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A monotonicity version of a concavity theorem of Lieb

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Abstract. We give a simple proof of a strengthened version of a theorem of Lieb that played a key role in the proof of strong subadditivity of the quantum entropy.

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1. Introduction. Let $M_n(\mathbb{C})$ denote the $n \times n$ complex matrices, $M_n^+(\mathbb{C})$ the subset consisting of positive semidefinite matrices, and $M_n^{++}(\mathbb{C})$ the subset consisting of positive definite matrices. The following theorem was proved by Lieb in 1973 [8, Theorem 6]:

Theorem 1.1 (Lieb). *For all self-adjoint $H \in M_n(\mathbb{C})$, the function*

$$Y \mapsto \mathrm{Tr}\left[\exp(H + \log Y)\right] \tag{1.1}$$

is concave on $M_n^{++}(\mathbb{C})$.

As a simple consequence of this, Lieb deduced his *triple matrix inequality*, a generalization of the Golden-Thompson inequality to three self adjoint matrices. This played a fundamental role in the proof of strong subadditivity of the quantum entropy [9]. For more recent applications of Theorem 1.1, see the influential paper of Tropp [16].

We now prove a stronger version of this theorem in terms of monotonicity instead of concavity. Recall that a linear map $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is *positive* if $\Phi(A) \in M_m^+(\mathbb{C})$ whenever $A \in M_n^+(\mathbb{C})$, and is unital if $\Phi(I) = I$, and *trace preserving* if $\mathrm{Tr}[\Phi(X)] = \mathrm{Tr}[X]$ for all $X \in M_n(\mathbb{C})$. We equip $M_n(\mathbb{C})$ with the Hilbert-Schmidt inner product, and we use $\Phi^\dagger$ to denote the corresponding adjoint of a linear map on $M_n(\mathbb{C})$. Note that Φ is unital if and only if $\Phi^\dagger$ is trace preserving. Our main result is:

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Theorem 1.2. *Let $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ be unital and positive. Then for all self-adjoint $H \in M_n(\mathbb{C})$ and all $Y \in M_m^{++}(\mathbb{C})$,*

$$\mathrm{Tr}\left[\exp(H + \log \Phi^\dagger(Y))\right] \geq \mathrm{Tr}\left[\exp(\Phi(H) + \log Y)\right]. \tag{1.2}$$

Before giving the very simple proof, we explain how Theorem 1.2 implies Theorem 1.1. Let $\Phi : M_n(\mathbb{C}) \to M_{2n}(\mathbb{C})$ be defined by

$$\Phi(H) = \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} \quad \text{and hence} \quad \Phi^\dagger\left(\begin{bmatrix} Y_1 & 0 \\ 0 & Y_2 \end{bmatrix}\right) = Y_1 + Y_2. \tag{1.3}$$

Thus,

$$\mathrm{Tr}\left[e^{H+\log(Y_1+Y_2)}\right] \geq \mathrm{Tr}\left[e^{H+\log Y_1}\right] + \mathrm{Tr}\left[e^{H+\log Y_2}\right].$$

Since $Y \mapsto \mathrm{Tr}\left[e^{H+\log Y}\right]$ is homogeneous of degree one, this is the same as concavity. Note that this particular map Φ is not only positive; it is completely positive.

Our proof is based on the well-known and elementary Gibbs variational principle for the free energy in terms of the entropy $S(X) = -\mathrm{Tr}[X \log X]$ of a density matrix X. This states that for all self-adjoint $K \in M_n(\mathbb{C})$,

$$\log\left(\mathrm{Tr}\left[e^K\right]\right) = \sup\left\{\mathrm{Tr}[XK] - \mathrm{Tr}[X \log X] \ : \ X \in M_n^{++}(\mathbb{C}) \, , \mathrm{Tr}[X] = 1\right\}. \tag{1.4}$$

