$K(\mathcal{C}) \rightarrow K(\mathcal{M}) \rightarrow K(\mathcal{M}/\mathcal{C}).$$

This is of course a special case of the localization sequence of [3, 8.6].

This paper organized as follows. In the first few sections, we recall the basic definitions of A -sets, quasi-exact categories (such as A -sets) and their K -theory. Section 3 recalls the Campbell–Zakharevich notions of CGW and $ACGW$ categories which are needed for K -theory localization sequences. Our version of the localization sequence is summarized in Section 4, with the technical details relegated to Sections 7 and 8. Sections 5–6 construct the analogues of the Gersten–Quillen spectral sequence associated to the coniveau filtration, and establish an analogue of Quillen’s proof of Gersten’s Conjecture.

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1. Monoids and monoid sets

By an (abelian) *monoid* A we mean a pointed set with a commutative product and distinguished elements $\{0, 1\}$ such that $a \cdot 1 = a$ and $a \cdot 0 = 0$ for all $a \in A$; these are sometimes called “pointed monoids.” For example, the free monoid \mathbb{N} is $\{0, 1, t, t^2, \dots\}$, and if A and B are monoids, so is $A \wedge B$. The initial pointed monoid $\{0, 1\}$ is called \mathbb{F}_1 in [5]. We call the set A^\times of invertible elements of a monoid its *units*; these form a group.

The monoid spectrum $\mathrm{MSpec}(A)$ is the space of prime ideals in A , together with the sheaf whose stalk at a prime s is the localization A_s of the monoid A . A *monoid scheme* (S, \mathcal{A}_S) is a topological space S with a sheaf of monoids \mathcal{A}_S which is locally MSpec of a monoid.

If A is a (pointed) monoid, an A -set is a pointed set $(X, *)$ with an action of A ; in particular, $1 \cdot x = x$ and $0 \cdot x = *$ for all $x \in X$. When we regard A as an A -set, we write the element 0 as $*$. For example, a free A -set is just a wedge of copies of A . We write $A\text{-}\mathbf{Sets}$ for the evident category of A -sets.

If (S, \mathcal{A}_S) is a monoid scheme, we write $\mathcal{A}_S\text{-}\mathbf{Sets}$ for the category of (quasi-coherent) sheaves of \mathcal{A}_S -sets. If $S = \mathrm{MSpec}(A)$, $\mathcal{A}_S\text{-}\mathbf{Sets}$ is equivalent to $A\text{-}\mathbf{Sets}$.

A monoid A is *partially cancellative*, or *pc* for short, if $ac = bc \neq 0$ implies $a = b$ for all a, b, c in A . If A is a cancellative monoid then A and A/I are pc for every ideal I of A ; the prototypical finite pc monoid is $\{1, t, \dots, t^N = t^{N+1} = 0\}$.

Definition 1.1. If A is a pc monoid, we say that a pointed A -set X is *partially cancellative* if for every $x \in X$ and a, b in A , if $a \cdot x = b \cdot x \neq *$ then $a = b$. The subcategory of pc monoid sets is closed in $\mathcal{A}\text{-}\mathbf{Sets}$ under subobjects and quotients. We do not impose the condition that $a \cdot x = a \cdot y \neq *$ implies $x = y$, as this rules out some key examples (see Example 1.2).

Similarly, we say a monoid scheme (S, \mathcal{A}_S) is *pc* if it is locally pc, and a sheaf of \mathcal{A}_S -sets is pc if it is locally pc. If A is pc, open subschemes of $\mathrm{MSpec}(A)$ are pc but are not usually affine.

Example 1.2. A (pointed) \mathbb{N} -set is just a pointed set X with a successor function $x \mapsto tx$. Every finite rooted tree is a pc \mathbb{N} -set; the successor of x is the adjacent vertex closer to the root vertex $*$. In fact, a finite \mathbb{N} -set is partially cancellative if and only if it is a rooted tree, because for every $x \in X$, the sequence $\{x, tx, t^2x, \dots\}$ terminates at the root vertex.

An \mathbb{N} -set is pc if and only if it contains no loop, i.e., for each element $x \neq *$ and integer $d > 1$ we have $t^d x \neq x$. A typical non-pc set is $\{1, t, \dots, t^d, \dots, t^N, * : t^N = t^d\}$.

Example 1.3. Let A be Γ_+ , where Γ is an abelian group, and $+$ is a disjoint basepoint. Then every A -set is a wedge of copies of cosets $(\Gamma/H)_+$. If H is a proper subgroup of Γ , then $(\Gamma/H)_+$ is not pc because $h \cdot x = 1 \cdot x$ for $h \in H$. Thus a Γ_+ -set is pc if and only if it is free.

2. K -theory of quasi-exact categories

The notion of exact category [13] has the following generalization.

Definition 2.1. ([8], [16, Ex. IV.6.14]) A *quasi-exact category* is a category \mathcal{M} with a distinguished zero object, and a coproduct \vee , equipped with a family \mathcal{S} of sequences of the form

$$X \xrightarrow{i} Y \xrightarrow{p} Z, \quad (2.2)$$

called “admissible,” such that:

- (i) any sequence isomorphic to an admissible sequence is admissible;
- (ii) for any admissible sequence (2.2), p is a cokernel for i and i is a kernel for p ;
- (iii) \mathcal{S} contains all split sequences (those with $X \cong Y \vee Z$); and
- (iv) the class of admissible epimorphisms (resp., admissible monics) is closed under composition and pullback along admissible monics (resp., pullback along admissible epimorphisms).

We will write admissible monics as $X \hookrightarrow Y$ and admissible epics as $Y \twoheadrightarrow Z$, and will often write Y/X for the cokernel of $X \hookrightarrow Y$.

The prototype of a quasi-exact category is a *regular* category; see Definition 7.1. The exact sequences are the sequences (2.2) for which p is a cokernel for i and i is a kernel for p .

Example 2.2.1. The category **Sets** of pointed sets is quasi-exact. More generally, if A is a (pointed) monoid, the category $A\text{-Sets}$ is quasi-exact; a sequence (2.2) is admissible if $X \hookrightarrow Y$ is an injection, and Z is isomorphic to the quotient A -set Y/X . If A is a noetherian monoid, the category $\mathcal{M}(A)$ of finitely generated pointed A -sets is quasi-exact. (See [16, Ex. IV.6.16].)

Similarly, if (S, \mathcal{A}_S) is a monoid scheme, $\mathcal{A}_S\text{-Sets}$ is a quasi-exact category; the admissible sequences are defined locally.

The wedge $X \wedge_{\mathcal{A}} Y$ of two \mathcal{A} -sets is defined to be the quotient of $X \wedge Y$ by the equivalence relation that $(xa, y) \sim (x, ay)$ for all $a \in A$, $x \in X$ and $y \in Y$. As noted in [5, 5.10], $X \wedge_{\mathcal{A}} Y$ is exact in both variables; in fact, $\mathcal{A}\text{-Sets}$ is a symmetric monoidal category.

Definition 2.3. If \mathcal{M} is quasi-exact, Quillen’s construction in [13] yields a category $Q\mathcal{M}$, and $K(\mathcal{M})$ is the connective spectrum with initial space $\Omega BQ\mathcal{M}$; we write $K_n(\mathcal{M})$ for $\pi_n K(\mathcal{M})$. The group $K_0(\mathcal{M})$ is generated by the objects, modulo the relations that $[X] = [Y] + [Z]$ for every sequence (2.2).

Example 2.3.1. The category **Sets**_{fin} of finite pointed sets is quasi-exact; every admissible sequence is split exact. It is well known that the Barratt–Priddy–Quillen theorem implies that $K(\mathbf{S}\mathbf{ets}_{\text{fin}})$ is homotopy equivalent to the sphere spectrum \mathbb{S} . (See [8], [5], [16, Ex. IV.6.15].)

Definition 2.4. If A is noetherian, the category $\mathcal{M}(A)$ of noetherian A -sets is quasi-exact; following [5], we write $G(A)$ for $K(\mathcal{M}(A))$. Similarly, if S is a noetherian monoid scheme, the category $\mathcal{M}(S)$ of sheaves of noetherian \mathcal{A}_S -sets is quasi-exact, and we write $G(S)$

for $K(\mathcal{M}(S))$. Since \wedge_A is biexact, $G(S)$ is an E_∞ -ring spectrum. See [16, IV.6.6] and [1, 3.8.2].

G -theory is contravariantly functorial with respect to monoid maps $A \rightarrow B$ for which B is noetherian as an A -module, and covariant for flat maps (a monoid map $A \rightarrow B$ is *flat* if the base extension $X \mapsto B \wedge_A X$ is exact).

If A is a noetherian pc monoid, the category $\mathcal{M}^{\text{pc}}(A)$ of noetherian pc A -sets is quasi-exact; following [11], we write $K'(A)$ for $K(\mathcal{M}^{\text{pc}}(A))$. Similarly, if (S, \mathcal{A}_S) is a noetherian pc monoid scheme, the category $\mathcal{M}^{\text{pc}}(S)$ of sheaves of noetherian pc \mathcal{A}_S -sets is quasi-exact, and we write $K'(S)$ for $K(\mathcal{M}^{\text{pc}}(S))$. $\mathcal{M}^{\text{pc}}(S)$ is symmetric monoidal, since $X \wedge_A Y$ is pc when both X and Y are, and $K'(A)$ and $K'(S)$ are E_∞ -ring spectra.

As noted in [11, 3.1.1], K' is covariantly functorial for flat maps.

