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A Kac model with exclusion

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Abstract. We consider a one dimensional Kac model with conservation of energy and an exclusion rule. Fix a number of particles n, and an energy $E > 0$. Let each of the particles have an energy $x_j \geq 0$, with $\sum_{j=1}^{n} x_j = E$. For ϵ positive, the allowed configurations $(x_1, \ldots, x_n)$ are those that satisfy $|x_i - x_j| \geq \epsilon$ for all $i \neq j$. At each step of the process, a pair (i, j) of particles is selected uniformly at random, and then they "collide", and there is a repartition of their total energy $x_i + x_j$ between them producing new energies x_i^* and x_j^* with $x_i^* + x_j^* = x_i + x_j$, but with the restriction that exclusion rule is still observed for the new pair of energies. This process bears some resemblance to Kac models for Fermions in which the exclusion represents the effects of the Pauli exclusion principle. However, the "non-quantized" exclusion rule here, with only a lower bound on the gaps, introduces interesting novel features, and a detailed notion of Kac's chaos is required to derive an evolution equation for the rescaled empirical measures for the process, as we show here.

Résumé. Nous considerons un modèle de Kac unidimensionnel avec conservation de l'énergie et une règle d'exclusion. Pour un nombre de particules n, et une énergie $E > 0$ fixes, soit $x_j \geq 0$ l'énergie de la particule j avec $\sum_{j=1}^{n} x_j = E$. Pour $\epsilon > 0$ les configurations admises de $(x_1, \ldots, x_n)$ sont celles qui satisfont $|x_i - x_j| \geq \epsilon$, pour tout $i \neq j$. À chaque pas du processus, une paire (i, j) de particules est sélectionnée uniformément au hasard, puis les particules « collisionnent ». Leur énergie totale $x_i + x_j$ est ensuite redistribuée produisant de nouvelles énergies x_i^* et x_j^* avec $x_i^* + x_j^* = x_i + x_j$, de telle sorte que la règle d'exclusion soit toujours observée pour la nouvelle paire. Ce processus présente des ressamblances avec modèles de Kac pour Fermions dans lesquels l'exclusion représente les effets du principe d'exclusion de Pauli. Cependant, la règle d'exclusion « non quantifié » ici, avec seulement une borne inférieure sur les écarts, introduit des nouveautées intéressantes, et une notion détaillée du chaos de Kac nécessaire pour dériver une équation d'évolution pour des mesures empiriques réchelonnée pour la processus, comme nous le montrons ici.

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1. Introduction

The first attempts to formulate kinetic equations for colliding particles that satisfy Boson or Fermion statstics go back at least to the works of Nordheim [10] and Uehling and Uhlenbeck [16]. To make a rigorous derivation of these equations starting from the Schrödinger equation for a large system of particles has proven very difficult. See [2] for a review. To understand the classical spatially homogeneous Boltzmann equation Mark Kac introduced a Markov jump process to mimic a real n-particle system and from this model he could rigorously derive a simplified (one dimensional) Boltzmann equation [9]. A similar kind of jump process has been studied by Colangeli et al. [6], who derive a kinetic equation from a particle system with discretized phasespace with exclusion.

We investigate a Kac model on the simplex with exclusion, but without dividing the simplex into cells, paying close attention to questions concerning Kac's notion of chaos for the model. Before we introduce our model with exclusion, it will be helpful to recall Kac's notion of chaos in the context of the corresponding model, in which states are characterized by their energy only, without exclusion.

Consider a system of n (indistinguishable) particles with a total energy E_n, and assume that the state of a particle is determined by its energy $x_j \geq 0$. The phase space of this system is then the simplex

$$(1) \qquad S_{E_n} := \left\{ (x_1, \ldots, x_n) \in \mathbb{R}_+^n : \sum_{j=1}^{n} x_j = E_n \right\}.$$

Let σ_n denote the uniform probability measure in S_{E_n}. In its simplest form, the Kac walk on the simplex S_{E_n} is the process in which binary collisions occur in a Poisson stream of jump times, with the expected waiting time between jumps being $1/n$, and when a jump occurs, a pair (i, j), $1 \leq i < j \leq n$ is selected uniformly at random, and then the energy of the pair is redistributed by the "collision", a new energy x_i^* for the i-th particle is selected uniformly at random from $[0, x_i + x_j]$, and then x_j^* is fixed by $x_i^* + x_j^* = x_i + x_j$. It is easy to see that the uniform probability measure is the unique invariant measure for this process, and the single particle marginals in equilibrium are certain beta distributions. The rate of approach to equilibrium has been studied by Giroux and Ferland [8]. The original Kac process [9] takes place on the $n - 1$-dimensional sphere consisting of vectors $(v_1, \ldots, v_n)$ such that $\sum_{j=1}^{n} v_j^2 = E_n$. The process described above is the image of the process on the sphere under the change of variables $x_j = v_j^2$.

As Kac discovered, the Kac process on the sphere *propagates chaos*, and it follows readily that the process on the simplex does as well. This means the following: For any probability density F_n with respect to the uniform probability measure on S_{E_n}, consider the empirical distribution

$$(2) \qquad \mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta(x - \tilde{x}_j) \quad \text{where } \tilde{x}_j = \frac{n}{E_n} x_j,$$

where $(x_1, \ldots, x_n)$ is distributed according to F_n. Note that

$$(3) \qquad \int_0^\infty x \, \mathrm{d}\mu_n = 1$$

for every $(\tilde{x}_1, \ldots \tilde{x}_n) \in S_{E_n}$. The notion of chaos concerns sequences, indexed by n, of probability measures or densities on S_{E_n}. We use upper case letters, e.g. F_n, to denote such densities. While in some contexts it is natural to reserve upper-case lets for the distribution functions of probability densities on the line, here such distribution functions do not come into play, and it is more natural to use upper and lower case to distinguish between probability densities on the high dimensional space S_{E_n}, and probability densities on $\mathbb{R}_+$.

Now let $g(x)$ be any probability density on $\mathbb{R}_+$ with $\int_0^\infty x g(x) \, \mathrm{d}x = 1$. A sequence $\{F_n\}$ of probability densities on S_{E_n} is called *g-chaotic* in the sense of Kac in case the sequence $\{\mu_n\}$ of empirical distributions as specified above converges in probability to $g(x) \, \mathrm{d}x$. The notion of chaos is often presented directly in terms of the probability measures: Consider a sequence of probability measures $\{m_n\}_n^\infty$ where the m_n are symmetric probability distributions on E^n, the n-fold product of a metric space E. Then $\{m_n\}_n^\infty$ is said to be *m-chaotic* for some probability measure m on E, if for every $k \geq 1$ and functions $\phi_j \in C(E)$, $j = 1, \ldots, k$ the following limit holds:

$$(4) \qquad \lim_{n \to \infty} \int_E \cdots \int_E \phi_1(x_1) \cdots \phi_k(x_k) m_n(\mathrm{d}x_1, \ldots, \mathrm{d}x_n) - \prod_{j=1}^{k} \int_E \phi_j(x) m(\mathrm{d}x) = 0.$$

In fact, this definition is equivalent to the definition given in terms of the empirical measures, as proven e.g. in [15].

Kac's main result in [9] (for the spherical case) is that if one starts with a chaotic sequence $\{F_n\}$ of initial data that is g-chaotic, and if for each $t > 0$ one lets $\{F_{n,t}\}$ denote the sequence of densities resulting from running the evolution for a time t, then this sequence is g_t-chaotic for some density g_t, and moreover, g_t is the unique solution of a certain non-linear Boltzmann-like equations starting from the initial data g. Thus, this Boltzmann-like equation gives a complete description, in the large n limit of the evolution of the scaled empirical distribution under the Kac process provided one starts with chaotic initial data.

Since S_{E_n} is very close to being a product space, it is possible to construct g-chaotic initial data for any probability density g satisfying $\int_0^\infty x g(x) \, \mathrm{d}x = 1$, $\int_0^\infty x^2 g(x) \, \mathrm{d}x < \infty$ and $g \in L^p(\mathbb{R}_+)$ for some $p > 1$: One takes $\prod_{j=1}^{n} g(\tilde{x}_j)$, and restricts it to simplex S_{E_n}, and normalizes [4]. By the Central Limit Theorem, under $\prod_{j=1}^{n} g(x_j)$, $\sum_{j=1} \tilde{x}_j$ is with high probability very close to n, and so the mass is tightly concentrated on S_{E_n}. As long as one does not look at too many coordinates at once, one cannot see the effects of the restriction. In the physics literature, this is known as the *equivalence of ensembles*. A related result can be found in [15], where g is a density on $\mathbb{R}^k$, and the simplex is replaced by a set $x_1 + \cdots + x_n = na \in \mathbb{R}^k$.

The restrictions that $\int_0^\infty x^2 g(x) \, \mathrm{d}x < \infty$ and $g \in L^p(\mathbb{R}_+)$ for some $p > 1$ may then be removed in a limiting process [4], and thus one has a construction of chaotic initial data for every meaningful initial density g. The corresponding nonlinear Boltzmann-like equation that governs the evolution of the large n empirical measure may then be studied in terms of the linear Kolmogorov equation associated to the Kac process on S_{E_n}.

That is, Kac had found an interesting way to study, by probabilistic means, a class of non-linear equations of a type that arise in kinetic theory. The method relies on the introduction of a family of stochastic processes indexed by n, the

number of particles. Because of constraints such as $\sum_{j=1}^{n} x_j = E_n$ that correspond to conservation laws in the particle system, the x_j are not independent, but their dependence is weak enough, for a wide class of sequences of probability measures including $\{\sigma_n\}$, that the empirical measure in (2) becomes non-random as $n \to \infty$.