A short proof from scratch can be found in [4, Appendix A]. There is a simple variant involving the relative entropy

$$D(X||Y) = \mathrm{Tr}\left[X\left(\log X - \log Y\right)\right] \tag{1.5}$$

of two density matrices: For $W \in M_n^{++}(\mathbb{C})$, replace K with $K + \log W$ in (1.4) to conclude that for all self-adjoint $K \in M_n(\mathbb{C})$ and all $W \in M_m^{++}(\mathbb{C})$,

$$\begin{aligned} &\log(\mathrm{Tr}[e^{K+\log W}]) \\ &= \sup\{\mathrm{Tr}[XK] - D(X||W) \ : \ X \in M_n^{++}(\mathbb{C}), \mathrm{Tr}[X] = 1\}. \end{aligned} \tag{1.6}$$

We shall also use the result due to Müller-Hermes and Reeb [11] that the relative entropy is monotone under positive trace-preserving maps $\Phi^\dagger$. That is, for all such maps $\Phi^\dagger$, and all density matrices X, Y,

$$D(\Phi^\dagger(X)||\Phi^\dagger(Y)) \leq D(X||Y). \tag{1.7}$$

A somewhat weaker result is due to Uhlmann, who proved in 1977 that (1.7) is true whenever $\Phi^\dagger$ is the dual of a unital *Schwarz map*; i.e., a unital map Φ such that $\Phi(A^*A) \geq \Phi(A)^*\Phi(A)$. Earlier still in 1973, Lindblad had shown that (1.7) was valid for all unital completely positive Φ. Lindblad's proof relied on the Lieb concavity theorem [8]. It is well known that the set of unital completely positive maps is a proper subset of the set of Schwarz maps, which in turn is a proper subset of the set of positive maps. Thus the theorem of Müller-Hermes and Reeb extended that of Uhlmann, which in turn extended that of Lindblad. By now, quite elementary proofs of Lindblad's theorem are known, especially after the work of Pusz and Woronowicz [12,13] and then

Donald [6] provided an elementary proof of the joint convexity of the relative entropy functional (1.5), essentially by displaying it as a Legendre transform.

The duality argument that we now provide translates any of these monotonicity results for the relative entropy into a monotonicity result for the functional (1.1), and of course, starting from the result in [11], we obtain the strongest conclusion.

Proof of Theorem 1.2. By (1.6),

$$\begin{aligned}
&\log(\mathrm{Tr}[e^{H+\log\Phi^\dagger(Y)}]) \\
&\quad = \sup\{\mathrm{Tr}[XH] - D(X||\Phi^\dagger(Y)) \ : \ X \in M_n^{++}(\mathbb{C}), \mathrm{Tr}[X] = 1\} \\
&\quad \geq \sup\{\mathrm{Tr}[\Phi^\dagger(W)H] - D(\Phi^\dagger(W)||\Phi^\dagger(Y)) \ : \ W \in M_m^{++}(\mathbb{C}), \mathrm{Tr}[W] = 1\} \\
&\quad \geq \sup\{\mathrm{Tr}[W\Phi(H)] - D(W||Y) \ : \ W \in M_m^{++}(\mathbb{C}), \mathrm{Tr}[W] = 1\} \\
&\quad = \log \mathrm{Tr}\left[e^{\Phi(H)+\log Y}\right],
\end{aligned}$$

where we used $\{\Phi^\dagger(W) \ : \ W \in M_m^{++}(\mathbb{C}), \mathrm{Tr}[W] = 1\} \subset \{X \in M_n^{++}(\mathbb{C}), \mathrm{Tr}[X] = 1\}$ and (1.7). Exponentiating both sides of the inequality yields (1.2). □

While Theorem 1.2 is strictly stronger than Theorem 1.1, the greatest interest in it may lie in the very simple proof that its proof provides of Theorem 1.1. This is an interesting example of how it may be easiest to prove a concavity result by first proving a monotonicity result, and then applying that to the particular map Φ that is defined in (1.3).

