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Reversed Strichartz estimates for wave on non-trapping asymptotically hyperbolic manifolds and applications

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ABSTRACT

We provide reversed Strichartz estimates for the shifted wave equations on non-trapping asymptotically hyperbolic manifolds using cluster estimates for spectral projectors proved previously in such generality. As a consequence, we solve a problem left open in Sire et al [Trans. AMS 373(2020):7639–7668] about the endpoint case for global well-posedness of nonlinear wave equations. We also provide estimates in this context for the maximal wave operator.

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1. Introduction and main results

The goal of this note is to obtain new reversed Strichartz estimates for the (shifted) wave operator on some complete manifolds with bounded geometry. We are concerned in particular with asymptotically hyperbolic manifolds. The argument builds on L^p -estimates for the spectral projectors (on the continuous spectrum). We draw also two consequences of the reverse Strichartz estimates: first, the endpoint version of classical Strichartz estimates and an application to global well-posedness for nonlinear wave equations on those manifolds; second, a sharp (in L^p spaces) estimate of the maximal function for the wave operator.

We work on an $n+1$ -dimensional complete non-compact Riemannian manifold (M, g) where $n \geq 1$ and the metric g is an asymptotically hyperbolic metric. This setting is the same as in Chen-Hassell [1, 2], Mazzeo [3] and Mazzeo-Melrose [4]. Let x be a boundary defining function for the compactification $\bar{M}$ of M : We say a metric g is conformally compact if x^2g is a Riemannian metric and extends smoothly up to the boundary ∂M : Mazzeo [3] showed that its sectional curvature tends to $-\frac{n-1}{4}$ as $x \rightarrow 0$; In particular, if the limit is such that $\lim_{x \rightarrow 0} \frac{1}{x^2} \text{Ric}(x^2g) = \frac{n-1}{4} \text{Id}$, we say that the conformally compact metric g is asymptotically hyperbolic. More specifically, let $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ be local coordinates on $Y \subset \partial M$, and (x, y) be the local coordinates on M near ∂M ; the metric g in a collar neighbourhood $(0, \epsilon) \times Y$ of ∂M takes the form

$$g \sim \frac{dx^2}{x^2} \oplus \frac{h(y)}{x^2} \sim \frac{dx^2}{x^2} \oplus \frac{h_{jk}(y) dy^j dy^k}{x^2}, \quad (1.1)$$

where $x \in C^1(\partial M)$ is a boundary defining function for ∂M and h is a smooth family of metrics on $Y \times \partial M$. In addition, if every geodesic in M reaches ∂M both forwards and backwards, we say M is nontrapping. The Poincaré disc $(\mathbb{D}^{n+1}, g_B)$ is a typical example of such manifold. Indeed, considering the ball $B^{n+1} \times \mathbb{R}^{n+1} : |z| < 1, |z| < 1$ endowed with the metric

$$g_B \approx \frac{4dz^2}{(1-|z|^2)^2}, \quad (1.2)$$

one can take $x \approx (1-|z|^2)^{1/2}$ as the boundary defining function and x the coordinates on S^n . Then the Poincaré metric takes the form

$$g_B \approx \frac{dx^2}{x^2} + \frac{1}{4} \frac{dx^2 + d\tilde{x}^2}{x^2},$$

where dx^2 is the standard metric on the sphere S^n .

Let $H \approx D_g^{-n/2}$ where D_g is the Laplace-Beltrami operator on $(\partial M, g_B)$. We recall the following result about estimates for the spectral measure and the spectral projectors by Chen and Hassell in [12, Theorem 1.6]. We denote by $dE_{\partial M}$ the spectral measure associated with the operator H , such that for every F a Borel function on $\mathbb{R}$, we have

$$\int_{\mathbb{R}} F(\lambda) dE_{\partial M}(\lambda) = \int_{\mathbb{R}} F(\lambda) dE_{\partial M}(\lambda)$$

with domain

$$\begin{aligned} \mathcal{D} &= \{w : \int_{\mathbb{R}} F(\lambda) dE_{\partial M}(\lambda) w < \infty\} \\ &= \mathcal{D} \end{aligned}$$