In the model introduced next, we consider another type of kinematic constraint. In addition to the conservation of energy, we impose an exclusion condition. This brings dependencies of a new type into consideration, and we show that Kac's notion of chaos is not enough to identify the evolution of a limiting density. Therefore a new approach is required, and a stronger notion of chaos, and one such approach is developed here.

1.1. *The incorporation of exclusion*

For Fermions, the Pauli exclusion principle asserts that a state (here characterized by its energy) only can be occupied by at most one particle. In this continuous setting, we model this without "quantizing" the state space, by requiring that for all pairs of particles, we have $|x_j - x_k| > \epsilon$ for some $\epsilon > 0$. We define

$$(5) \qquad S_{E_n,\epsilon} := \left\{ (x_1, \ldots, x_n) \in \mathbb{R}_+^n : \sum_{j=1}^{n} x_j = E_n, |x_j - x_k| > \epsilon \text{ for all } i \neq j \right\},$$

and assuming that $E_n > \epsilon n(n-1)/2$ so that $S_{E_n,\epsilon} \neq \varnothing$, we let $d\sigma_{n,\epsilon}$ denote the uniform probability measure on $S_{E_n,\epsilon}$.

The process that we consider is the following: Again, the collision times arrive in a Poisson stream with expected waiting time equal to $1/n$, and again, when a jump time occurs, a pair (i, j), $1 \leq i < j \leq n$ is selected uniformly at random. The energy of the two particles is then reapportioned as before, with x_i^* chosen uniformly from $[0, x_i + x_j]$ and then $x_j^* = x_i + x_j - x_i^*$, except the jump only occurs if the new configuration $(x_1, \ldots, x_i^*, \ldots, x_j^*, \ldots, x_n)$ of energy levels satisfies the exclusion condition; i.e., only if it belongs to $S_{E_n,\epsilon}$. It is easy to see that $\sigma_{n,\epsilon}$ is the invariant measure for this process, and since the process is reversible, it is natural to refer to it as the *equilibrium measure*.

While $S_{E_n,\epsilon}$ is non-empty whenever $E_n > \epsilon n(n-1)/2$, if E_n is not too much larger than this value, the spacing between most levels will be very close to ϵ. Think of a long line of parked cars with no marked spaces. For a new pair of cars to park, they must both find gaps of sufficient width. If there is a constraint on the sum of their distances from the start of the line, there may be no way for them to park. In terms of our model, if two cars pull out and look for different spaces, it may be that their only option is to return to the spaces they had (or to swap).

We shall find interesting large n limits only if the energies E_n grow with n in a certain way. Define

$$(6) \qquad \alpha_n := \frac{\epsilon n(n-1)}{E_n}.$$

Then

$$(7) \qquad E_n - \frac{\epsilon n(n-1)}{2} = \left(1 - \frac{\alpha_n}{2}\right) E_n$$

is the *excess energy*, the difference between the minimum energy for a configuration of n particles satisfying the exclusion constraint and the available energy. Clearly we must have $0 \leq \alpha_n \leq 2$. We shall be studying sequences of probability measures $\{F_n \sigma_n\}$ on $S_{E_n,\epsilon}$ with E_n and n related by

$$(8) \qquad \lim_{n \to \infty} \alpha_n = \alpha \in]0, 2[.$$

As before, we rescale the variables with the average energy,

$$(9) \qquad \tilde{x}_j = \frac{n}{E_n} x_j,$$

and define the empirical distribution

$$(10) \qquad \mu_n := \frac{1}{n} \sum_{j=1}^{n} \delta(x - \tilde{x}_j).$$

We also need to rescale ϵ, and set

$$(11) \qquad \tilde{\epsilon}_n = \frac{\epsilon n}{E_n} = \frac{\alpha_n}{n-1}.$$

Because, $\sum_{j=1}^{n} \tilde{x}_j = n$ for every $(x_1, \ldots, x_n) \in S_{E_n,\epsilon}$, one always has that

$$\int_0^\infty x \, d\mu_n = 1. \tag{12}$$

The exclusion limits the amount of mass the μ_n can assign to any half open interval $]a, b]$ in $\mathbb{R}_+$: There can be at most $(b - a)/\tilde{\epsilon}_n$ particles in this interval, and hence

$$\int_{]a,b]} d\mu_n \leq \frac{1}{n\tilde{\epsilon}_n}(b - a) = \frac{n-1}{n\alpha_n}(b - a). \tag{13}$$

It follows from (13) that if μ_n almost surely converges vaguely to $g(x) \, dx$ along a sequence with $\alpha_n \to \alpha$, then

$$g(x) \leq \frac{1}{\alpha}, \tag{14}$$

and provided no mass escapes,

$$\int_0^\infty g(x) \, dx = 1. \tag{15}$$

In what follows we will only use the rescaled variables $\tilde{x}_j$ and $\tilde{\epsilon}_n$, but suppress the tildes from the notation. Moreover, in this scaling $E_n = n$, and therefore $S_{E_n,\epsilon}$ becomes S_{n,ϵ_n}.

At this point we can define a notion of chaos for our class of models:

Definition 1.1. Let $\alpha > 0$ and let $f(x)$ be a probability density on $\mathbb{R}_+$. We define a sequence $\{F_n\}_{n\geq 2}$ of probability measures on S_{n,ϵ_n} to be (α, f)-*chaotic* if $(x_1, \ldots, x_n)$ is random with distribution F_n, and the empirical measures $\mu_n = \frac{1}{n}\sum_{j=1}^{n} \delta(x - x_j)$ converge in probability to $f(x) \, dx$ as $n \to \infty$ and $\alpha_n := \epsilon n(n-1)/E_n \to \alpha$.

Let $\mathcal{P}_t$ be the semigroup associated to a Markov process on S_{n,ϵ_n}, that is, $F_n(x, t) = \mathcal{P}_t F_n(x, 0)$. Following Kac, we say that the semigroup $\mathcal{P}_t$ *propagates chaos with parameter* α in case whenever $\{F_n(x, 0)\}$ is (α, f_0)-chaotic, then $\{F_n(x, t)\}$ is (α, f_t)-chaotic for some probability density f_t on $\mathbb{R}_+$.

In the Kac process that we study here, pairs of particle will interact by redistributing their energies x_i and x_j to a new pair x_i^* and x_j^* with $x_i + x_j = x_i^* + x_j^*$ provided the gaps around x_i^* and x_j^* are large enough for the exclusion constraint to be satisfied. Let $x \in \mathbb{R}_+$. Then for all sufficiently large n, and all $(x_1, \ldots x_n) \in S_{n,\epsilon_n}$, $x < \max_{1 \leq k \leq n}\{x_k\}$. Let $x_{(j)}$ and $x_{(j+1)}$ be the pair of *consecutive* energies such that $x \in [x_{(j)}, x_{(j+1)}[$. Define the *gap at energy* x to be

$$\zeta(x) := x_{(j+1)} - x_{(j)} - \frac{\alpha}{n-1}.$$

Only when $\zeta \geq \frac{\alpha}{n-1}$ is it possible for an interaction to result in either $x_i^* \in [x_{(j)}, x_{(j+1)}[$ or $x_j^* \in [x_{(j)}, x_{(j+1)}[$ since only in this case is the minimum spacing $\frac{\alpha}{n-1}$ (in the scaled variable) available above and below some energy in this interval.

It is probably intuitively clear, and will be shown later on, that the evolution of the empirical density depends strongly on distribution of the energy gaps: For a given probability density $f(x)$ as in Definition 1.1, and any $0 < \alpha < 2$, there are different (α, f)-chaotic sequences $\{F_n\}_{n\geq 2}$ that have very different gap distributions, and this will result in different sorts of interactions being favored in the process, and thus to different results for f_t under the time evolution. Thus, this definition as it stands will not lead to a well-defined evolution equation for the limiting density f_t. We must bring in information on the gaps.

Definition 1.2. Let a sequence $\{F_n\}_{n\geq 2}$ be (α, f)-*chaotic* according to Definition 1.1. We say that $\{F_n\}_{n\geq 2}$ is (α, f)-*chaotic in detail* if for any $x \in R_+$, the random interval $]x_{(j)}, x_{(j+1)}[$ that contains x, the gap length $\zeta_{x,n} = x_{(j+1)} - x_{(j)} - \alpha/(n-1)$ satisfies

$$\lim_{n \to \infty} \mathbb{P}\left[(n-1)\zeta_{x,n}/\alpha > r\right] \to e^{-\frac{\alpha f(x)}{1-\alpha f(x)}r}. \tag{16}$$

We say that the semigroup $\mathcal{P}_t$ *propagates detailed chaos with parameter* α in case whenever $\{F_n(x, 0)\}$ is (α, f_0) chaotic in detail, then the same holds for $\{F_n(x, t)\}$ for some probability density f_t on $\mathbb{R}_+$.

As we shall show below, this particular gap distribution specified in (16) is the only one that is possible: If the gap lengths are asymptotically exponential, and the empirical distribution is asymptotically deterministic with density f, then the exponential rates must be related to f as specified in (16). Thus one could formulate the definition less specifically, only requiring that the gap lengths are asymptotically exponential with *some* rate.

This is probably the simplest generalization of the notion of chaos to our class of models with the exclusion constraint. We consider four questions concerning the Kac model on the simplex with exclusion:

(1) Does the $\{\sigma_{n,\epsilon}\}$ of equilibrium measures satisfy the detailed chaos condition when $\alpha_n \to \alpha$? If so, what is the limiting density f_α for which this sequence is (α, f_α)-chaotic, and how does f_α compare with the Fermi–Dirac distribution, which one might expect in a "quantized" model; i.e., one in which parking spaces are marked with lines?
(2) For which probability densities g on $\mathbb{R}_+$ that satisfy (14) and (15) do there exist sequences that satisfy (α, g)-chaos and detailed (α, g)-chaos conditions?
(3) Is detailed chaos propagated, and if so, what is the equation that governs the evolution of the limiting marginal densities?
(4) At which energy levels in equilibrium do collisions occur a rate bounded away from zero, and at which energy levels are the collisions "frozen out"?