Example 2.4.1. Let Γ be a group; the K -theory of the category of finitely generated free Γ -sets is written as $K(\Gamma)$. When $A = \Gamma_+$, we saw in Example 1.3 that every pc A -set is a free A -set, so $K'(\Gamma_+) \simeq K(\Gamma)$. If $X = A^{\vee r}$, then $\text{Aut}(X)$ is the wreath product $\Gamma \wr \Sigma_r$. By the Barratt–Priddy–Quillen theorem,

$$K'(\Gamma_+) \simeq K(\Gamma) \simeq S^\infty(B\Gamma_+) = S^\infty(B\Gamma) \vee \mathbb{S}.$$

Similarly, if Γ is abelian, the category of Γ -sets is the product over all subgroups $[H]$ of the category of Γ/H -sets.² By the equivariant Barratt–Priddy–Quillen theorem [4, 5.1],

$$G(\Gamma_+) \simeq \bigvee_{H \leq \Gamma} S^\infty[B(\Gamma/H)_+].$$

In particular, $G_0(\Gamma_+) = \pi_0 G(\Gamma_+)$ is the Burnside ring of Γ .

We say that an A -set X has *finite length* if it has a finite filtration

$$1 \twoheadrightarrow F_1 X \twoheadrightarrow \cdots \twoheadrightarrow F_n X = X$$

such that each $F_i X / F_{i-1} X$ is irreducible, i.e., isomorphic to $(A^\times)_+$. If A is noetherian, the category $\mathcal{M}_{\text{fin}}(A)$ of A -sets of finite length is quasi-exact, as is the category $\mathcal{M}_{\text{fin}}^{\text{pc}}(A)$ of pc A -sets of finite length. By Dévissage (see [3, 6.2] or [11, 3.3, 3.3.1]), we have:

Theorem 2.5. *If A is a noetherian monoid, with units $\Gamma = A^\times$, then $K(\mathcal{M}_{\text{fin}}(A)) \simeq G(\Gamma_+)$ and if A is a pc monoid then:*

$$K(\mathcal{M}_{\text{fin}}^{\text{pc}}(A)) \cong K(\Gamma) \simeq S^\infty(B\Gamma_+) \simeq S^\infty(B\Gamma) \vee \mathbb{S}.$$

If A has finite length then $G(A) \simeq G(\Gamma_+)$ and $K'(A) \simeq K(\Gamma)$.

² If Γ is not abelian, replace Γ/H by the Weyl group $N_\Gamma(H)/H$.

The following result is taken from [11, 5.3].

Theorem 2.6. *Let S be a pc monoid scheme and $Z \xrightarrow{i} S$ an equivariant closed subscheme with open complement $U \xrightarrow{j} S$. Then there is a fibration sequence of spectra*

$$K'(Z) \xrightarrow{i_*} K'(S) \xrightarrow{j^*} K'(U).$$

Remark 2.7. By [9, 1.3], every ideal I in a noetherian monoid has a finite number of associated primes. It follows that the support of an equivariant closed subscheme Z of a noetherian monoid scheme has a finite number of minimal points.

3. CGW and ACGW categories

Campbell and Zakharevich have defined a *CGW-category* to be a double category satisfying a certain list of axioms; we refer the reader to [3, 2.5] for the precise definition. Here is our main example:

If \mathcal{M} is a quasi-exact category, we can form a double category $\mathcal{M}^{[d]}$ with the same objects as \mathcal{M} ; the horizontal and vertical maps are the admissible monics and epics (composed backwards), respectively, and the 2-cells are commutative diagrams of the form

$$\begin{array}{ccc} X & \xrightarrow{\quad} & Y \\ \uparrow & & \uparrow \\ X' & \xrightarrow{\quad} & Y' \end{array} \quad (3.1)$$

We say that a square (3.1) is *distinguished* if the natural map of cokernels $Y'/X' \rightarrow Y/X$ is an isomorphism. Thus distinguished squares are both pushout squares and pullback squares.

As pointed out in [11, 2.3], if \mathcal{M} is a quasi-exact category, $\mathcal{M}^{[d]}$ is a CGW-category, and \mathcal{M} is an “ambient category” for $\mathcal{M}^{[d]}$ in the sense of [3, 2.3], where $k(Y \rightarrow Y/X)$ is its kernel $X \rightarrow Y$ and $c(X \rightarrow Y)$ is its cokernel $Y \rightarrow Y/X$. Moreover, the K -theory $K(\mathcal{M}^{[d]})$ agrees with $K(\mathcal{M})$.

There is a stronger notion, that of an ACGW-category, defined in [3, 5.6]. If \mathcal{M} is an ACGW-category and \mathcal{C} is an ACGW-subcategory closed under subobjects, quotients, and extensions (see [3, 2.12]), the double category $\mathcal{M} \setminus \mathcal{C}$ is defined in [3, 8.1]. There is a canonical morphism of double categories $\mathcal{M} \rightarrow \mathcal{M} \setminus \mathcal{C}$.

A key construction in the language of ACGW-categories is [3, 8.6], which we now cite in modified form; see Remark 3.2.1. The condition (CGW) is that $\mathcal{M} \setminus \mathcal{C}$ is a CGW category, and condition (W) is described in Section 8 below.

Theorem 3.2. *Let \mathcal{M} be an ACGW-category and let \mathcal{C} be an ACGW-subcategory satisfying the technical conditions (CGW) and (W). Then*

$$K(\mathcal{C}) \rightarrow K(\mathcal{M}) \rightarrow K(\mathcal{M} \setminus \mathcal{C})$$

is a homotopy fiber sequence.

Proof. This is Theorem [3, 8.6], except that we have removed the superfluous hypothesis that Condition (E) holds. Indeed, Condition (E) in [3] is used only in the proof of [3, 10.22], a lemma in service of their main localization theorem. In the absence of condition (E), the *isomorphism* of categories in [3] may be weakened to an *equivalence* of categories, which still induces the requisite homotopy equivalence for the main theorem. \square

Remark 3.2.1. We do not know whether condition (E) holds in general for the CGW-categories associated to a Serre subcategory of an adherent category (see Definition 7.4), but fortunately this issue is extraneous.

Example 3.3. By [11, 2.5], if \mathcal{M} is a quasi-exact subcategory of $A\text{--Sets}$ (or $\mathcal{A}_S\text{--Sets}$) closed under pushouts along pairs of monics, and pullbacks along pairs of epics, then the associated double category $\mathcal{M}^{[d]}$ is an ACGW-category.

In particular, when A is a noetherian monoid, the associated double categories of $\mathcal{M}(A)$ and $\mathcal{M}^{\text{pc}}(A)$ are ACGW-categories; the same is true for the subcategories $\mathcal{M}(S)$ and $\mathcal{M}^{\text{pc}}(S)$ over a monoid scheme.

The difficulty with Theorem 3.2 is that $\mathcal{M} \setminus \mathcal{C}$ is hard to work with. Since we are working with ambient categories having more structure, we have access to a different kind of localization, viz., a quotient category \mathcal{M}/\mathcal{C} , which we establish in Theorem 7.8, and summarize in the next section.

4. The localization \mathcal{M}/\mathcal{C}

To define \mathcal{M}/\mathcal{C} , we need to introduce some category-theoretic vocabulary. We say that a pointed category is *regular* if it admits all finite limits and has a good intrinsic notion of short exact sequence, where the admissible monics are always kernels and the admissible epics are always cokernels. (See Definition 7.1.) A *regular functor* is one that preserves this structure, i.e., finite limits and short exact sequences.

Every regular category is quasi-exact; see Remark 7.2. The categories $A\text{--Sets}$ and $\mathcal{M}(A)$ are regular, as are $A\text{--Sets}^{\text{pc}}$ and $\mathcal{M}^{\text{pc}}(A)$.

Definition 4.1. A full pointed subcategory \mathcal{C} of a regular category \mathcal{M} is a *Serre subcategory* if it is closed under finite limits and for every exact sequence (2.2), Y is in \mathcal{C} if and only if both X and Z are in \mathcal{C} .

Proposition 4.2. *If $F: \mathcal{M} \rightarrow \mathcal{N}$ is a regular functor between regular categories, then the full subcategory $\ker(F) = F^{-1}(0)$ is a Serre subcategory of \mathcal{M} .*

Proof. Clearly, $\ker(F)$ is closed under finite limits. Consider an exact sequence (2.2) in \mathcal{M} ; applying F yields an exact sequence in \mathcal{N} ,

$$F(X) \rightarrowtail F(Y) \twoheadrightarrow F(Z).$$

It is clear that $F(Y) \cong 0$ if and only if $F(X) \cong F(Z) \cong 0$. \square

Example 4.3. Let S be a multiplicatively closed subset of an (abelian) monoid A . Then the localization $A\text{-Sets} \rightarrow S^{-1}A\text{-Sets}$ is a regular functor, because it preserves exact sequences and finite limits. Indeed, the ring-theoretic proofs go through; for example, if $X \xrightarrow{i} Y$ is an injection and $i(x)/s = 0$ in $S^{-1}Y$ for $x \in X$ then some $i(s_1x) = 0$ in Y and hence $s_1x = 0$ in X .

By Proposition 4.2, the category of S -torsion A -sets is a Serre subcategory of $A\text{-Sets}$; this description restricts to Serre subcategories of $\mathcal{M}(A)$ and $\mathcal{M}^{\text{pc}}(A)$ (see Definition 2.4).