Theorem 1.1. Let $n \geq 1$. Suppose $(\partial M, g_B)$ is a non trapping asymptotically hyperbolic manifold of dimension $n+1$. Assume that there has no pure point eigenvalue and has no resonance at the bottom of the continuous spectrum of $H \approx D_g^{-n/2}$. Then for some constant C , we have: for $k \geq 1$

$$\|dE_{\partial M}(\lambda) f\|_{L^p} \leq Ck^2, \quad 1 < p < 2. \quad (1.3)$$

For $k \geq 1$, we have

$$\begin{aligned} \|dE_{\partial M}(\lambda) f\|_{L^p} &\geq Ck^{\frac{2n-1}{2} - \frac{1}{p}}, \quad 1 < p < \frac{2n-1}{n-1} \\ &\geq Ck^{\frac{2n-1}{2} - \frac{1}{p}}, \quad \frac{2n-1}{n-1} < p < 2. \end{aligned} \quad (1.4)$$

Corollary 1.2. Suppose $(\partial M, g_B)$ is a non trapping asymptotically hyperbolic manifold of dimension $n+1$ with $n \geq 1$. Assume that there has no pure point eigenvalue and has no resonance at the bottom of the continuous spectrum of H . Let $d \in (0, 1)$ and define the spectral projector on the frequencies $\frac{1}{2}k \leq d$

$$v_k^{df} \approx \int_0^d v_{\frac{1}{2}k, \lambda} dE_{\partial M}(\lambda) f. \quad (1.5)$$

Then for $k \geq 0$

$$jjv_k^d jj_{p|p^0} C d \delta d \leq k p^{2\delta} l \leq k p^{\delta n p | \delta l = p^0 p^3}, \quad 1 \leq p \leq \frac{2\delta n p | 2p}{n p | 4} \quad (1.6)$$

In particular, we have

$$jjv_k^d jj_{p|p^0} C d \delta d \leq k p^{\delta n p | \delta l = p^0 p^1}, \quad \max_{1 \leq p \leq \frac{2\delta n p | 1p}{n p | 4}} \frac{2\delta n p | 1p}{n p | 4} \leq \frac{2\delta n p | 2p}{n p | 4} \quad (1.7)$$

Proof. By [Theorem 1.1](#) and integrating the spectral measure on $\mathbb{R}^d$, one gets the desired result. w

Remark. In the case of high frequencies $k \gg 1$, the previous theorem is known to hold on asymptotically conic manifold and even on manifolds with bounded geometry (see [5]). In particular, asymptotically hyperbolic manifolds are examples of manifolds with bounded geometry in the following sense: suppose (M, g) to be a complete Riemannian manifold of dimension $n \geq 1$ with $n \geq 2$: We assume that M has C^1 bounded geometry, i.e. the local injectivity radius of M has a positive lower bound e , the metric tensor g_{ij} , expressed in normal coordinates in the ball of radius $e/2$ around any point $z \in M$ is uniformly bounded in $C^1(B_{e/2}, e/2)$ as z ranges over M ; and the inverse metric g^{ij} is uniformly bounded in supremum norm. The geometry properties (e.g. Kunze-Stein phenomenon) are implicitly contained in the spectral measure estimate of [Theorem 1.1](#) proved by Chen and Hassell [12].

The previous corollary has the following applications.

Theorem 1.3 (Reversed local-in-time Strichartz estimates). Suppose (M, g) is a non trapping asymptotically hyperbolic manifold of dimension $n \geq 1$ with $n \geq 1$. Assume that there has no pure point eigenvalue and has no resonance at the bottom of the continuous spectrum of H . Let u be the unique solution of

$$\begin{aligned} \partial_t^2 u &= H u \quad \text{in } \mathbb{R}_t \times M, \\ u|_{t=0} &= f, \quad \partial_t u|_{t=0} = 0; \end{aligned} \quad (1.8)$$