Theorem 2.1 gives a positive answer to the first question, explicitly identifying f_α, which is not the Fermi–Dirac distribution; see Figure 1. Theorem 2.1 provides quantitative bounds on the rate at which $W_1(\mu_n, f_\alpha \, dx) \to 0$ in probability, where W_1 is the Kantorovich–Rubinstein transport metric. Mass transport methods are the basis of a number of our proofs.

Theorem 3.10 answers the second question – such chaotic sequences exist for *all* densities satisfying the two necessary requirements (14) and (15). Such sequences can be constructed in qualitatively different ways, and we provide two examples of constructions, the second one given in Theorem 3.13. Other results in this section provide quantitative chaos estimates, again in the W_1 metric for a broad class of densities g satisfying mild regularity hypotheses.

In Section 4 we derive under the assumption of propagation of detailed chaos, the Boltzmann-like equation that governs the evolution of the limiting empirical measure. The equation resembles the Uehling–Uhlenbeck equation of quantum kinetic theory, but with a different "exclusion factor" corresponding to our different exclusion model. But this exclusion factor turns out to depend on the chaotic sequence: Definition 1.1 is not restrictive enough to uniquely determine the evolution of the limiting empirical measure, though the additional information on the gaps provided in Definition 1.2 is enough.

We prove that the limiting densities f_α obtained from equilibrium measures $\{\sigma_{n,\epsilon}\}$ are stationary solutions to the Boltzmann–Kac equation. We do not prove that propagation of chaos according to either Definition 1.1 or Definition 1.2 holds, but we do provide numerical evidence that detailed (α, f)-chaoticity is propagated, and also that if initial data are only (α, f_0)-chaotic, without the correct exponential gap distribution (16) for α and f_0, this is actually improved by the evolution: The gap distribution converges rapidly to the correct exponential distribution, so that in this sense it appear that not only is chaos propagated, but it strengthens. The numerical evidence for this is presented in Section 5, and further results are available as supplementary material [3].

2. The empirical distribution with exclusion

Equip the rescaled state space, still denoted S_{n,ϵ_n} and defined in (5), with the uniform probability measure $\sigma_{n,\epsilon}$. Let $\mathbb{E}$ denote expectation with respect to this probability measure. Then $x_1, \ldots, x_n$ become random variables.

For two probability measures μ and ν on $\mathbb{R}_+$, let $W_1(\mu, \nu)$ denote the Kantorovich–Rubinstein distance between μ and ν. Recall that

$$(17) \qquad W_1(\mu, \nu) = \sup\left\{ \left| \int_0^\infty \chi \, d\mu - \int_0^\infty \chi \, d\nu \right| : \chi \in \mathrm{Lip}_1 \right\}.$$

Lip_1 denotes the class of 1-Lipschitz functions; i.e., functions χ such that $|\chi(x) - \chi(y)| \le |x - y|$ for all x, y. Note that we may restrict to $\chi \in \mathrm{Lip}_1$ and $\chi(0) = 0$ without changing anything.

Theorem 2.1. *For* $\alpha_n = \epsilon n(n - 1)/E_n \to \alpha$, *the sequence of uniform probability measures on* S_{n,ϵ_n} *is* (α, f_α)-*chaotic in detail*, *where*

$$(18) \qquad f_\alpha(x) = \frac{d}{dx}\phi^{-1}(x) = \frac{1}{\phi'(\phi^{-1}(x))},$$
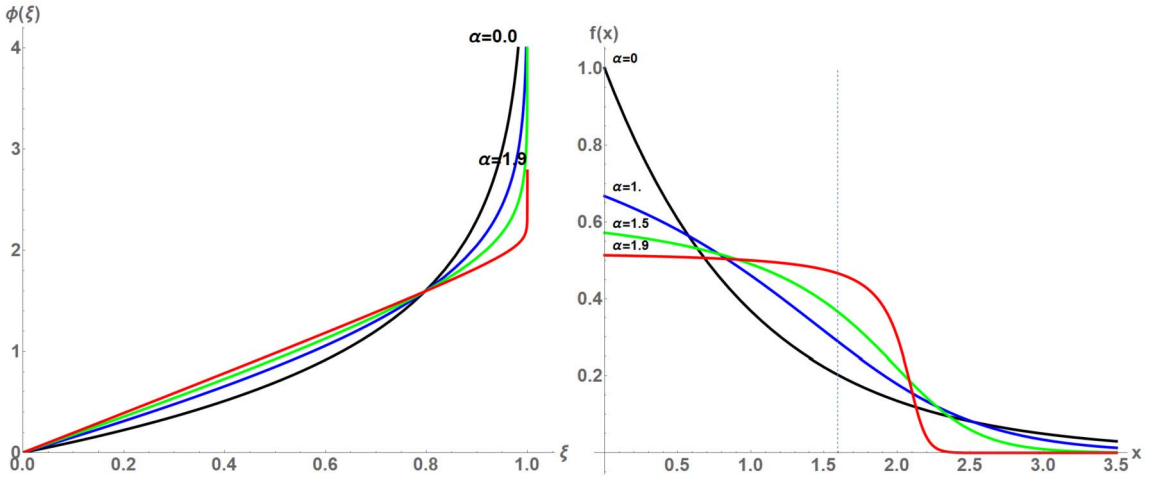
Fig. 1. The function $\phi(\xi)$ for $\alpha = 0.0, 1.0, 1.5, 1.9$ (left), and the density $f(x)$ for the same values of α. Curiously, for $\xi = \xi_0 \approx 0.797$ (the solution of $1 - \xi - e^{-2\xi} = 0$), $\phi(\xi)$ is independent of α, and hence the fraction of the mass of $f(x)$ in $0 \leq x \leq 2\xi_0$ (indicated by the dashed line) is ξ_0 for all values of α.

and

$$\phi(\xi) := (1 - \alpha/2) \log\left(\frac{1}{1-\xi}\right) + \alpha\xi. \tag{19}$$

Moreover, the sequence of empirical measures $\{\mu_n\}$, defined as in (10), is such that there is a constant C such that for any $\delta > 0$ and all sufficiently large n,

$$\mathbb{P}\{W_1(\mu_n, f_\alpha\,\mathrm{d}x) > \delta\} \leq \frac{1}{\delta}\left(\frac{C}{\sqrt{n}} + \frac{3}{2}|\alpha_n - \alpha|\right). \tag{20}$$

The theorem is a corollary of Theorem 3.5, except for the statement of detailed chaoticity, which is proven in Section 3.2. The function ϕ in equation (19) is known as the quantile function for the distribution with density f_α and it is derived as a limit of explicitly constructed quantile functions for each n. We refer to [5] for similar functions related to Young diagrams and shape functions associated with random permutations.

Theorem 2.1 shows that although the exclusion introduces new dependencies between the random variables $x_1, \ldots, x_n$ that are far more complicated that those induced by $\sum_{j=1}^n x_j = n$ which would be the only constraint in the absence of exclusion, these new dependencies are not an obstacle to chaos in the sense of Kac: If $\sigma_{n,\epsilon}$ denote the law of $(x_1, \ldots, x_n)$, and $\alpha_n \to \alpha$, then $\{\sigma_{n,\epsilon}\}$ is (α, f_α)-chaotic.

While the form of $\phi(\xi)$ is simple, it seems difficult to express the function f in closed form, but it clearly differs from the Fermi–Dirac density that is the relevant expression in a quantized setting, although it does resemble it for large values of α. The function f is plotted for some different values of α in Figure 1.

2.1. Parameterization of S_{n,ϵ_n} by the standard simplex

We shall make use of a parameterization of the state space S_{n,ϵ_n} in terms of the standard simplex

$$S_1 := \left\{(x_1, \ldots, x_n) \in \mathbb{R}_+^n : \sum_{j=1}^n x_j = 1\right\}. \tag{21}$$

We first define S_{n,ϵ_n}^* to be the subset consisting of all $(x_1, \ldots, x_n)$ with $x_1 < x_2 < \cdots < x_n$. Up to a set of measure zero, one recovers S_{n,ϵ_n} by taking the union over all permutations

$$\bigcup_{\pi \in \mathfrak{S}_n} \left\{(x_{\pi(1)}, \ldots, x_{\pi(n)}) : (x_1, \ldots, x_n) \in S_{n,\epsilon_n}^*\right\}.$$

The measures we study are all symmetric under interchange of particles, and hence it suffices to parameterize S_{n,ϵ_n}^*.

Lemma 2.2. *For $(\tilde{z}_1, \ldots, \tilde{z}_n) \in S_1$, define $T_n(\tilde{z}_1, \ldots, \tilde{z}_n)$ to be the vector in $\mathbb{R}_+^n$ whose jth component x_j is given by*

$$(22) \qquad x_j = n\left(1 - \frac{\alpha_n}{2}\right)\left(\frac{\tilde{z}_1}{n} + \frac{\tilde{z}_2}{(n-1)} + \cdots + \frac{\tilde{z}_j}{n+1-j}\right) + \frac{j-1}{n-1}\alpha_n.$$

Then T_n provides a one-to-one parameterization of S_{n,ϵ_n}^* by S_1, and moreover $\sigma_{n,\epsilon}$ is the pushforward of the uniform probability measure on S_1 under T_n, averaged over permutations.