More generally, if U is open in a monoid scheme (S, \mathcal{A}_S) then the localization $\mathcal{A}_S\text{-Sets} \rightarrow \mathcal{A}_U\text{-Sets}$ is a regular functor. Indeed, exact sequences and finite limits are determined locally on affine open subschemes. By Proposition 4.2, the category of \mathcal{A}_S -sets vanishing on U is a Serre subcategory of $\mathcal{A}_S\text{-Sets}$, and this restricts to Serre subcategories of $\mathcal{M}(S)$ and $\mathcal{M}^{\text{pc}}(S)$ as well.

If \mathcal{C} is a Serre subcategory of an adherent category \mathcal{M} , there exists a quotient category \mathcal{M}/\mathcal{C} , which is essentially a non-additive version of Gabriel's construction [10, Chapitre III]; the construction is given in Theorem 7.8 below. The definition of an *adherent* category is given in Definition 7.4; it is a regular category satisfying some technical axioms.

Here are the two theorems about \mathcal{M}/\mathcal{C} which we need.

Theorem 4.4. *Let \mathcal{C} be a Serre subcategory of an adherent category \mathcal{M} . Then there exists a quotient category \mathcal{M}/\mathcal{C} , which is regular, and a regular functor $\mathcal{M} \rightarrow \mathcal{M}/\mathcal{C}$, which is initial among all regular functors from \mathcal{M} which send \mathcal{C} to zero.*

The full statement of Theorem 4.4, and the proof, can be found in Theorem 7.8. The category \mathcal{M}/\mathcal{C} has the same objects as \mathcal{M} , but all maps with kernel and cokernel in \mathcal{C} are forced to be isomorphisms and thus all objects in \mathcal{C} are isomorphic to zero.

In Lemma 8.1, we point out that the double category $\mathcal{M}^{[d]}$ associated with an adherent category \mathcal{M} is an ACGW category. A fuller statement of the following theorem (and its proof) is given in Lemma 8.2 and Theorem 8.4.

Theorem 4.5. *Let \mathcal{C} be a Serre subcategory of an adherent category \mathcal{M} . Then the double category associated to \mathcal{M}/\mathcal{C} is a CGW-category, equivalent to the CGW-localization $\mathcal{M}\backslash\mathcal{C}$ of $\mathcal{M}^{[d]}$ by $\mathcal{C}^{[d]}$, and there is a fibration sequence:*

$$K(\mathcal{C}) \rightarrow K(\mathcal{M}) \rightarrow K(\mathcal{M}/\mathcal{C}).$$

Example 4.6. Let (S, \mathcal{A}_S) be a monoid scheme. For brevity, we write $\mathbf{M}(S)$ for the category \mathcal{A}_S –**Sets**. If $S = \mathrm{MSpec}(A)$, we will write $\mathbf{M}(A)$ for $\mathbf{M}(S)$. If $U \subset S$ is an open subscheme, with complement Z , then $\mathbf{M}(U)$ is $\mathbf{M}(S)/\mathbf{M}_Z(S)$, where $\mathbf{M}_Z(S)$ is the subcategory of all M in $\mathbf{M}(S)$ with $M|_U = 0$. The proof is the same as Gabriel’s in [10, p. 380]; use the fact that the exact functor $j^*: \mathbf{M}(S) \rightarrow \mathbf{M}(U)$ has a right adjoint, the section functor $j_*: \mathbf{M}(U) \rightarrow \mathbf{M}(S)$, and the universal map $j_*: \mathbf{M}(U) \rightarrow \mathbf{M}(S)/\mathbf{M}_Z(S)$ is an equivalence.

Suppose S is noetherian. As in Gabriel [10, p. 379], there is a bijection between the Serre subcategories of $\mathbf{M}(S)$ and $\mathcal{M}(S)$ (noetherian \mathcal{A}_S –sets), given by $\mathbf{M}'(S) \mapsto \mathcal{M}'(S) = \mathbf{M}'(S) \cap \mathcal{M}(S)$. Hence $\mathcal{M}_Z(S)$ is a Serre subcategory of $\mathcal{M}(S)$. By Theorem 4.5 and Dévissage [3, 6.2], there is a fibration sequence

$$G(Z) \rightarrow G(S) \rightarrow G(U).$$

Similarly, if S is pc, then $\mathcal{M}_Z^{\mathrm{pc}}(S)$ is a Serre subcategory of $\mathcal{M}^{\mathrm{pc}}(S)$, and we recover the fibration sequence of Theorem 2.6.

Application 4.7. If (S, \mathcal{A}_S) is a monoid scheme, then the subcategory $\mathbf{M}_d(S)$ of sheaves in $\mathbf{M}(S)$ which are supported in codimension $\geq d$ is a Serre subcategory. It contains $\mathbf{M}_{d+1}(S)$ and

$$\mathbf{M}_d(S)/\mathbf{M}_{d+1}(S) \cong \bigvee_{\mathrm{ht}(s)=d} \mathbf{M}_s(\mathcal{A}_s),$$

where $\mathbf{M}_s(\mathcal{A}_s)$ is the category of torsion \mathcal{A}_s –sets. Similarly, if S is noetherian then $\mathcal{M}_d(S)/\mathcal{M}_{d+1}(S)$ is equivalent to the wedge of the categories $\mathcal{M}_{\mathrm{fin}}(\mathcal{A}_s)$ of finite length \mathcal{A}_s –sets. The parallel assertion holds for $\mathcal{M}_d^{\mathrm{pc}}(S)/\mathcal{M}_{d+1}^{\mathrm{pc}}(S)$.

For example, if S is noetherian, then $\mathbf{M}_1(S)$ is the category of \mathcal{A}_S –sets vanishing on the open subscheme U consisting of the (finitely many) points of codimension 0. In this case, we get the equivalences

$$\mathbf{M}(S)/\mathbf{M}_1(S) \cong \mathbf{M}(U) = \bigvee_{\mathrm{ht}(s)=0} \mathbf{M}(\mathcal{A}_s)$$

and $\mathcal{M}(S)/\mathcal{M}_1(S) \cong \mathcal{M}(U) = \bigvee_{\mathrm{ht}(s)=0} \mathcal{M}(\mathcal{A}_s)$.

If S has a unique minimal point, j_* identifies $\mathbf{M}(S)/\mathbf{M}_1(S)$ with the category Q of torsionfree objects in $\mathbf{M}(S)$, and identifies $\mathcal{M}(S)/\mathcal{M}_1(S)$ with the noetherian objects in Q .

5. The coniveau filtration

We will apply the localization Theorem 4.5 to $\mathcal{M}_{d+1}(S) \subset \mathcal{M}_d(S)$ and its pc analogue. By 4.7, $\mathcal{M}_d^{\text{pc}}(S)/\mathcal{M}_{d+1}^{\text{pc}}(S)$ is equivalent to the coproduct over height d points s of torsion pc \mathcal{A}_s -modules.

Let (S, \mathcal{A}_S) be a noetherian monoid scheme, such as $\text{MSpec}(A)$. If s is a point in S , we write Γ_s for the group of units of the pointed monoid \mathcal{A}_s , so $(\Gamma_s)_+ = \mathcal{A}_s/p_s$. If $S = \text{MSpec}(A)$, then p_s is a prime ideal in the monoid A , and Γ_s is the group completion of the cancellative monoid A/p_s . By Example 2.4.1, $K'(\Gamma_s) = S^\infty(B\Gamma_s) \vee \mathbb{S}$.

Let S be a noetherian monoid scheme. By Theorem 4.5, the quasi-exact category $\mathcal{M}_d(S)/\mathcal{M}_{d+1}(S)$ is the ambient category of the double category $\mathcal{M}_d(S) \backslash \mathcal{M}_{d+1}(S)$. By 4.7, the quotient $\mathcal{M}_d(S)/\mathcal{M}_{d+1}(S)$ is the coproduct over the height d points $s \in S$ of the category $\mathcal{M}_{\text{fin}}^{\text{pc}}(\mathcal{A}_s) = \bigcup_n \mathcal{M}(\mathcal{A}_s/p_s^n)$ of finite length \mathcal{A}_s -sets.

Now $K(\mathcal{M}_{\text{fin}}^{\text{pc}}(\mathcal{A}_s)) \simeq K(\Gamma_s)$ by Theorem 2.5 (i.e., by Dévissage), so we have long exact sequences

$$\begin{aligned} \cdots \rightarrow K_*\mathcal{M}_{d+1}^{\text{pc}}(S) \rightarrow K_*\mathcal{M}_d^{\text{pc}}(S) \rightarrow \bigoplus K_*(\Gamma_s) \rightarrow \cdots \\ \cdots \rightarrow K_*\mathcal{M}_{d+1}(S) \rightarrow K_*\mathcal{M}_d(S) \rightarrow \bigoplus G_*(\Gamma_s) \rightarrow \cdots \end{aligned} \quad (5.1)$$

The direct sums in these sequences are indexed by the points $s \in S$ of codimension d . Each of these sequences form exact couples, yielding fourth quadrant spectral sequences:

$$\begin{aligned} E_1^{p,q} &= \bigoplus_{\text{ht}(s)=p} K_{-p-q}(\Gamma_s) \Rightarrow K'_{-p-q}(S) \\ E_1^{p,q} &= \bigoplus_{\text{ht}(s)=p} G_{-p-q}(\Gamma_s) \Rightarrow G_{-p-q}(S) \end{aligned} \quad (5.2)$$

These are the analogues of the usual coniveau spectral sequence for the K -theory of coherent sheaves over a noetherian scheme; see [13, 5.4] or [16, V.9].