Then for $\frac{2\delta n p | 2p}{n} \leq q \leq 1$, $2 \leq p < 1$ and $s \geq \delta n p | 1p - \frac{1}{2} \frac{1}{p} - \frac{1}{1}$

$$jj u jj_{L^q(M; L^p(\delta^{-1/2} \cdot, 1) \cdot)} C jj \partial_t^s u jj_{L^2(M)}^s \quad (1.9)$$

Theorem 1.4 (Reversed global-in-time Strichartz estimates). Assume (M, g) and H be in [Theorem 1.3](#). Let $q_n \geq 1$ when $n \geq 1, 2$ and $q_n \geq \frac{2\delta n p | 1p}{n^2}$ when $n \geq 3$. For $\frac{2\delta n p | 2p}{n} \leq q \leq \frac{2}{p} - 2 \leq p < 1$ and $s \geq \delta n p | 1p - \frac{1}{2} \frac{1}{p}$, the global-in-time estimates hold

$$jj u jj_{L^q(M; L^p(\delta R^p \cdot))} C jj H^s f jj_{L^2(M)}^s \quad (1.10)$$

Furthermore, if $q_n < q \leq 1$ with $n \geq 3$, we have the global-in-time estimates

$$jj u jj_{L^q(M; L^p(\delta R^p \cdot))} C jj \partial_t^s H u jj_{L^2(M)}^{s=2} \quad (1.11)$$

The standard Strichartz estimates were established in [13] but with an arbitrary small loss of regularity due to the lack of Littlewood-Paley theory in the non-doubling

setting. Compared with the standard Strichartz estimates, these ones reverse the order of space-time integration with no loss. Estimates in this form have been extensively studied in [6, 7] for waves on manifolds, [8] for waves with potentials.

Another application is a sharp maximal function estimate. In [9], Rogers and Villarroya proved the following sharp maximal estimate for the wave operator in $\mathbb{R}^d$:

$$k \sup_{\mathbb{R}^d} |e^{it\sqrt{-\Delta}} f|_{L^q(\mathbb{R}^d)} \leq C \|f\|_{H^s(\mathbb{R}^d)} \quad (1.12)$$

provided $q \geq 2$ and $s > \frac{d-1}{2}$. We get an analogous statement in our context:

Corollary 1.3 (Maximal estimate). Let the manifold (M, g) and the operator H be in Theorem 1.3. Then the following holds

$$k \sup_{\mathbb{R}^d} |e^{itH} f|_{L^q(M)} \leq C \|f\|_{H^s(M)} \quad (1.13)$$

provided $q \geq 2$ and $s > \frac{d-1}{2}$.

Proof. This is just a consequence of Theorem 1.4 with $p \geq 2$ and the Sobolev embedding $H^{\frac{1}{2}}(\mathbb{R}^d) \hookrightarrow L^1(\mathbb{R}^d)$ for any $\epsilon > 0$. $\square$

2. Proof of Theorem 1.3

In [10, Theorem 2.1], the authors show that the bounds (1.9) follows from Corollary 1.2, and their proof works equally well in our circumstances. Nonetheless, we include a proof for the sake of completeness which will serve as a model for certain global Strichartz estimates.

To prove (1.9), it suffices to show that

$$k e^{itH} f \|_{L^q(M; L^p(\mathbb{R}^d))} \leq C \|f\|_{H^s(M)} \quad (2.1)$$

There exists a $q \geq 2$ satisfying $\text{supp } f \subset \mathbb{R}^d$ such that

$$k e^{itH} f \|_{L^q(M; L^p(\mathbb{R}^d))} \leq C \|f\|_{H^s(M)} \quad (2.2)$$

Then to prove (2.1), it suffices to prove

$$k e^{itH} f \|_{L^q(M; L^p(\mathbb{R}^d))} \leq C \|f\|_{H^s(M)} \quad (2.3)$$

To prove this, we shall change notation a bit, and in particular $d \geq 1$ in (1.5), let

$$v_k = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^d} e^{ik \cdot x} f(x) dx \quad (2.4)$$

so that $f = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^d} e^{-ik \cdot x} v_k dx$. Then, for $\frac{2\delta n p}{n} \leq q \leq 1$, (1.7) yields

$$\|v_k\|_{L^q(M)} \leq C \|f\|_{H^s(M)} \quad k \geq 0, 1, 2, \dots \quad (2.5)$$