Proof. First note that

$$\sum_{j=1}^{n}\left(\frac{\tilde{z}_1}{n} + \frac{\tilde{z}_2}{(n-1)} + \cdots + \frac{\tilde{z}_j}{n+1-j}\right) = 1 \quad \text{and} \quad \sum_{j=1}^{n} \frac{j-1}{n-1}\alpha_n = \frac{n\alpha_n}{2},$$

so that $\sum_{j=1}^{n} x_j = n$, and for $j > 1$,

$$(23) \qquad x_j - x_{j-1} = n\left(1 - \frac{\alpha_n}{2}\right)\frac{\tilde{z}_j}{n+1-j} + \epsilon_n \geq \epsilon_n.$$

Thus the image of T_n lies in S_{n,ϵ_n}. Moreover, (23) shows that T_n is invertible, and gives an explicit formula for the inverse from which one sees, by the same computations that $T_n^{-1}(S_{n,\epsilon_n}) \subset S_1$. This proves the statements about the parameterization. The proof of the description of $\sigma_{n,\epsilon}$ in terms of T_n is somewhat more involved.

We begin by considering the case with no exclusion ($\epsilon = 0$): The uniform density is also the equilibrium distribution of a set of particles at equilibrium, so that for $\phi \in C(\mathbb{R}^n)$,

$$(24) \qquad \begin{aligned}
\mathbb{E}\big(\phi(x_1, \ldots, x_n)\big) &= \frac{1}{\mathcal{Z}} \int_{0 < x_1 + \cdots + x_{n-1} < n} \phi(x_1, \ldots, x_n) \, dx_1 \, dx_2 \cdots dx_{n-1} \\
&= \frac{1}{\tilde{\mathcal{Z}}} \int_{\substack{0 < x_1 + \cdots + x_{n-1} < n \\ x_1 < \cdots < x_n}} \sum_{\pi} \phi_{\pi}(x_1, \ldots, x_n) \, dx_1 \, dx_2 \cdots dx_{n-1},
\end{aligned}$$

where $x_n = n - x_1 - \cdots - x_{n-1}$, and, in the second row, ϕ_{π} denotes the composition of ϕ with the permutation operator $\pi : (x_1, \ldots, x_n) \mapsto (x_{\pi_1}, x_{\pi_2}, \ldots, x_{\pi_n})$, and the sum is taken over all permutations. The normalizing factor $\tilde{\mathcal{Z}}$ is given by

$$(25) \qquad \tilde{\mathcal{Z}} := \int_{\substack{0 < x_1 + \cdots + x_{n-1} < n \\ 0 < x_1 < \cdots < x_n}} dx_1 \, dx_2 \cdots dx_{n-1}.$$

Here we have parameterized S_n with its projection on $\{(x_1, \ldots, x_{n-1}) \mid x_j > 0, x_1 + \cdots + x_{n-1} < n\}$, and set $d\sigma(x_1, \ldots, x_{n-1}) = dx_1 \cdots dx_{n-1}$ without the factor $\sqrt{n}$ which may anyway be absorbed into $\tilde{\mathcal{Z}}$.

Now consider the case $\epsilon > 0$: The expectation in equation (24) can then be computed with the same integrals, but adding the restriction that $x_j - x_{j-1} > \epsilon_n$ for all $j > 1$. Therefore we set $z_j = x_j - x_{j-1} - \epsilon_n > 0$ for $1 < j < n$ and set $z_1 = x_1$. This yields the following change of variables:

$$(26) \qquad \begin{aligned}
x_1 &= z_1, \\
x_2 &= x_1 + \epsilon_n + z_2 = z_1 + z_2 + \epsilon_n, \\
&\cdots \\
x_{n-1} &= z_1 + \cdots + z_{n-1} + (n-2)\epsilon_n, \\
x_n &= n - (n-1)z_1 - (n-2)z_2 - \cdots - z_{n-1} - \epsilon_n(n-1)(n-2)/2.
\end{aligned}$$

The Jacobian of $(z_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{n-1})$ has determinant one, and hence to compute the integrals in equation (24), it is enough to find the domain of $(z_1, \ldots, x_{n-1})$. For each $j < k \leq n$, and each $1 \leq m \leq n - j$, we have $x_{j+m} \geq x_j + m\epsilon_n$, and evidently this is the smallest value x_{j+m} can take, given x_j. Therefore,

$$n - \sum_{k=1}^{j} x_k = \sum_{m=1}^{n-j} x_{j+m} \geq (n-j)x_j + \frac{(n+1-j)(n-j)}{2}\epsilon_n.$$

Rearranging terms,

$$(27) \qquad x_j < \frac{1}{n+1-j}\left(n - x_1 - x_2 - \cdots - x_{j-1} - \epsilon_n \frac{(n+1-j)(n-j)}{2}\right).$$

Since $x_k = z_1 + \cdots + z_k + (k-1)\epsilon_n$,

$$\sum_{k=1}^{j-1} x_k = \sum_{k=1}^{j-1}(j-k)z_k + \epsilon_n \frac{(j-1)(j-2)}{2},$$

and since

$$(j-1)(j-2) + (n+1-j)(n-1) + 2(j-1)(n+1-j) = n(n-1),$$

$$
\begin{aligned}
(28) \qquad z_j &< \frac{1}{n+1-j}\left(n - \sum_{k=1}^{j-1}(j-k)z_k - \epsilon_n \frac{(j-1)(j-2)}{2} - \epsilon_n \frac{(n+1-j)(n-j)}{2}\right) \\
&\quad - z_1 - \cdots - z_{j-1} - (j-1)\epsilon_n \\
&= \frac{1}{n+1-j}\left(n\left(1 - \frac{\alpha_n}{2}\right) - \sum_{k=1}^{j-1}(n+1-k)z_k\right).
\end{aligned}
$$

Define

$$(29) \qquad \tilde{z}_j := \frac{n+1-j}{n}\frac{1}{1-\alpha_n/2}z_j.$$

Then (28) becomes $\tilde{z}_j \leq 1 - \sum_{k=1}^{j-1}\tilde{z}_k$. Using this notation, the version of equation (24) with exclusion can be written

$$
\begin{aligned}
(30) \qquad \mathbb{E}\big(\phi(x_1, \ldots, x_n)\big) &= \frac{\mathcal{J}_{n,\epsilon}}{\mathcal{Z}_{n,\epsilon}}\int_0^1 d\tilde{z}_1 \int_0^{1-\tilde{z}_1} d\tilde{z}_2 \cdots \int_0^{1-\tilde{z}_1-\cdots-\tilde{z}_{j-1}} d\tilde{z}_j \cdots \\
&\qquad \cdots \int_0^{1-\tilde{z}_1-\cdots-\tilde{z}_{n-2}} d\tilde{z}_{n-1} \sum_\pi \phi_\pi(x_1, x_2, \ldots, x_n),
\end{aligned}
$$

where $\mathcal{J}_{n,\epsilon}$ is the Jacobian corresponding to the change of variables given in (29). Taking φ to be the constant function 1, it is evident that

$$\mathcal{Z}_{n,\epsilon} = \mathcal{J}_{n,\epsilon} = \frac{1}{n!}\left(n - \frac{n\alpha_n}{2}\right)^{n-1},$$

which gives the value of $\mathcal{Z}_{n,\epsilon}$. However, we only need to know that $\mathcal{J}_{n,\epsilon}/\mathcal{Z}_{n,\epsilon} = 1$, and then observe that the substitution (29) transforms $x_j = z_1 + \cdots + z_{n-1} + (j-1)\epsilon_n$ into (22) for all $j < n$. $\qquad \square$

Remark 2.3. Lemma 2.2 provides a convenient method for sampling $(x_1, \ldots, x_n)$: simply take $(\tilde{z}_1, \ldots, \tilde{z}_n)$ uniformly from the standard n-simplex, i.e. $\tilde{z}_1 + \cdots + \tilde{z}_n = 1$, and compute the x_j according to the formula (22).

Remark 2.4. Because T_n is continuous and invertible with a continuous inverse, it sets up a one-to one correspondence between symmetric Borel probability measures on S_{n,ϵ_n} and Borel probability measures on S_1. This correspondence provides a useful way to think about symmetric Borel probability measures on S_{n,ϵ_n} in terms of partitions of the excess energy. In the rescaled variables, the excess energy is

$$n\left(1 - \frac{\alpha_n}{2}\right) = \sum_{j=1}^n n\left(1 - \frac{\alpha_n}{2}\right)\tilde{z}_j.$$

Thus one may think of $\{\tilde{z}_1, \ldots, \tilde{z}_n\}$ as specifying a partition of the excess energy into n components

$$(31) \qquad \left\{n\left(1 - \frac{\alpha_n}{2}\right)\tilde{z}_j\right\}_{j=1}^n.$$

The first term in the partition may be understood as making an equal contribution of

$$n\left(1 - \frac{\alpha_n}{2}\right)\frac{\tilde{z}_1}{n}$$

to the energy of each particle, and in the same way the second component makes equal contributions of

$$n\left(1 - \frac{\alpha_n}{2}\right)\frac{\tilde{z}_2}{n-1}$$

to the energy of each of the last $n-1$ particles and so forth. Adding these up, together with the total excluded energies, one arrives at (22).

3. Pre-chaotic sequences of probability measures on S_1

We now identify a class of sequences of measures on S_1 whose pushforwards under T_n will be shown to be (α, g)-chaotic on S_{n,ϵ_n} in the sense of Definition 1.1.