Example 5.3. Let D be a discrete valuation monoid with group of units $\Gamma = D^\times$ and parameter t , i.e., $D = \Gamma_+ \wedge \mathbb{N}$. Since the group completion of D is $D_0 = (\Gamma \times \mathbb{Z})_+$, the localization sequence [11, 3.5] becomes

$$\cdots K_n(\Gamma) \xrightarrow{0} K'_n(D) \rightarrow K_n(\Gamma \times \mathbb{Z}) \xrightarrow{d} K_{n-1}(U) \cdots$$

(Recall that $K_n(\Gamma) = K'_n(\Gamma_+)$.) The left map is zero by Additivity [3, 7.15], because for every Γ_+ -set X there is a characteristic exact sequence

$$X \times \mathbb{N} \xrightarrow{t} X \times \mathbb{N} \twoheadrightarrow X.$$

Thus the E_1 page of the spectral sequence (5.2) for $K'(D)$ has only 2 nonzero columns ($p = 0, 1$), and looks like:

$$\begin{aligned} E_1^{p,0} : \quad & K_0(\Gamma \times \mathbb{Z}) \longrightarrow 0 \\ E_1^{p,-1} : \quad & K_1(\Gamma \times \mathbb{Z}) \xrightarrow{d} K_0(\Gamma) \rightarrow 0 \\ E_1^{p,-2} : \quad & K_2(\Gamma \times \mathbb{Z}) \rightarrow K_1(\Gamma) \rightarrow 0 \rightarrow 0 \\ E_1^{p,-3} : \quad & K_3(\Gamma \times \mathbb{Z}) \rightarrow K_2(\Gamma) \rightarrow 0 \rightarrow 0 \end{aligned}$$

Now $K_n(\Gamma \times \mathbb{Z}) \cong K_n(\Gamma) \oplus K_{n-1}(\Gamma)$ by 2.4.1, because $B\mathbb{Z} = S^1$, and

$$K(\Gamma \times \mathbb{Z}) = S^\infty B(\Gamma \times \mathbb{Z})_+ = S^\infty (S^1_+ \wedge B\Gamma_+) = S^1 S^\infty (B\Gamma)_+ \vee S^\infty (B\Gamma)_+.$$

Hence $K_n(\Gamma \times \mathbb{Z})$ maps onto $K_{n-1}(\Gamma) = \pi_{n-1} S^\infty (B\Gamma) \vee \mathbb{S}$ with kernel $K_n(\Gamma)$. Thus E_2 is concentrated on the column $p = 0$, with $K'_q(D) = E_2^{0,-q} = K_q(\Gamma)$.

Lemma 5.4. *If D is a discrete valuation monoid with units Γ , $d_1^{0,-1}: K'_1(D_0) \rightarrow K_0(\Gamma) = \mathbb{Z}$ is the surjection sending $[t]$ to ± 1 .*

Proof. By Example 5.3, $K'_1(D_0) = (\Gamma \times \mathbb{Z}) \oplus \pi_1^s$ and $K'_1(D) = \Gamma \oplus \pi_1^s$ and $K'_1(D_0)/K'_1(D) \cong K_0(\Gamma) = \mathbb{Z}$. \square

Remark 5.4.1. The analysis of the spectral sequence for $G_*(D)$ is more complicated because the formula for $G_*(\Gamma \times \mathbb{Z})_+$ is more complicated.

Weil divisors and $K'_0(S)$

We now connect the $E_\infty^{1,-1}$ term to the divisor class group, in the same way that the Chow groups are related to the groups $E_\infty^{p,-p}$ in the classical Gersten spectral sequence for $G_*(X)$.

Recall that a monoid scheme is *normal* if each \mathcal{A}_s is cancellative and if $a, b \in \mathcal{A}_s$ satisfy $(a/b)^n \in \mathcal{A}_s$ then $a/b \in \mathcal{A}_s$. If (S, \mathcal{A}_S) is a normal (pc) monoid scheme then for each height 1 point x , \mathcal{A}_s is a Discrete Valuation monoid by [9, 2.6]. The following definition is taken from [9, 4.2].

Definition 5.5. Let (S, \mathcal{A}_S) be a normal (pc) monoid scheme. A *Weil divisor* is an element of $\text{Div}(S)$, the free abelian group on the height 1 points of S . If a is a nonzero element of \mathcal{A}_0 , we can define the principal divisor $\text{div}(a)$ in $\text{Div}(S)$ by the same formula as in Algebraic Geometry. The divisor class group $\text{Cl}(S)$ is the quotient of $\text{Div}(S)$ by the subgroup of principal divisors. Thus there is an exact sequence

$$1 \rightarrow \mathcal{A}_S(X)^\times \rightarrow \mathcal{A}_0^\times \xrightarrow{\text{div}} \text{Div}(S) \rightarrow \text{Cl}(S) \rightarrow 0.$$

Remark 5.5.1. A monoid scheme S is *factorial* if and only if $\mathrm{Cl}(\mathcal{A}) = 0$; if the stalks \mathcal{A}_s of \mathcal{A} are factorial then $\mathrm{Cl}(S) = \mathrm{Pic}(S)$ by [9, 6.4].

By Lemma 5.4, the component $K'_1(\mathcal{A}_0) \xrightarrow{d} K'_0(\mathcal{A}_s) = \mathbb{Z}$ of the differential d_1 is the projection $K'_1(\mathcal{A}_0) \rightarrow \mathcal{A}_0^\times$, followed by the map div . Thus

Theorem 5.6. *If S is a normal (pc) monoid scheme, the differential $E_1^{0,-1} \xrightarrow{d_1} E_1^{1,-1}$ is the principal divisor map div , and $E_\infty^{-1,-1} \cong \mathrm{Cl}(S)$. Hence $K'_0(S)$ surjects onto $\mathbb{Z} \oplus \mathrm{Cl}(S)$.*

Remark 5.6.1. If A is a 2-dimensional normal monoid, the kernel of $K'_0(A) \rightarrow \mathbb{Z} \oplus \mathrm{Cl}(X)$ is the subgroup generated by A/\mathfrak{m} . In general, the groups $E_2^{p,-p}(S)$ are the higher class groups W_p studied by Claborn and Fossum; see [6] or [16, Ex. II.6.4.3].

The following example shows that the augmented rows of the spectral sequence aren't always exact when A is normal.

Example 5.7. Let A be the (2-dimensional normal) submonoid of \mathbb{N}^2 generated by $x = (1, 0)$, $z = (1, 1)$ and $y = (1, 2)$. By [9, 4.2.1], $\mathrm{Cl}(A) = \mathbb{Z}/2$ for A . Since $A/(y, z) \cong \mathbb{N}$, and $A/(x, y, z) = *$, it follows that $E_2^{2,-2} = 0$ and $K'_0(A) \cong \mathbb{Z} \oplus \mathbb{Z}/2$.

6. Gersten's conjecture for pc monoid schemes

Quillen proved [13, 5.1] that if R is a semilocal ring, smooth/ k , then the K -theory spectral sequence degenerates at E_2 . (This result is known as “Gersten's conjecture.”) In this section, we give the analogue for monoids.

The following definition is taken from [7, 6.4]; the terminology reflects Theorem 6.5 in [7] that a (cancellative) monoid A is 0-smooth if and only if its monoid ring $k[A]$ is smooth over k for every field k of characteristic 0.

Definition 6.1. A monoid A is *0-smooth* if it is a product $\Gamma_+ \wedge (\mathbb{N}^n)$, where Γ is a finitely generated abelian group.

Theorem 6.2. *If A is a 0-smooth monoid, the sequence*

$$0 \rightarrow K'_n(A) \rightarrow \bigoplus_{\mathrm{ht}(s)=0} K'_n(\mathcal{A}_s) \xrightarrow{d} \bigoplus_{\mathrm{ht}(s)=1} K'_{n-1}(\mathcal{A}_s) \xrightarrow{d} \cdots$$

is exact for all n , where d is the E_1 differential in the spectral sequence (5.2). The sequence remains exact if K'_n is replaced by G_n .

Let \mathcal{K}'_n denote the sheaf associated to the K -theory of partially cancellative \mathcal{A}_S -sets. As in [13, (5.9)], this gives a flasque resolution of the sheaf \mathcal{K}'_n on any 0-smooth monoid scheme S :

$$0 \rightarrow \mathcal{K}'_n \rightarrow \bigoplus_{\text{ht}(s)=0} (i_s)_* K_n(\Gamma_s) \rightarrow \bigoplus_{\text{ht}(s)=1} (i_s)_* K_{n-1}(\Gamma_s) \rightarrow \cdots$$

and gives a canonical isomorphism

$$E_2^{pq}(S) \cong H^p(S, \mathcal{K}'_q). \quad (6.3)$$

Formula (6.3) remains valid if K' -theory is replaced by G -theory.

Proof. We may suppose that $A = \Gamma_+ \wedge \mathbb{N}^{m+1}$, with the maximal ideal generated by x_0, \dots, x_m . As noted by Quillen [13], it suffices to show that for every non-zero-divisor $t \in A$ that $\mathcal{M}_d(A/tA) \rightarrow \mathcal{M}_d(A)$ induces the zero map on K -theory. Since A is 0-smooth, $t = \gamma x_0^{n_0} \cdots x_m^{n_m}$ for some $\gamma \in \Gamma$, where some $n_i > 0$; we may suppose that $n_0 > 0$.