It is interesting to observe another advantage of the monotonicity approach to convexity or concavity inequalities. A duality method for proving convexity and concavity inequalities was introduced by myself and Lieb in [3] which uses the following lemma:

Lemma 1.3. *If $f(x, y)$ is jointly convex in x, y, then $g(x) := \inf_y f(x, y)$ is convex. If $f(x, y)$ is jointly concave in x, y, then $g(x) := \sup_y f(x, y)$ is concave.*

This may be found in [14, Theorem 1], and the simple proof is also given in [3]. Since $(X, W) \mapsto D(X||W)$ is jointly convex, for fixed K, $(X, W) \mapsto \mathrm{Tr}[XK] - D(X||W)$ is jointly concave. Then by Lemma 1.3 and (1.6), $W \mapsto \log \mathrm{Tr}\left[e^{K+\log W}\right]$ is concave. However, this is a weaker statement than Theorem 1.1 since if $g(x) = \log(f(x))$ with f positive and twice continuously differentiable,

$$g''(x) = -\left(\frac{f'(x)}{f(x)}\right)^2 + \frac{1}{f(x)}f''(x).$$

Thus, concavity of f implies concavity of $\log f$, but not the other way around. However, monotonicity of f is equivalent to monotonicity of $\log f$. For this reason, we could simply use the Gibbs variational principle to prove our Theorem 1.2. Tropp [15] found an ingenious variational representation of $W \mapsto \mathrm{Tr}\left[e^{K+\log W}\right]$ which allowed him to give a proof of Theorem 1.1 using the joint convexity of the relative entropy and Lemma 1.3, along the lines of [3]. However, when using duality to prove the monotonicity theorem, the logarithm is not an issue because log monotonicity is the same as monotonicity, and we

do not need Lemma 1.3. Already in 1973, Epstein [7] gave a second proof of Theorem 1.1 using the theory of Herglotz functions, but this is considerably more involved than the present proof. For more information, see [2].

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References

[1] Beigi, S.: Sandwiched Rényi divergence satisfies data processing inequality. J. Math. Phys. **54**, 122202 (2013)

[2] Carlen, E.A.: On some convexity and monotonicity inequalities of Elliott Lieb. arXiv:2202.03591 (2022)

[3] Carlen, E.A., Lieb, E.H.: A Minkowski-type trace inequality and strong subadditivity of quantum entropy II: convexity and concavity. Lett. Math. Phys. **83**, 107–126 (2008)

[4] Carlen, E.A., Lieb, E.H.: Some trace inequalities for exponential and logarithmic functions. Bull. Math. Sci. **1**(2), 1950008, 39 pp. (2019)

[5] Choi, M.D.: Some assorted inequalities for positive linear maps on C^* algebras. J. Oper. Theory **4**, 271–285 (1980)

[6] Donald, M.J.: On the relative entropy. Comm. Math. Phys. **105**, 13–34 (1986)

[7] Epstein, H.: On a concavity theorem of Lieb. Comm. Math. Phys. **31**, 317–327 (1973)

[8] Lieb, E.H.: Convex trace functions and the Wigner-Yanase-Dyson conjecture. Adv. Math. **11**, 267–288 (1973)

[9] Lieb, E.H., Ruskai, M.B.: Proof of the strong subadditivity of quantum-mechanical entropy. J. Math. Phys. **14**, 1938–1941 (1973)

[10] Lindblad, G.: Expectations and entropy inequalities for finite quantum systems. Comm. Math. Phys. **39**, 111–119 (1974)

[11] Müller-Hermes, A., Reeb, D.: Monotonicity of the quantum relative entropy under positive maps. Ann. Henri Poincaré **18**(2107), 1777–1788 (2020)

[12] Pusz, W., Woronowicz, S.L.: Functional calculus for sesquilinear forms and the purification map. Rep. Math. Phys. **8**(2), 159–170 (1975)

[13] Pusz, W., Woronowicz, S.L.: Form convex functions and the WYDL and other inequalities. Lett. Math. Phys. **2**, 505–512 (1978)

[14] Rockafellar, R.T.: Conjugate Duality and Optimization. Lectures given at the Johns Hopkins University, Baltimore, Md., June, 1973. Conference Board of the

Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 16. Society for Industrial and Applied Mathematics, Philadelphia, Pa. (1974)

[15] Tropp, J.: From joint convexity of quantum relative entropy to a concavity theorem of Lieb. Proc. Amer. Math. Soc. **140**, 1757–1760 (2012)

[16] Tropp, J.: User-friendly tail bounds for sums of random matrices. Found. Comput. Math. **12**, 389–434 (2012)

[17] Uhlmann, A.: Relative entropy and the Wigner-Yanase-Dyson-Lieb concavity in an interpolation theory. Comm. Math. Phys. **54**, 21–32 (1977)

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