For each n, let τ_n be a Borel probability measure on S_1. Also define the function

$$(32) \qquad w_n(\xi) = n\mathbb{E}_{\tau_n}[\tilde{z}_k] \quad \text{for } \frac{k-1}{n} < \xi \le \frac{k}{n}, 1 \le k \le n,$$

and $w(0) = n\mathbb{E}_{\tau_n}[\tilde{z}_1]$. Here $\mathbb{E}_{\tau_n}$ is the expectation with respect to the measures τ_n. That is, for $\xi > 0$, $w_n(\xi) = n\mathbb{E}[\tilde{z}_{\lceil n\xi \rceil}]$ where $\lceil n\xi \rceil$ is the least integer k such that $n\xi \le k$. Note that for each n,

$$\int_0^1 w_n(\xi)\,d\xi = \sum_{k=1}^n \mathbb{E}_{\tau_n}[\tilde{z}_k] = 1,$$

so that w_n is a probability density.

Definition 3.1. Let $w : [0, 1[\to \mathbb{R}_+$ be a continuous probability density. A sequence $\{\tau_n\}$ of probability measures on S_1 is *w-pre-chaotic* in case

$$(33) \qquad n\mathbb{E}_{\tau_n}[\tilde{z}_j] = w(j/n) + r(j/n)$$

for a continuous function $r(s)$ decreasing to zero when $s \to 0$, where for each $0 < \xi_* < 1$, and each $\epsilon > 0$, there is an n_ϵ so that

$$(34) \qquad \left|r(j/n)\right| < \epsilon \quad \text{for all } n > n_\epsilon, j < n\xi_*,$$

and moreover, for some constant $C < \infty$ depending only on ξ_*,

$$(35) \qquad \text{Var}[\tilde{z}_i] \le \frac{C}{n}\epsilon \quad \text{and} \quad \left|\text{Cov}(\tilde{z}_j, \tilde{z}_k)\right| \le \frac{C}{n^2}\epsilon \quad \text{for all } n > n_\epsilon, j, k < n\xi_*,$$

for all n, j and k.

Remark 3.2. By Lemma 2.2, the equilibrium distribution $\sigma_{n,\epsilon}$ arises when the random partition in (31) is determined by choosing $(\tilde{z}_1, \ldots, \tilde{z}_n)$ from a flat Dirichlet distribution; i.e., the uniform density on S_1, and then the random variables $\tilde{z}_j$ satisfy

$$\mathbb{E}[\tilde{z}_i] = \frac{1}{n},$$

$$(36) \qquad \text{Var}[\tilde{z}_i] = \frac{(n-1)}{n^2(n+1)},$$

$$\text{Cov}[\tilde{z}_i, \tilde{z}_j] = \frac{-1}{n^2(n+1)}.$$

Moreover, it is clear that for each n, $w_n(\xi) = 1$ for all n and ξ. In this case, w is continuous on the closed interval $[0, 1]$ and hence is bounded at 1 also, though the definition allows for $w(t)$ to diverge as $t \uparrow 1$. Later, we shall see that we need this generality.

The next lemma will be used several times in what follows.

Lemma 3.3. *Let f and g be two non-negative integrable functions on $[0, 1]$ such that*

$$\left| \int_0^1 \left(f(\xi) - g(\xi) \right) d\xi \right| \le a.$$

Then for all $0 < \xi_* < 1$,

$$\int_0^1 \left| f(\xi) - g(\xi) \right| dx \le 2 \int_0^{\xi_*} \left| f(\xi) - g(\xi) \right| dx + 2 \int_{\xi_*}^1 g(\xi) d\xi + a.$$

Proof. We have

$$\int_0^1 \left| f(\xi) - g(\xi) \right| dx = \int_0^{\xi_*} \left| f(\xi) - g(\xi) \right| dx + \int_{\xi_*}^1 \left| f(\xi) - g(\xi) \right| dx$$

$$\le \int_0^{\xi_*} \left| f(\xi) - g(\xi) \right| dx + \int_{\xi_*}^1 f(\xi) d\xi + \int_{\xi_*}^1 g(\xi) d\xi.$$

Next,

$$\int_{\xi_*}^1 f(\xi) d\xi = \int_{\xi_*}^1 g(\xi) d\xi + \int_{\xi_*}^1 \left(f(\xi) - g(\xi) \right) d\xi$$

$$\le \int_{\xi_*}^1 g(\xi) d\xi + a - \int_0^{\xi_*} \left(f(\xi) - g(\xi) \right) d\xi$$

$$\le \int_{\xi_*}^1 g(\xi) d\xi + a + \int_0^{\xi_*} \left| f(\xi) - g(\xi) \right| d\xi. \qquad \square$$

Our first application is the following:

Lemma 3.4. *Let $\{\tau_n\}$ be a w pre-chaotic sequence, and let w_n be defined in terms of τ_n as in (32). Then*

$$(37) \qquad \qquad \lim_{n \to \infty} \int_0^1 \left| w_n(\xi) - w(\xi) \right| d\xi = 0.$$

Proof. Pick $\epsilon > 0$, and choose $0 < \xi_* < 1$ such that $\int_{\xi_*}^1 w(t) dt < \epsilon$. By hypothesis w is continuous on $[0, \xi_*]$, and for all $n > n_\epsilon$,

$$\int_0^{\xi_*} \left| w_n(\xi) - w(\xi) \right| d\xi = \sum_{k < n\xi_*} \int_{(k-1)/n}^{k/n} \left| w(k/n) + r(k, n) - w(\xi) \right| d\xi$$

$$\le \sum_{k < n\xi_*} \int_{(k-1)/n}^{k/n} \left| w(k/n) - w(\xi) \right| d\xi + \epsilon.$$

If ω denotes the modulus of continuity of w on $[0, \xi_*]$,

$$\left| w(k/n) - w(\xi) \right| \le \omega(1/n) \quad \text{on} \quad \left[\frac{k-1}{n}, \frac{k}{n} \right].$$

Thus

$$\int_0^{\xi_*} \left| w_n(\xi) - w(\xi) \right| d\xi \le \omega(1/n) + \epsilon$$

for all sufficiently large n. By Lemma 3.3, for all sufficiently large n,

$$\int_0^1 \big|w_n(\xi) - w(\xi)\big|\,d\xi \le 2\omega(1/n) + 4\epsilon.$$

Since $\epsilon > 0$ is arbitrary, the lemma is proved. $\qquad\square$

3.1. *Chaotic sequences of probability measures on S_{n,ϵ_n}*

In this section we prove the following:

Theorem 3.5. *Let w be a probability density on $[0, 1]$ that is continuous on $[0, 1[$, and let $\{\tau_n\}$ be a w pre-chaotic sequence of probability densities on S_1. Fix a sequence of energies $\{E_n\}$ with $\alpha_n = \epsilon n(n-1)/E_n \to \alpha$, and define the maps T_n in terms of α_n. Let $\hat\tau_n$ denote the pushforward of τ_n onto S_{n,ϵ_n}, averaged over permutations. Let $\{\mu_n\}$ be the sequence of empirical measures associated to $\{\hat\tau_n\}$. Then*

$$(38) \qquad \lim_{n\to\infty} W_1\big(\mu_n, g(x)\,dx\big) = 0,$$

where g is a probability density on $\mathbb{R}_+$ related to w as follows: *Define the increasing function ϕ on $[0, 1]$ by*

$$(39) \qquad \phi(\xi) = (1 - \alpha/2)\int_0^\xi \frac{w(t)}{1 - t}\,dt + \alpha\xi$$

and then

$$(40) \qquad g(x) = \frac{1}{\phi'(\phi^{-1}(x))}.$$

Theorem 3.5 gives conditions for $\{\hat\tau_n\}$ to be (α, g) chaotic for a probability density g on $\mathbb{R}_+$ that is determined by α and w. Notice that as long as $w(1) \ne 0$ we have $\lim_{\xi\to 1}\phi(\xi) = \infty$, and if in addition $\alpha > 0$ or if w does not vanish on any interval, then ϕ is strictly increasing, so that ϕ is invertible from $[0, 1]$ to $[0, \infty[$, and evidently it is differentiable. It is also possible to invert the relation between g and w, so that given an appropriate density g, one can find the w for which (39) and (40) yield g:

Theorem 3.6. *Let $\alpha \in \,]0, 2[$. Let $g(x)$ be a probability density on $\mathbb{R}_+$ such that*

$$(41) \qquad g(x) < \frac{1}{\alpha} \quad a.e. \quad and \quad \int_0^\infty xg(x)\,dx = 1.$$

Let $G(x) = \int_0^x g(t)\,dt$ denote the distribution function of g, and for $\xi \in [0, 1]$ define

$$(42) \qquad w(\xi) := \frac{1}{1 - \alpha/2}\left(\frac{1}{g(G^{-1}(\xi))} - \alpha\right)(1 - \xi).$$

Then w is a probability density on $[0, 1]$, and with ϕ defined as in (39)

$$(43) \qquad g(x) = \frac{1}{\phi'(\phi^{-1}(x))},$$

and

$$(44) \qquad \frac{\alpha g(x)}{1 - \alpha g(x)} = \frac{2\alpha}{2 - \alpha}\frac{1 - \phi^{-1}(x)}{w(\phi^{-1}(x))}.$$

Proof. We compute, using the change of variables $x := G^{-1}(\xi)$,

$$
\begin{aligned}
(1 - \alpha/2) \int_0^1 w(\xi)\, d\xi &= \int_0^\infty \big(1 - \alpha g(x)\big)\big(1 - G(x)\big)\, dx \\
&= \int_0^\infty \big(1 - G(x)\big)\, dx - \alpha \int_0^\infty g(x)\big(1 - G(x)\big)\, dx \\
&= \int_0^\infty x g(x)\, dx - \frac{1}{2}\alpha.
\end{aligned}
\tag{45}
$$

Thus, whenever, $g(x) < 1/\alpha$ almost everywhere and $\int_0^\infty x g(x)\, dx = 1$, $w(x)$ is a probability density on $[0, 1]$.