Set B be the submonoid $\Gamma_+ \wedge \mathbb{N}^m$ on x_1, \dots, x_m . Then A is flat over B , so $A' = A \wedge_B (A/tA)$ is flat over A/tA , and $A \rightarrow A/tA$ factors through $p : A \rightarrow A'$. There is a canonical splitting $u : A' \rightarrow A/tA$, induced by $A \rightarrow A/tA$, so that

$$\mathcal{M}_d(A/tA) \xrightarrow{u_*} \mathcal{M}_d(A') \xrightarrow{p^*} \mathcal{M}_d(A)$$

is the natural map. Now for any A/tA -set X , there is an exact functor $v : \mathcal{M}_d(A/tA) \rightarrow \mathcal{M}_d(A')$ sending X to $v(X) = X \wedge \langle x_0 \rangle$ and an exact sequence in $\mathcal{M}_d(A')$:

$$X \wedge \langle x_0 \rangle \xrightarrow{x_0} X \wedge \langle x_0 \rangle \rightarrow u_*(X).$$

By Additivity, $u_* \simeq 0$. Hence the map $K(\mathcal{M}_d(A/tA)) \rightarrow K(\mathcal{M}_d(A))$ is zero, as asserted. \square

7. Regular localization

We now develop the machinery promised in Section 4. Recall that our goal is to employ the localization theorem of [3], which is phrased in the language of (A)CGW-categories. Many cases of interest (ours included) arise not from arbitrary double categories, but from ordinary categories equipped with distinguish classes of morphisms and squares, see [3, 4.1-6].

Were we to work purely in CGW-language, we would need to invoke arguments that are needlessly complicated for our purposes. The quotient construction for CGW-categories does not immediately identify the nature of the quotient category; we are able to give a much clearer description of the quotient category (in our cases) and prove that it agrees with the CGW-version. Our version is also familiar: the localization construction $\mathcal{M} \rightarrow \mathcal{M}/\mathcal{C}$ generalizes from abelian categories (due to Gabriel [10]) to regular categories, including A -Sets and A -Sets^{pc}. As we generalize constructions in algebraic geometry to the world of monoid schemes, we show that we can “de-additivize” classical

proofs and obtain identical results without requiring the reader to first digest the entire theory of CGW-categories.

The following definition is taken from [2, A.5]; see also (Reference [15]).

Definition 7.1. Let \mathcal{M} be a category with finite limits. An epimorphism $Y \xrightarrow{p} Z$ in \mathcal{C} is *regular* if it is a coequalizer $X \rightrightarrows Y \xrightarrow{p} Z$.

We say that \mathcal{M} itself is *regular* if regular epimorphisms are stable under pullback by arbitrary morphisms and every *kernel pair* admits a coequalizer. Recall that the *kernel pair* for a morphism $f: X \rightarrow Y$ is the diagram $X \times_Y X \rightrightarrows X$ given by the canonical maps in the fiber product

$$\begin{array}{ccc} X \times_Y X & \rightarrow & X \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

The coequalizer of the kernel pair in a (concrete) regular category is the quotient of X given by identifying elements which become equal under f , i.e., the image of the morphism f .

A sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ is called *exact* if $i = \ker(p)$ and $p = \operatorname{coker}(i)$. Kernels always exist in a regular category, but cokernels do not necessarily exist; when they do, they are regular epimorphisms and the coequalizer of their kernel pair. Thus given a kernel $i: X \rightarrow Y$ we can always extend it to a short exact sequence.

A functor between regular categories is *regular* if it preserves finite limits and regular epimorphisms (thus short exact sequences as well).

The category of pointed sets is the prototype of a regular category; every surjection is a regular epimorphism (see [2, A.5.6]). In fact, the category of pointed objects in any elementary topos is regular.

An abelian category is another (particularly nice) example of a regular category. The regular epimorphisms are just cokernels. A regular functor between abelian categories is just an exact functor.

Remark 7.2. Every regular category is quasi-exact, using the intrinsic notion of short exact sequence in the regular category, and its associated double category is a CGW category by [11, 2.3]. All of the quasi-exact categories considered in this paper are actually regular.

One main feature of a regular category is that every morphism can be factored uniquely (up to isomorphism) as a regular epimorphism (onto the image) followed by a monomorphism.

Example 7.3. If A is a (pointed) monoid, $A\text{-}\mathbf{Sets}$ is regular, and the forgetful functor $A\text{-}\mathbf{Sets} \rightarrow \mathbf{Sets}$ is a regular functor (because it has both a left and right adjoint; see [11, 2.5.1]). If A is a pc monoid, $A\text{-}\mathbf{Sets}^{\text{pc}}$ is a regular subcategory of $A\text{-}\mathbf{Sets}$.

Every monomorphism in $A\text{-}\mathbf{Sets}$ (and hence $A\text{-}\mathbf{Sets}^{\text{pc}}$) is a kernel, but not every epimorphism is a cokernel. In fact, the short exact sequences in $A\text{-}\mathbf{Sets}$ and $A\text{-}\mathbf{Sets}^{\text{pc}}$ have the form

$$X \rightharpoonup^i Y \twoheadrightarrow^p Y/X.$$

The following definition is weaker than the related notions of a *coherent* category and an *adhesive* category, but is just right for our purposes; see [12, A.1] for more details.

Definition 7.4. A regular category \mathcal{M} is called *adherent* if it satisfies the following axioms:

- (A1) Pushouts along kernels in \mathcal{M} are also pullbacks;
- (A2) the image of every map is a kernel;
- (A3) for each X in \mathcal{M} , the category of monics $X' \rightharpoonup X$ has both pushouts and pullbacks.

In particular, (A2) implies that every monomorphism is a kernel.

Using De Morgan’s laws on the underlying sets, it is easy to see that $A\text{-}\mathbf{Sets}$ and $A\text{-}\mathbf{Sets}^{\text{pc}}$ are adherent categories. The category of pointed objects in any elementary topos, e.g., pointed finite sets, is adherent; this gives a large class of examples.

Construction 7.5. Let \mathcal{M} be a adherent category, and suppose that $X'_i \rightharpoonup X \twoheadrightarrow X''_i$ ($i = 1, 2$) are exact sequences in \mathcal{M} . Let X'_{12} denote the pullback $X'_1 \times_X X'_2$. Write X' for the pushout of X'_1 and X'_2 along X'_{12} , which by abuse of notation we will also write $X'_1 \cup X'_2$. Note that X' exists and $X' \rightharpoonup X$ is monic by axiom (A3).

Let X'' denote the cokernel of $X' \rightharpoonup X$ and write X''_{12} for the pullback of X''_1 and X''_2 along X'' . Then we have exact sequences

$$X'_{12} \rightharpoonup X \twoheadrightarrow X''_{12} \quad \text{and} \quad X' \rightharpoonup X \twoheadrightarrow X''$$

which fit into diagram (7.5.1), where both squares are bicartesian.

$$\begin{array}{ccccccc}
 X'_{12} & \rightharpoonup & X'_1 & & & & \\
 \downarrow & & \downarrow & \searrow & & & \\
 X'_2 & \rightharpoonup & X' & \twoheadrightarrow & X & \twoheadrightarrow & X''_{12} \twoheadrightarrow X'' \\
 & & & & \searrow & & \downarrow \\
 & & & & & & X''_1 \twoheadrightarrow X''
 \end{array} \tag{7.5.1}$$

The top-left square is bicartesian by construction. The bottom-right square is cartesian, so by [12, A.1.4.3] it is also cocartesian because cokernels are regular epimorphisms. Note that $X' \twoheadrightarrow X \twoheadrightarrow X''_{12}$ is *not* claimed to be an exact sequence.

We obtain Diagram (7.5.1) as follows: by taking iterated pushouts along our string of monomorphisms, we obtain

$$\begin{array}{ccccccc}
 X'_{12} & \twoheadrightarrow & X'_1 & \twoheadrightarrow & X' & \twoheadrightarrow & X \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 * & \twoheadrightarrow & X'_1/X'_{12} & \twoheadrightarrow & X'/X'_{12} & \twoheadrightarrow & X''_{12} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & * & \twoheadrightarrow & X'/X'_1 & \twoheadrightarrow & X''_1 \\
 & & & & \downarrow & & \downarrow \\
 & & & & * & \twoheadrightarrow & X''
 \end{array}$$

Note that all instances of \twoheadrightarrow are cokernels. We can repeat this process with the chain

$$X'_{12} \twoheadrightarrow X'_2 \twoheadrightarrow X' \twoheadrightarrow X$$

to obtain the square

$$\begin{array}{ccc}
 X''_{12} & \twoheadrightarrow & X''_1 \\
 \downarrow & & \downarrow \\
 X''_2 & \twoheadrightarrow & X''
 \end{array}$$

By construction, this is a pushout; as we have also identified it as a pullback, this completes the construction.

Definition 7.6. Let \mathcal{M} be an adherent category. A full subcategory $\mathcal{C} \subset \mathcal{M}$ is called *Serre* if it is closed under finite limits and, for every short exact sequence $X \twoheadrightarrow Y \twoheadrightarrow Z$ in \mathcal{M} , Y is in \mathcal{C} if and only if X, Z are both in \mathcal{C} .

Note that \mathcal{C} also forms an adherent category. It admits coequalizers of kernel pairs as, for any $f: X \rightarrow Y$ in \mathcal{C} , the coequalizer of $X \times_Y X \rightrightarrows X$ is just $\text{im}(f)$, which is a subobject of Y . Thus \mathcal{C} is regular, and axioms (A1), (A2), and (A3) hold because \mathcal{C} is a full subcategory closed under subobjects.

Remark 7.6.1. The definition of a Serre subcategory of an abelian category does not explicitly state that it be closed under finite limits. We need this assumption because we do not have biproducts; while there is a canonical short exact sequence $X \twoheadrightarrow X \vee Z \twoheadrightarrow Z$

which makes \mathcal{C} closed under coproducts (and therefore pushouts, which are quotients of coproducts), the product does not sit naturally in a short exact sequence. To prove that the quotient category \mathcal{M}/\mathcal{C} (to be defined shortly) has finite limits, we need \mathcal{C} to be closed under finite limits.