To use this, we first note that by Sobolev estimates

$$k e^{itH} f \|_{L^q(M; L^p(\mathbb{R}^d))} \leq C \|f\|_{H^s(M)} \quad (2.6)$$

3. The proof of Theorem 1.4

In this section, we prove the global result about Theorem 1.4 which is direct consequence of the following Proposition.

Proposition 3.1. Suppose the operator H , q_n and the manifolds $\partial M, g_P$ to be in Theorem 1.4. For $\frac{2\delta n p^2}{n} - q_n, 2 < p < 1$ and $s \geq \delta n p - 1 p_{2-q_p}$, the global-in-time estimates hold

$$\|j\|_{L^q(\partial M; L^p(\partial R p))} C \|j\|_{L^2(\partial M p)}^{s-1} H^s f \|j\|_{L^2(\partial M p)}^s \quad (3.1)$$

Furthermore, if $q_n < q_p - 1$ with $n \geq 3$, we have the global-in-time estimates

$$\|j\|_{L^q(\partial M; L^p(\partial R p))} C \|j\|_{L^2(\partial M p)}^{3-4\delta n p} H^{3-4\delta n p} \|j\|_{L^2(\partial M p)}^{s-2\delta n p+4} f \|j\|_{L^2(\partial M p)}^s \quad (3.2)$$

Proof. Consider first the case with $q = q_n$: To prove (3.1), it suffices to show that, for $0 < \epsilon < 1$, there is a uniform constant C independent of ϵ so that

$$\|j\|_{L^q(\partial M; L^p(\partial R p))}^{p-1} e^{it} H^s f \|j\|_{L^q(\partial M; L^p(\partial R p))} C \|j\|_{L^2(\partial M p)}^{p-1} H^s f \|j\|_{L^2(\partial M p)}^s \quad (9.17^{0p})$$

To this end, similar to the proof of Theorem 1.3, it suffices to show that if we fix $q \geq 2$ $\partial R p$ with $\text{supp } q \subset \partial_2, 2p$, then we have the uniform bounds

$$\|j\|_{L^q(\partial M; L^p(\partial R p))}^{p-1} e^{it} H^s f \|j\|_{L^q(\partial M; L^p(\partial R p))} C \|j\|_{L^2(\partial M p)}^{p-1} H^{s-2} f \|j\|_{L^2(\partial M p)}^s \quad (3.3)$$

As before, we use $H_t^{\frac{1}{2}-\frac{1}{p}} \partial R p \rightarrow L_t^p \partial R p$ Sobolev estimates to deduce that

$$\|j\|_{L^q(\partial M; L^p(\partial R p))}^{p-1} e^{it} H^s f \|j\|_{L^q(\partial M; L^p(\partial R p))} \|j\|_{L^2(\partial M p)}^{1-2/p} \partial q \partial e t p e^{it} H^s f \|j\|_{L^q(\partial M; L^2(\partial R p))}^p$$

Similarly as before, let

$$F_k^e \partial t, x p \geq \|j\|_{L^2(\partial M p)}^{1-2/p} \partial q \partial e t p e^{it} H^s f p$$

If we take the Fourier transform in t , we deduce that

$$\|F_k^e \partial s, x p \geq \|j\|_{L^2(\partial M p)}^{1-2/p} e^{1-s} \partial q \partial e s^1 H^s f p \partial x p \geq \|F_k^1 \partial s, x p \geq 0$$

where

$$\|F_k^e \partial s, x p \geq \|j\|_{L^2(\partial M p)}^{1-2/p} e^{1-s} \partial q \partial e s^1 H^s f p v_k^e f p \partial x p$$

and v_{ke}^e is the spectral projection operator for I_k associated with H^p which is given by

$$v_{ke}^e f \geq \frac{1}{\delta_1} v_{I_k} \partial k p d E \partial k p f, \quad I_k \geq \frac{1}{2} k e, \partial k p - 1 p e p, \quad k \geq 0, 1, 2, 3, \dots \quad (3.4)$$