With this choice of $w(\xi)$ in (39), we find

$$
\phi(\xi) = \int_0^\xi \left(\frac{1}{g(G^{-1}(t))} - \alpha \right) dt + \alpha \xi = \int_0^\xi \frac{1}{g(G^{-1}(t))}\, dt.
\tag{46}
$$

It follows that $\phi'(x) = 1/g((G^{-1}(\xi)) = (G^{-1}(\xi))'$ and then since $\phi(0) = G^{-1}(0) = 0$, $\phi(\xi) = G^{-1}(\xi)$. Thus, $G(x) = \phi^{-1}(x)$, and (43) is valid.

Finally, by (43),

$$
\frac{\alpha g(x)}{1 - \alpha g(x)} = \frac{\alpha}{\phi'(\phi^{-1}(x)) - \alpha},
$$

and then since $\phi'(\xi) = (1 - \alpha/2)\frac{w(\xi)}{1-\xi} + \alpha$, (44) follows. $\qquad\square$

As an example, consider $g(x) = e^{-x}$, which satisfies (41) as long as $\alpha \le 1$. Then $G(x) = 1 - e^{-x}$, and then $G^{-1}(\xi) = -\log(1 - \xi)$. Therefore,

$$
w(\xi) = \frac{1}{1 - \alpha/2}\big(1 - \alpha(1 - \xi)\big),
$$

which is bounded on all of $[0, 1]$. By Theorem 3.5 and Theorem 3.6, for all $\alpha \le 1$, there exists an (α, g)-chaotic sequence.

We now prepare to prove Theorem 3.5. The first step is to encode the empirical distribution into a random function as follows: Define a random function $\psi_n : [0, 1] \to \mathbb{R}^+$ by setting $x_0 = 0$, and then

$$
\psi_n(\xi) := x_{k-1} \quad \text{for } \frac{k-1}{n} \le \xi < \frac{k}{n},\ 1 \le k \le n.
$$

Explicitly,

$$
\psi_n(\xi) = x_{\lfloor n\xi \rfloor} = \left(1 - \frac{\alpha_n}{2} \right) \sum_{j=1}^{\lfloor \xi n \rfloor} \frac{\tilde{z}_j}{1 - \frac{j-1}{n}} + \frac{\alpha_n(\lfloor n\xi \rfloor - 1)_+}{n - 1},
\tag{47}
$$

where $\lfloor n\xi \rfloor$ is the largest integer k such that $k \le n\xi$. The point of the definition is this: Let χ be any 1-Lipschitz function on $\mathbb{R}_+$ with $\chi(0) = 0$. Then on account of (12), χ is, with probability 1, integrable with respect to the empirical distribution μ_n, and one has

$$
\int_0^\infty \chi\, d\mu_n = \int_0^1 \chi\big(\psi_n(\xi)\big)\, d\xi + \frac{1}{n}\chi(x_n).
\tag{48}
$$

Define ρ_n to be the pushforward under ψ_n of the uniform measure on $[0, 1]$, so that we can rewrite (48) as

$$
\mu_n = \rho_n + \frac{1}{n}\delta(x - x_n) - \frac{1}{n}\delta(x).
$$

Had we used the ceiling function $\lceil \cdot \rceil$ in place of the floor function $\lfloor \cdot \rfloor$, we would have had $\mu_n = \rho_n$, and then we would have

$$
\int_0^1 \psi_n(\xi)\, d\xi = \frac{1}{n} \sum_{j=1}^n x_j = 1,
$$

so that ψ_n would be a random probability distribution. This would be convenient, but then some estimates that follow would be more complicated. It is easy to estimate the small difference:

Lemma 3.7. *We have*

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{E}[x_n] = 0, \tag{49}$$

and for all $\delta > 0$,

$$\lim_{n \to \infty} \mathbb{P}\big\{ W_1(\mu_n, \rho_n) > \delta \big\} = 0. \tag{50}$$

Proof. Since χ is 1-Lipschitz with $\chi(0) = 0$,

$$\left| \int_0^\infty \chi \, d\mu_n - \int_0^\infty \chi \, d\rho_n \right| = \frac{1}{n} \big| \chi(x_n) \big| \leq \frac{1}{n} x_n,$$

and hence $W_1(\mu_n, \rho_n) \leq \frac{1}{n} x_n$. Thus, once we have proved (49), (50) follows by Markov's inequality.

Now (36) yields

$$\mathbb{E}x_n = \mathbb{E}\left(n\left(1 - \frac{\alpha_n}{2}\right)\left(\frac{\tilde{z}_1}{n} + \frac{\tilde{z}_2}{(n-1)} + \cdots + \frac{\tilde{z}_n}{1}\right) + \alpha_n \right).$$

Pick $0 < \xi_* < 1$, and split the sum into two pieces

$$\sum_{k=1}^n \frac{\mathbb{E}[\tilde{z}_k]}{n-k+1} = \sum_{k \leq \lfloor n\xi_* \rfloor} \frac{\mathbb{E}[\tilde{z}_k]}{n-k+1} + \sum_{k > \lfloor n\xi_* \rfloor} \frac{\mathbb{E}[\tilde{z}_k]}{n-k+1}.$$

The last term satisfies

$$\sum_{k > \lfloor n\xi_* \rfloor} \frac{\mathbb{E}[\tilde{z}_k]}{n-k+1} \leq \sum_{k > \lfloor n\xi_* \rfloor} \mathbb{E}[\tilde{z}_k] \leq \int_{\xi_* - 1/n}^1 w_n(\xi) \, d\xi,$$

and by Lemma 3.4, for and $\epsilon > 0$ sufficiently large n, this is bounded above by

$$\int_{\xi_* - 1/n}^1 w(\xi) \, d\xi + \epsilon$$

uniformly in ξ_*. Because

$$\lim_{\xi_* \uparrow 1} \int_{\xi_*}^1 w(\xi) \, d\xi = 0,$$

we can choose $\xi_* < 1$ so that

$$\sum_{k > \lfloor n\xi_* \rfloor} \frac{\mathbb{E}[\tilde{z}_k]}{n-k+1} < \epsilon$$

for all sufficiently large n. Next,

$$n \sum_{k \leq n\xi_*} \frac{\mathbb{E}[\tilde{z}_k]}{n-k+1} = \sum_{k \leq n\xi_*} \frac{n\mathbb{E}[\tilde{z}_k]}{1 - (k-1)/n} \frac{1}{n} = \sum_{k \leq n\xi_*} \frac{(w_n(k-1)/n)}{1 - (k-1)/n} \frac{1}{n}.$$

By (33), for all $\epsilon > 0$ this is bounded by

$$\int_0^{\xi_*} \frac{w(t)}{1-t} \, dt + 2\epsilon$$

for all sufficiently large n. Altogether, for all $\epsilon > 0$ and all sufficiently large n,

$$\frac{1}{n} \mathbb{E}[x_n] \leq (1 - \alpha_n/2)\left(\frac{1}{n}\left(\int_0^{\xi_*} \frac{w(t)}{1-t} \, dt + 2\epsilon + \alpha_n\right) + \epsilon\right).$$

Since $\epsilon > 0$ is arbitrary, this proves (49). $\qquad\square$

Now let $n \to \infty$ with $\alpha_n \to \alpha$. We shall show below that if $\{\hat{\tau}_n\}$ is the pushforward of a w-pre-chaotic sequence $\{\tau_n\}$ of probability densities on S_1, and μ_n is the corresponding sequence of empirical measures, then along this limit, the variance of $\psi_n(\xi)$ converges to zero, and moreover, its expectation $\phi_n(\xi) := \mathbb{E}[\psi_n(\xi)]$ converges to a limiting function $\phi := \lim_{n\to\infty} \phi_n$. In this case

$$\lim_{n\to\infty} \int_0^\infty \chi \, \mathrm{d}\mu_n = \int_0^1 \chi\big(\phi(\xi)\big) \, \mathrm{d}\xi = \int_0^\infty \chi(x) f_\alpha(x) \, \mathrm{d}x,$$

with convergence in probability, where $f_\alpha(x) := 1/\phi'(\phi^{-1}(x))$, as in Theorem 2.1.

Computing the expectation of $\psi_n(\xi)$, we see that

$$
\begin{aligned}
\phi_n(\xi) &= \left(1 - \frac{\alpha_n}{2}\right) \sum_{j=1}^{\lfloor \xi n \rfloor} \frac{\mathbb{E}[\tilde{z}_j]}{1 - \frac{j-1}{n}} + \frac{\alpha_n(\lfloor n\xi \rfloor - 1)_+}{n-1} \\
&= \left(1 - \frac{\alpha_n}{2}\right) \frac{1}{n} \sum_{j=1}^{\lfloor \xi n \rfloor} \frac{w_n(j/n)}{1 - \frac{j-1}{n}} + \frac{(\lfloor \xi n \rfloor - 1)_+}{(n-1)\xi} \alpha_n \xi.
\end{aligned}
$$
(51)

For $\frac{j-1}{n} < t \le \frac{j}{n}$

$$\frac{1}{1-t} - \frac{1}{n}\frac{1}{(1-t)^2} \le \frac{1}{1 - \frac{j-1}{n}} \le \frac{1}{1-t},$$
(52)

and therefore

$$\int_0^\xi \frac{w_n(t)}{1-t} \, \mathrm{d}t - \frac{1}{n} \int_0^\xi \frac{w_n(t)}{(1-t)^2} \, \mathrm{d}t \le \frac{1}{n} \sum_{j=1}^{\lfloor \xi n \rfloor} \frac{w_n(j/n)}{1 - \frac{j-1}{n}} \le \int_0^\xi \frac{w_n(t)}{1-t} \, \mathrm{d}t.$$
(53)

Setting

$$\widetilde{\phi}_n(\xi) := (1 - \alpha_n/2) \int_0^\xi \frac{w_n(t)}{1-t} \, \mathrm{d}t + \frac{(\lfloor \xi n \rfloor - 1)_+}{(n-1)\xi} \alpha_n \xi \ge \phi_n(x),$$
(54)

we have

$$\left|\phi_n(\xi) - \widetilde{\phi}_n(\xi)\right| \le (1 - \alpha_n/2) \frac{1}{n} \int_0^\xi \frac{w_n(t)}{(1-t)^2} \, \mathrm{d}t \le (1 - \alpha_n/2) \frac{C}{n} \frac{1}{1-\xi}.$$
(55)

Note also that

$$1 - \frac{2}{\xi}\frac{1}{n-2} \le \frac{(\lfloor \xi n \rfloor - 1)_+}{(n-1)\xi} \le 1,$$
(56)

and therefore, if we assume that $\alpha_n \to \alpha$,

$$\phi_n(\xi) \to (1 - \alpha/2) \int_0^\xi \frac{w(t)}{1-t} \, \mathrm{d}t + \alpha\xi =: \phi(\xi)$$
(57)

when $n \to \infty$.