Before we define the quotient of an adherent category by a Serre subcategory, we have the following technical lemma.

Let X, Y be two objects in an adherent category \mathcal{M} , and let \mathcal{C} be a Serre subcategory of \mathcal{M} . Let $I = I_{X,Y}$ be the following category: its objects are the pairs (X', Y'') for every pair of short exact sequences

$X' \rightarrowtail X \twoheadrightarrow X''$ and $Y' \rightarrowtail Y \twoheadrightarrow Y''$ such that $X'', Y' \in \mathcal{C}$. The maps $(X'_1, Y''_1) \rightarrow (X'_2, Y''_2)$ are induced by composition with maps in \mathcal{M} of the form $X'_1 \rightarrowtail X'_2 \rightarrowtail X$ and $Y \twoheadrightarrow Y'_1 \twoheadrightarrow Y'_2$. Note that I is equivalent to a partially-ordered set.

Lemma 7.7. *Let X, Y be two objects in an adherent category \mathcal{M} , and let \mathcal{C} be a Serre subcategory of \mathcal{M} . Then the category $I = I_{X,Y}$ is filtered, and $\mathrm{Hom}_{\mathcal{M}}: I \rightarrow \mathbf{Sets}$ is a functor.*

Proof. To begin, fix two short exact sequences $X'_i \rightarrowtail X \twoheadrightarrow X''_i$ with X''_i in \mathcal{C} . Then the inclusions $X'_i \rightarrowtail X$ yield a span

$$\mathrm{Hom}_{\mathcal{M}}(X'_2, Y'') \longleftarrow \mathrm{Hom}_{\mathcal{M}}(X, Y'') \longrightarrow \mathrm{Hom}_{\mathcal{M}}(X'_1, Y'')$$

Setting $X'_{12} = X'_1 \times_X X'_2$, we claim that (X'_{12}, Y'') is an upper bound of (X'_1, Y'') and (X'_2, Y'') in I , i.e., that the cokernel X''_{12} of $X'_{12} \rightarrowtail X$ is in \mathcal{C} . By Construction 7.5, X''_{12} is the pullback of X''_1 and X''_2 along X'' . By assumption, X''_1 and X''_2 are in \mathcal{C} , and X'' is a quotient of X''_2 (or X''_1) so is also in \mathcal{C} . Thus X''_{12} is a (finite) limit of objects in \mathcal{C} and is also in this subcategory.

Dually, given short exact sequences $Y'_i \rightarrowtail Y \twoheadrightarrow Y''_i$ with Y'_i in \mathcal{C} , we have the span

$$\mathrm{Hom}_{\mathcal{C}}(X, Y''_1) \longleftarrow \mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Y''_2)$$

Setting $Y'' = Y''_1 \vee_Y Y''_2$, it suffices to prove that the kernel Y' of $Y \rightarrow Y''$ is in \mathcal{C} . Appealing to Construction 7.5, $Y' = Y'_1 \cup Y'_2$, and we have an identification of cokernels

$$Y'/Y'_1 = (Y'_1 \cup Y'_2)/Y'_1 \cong Y'_2/(Y'_1 \cap Y'_2) = Y'_2/Y'_{12}.$$

Since Y'_2 and hence Y'_2/Y'_{12} is in \mathcal{C} , this shows that Y'/Y'_1 is in \mathcal{C} . Since Y'_1 is also in \mathcal{C} , Y' is in the Serre subcategory \mathcal{C} , as desired. \square

Theorem 7.8. *Let \mathcal{C} be a Serre subcategory of an adherent category \mathcal{M} . Then we can define the quotient category \mathcal{M}/\mathcal{C} in the following way: the objects are the objects of \mathcal{M} , and the morphisms are the sets*

$$\mathrm{Hom}_{\mathcal{M}/\mathcal{C}}(X, Y) = \mathrm{colim}_{I_{X,Y}} \mathrm{Hom}_{\mathcal{M}}(X', Y'')$$

defined above. Then \mathcal{M}/\mathcal{C} is a regular category, and $\mathcal{M} \rightarrow \mathcal{M}/\mathcal{C}$ is a regular functor, initial amongst those sending \mathcal{C} to zero.

We take inspiration from Gabriel's thesis [10], where this result is proved for abelian categories. We will prove our theorem in a series of steps.

Lemma 7.9. *The construction of \mathcal{M}/\mathcal{C} has well-defined hom-sets, and thus is a category.*

Proof. We first prove that our definition of $\mathrm{Hom}_{\mathcal{M}/\mathcal{C}}$ behaves well with respect to composition. Suppose that $f \in \mathrm{Hom}_{\mathcal{M}/\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{M}/\mathcal{C}}(Y, Z)$. Then we can take specific representatives $f: X' \rightarrow Y_1''$ and $g: Y_2' \rightarrow Z''$. To compose these, we need to match the codomain of f to the domain of g by taking equivalent representatives. Consider $Y_1' \twoheadrightarrow Y_1' \cup Y_2'$, where Y_i' is the kernel of $Y \twoheadrightarrow Y_i''$ ($i = 1, 2$). Then the inverse image $X^* = f^{-1}(Y_1' \cup Y_2'/Y_1')$ is a subobject of X' (and hence of X), fitting into the pullback diagram

$$\begin{array}{ccc} X^* & \twoheadrightarrow & X' \\ f^* \downarrow & & \downarrow f \\ (Y_1' \cup Y_2')/Y_1' & \twoheadrightarrow & Y_1'' \end{array}$$

The cokernel of $X^* \twoheadrightarrow X$ fits into a short exact sequence

$$X'/X^* \twoheadrightarrow X/X^* \twoheadrightarrow X/X' = X''.$$

Cf. [11, 2.4]. Because the source and quotient of this exact sequence are in \mathcal{C} , so is the middle; thus X^* is a permissible domain in the colimit defining $\mathrm{Hom}_{\mathcal{M}/\mathcal{C}}(X, Y)$. The codomain has kernel Y_1' , so it is also permissible. The composition $f^*: X^* \rightarrow (Y_1' \cup Y_2')/Y_1'$ gives the same map as f in the colimit.

For g , consider the pullback $Y_{12}' = Y_1' \times_Y Y_2'$, and form the quotient $Y_{12}' \twoheadrightarrow Y_2' \twoheadrightarrow Y_2'/Y_{12}'$; taking the pushout gives a map

$$\begin{array}{ccc} Y_2' & \xrightarrow{g} & Z'' \\ \downarrow & & \downarrow \\ Y_2'/Y_{12}' & \xrightarrow{g^{**}} & Z^{**} \end{array}$$

This pushout is constructed as follows: take the image factorization of the map $Y_{12}' \twoheadrightarrow Z''$:

$$\begin{array}{ccc} Y'_{12} & \dashrightarrow & J \\ \downarrow & & \downarrow \\ Y'_2 & \xrightarrow{g} & Z'' \end{array}$$

Because the image of every map in \mathcal{M} is a kernel by Axiom (A2), $J \twoheadrightarrow Z''$ is still a kernel. The pushout and the map g^{**} are created by taking cokernels vertically.

By assumption, Y'_1 and Y'_2 are in \mathcal{C} , so Y'_{12} and Y'_2/Y'_{12} are in \mathcal{C} as well. Thus g^{**} is part of the colimit and represents the same map as g . But by Construction 7.5 we have an identification

$$Y'_1 \cup Y'_2/Y'_1 \cong Y'_2/Y'_{12}$$

which allows us to take the actual composition $g^{**} \circ f^*$, which gives the requisite composition law. It is clear that composition is associative. \square

Having established how composition works, we have the following useful fact: suppose that $X' \twoheadrightarrow X \twoheadrightarrow X''$ is a short exact sequence with X'' in \mathcal{C} . Then $X' \cong X$ in \mathcal{M}/\mathcal{C} . Similarly, for any exact sequence $Y' \twoheadrightarrow Y \twoheadrightarrow Y''$, if Y' is in \mathcal{C} , then $Y \cong Y''$. Therefore given a map $f: X \rightarrow Y$ in \mathcal{M}/\mathcal{C} , the domain and codomain of the representative $X' \rightarrow Y''$ are isomorphic to the original ones.

Lemma 7.10. *The category \mathcal{M}/\mathcal{C} admits finite limits, and moreover these are computed in \mathcal{M} .*

Proof. Note that \mathcal{M}/\mathcal{C} still has a zero object, as the filtered colimits defining $\text{Hom}_{\mathcal{M}/\mathcal{C}}(X, *)$ and $\text{Hom}_{\mathcal{M}/\mathcal{C}}(*, Y)$ are constant on the singleton set. Therefore the existence of finite limits is equivalent to the existence of pullbacks.

Consider a cospan in \mathcal{M}/\mathcal{C} of the form

$$X_1 \xrightarrow{f_1} Z \xleftarrow{f_2} X_2$$

To find the pullback in \mathcal{M}/\mathcal{C} , we pick maps in \mathcal{M} that represent f_1 and f_2 , and form a common codomain Z'' .

$$\begin{array}{ccccc} & Z_1 & & X'_2 & \\ & \downarrow & & \downarrow f'_2 & \\ Z_2 & \twoheadrightarrow & Z & \twoheadrightarrow & Z''_2 \\ & & \downarrow & & \downarrow \\ X'_1 & \xrightarrow{f'_1} & Z''_1 & \twoheadrightarrow & Z'' \end{array}$$

Here, as in (7.5.1), Z'' is the quotient of Z by the pushout $Z' = Z'_1 \cup Z'_2$ of the kernels Z'_1 and Z'_2 , along their intersection inside of Z ; Z' is still in \mathcal{C} . Thus Z'' is a common codomain.