Since $\text{supp } q \subset \partial_2, 2p$, one sees the fact that $q \partial e t s^1 H^s f p v_{ke}^e$ vanishes if $s \geq \frac{1}{2} \partial k$ $100p, \partial k p - 100p$: Consequently,

$$\frac{\delta_1}{1} F_k^e \partial t, x p \overline{F_k^e \partial t, x p} dt \geq \frac{\delta_1}{2p-1} \|F_k^e \partial s, x p \overline{F_k^e \partial s, x p} ds \geq 0 \quad \text{if } |j-k| > 100: 1$$

4. Small data well-posedness and Strauss conjecture on asymptotically hyperbolic manifolds

We now draw some consequences of the previous estimates. We provide an application of the global reversed Strichartz estimates established above about global existence for nonlinear waves. Consider the wave equation with a nonlinearity satisfying

$$jF_p \delta u \leq C |jF_p \delta u|^p,$$

for some constant $C > 0$,

$$\begin{aligned} \partial_t \delta u &\leq C \|\delta u\|_{L^2_{\delta M}}^2, \\ \partial_t \delta u &\leq C \|\delta u\|_{L^2_{\delta M}}^2, \end{aligned} \quad (4.1)$$

In [13], the authors addressed the small data well-posedness for any power $p \geq 1$, leaving the end-point case open. This issue was raised by the methods we used which was not allowing us to get the suitable Strichartz estimates for $p \geq 1$. We address here the latter and then focus only on $p \geq 1$: We prove

Theorem 4.1. Let (M, g) be a non-trapping asymptotically hyperbolic manifold of dimension $n \geq 1$. Assume that there has no pure point eigenvalue and has no resonance at the bottom of the continuous spectrum of H . Then there exists a constant $\epsilon_0 > 0$ such that the Cauchy problem

$$\begin{aligned} \partial_t u &\leq C \|F_p \delta u\|_{L^2_{\delta M}}, \quad \partial_t u \leq C \|F_p \delta u\|_{L^2_{\delta M}}; \\ u &\leq C \|u\|_{L^2_{\delta M}}, \quad u \leq C \|u\|_{L^2_{\delta M}}, \end{aligned} \quad (4.2)$$

has a global solution, provided $\epsilon_0 > 0$ and

$$\|H^{\frac{1}{2}} u_0\|_{L^2_{\delta M}} \leq \epsilon_0 \|H^{\frac{1}{2}} u_1\|_{L^2_{\delta M}}.$$

Proof. Consider the complete space $L^q_{\delta M}; L^p_{\delta R}$ for $q \geq p \geq \frac{2n+2}{n-1}$, i.e. $L^{\frac{2n+2}{n-1}}_{\delta R}$. Define the map T by $v = Tu$ where v solves, given $u \in L^{\frac{2n+2}{n-1}}_{\delta R}$, M :

$$\begin{aligned} \partial_t v &\leq C \|F_p \delta u\|_{L^2_{\delta M}}, \quad \partial_t v \leq C \|F_p \delta u\|_{L^2_{\delta M}}; \\ v &\leq C \|u\|_{L^2_{\delta M}}, \quad v \leq C \|u\|_{L^2_{\delta M}}. \end{aligned} \quad (4.3)$$

Estimate (3.1) then gives

$$\|jv\|_{L^2_{\delta M}} \leq C \|ju\|_{L^2_{\delta M}} + C \|ju\|_{L^2_{\delta M}}^p.$$

A standard computation shows that if ϵ_0 is small enough, T maps a ball of $L^{\frac{2n+2}{n-1}}_{\delta R}$ into itself and is actually a contraction, hence by the Banach fixed point theorem this leads to the desired result (see for instance [11] for more details). $\square$

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