We then have from (53), (55) and (56) for all ξ,

$$
\begin{aligned}
\left|\phi(\xi) - \phi_n(\xi)\right| &\le \left|\phi(\xi) - \widetilde{\phi}_n(\xi)\right| + \left|\widetilde{\phi}_n(\xi) - \phi_n(\xi)\right| \\
&\le (1 - \alpha/2) \int_0^\xi \frac{|w(t) - w_n(t)|}{1-t} \, \mathrm{d}t + \left(1 - \frac{(\lfloor \xi n \rfloor - 1)_+}{(n-1)\xi}\right)\alpha \\
&\quad + \widetilde{\phi}_n(\xi) - \phi_n(\xi).
\end{aligned}
$$
(58)

Now define ν_n to be the probability measure on $\mathbb{R}_+$ that is the pushforward of the uniform probability measure on $[0, 1]$ under ϕ_n, and let ν be determined by ϕ in the same way.

Lemma 3.8. *We have*

$$\lim_{n\to\infty} \int_0^1 \left|\phi_n(\xi) - \phi(\xi)\right| d\xi = 0, \tag{59}$$

and

$$\lim_{n\to\infty} W_1(\nu_n, \nu) = 0. \tag{60}$$

Proof of Lemma 3.8. Let $\chi \in \mathrm{Lip}_1$. Then

$$\left|\int_0^1 \chi\big(\phi_n(\xi)\big) d\xi - \int_0^1 \chi\big(\phi(\xi)\big) d\xi\right| \le \int_0^1 \left|\phi_n(\xi) - \phi(\xi)\right| d\xi.$$

It remains to show (59). We estimate each of the terms coming from (58).

Suppose first that $\alpha_n = \alpha$ for all n. To bound the integral of the first term on the right in (58), change the order of integration:

$$\int_0^1 \left(\int_0^\xi \frac{|w(t) - w_n(t)|}{1-t} \, dt\right) d\xi = \int_0^1 \left(\int_t^1 \frac{|w(t) - w_n(t)|}{1-t} \, d\xi\right) dt$$
$$= \int_0^1 \left|w(t) - w_n(t)\right| dt,$$

and by Lemma 3.4, the right side vanishes in the limit $n \to \infty$. Making the obvious addition and subtraction argument, we see that the same conclusion holds under the assumption that $\lim_{n\to\infty} \alpha_n = \alpha$.

Next, to estimate $\int_0^1 (1 - \frac{(\lfloor \xi n\rfloor - 1)_+}{(n-1)\xi}) d\xi$ we break the integral up into two pieces, and use (56) away from $\xi = 0$:

$$\int_0^1 \left(1 - \frac{(\lfloor \xi n\rfloor - 1)_+}{(n-1)\xi}\right) d\xi \le \int_0^{1/n} 1 \, d\xi + \int_{1/n}^1 \frac{2}{\xi} \frac{1}{n-2} \, d\xi = \frac{1}{n} + \frac{2\log n}{n-2},$$

and this too vanishes in the limit $n \to \infty$.

Finally, to estimate $\int_0^1 (\widetilde{\phi}_n(\xi) - \phi_n(\xi)) d\xi$, we break the integral up into two pieces, but at the other end, and use (55):

$$\int_0^{1-1/n} \big(\widetilde{\phi}_n(\xi) - \phi_n(\xi)\big) d\xi \le (1 - \alpha/2)\frac{C}{n} \int_0^{1-1/n} \frac{1}{1-\xi} \, d\xi = (1-\alpha/2)\frac{C\log n}{n}$$

while

$$\int_{1-1/n}^1 \big(\widetilde{\phi}_n(\xi) - \phi_n(\xi)\big) d\xi \le \int_{1-1/n}^1 \widetilde{\phi}_n(\xi) \, d\xi \le (1-\alpha/2)\int_{1-1/n}^1 \log(1-\xi) \, d\xi + \frac{1}{n}\alpha$$
$$= (1-\alpha/2)\frac{\alpha + 1 + \log n}{n}.$$

To pass to the general case, let $\tilde{\phi}$ denote the function ϕ with α replaced by some $\alpha_n \in]0, 2[$. Then it is easy to see that $\int_0^1 |\phi - \tilde{\phi}| \, d\xi \le \frac{3}{2}|\alpha - \alpha_n|$. Now one more application of the triangle inequality yields (59) in general. $\square$

Lemma 3.9. $\lim_{n\to\infty} \mathbb{P}\{W_1(\rho_n, \nu_n) > \delta\} = 0$. *Recall here that ρ_n is the pushforward of the uniform measure on $[0, 1]$ by the map ψ_n from equation (47).*

Proof. Take $\chi \in \mathrm{Lip}_1$ and estimate

$$\left|\int_0^1 \chi\big(\psi_n(\xi)\big) d\xi - \int_0^1 \chi\big(\phi_n(\xi)\big) d\xi\right| \le \int_0^1 \left|\chi\big(\psi_n(\xi)\big) - \chi\big(\phi_n(\xi)\big)\right| d\xi,$$

uniformly in χ, and hence $W_1(\rho_n, \nu_n) \le \int_0^1 |\psi_n(\xi) - \phi_n(\xi)| \, d\xi$. By Markov's inequality, for any $\delta > 0$,

$$\mathbb{P}\{W_1(\rho_n, \nu_n) > \delta\} \le \frac{1}{\delta}\mathbb{E}\left(\int_0^1 \left|\psi_n(\xi) - \phi_n(\xi)\right| d\xi\right). \tag{61}$$

Now note that $\int_0^1 \psi_n(\xi)\,d\xi = \frac{1}{n}\sum_{j=0}^{n-1} x_j = 1 - \frac{1}{n}x_n$, and likewise

$$\int_0^1 \phi_n(\xi)\,d\xi = \frac{1}{n}\sum_{j=0}^{n-1}\mathbb{E}[x_j] = 1 - \frac{1}{n}\mathbb{E}[x_n].$$

Therefore,

$$\left|\int_0^1 \psi_n(\xi)\,d\xi - \int_0^1 \phi_n(\xi)\,d\xi\right| \le \frac{1}{n}|x_n - \mathbb{E}[x_n]|.$$

Then by Lemma 3.3,

$$\int_0^1 |\psi_n(\xi) - \phi_n(\xi)|\,d\xi \le 2\int_0^{\xi_*} |\psi_n(\xi) - \phi_n(\xi)|\,d\xi + 2\int_{\xi_*}^1 \phi_n(\xi)\,d\xi + \frac{1}{n}|x_n - \mathbb{E}[x_n]|.$$

Next, by Lemma 3.8

$$\limsup_{n\to\infty}\int_{\xi_*}^1 \phi_n(\xi)\,d\xi \le \int_{\xi_*}^1 \phi(\xi)\,d\xi + \limsup_{n\to\infty}\int_0^1 |\phi_n(\xi) - \phi(\xi)|\,d\xi = \int_{\xi_*}^1 \phi(\xi)\,d\xi,$$

and $\frac{1}{n}\mathbb{E}[|x_n - \mathbb{E}[x_n]|] \le \frac{2}{n}\mathbb{E}[x_n]$ which tends to zero by Lemma 3.7. Therefore,

$$\limsup_{n\to\infty}\mathbb{E}\left[\int_0^1 |\psi_n(\xi) - \phi_n(\xi)|\,d\xi\right] \le 2\limsup_{n\to\infty}\mathbb{E}\left[\int_0^{\xi_*} |\psi_n(\xi) - \phi_n(\xi)|\,d\xi\right]$$
$$+ 2\int_{\xi_*}^1 \phi(\xi)\,d\xi.$$

We next show that the first term on the right is zero. Pick $\epsilon > 0$. Then by (35), there is a constant C depending only on ξ_* such that for some n_ϵ

$$\text{Var}[\tilde{z}_i] \le \frac{C}{n}\epsilon \quad \text{and} \quad |\text{Cov}(\tilde{z}_j, \tilde{z}_k)| \le \frac{C}{n^2}\epsilon \quad \text{for all } n > n_\epsilon,\ j, k < n\xi_*.$$