As the total righthand and bottom compositions also represent the maps f_1, f_2 , we may pretend (up to renaming the X'_i as X_i and Z'' as Z) that this was our original situation. We know the fiber product $P = X_1 \times_Z X_2$ exists in \mathcal{M} and we have projections $P \rightarrow X_1$ and $P \rightarrow X_2$ that give rise to maps in \mathcal{M}/\mathcal{C} . We just need to prove that P has the requisite universal property.

Suppose that we have $T \in \mathcal{M}/\mathcal{C}$ fitting into a diagram

$$\begin{array}{ccc} T & & \\ \downarrow & \searrow & \\ P & \longrightarrow & X_2 \\ \downarrow & & \downarrow f_2 \\ X_1 & \xrightarrow{f_1} & Z \end{array}$$

Then moving this picture over to \mathcal{M} , we can pick representatives h, k for the maps from T to obtain

$$\begin{array}{ccccc} T' & & & & \\ \downarrow & \searrow h & & & \\ P & \longrightarrow & X_2 & \longrightarrow & X_2'' \\ \downarrow & & \downarrow f_2 & & \downarrow \\ X_1 & \xrightarrow{f_1} & Z & & \\ \downarrow & & \searrow & & \\ X_1'' & \longrightarrow & Z'' & & \end{array}$$

There is a unique map from T' to $P'' := X_1'' \times_{Z''} X_2''$. Moreover, the kernel of $P \rightarrow P''$ is the pullback $X'_1 \times_{Z'} X'_2$ of the kernels of h and k . Since \mathcal{C} is closed under finite limits, P'' is an admissible quotient of P . Therefore the map $T' \rightarrow P''$ represents a map $T \rightarrow P$ in \mathcal{M}/\mathcal{C} . Note that any other choice of representative for h, k will lead to an equivalent map $T \rightarrow P$ in the filtered colimit, which proves uniqueness. Therefore the quotient category admits finite limits, and the quotient functor to \mathcal{M}/\mathcal{C} preserves these, as the limits are computed in \mathcal{M} . \square

Lemma 7.11. *The category \mathcal{M}/\mathcal{C} is a regular category.*

Proof. It remains to check that the quotient still admits coequalizers of kernel pairs which are stable under pullback. Suppose that $f: X \rightarrow Y$ is a map in \mathcal{M}/\mathcal{C} and that we have already replaced the domain and codomain so that $f: X \rightarrow Y$ is now in \mathcal{M} . Then we can take the coequalizer of its kernel pair in \mathcal{M} to obtain

$X \times_Y X \rightrightarrows Y \xrightarrow{p} Z$. We will show that $Y \xrightarrow{p} Z$ represents the coequalizer in \mathcal{M}/\mathcal{C} . By the previous argument, the kernel pair $X \times_Y X \rightrightarrows X$ in \mathcal{M}/\mathcal{C} is computed in \mathcal{M} , as it is a finite limit.

Suppose that $T \in \mathcal{M}/\mathcal{C}$ fits into the diagram

$$X \times_Y X \rightrightarrows Y \longrightarrow T.$$

Pick a representative $h: Y' \rightarrow T''$, and take the pullback in \mathcal{M} :

$$\begin{array}{ccc} X' & \xrightarrow{\quad} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{\quad} & Y. \end{array}$$

Now $X' \times_{Y'} X' \cong X \times_Y X$ in \mathcal{M}/\mathcal{C} , because $Y' \rightarrow Y$ and hence $X' \rightarrow X$ are isomorphisms in \mathcal{M}/\mathcal{C} . Thus the kernel pair for $f': X' \rightarrow Y'$ in \mathcal{M} also represents the kernel pair for f in \mathcal{M}/\mathcal{C} . In particular, the coequalizer

$$X' \times_{Y'} X' \rightrightarrows Y' \xrightarrow{p'} Z'$$

gives us a map $Z' \rightarrow T''$; because $Z' \cong Z$ we obtain a map $Z \rightarrow T$ in \mathcal{M}/\mathcal{C} as required. Since this map is unique up to picking a different representative for $Z \rightarrow T$, it is literally unique in $\text{Hom}_{\mathcal{M}/\mathcal{C}}(Z, T)$. Because we have already shown that finite limits and coequalizers in \mathcal{M}/\mathcal{C} are computed in \mathcal{M} , coequalizers of kernel pairs are stable under pullback in \mathcal{M}/\mathcal{C} . \square

Lemma 7.12. *The quotient functor $\mathcal{M} \rightarrow \mathcal{M}/\mathcal{C}$ is initial amongst those regular functors sending \mathcal{C} to zero.*

Proof. Suppose that $F: \mathcal{M} \rightarrow \mathcal{N}$ is another regular functor that sends every object in \mathcal{C} to zero. Then for any $X, Y \in \mathcal{M}$, we obtain a map of filtered systems $I \rightarrow \mathbf{Sets}$ (where $I = I_{X,Y}$ is the category of Lemma 7.7) given by

$$\text{Hom}_{\mathcal{M}}(X', Y'') \rightarrow \text{Hom}_{\mathcal{N}}(F(X'), F(Y''))$$

for each $(X', Y'') \in I$. Because both the cokernel of $X' \rightarrow X$ and the kernel of $Y \rightarrow Y''$ are in \mathcal{C} , we obtain isomorphisms $F(X') \cong F(X)$ and $F(Y) \cong F(Y'')$, so that the codomain filtered system is essentially constant with value $\text{Hom}_{\mathcal{N}}(F(X), F(Y))$. Taking

the colimits yields a natural map $\mathrm{Hom}_{\mathcal{M}/\mathcal{C}}(X, Y) \rightarrow \mathrm{Hom}_{\mathcal{N}}(F(X), F(Y))$, which defines the functor $\tilde{F}: \mathcal{M}/\mathcal{C} \rightarrow \mathcal{N}$ as required. \square

We do not expect the quotient to be adherent in general, nor do we require it for the application of [3, 8.6].

8. Comparison of localizations

We now prove the equivalence of the regular localization developed in the previous section and the CGW-localization from [3].

Lemma 8.1. *If \mathcal{M} is an adherent category, then, using the short exact sequences of \mathcal{M} , the associated double category $\mathcal{M}^{[d]}$ is an ACGW-category.*

Proof. By Remark 7.2, $\mathcal{M}^{[d]}$ is a CGW category. To add the letter ‘A’, we need to check a few more axioms, listed in [3, 5.5–5.6]. Axiom (P) is the assertion that monomorphisms are closed under pullbacks, and cokernels are closed under pushouts (as we showed in the proof of Lemma 7.7).

For axiom (U), we need to check that commutative squares of mono-morphisms give rise (upon taking cokernels) to pullback squares of cokernels; this was again proven in Construction 7.5 in the specific case of $X' = X'_1 \cup X'_2$, but the proof in the general case is identical. The mixed pullback square is defined using the factorizations of morphisms in a regular category; the object $X \odot_Y Z$, defined in [3, 5.6] for a composable pair $X \rightarrowtail Y \twoheadrightarrow Z$, is just the image of the composition. The compatibility condition follows from the uniqueness of such factorizations.

Axiom (S) concerns the pullback of monomorphisms and their associated cokernels. We have already checked everything required for this axiom in Construction 7.5. In particular, the restricted pushout of this axiom is just the pushout in the ambient category \mathcal{M} .

Finally, axiom (PP) uses the full strength of an adherent category. Because restricted pushouts for us are pushouts, Axiom (A3) guarantees that these always exist. Pushouts also exist along cokernels, as we demonstrated in Construction 7.5, and there is no further compatibility to check as restricted pushouts are just pushouts (so they are appropriately functorial). \square

Lemma 8.2. *Let \mathcal{C} be a Serre subcategory of an adherent category \mathcal{M} . Then the CGW-category $(\mathcal{M}/\mathcal{C})^{[d]}$ is equivalent to the CGW-quotient $\mathcal{M}^{[d]} \setminus \mathcal{C}^{[d]}$.*

As noted in Section 3, it follows that $K(\mathcal{M}/\mathcal{C}) \cong K(\mathcal{M}^{[d]} \setminus \mathcal{C}^{[d]})$.

Proof. The definition of the CGW-quotient can be found at [3, 8.1].

The m-morphisms $V \rightarrowtail Z$ of the CGW-quotient $\mathcal{M}^{[d]} \backslash \mathcal{C}^{[d]}$ are defined as compositions

$$V \xrightarrow{\bullet} W \xleftarrow{\bullet} X \xleftarrow{\bullet} Y \xrightarrow{\bullet} Z$$

where the decoration \bullet denotes that the (co)kernel of the map is in the Serre subcategory \mathcal{C} . The monomorphisms $V \rightarrowtail Z$ of \mathcal{M}/\mathcal{C} are those that have a representative $V' \rightarrow Z''$ which has its kernel in \mathcal{C} . Put another way, they are defined by

$$V \xleftarrow{\bullet} V' \xrightarrow{\bullet} Z'' \xleftarrow{\bullet} Z.$$

We can take the pushout Y of the span of monomorphisms to obtain an equivalent expression

$$V \xrightarrow{\bullet} Y \xleftarrow{\bullet} Z'' \xleftarrow{\bullet} Z$$

By inspection, kernel and the cokernel of $Z \rightarrow Y$ belong to \mathcal{C} . This means that any monomorphism $V \rightarrow Z$ can be described as a right fraction in a way identical to [3, 8.3]. The same reasoning goes through to identify the m-morphisms in \mathcal{M}/\mathcal{C} with those in $\mathcal{M}^{[d]} \backslash \mathcal{C}^{[d]}$; the argument for the e-morphisms follows by dual reasoning. \square

We will make use of the following lemma concerning isomorphisms in the quotient category \mathcal{M}/\mathcal{C} .

Lemma 8.3. *Let \mathcal{C} be a Serre subcategory of an adherent category \mathcal{M} . Let $h: X \dashrightarrow V$ be an isomorphism in \mathcal{M}/\mathcal{C} . Then there exists a representative $h: X' \rightarrow V''$ of h in \mathcal{M} which is a retract, and hence a monomorphism.*

Proof. Because h is an isomorphism, we know there must exist some inverse $p: V \dashrightarrow X$ such that $p \circ h = \text{id}_X$ in \mathcal{M}/\mathcal{C} . If we pick representatives in \mathcal{M} for h, p and compose them as in the proof of Theorem 7.8, we obtain

$$\begin{array}{ccccc} X' & \xrightarrow{h} & V'' & \xrightarrow{p} & X' \\ & \searrow & & \nearrow & \\ & & = & & \end{array}$$

for some $X' \rightarrowtail X$ and $V \twoheadrightarrow V''$. Because h is a retract, it is a monomorphism. \square

Theorem 8.4. *Let \mathcal{C} be a Serre subcategory of an adherent category \mathcal{M} . Then the double categories $\mathcal{C}^{[d]}$ and $\mathcal{M}^{[d]}$ satisfy conditions (W) and (CGW) of [3, 8.6]. We therefore obtain a homotopy fiber sequence on K-theory:*

$$K(\mathcal{C}) \rightarrow K(\mathcal{M}) \rightarrow K(\mathcal{M}/\mathcal{C}).$$

Special cases of Theorem 8.4 are given in [3, 8.3] and [11, 3.5].

Proof. We can apply [3, 8.6] to obtain the homotopy fiber sequence as soon as we verify these conditions. Condition (CGW) holds by Theorem 7.8 because \mathcal{M}/\mathcal{C} is a regular category which is the ambient category of $\mathcal{M}^{[d]}\backslash\mathcal{C}^{[d]}$.

For condition (W), we will prove that the categories \mathcal{I}_V^m and \mathcal{I}_V^e are filtered for all $V \in \mathcal{M}$. The category \mathcal{I}_V^m has objects (X, ϕ) with $\phi: X \dashrightarrow V$ an isomorphism in \mathcal{M}/\mathcal{C} ; the maps of \mathcal{I}_V^m (under our hypotheses) are maps $g: X_1 \rightarrow X_2$ in \mathcal{M} which are sent to isomorphisms over V in \mathcal{M}/\mathcal{C} . The category \mathcal{I}_V^m is isomorphic to \mathcal{I}_V^e by appealing to image factorizations, just as in the abelian case, cf. [3, 8.7]. Therefore it suffices to prove that \mathcal{I}_V^m is filtered.

Recall that a nonempty category I is filtered if two conditions hold. First, for any two objects $X, Y \in I$, there exists a third object Z and maps $X \rightarrow Z$ and $Y \rightarrow Z$, i.e., Z is an upper bound for X and Y . Second, any two parallel arrows $g_1, g_2: X \rightarrow Y$ admit a weak coequalizer, i.e., some $h: Y \rightarrow Z$ such that $h \circ g_1 = h \circ g_2$. These conditions may be combined by saying that any finite diagram in I admits a (non-unique) cocone.

Suppose we have two objects (X_1, ϕ_1) and (X_2, ϕ_2) of \mathcal{I}_V^m . If we choose representatives of ϕ_1 and ϕ_2 using Lemma 8.3, up to identifying their codomains we obtain

$$\begin{array}{ccccc}
 & X'_1 & \xrightarrow{\phi_1} & V'' & \xleftarrow{\phi_2} & X'_2 \\
 & \swarrow & & \uparrow & & \searrow \\
 X_1 & & & V & & X_2
 \end{array}$$

where the vertical arrows are maps in \mathcal{M} which become isomorphisms in \mathcal{M}/\mathcal{C} and the horizontal arrows are monomorphisms in \mathcal{M} . Taking pushouts on the left and right we obtain

$$\begin{array}{ccccccc}
 & X'_1 & \xrightarrow{\phi_1} & V'' & \xleftarrow{\phi_2} & X'_2 & \\
 & \swarrow & & \uparrow & & \searrow & \\
 X_1 & \longrightarrow & Y_1 & & & Y_2 & \longleftarrow & X_2 \\
 & & & \uparrow & & & & \\
 & & & V & & & &
 \end{array}$$

and finally pushing out the central span gives

$$\begin{array}{ccccccc}
 & X'_1 & \xrightarrow{\phi_1} & V'' & \xleftarrow{\phi_2} & X'_2 & \\
 & \swarrow & & \uparrow & & \searrow & \\
 X_1 & \longrightarrow & Y_1 & & & Y_2 & \longleftarrow & X_2 \\
 & & & \uparrow & & & & \\
 & & & V & & & & \\
 & & & \downarrow \psi & & & & \\
 & & & Y & & & &
 \end{array}$$

The map ψ does not exist until we pass to the quotient category. The object (Y, ψ) is then an upper bound for (X_1, ϕ_1) and (X_2, ϕ_2) , which proves the first condition of I_V^m being filtered.

Supposing now that we have two parallel morphisms in I_V^m

$$g_1, g_2: (X, \phi) \rightarrow (Y, \psi),$$

we need to find a weak coequalizer. Since these are maps over V , we have an equality of isomorphisms $\psi \circ g_1 = \psi \circ g_2 = \phi$ in \mathcal{M}/\mathcal{C} . Therefore ψ is nearly the map we want, but it does not exist in \mathcal{M} .

To remedy this, we consider the map $h = \psi^{-1} \circ \phi: X \dashrightarrow Y$ in \mathcal{M}/\mathcal{C} . By Lemma 8.3, we may take a monic representative $h: X' \rightarrow Y''$ with section $p: Y'' \rightarrow X'$ such that $p \circ h = \text{id}_{X'}$ in \mathcal{M} . Since p represents $\phi^{-1} \circ \psi$, we obtain a weak coequalizer by composing p with g_i :

$$X \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{array} Y \twoheadrightarrow Y'' \xrightarrow{p} X'.$$

Specifically, the weak coequalizer is $(Y, \psi) \rightarrow (X', \phi|_{X'})$ given by the canonical projection to Y'' followed by p .

We conclude that I_V^m and I_V^e are filtered, thus \mathcal{C} is both m- and e-well represented in \mathcal{M} . This proves that $\mathcal{C} \subset \mathcal{M}$ satisfies condition (W), so applying [3, 8.6] we complete the proof. \square

Remark 8.5. A recent preprint by Sarazola and Shapiro in [14] also studies $\mathcal{M} \backslash \mathcal{C}$ using an alternative approach to ours which focuses more on the double-categorical aspects of Campbell-Zakharevich's original work [3].

References

- [1] C. Barwick, Multiplicative structures on algebraic K-theory, *Doc. Math.* 20 (2015) 859–878.
- [2] F. Borceux, D. Bourn, Mal'cev, Protomodular, Homological and Semiabelian Categories, *Math. Appl.*, vol. 566, Kluwer, 2004.
- [3] J. Campbell, I. Zakharevich, Devissage and localization for the Grothendieck spectrum of varieties, preprint, arXiv:1811.08014v2, 2018.
- [4] G. Carlsson, C. Douglas, B. Dundas, Higher topological cyclic homology and the Segal conjecture for tori, *Adv. Math.* 226 (2011) 1823–1874.
- [5] C. Chu, O. Lorscheid, R. Santhanam, Sheaves and K -theory for \mathbb{F}_1 -schemes, *Adv. Math.* 229 (2012) 2239–2286.
- [6] L. Claborn, R. Fossum, Generalizations of the notion of class group, *Ill. J. Math.* 12 (1968) 228–253.
- [7] G. Cortiñas, C. Haesemayer, M.E. Walker, C. Weibel, Toric varieties, monoid schemes and cdh descent, *J. Reine Angew. Math.* 698 (2015) 1–54.
- [8] A. Deitmar, Remarks on zeta functions and K -theory over \mathbb{F}_1 , *Proc. Jpn. Acad., Ser. A, Math. Sci.* 82 (2006) 141–146.
- [9] J. Flores, C. Weibel, Picard groups and class groups of monoid schemes, *J. Algebra* 415 (2014) 247–263.
- [10] P. Gabriel, Des catégories abéliennes, *Bull. Soc. Math. Fr.* 90 (1962) 323–448.

- [11] C. Haesemeyer, C. Weibel, The K' -theory of monoid sets, *Proc. Am. Math. Soc.* 149 (2021) 2813–2824.
- [12] P. Johnstone, *Sketches of an Elephant: a Topos Theory Compendium*, Oxford Logic Guides, vol. 43, Oxford University Press, New York, 2002.
- [13] D. Quillen, Higher Algebraic K -Theory I, *Lecture Notes in Math.*, vol. 341, Springer, 1973, pp. 85–147.
- [14] M. Sarazola, B. Shapiro, A Gillet-Waldhausen theorem for chain complexes of sets, [arXiv:2107.07701](https://arxiv.org/abs/2107.07701).
- [15] P. Taylor, *Practical Foundations of Mathematics*, Cambridge Studies in Advanced Mathematics, vol. 59, Cambridge University Press, Cambridge, 1999.
- [16] C. Weibel, *The K-Book*, *Grad. Studies in Math.*, vol. 145, AMS, 2013.