We then have from equations (47) and (51) that for all $\xi < \xi_*$

$$
\begin{aligned}
\text{Var}[\psi_n(\xi)] &= \mathbb{E}[(\psi_n(\xi) - \phi_n(\xi)])^2] \\
&= \mathbb{E}\left[\left(\left(1 - \frac{\alpha_n}{2}\right)\sum_{j=1}^{\lfloor \xi n\rfloor} \frac{\tilde{z}_j - \mathbb{E}[\tilde{z}_j]}{1 - \frac{j-1}{n}}\right)^2\right] \\
&= \left(1 - \frac{\alpha_n}{2}\right)^2\left(\sum_{j=1}^{\lfloor n\xi\rfloor} \frac{\text{Var}[\tilde{z}_j]}{(1 - \frac{j-1}{n})^2} + \sum_{\substack{j,k=1 \\ j\ne k}}^{\lfloor n\xi\rfloor} \frac{\text{Cov}[\tilde{z}_j, \tilde{z}_k]}{(1 - \frac{j-1}{n})(1 - \frac{k-1}{n})}\right).
\end{aligned}
$$
(62)

Using the bounds on $\text{Var}[\tilde{z}_j]$ and $\text{Cov}[\tilde{z}_j, \tilde{z}_k]$ from equation (35), for all sufficiently large n,

$$\text{Var}[\psi_n(\xi)] \le \left(1 - \frac{\alpha_n}{2}\right)^2\left(\sum_{j=1}^{\lfloor n\xi\rfloor} \frac{C\epsilon}{n(1 - \frac{j-1}{n})^2} + \sum_{\substack{j,k=1 \\ j\ne k}}^{\lfloor n\xi\rfloor} \frac{C\epsilon}{n^2(1 - \frac{j-1}{n})(1 - \frac{k-1}{n})}\right).$$
(63)

The first of the terms in the parentheses is smaller than

$$\epsilon C \int_0^\xi \frac{1}{(1-\xi)^2}\,d\xi = \epsilon C\frac{1}{1-\xi},$$
(64)

and the second is smaller than

$$(65) \qquad \epsilon C\left(\frac{1}{n}\sum_{k=1}^{\lfloor n\xi\rfloor}\frac{1}{1-\frac{k-1}{n}}\right)^2 \leq \epsilon C\left(\log(1-\xi)\right)^2,$$

where, as above, we have used (53) and its analog for $(1-\xi)^{-2}$. It follows that for all $\xi < \xi_*$,

$$(66) \qquad \mathrm{Var}\left[\psi_n(\xi)\right] \leq \epsilon C\left((1-\xi)^{-1} + \left(\log(1-\xi)\right)^2\right).$$

Therefore, for all $n > n_\epsilon$.

$$\mathbb{E}\left[\int_0^{\xi_*}\left|\psi_n(\xi)-\phi_n(\xi)\right|\mathrm{d}\xi\right] = \int_0^{\xi_*}\mathbb{E}\left[\left|\psi_n(\xi)-\phi_n(\xi)\right|\right]\mathrm{d}\xi$$

$$\leq \int_0^{\xi_*}\left(\mathbb{E}\left[\left|\psi_n(\xi)-\phi_n(\xi)\right|^2\right]\right)^{1/2}\mathrm{d}\xi$$

$$\leq \left(\epsilon C\left((1-\xi_*)^{-1} + \left(\log(1-\xi_*)\right)^2\right)\right)^{1/2}.$$

Since $\epsilon > 0$ is arbitrary, this proves that

$$\limsup_{n\to\infty}\mathbb{E}\left[\int_0^1\left|\psi_n(\xi)-\phi_n(\xi)\right|\mathrm{d}\xi\right] \leq \int_{\xi_*}^1\phi(\xi)\,\mathrm{d}\xi$$

for all $0 < \xi_* < 1$. However, ϕ is integrable, we can choose ξ_* to make this arbitrarily small. Thus,

$$\lim_{n\to\infty}\mathbb{E}\left[\int_0^1\left|\psi_n(\xi)-\phi_n(\xi)\right|\mathrm{d}\xi\right] = 0.$$

The main assertion now follows from (61). □

Proof of Theorem 3.5. By the triangle inequality,

$$W_1(\nu,\mu_n) \leq W_1(\nu,\nu_n) + W_1(\nu_n,\rho_n) + W_1(\rho_n,\nu_n).$$

Now applying Lemma 3.7, Lemma 3.8 and Lemma 3.9 yields the result yields (38). Then since

$$\lim_{n\to\infty}\mu_n\left([0,x]\right) = \lim_{n\to\infty}\int_0^1 1_{[0,x]}(\psi_n(\xi))\,\mathrm{d}\xi = \lim_{n\to\infty}\int_0^1 1_{[0,x]}(\phi(\xi))\,\mathrm{d}\xi = \phi^{-1}(x),$$

the cumulative distribution function of the limiting empirical measure is $\phi^{-1}(x)$ and hence limiting empirical measure has the density

$$(67) \qquad g(x) = \frac{\mathrm{d}}{\mathrm{d}x}\phi^{-1}(x) = \frac{1}{\phi'(\phi^{-1}(x))}. \qquad \square$$

3.2. *Detailed chaoticity of the equilibrium sequence*

As our first application of Theorem 3.5, we identify the limiting equilibrium density f_α, and prove the detailed (α, f_α)-chaoticity of the equilibrium sequence:

Proof of Theorem 2.1. By Lemma 2.2, the sequence of uniform probability measures on S_{n,ϵ_n} are obtained by averaging over permutations the pushforwards under the map T_n described there of the flat Dirichlet measure on the standard simplices of the same dimension. In this case we have at fixed n that $w_j = 1/n$ for all j, and $w(t) = 1$. By Remark 3.2, the sequence of flat Dirichlet measures on the standard simplices is w-chaotic in the sense of Definition 3.1 for $w = 1$. For $w(t) = 1$ for all t, $\phi(\xi)$ is given by (19), and this identifies the limiting density f_α.

We next show that detailed (α, f_α) chaoticity holds for the sequence. The gap length $\zeta_{x,n} = x_{(j+1),n} - x_{j,n} - \alpha/(n-1)$ satisfies

$$\frac{n-1}{\alpha}\zeta_{x,n} = \frac{2-\alpha}{2\alpha}\frac{(n-1)\tilde{z}_j}{1-\frac{j-1}{n}} \quad \text{where } j = \lfloor n\phi^{-1}(x)\rfloor.$$

Each $\tilde{z}_j$ has a Beta$(1, n-1)$ distribution, and hence the probability density for $(n-1)\tilde{z}_j$ is $\frac{n}{n-1}(1 - z/(n-1))^{n-1}$ which converges to e^{-z} as $n \to \infty$. Therefore

$$
\lim_{n\to\infty} \mathbb{P}\big[(n-1)\zeta_{x,n}/\alpha > r\big] \to e^{-\frac{2\alpha(1-\phi^{-1}(x))}{2-\alpha}r}, \tag{68}
$$

and by (44), this is equivalent to (16). with $g = f_\alpha$.

The rate information is easily extracted from the Lemmas since all but Lemma 3.8 give rates. However, one can extract a rate from the proof of Lemma 3.8 by considering the last two displayed inequalities in the proof. The details are left to the reader. $\qquad\square$

3.3. Chaotic sequences by non-flat Dirichlet measures

The construction just provided leads to other chaotic sequences of probability measures on S_{n,ϵ_n}: Instead of pushing forward the flat Dirichlet measure on S_1, we can pushforward more general Dirichlet distributions, and as we show in this section this leads to the construction of (α, g)-chaotic sequences for all probability densities g on $\mathbb{R}_+$ that satisfy (14) and (15). However, except in the case of the flat Dirichlet measures, these sequences will not satisfy the detailed chaoticity.

Let $w = (w_1, \ldots, w_n)$ be a probability measure on $\{1, \ldots, n\}$. That is, $w_j \geq 0$ for all j and $\sum_{j=1}^n w_j = 1$. Then

$$
h_w(\tilde{z}_1, \ldots \tilde{z}_n) = \left(\prod_{j=1}^n \Gamma(nw_j)\right)^{-1} \Gamma(n) \prod_{j=1}^n \tilde{z}_j^{nw_j-1}, \tag{69}
$$

is the density for a Dirichlet distribution on S_1 with concentration parameters w. Random variables $(\tilde{z}_1, \ldots, \tilde{z}_n)$ with this distribution satisfy

$$
\begin{aligned}
\mathbb{E}[\tilde{z}_j] &= w_j, \\
\mathrm{Var}[\tilde{z}_i] &= \frac{w_j(1-w_j)}{n+1}, \\
\mathrm{Cov}[\tilde{z}_i, \tilde{z}_j] &= -\frac{w_i w_j}{n+1}.
\end{aligned} \tag{70}
$$

Now let $w(x)$ be a continuous probability density on $[0, 1]$, and suppose that for each n, we produce $(w_1, \ldots, w_n)$ by taking w_j to be the mass assigned by $w(x)\,dx$ to the jth interval on in the uniform partition of $[0, 1]$. Let τ_n denote the Dirichlet distribution on the standard simplex in n dimensions with the distribution given by (69) and this choice of $(w_1, \ldots, w_n)$. Then (37) holds on account of the continuity of w, and (35) holds with $C = \max_{\xi \in [0,1]}\{w(\xi)\}$ which is finite by the continuity of w. Thus $\{\tau_n\}$ is a w-pre-chaotic sequence.

Now fix $\alpha \in {]0, 2[}$, and a sequence $\alpha_n \subset [0, 2]$ with $\alpha_n \to \alpha$. Let $\hat{\tau}_n$ denote the probability measure on S_{n,ϵ_n} obtained by pushing forward τ_n under the map specified in Lemma 2.2 at the value α_n, and then averaging over permutations. By Theorem 3.5, $\{\hat{\tau}_n\}$ is (α, g) chaotic where

$$
g(x) = \frac{1}{\phi'(\phi^{-1}(x))} \quad \text{and} \quad \phi(\xi) = (1 - \alpha/2)\int_0^\xi \frac{w(t)}{1-t}\,dt + \alpha\xi. \tag{71}
$$

Provided $\int_0^1 (1-t)^{-1}w(t)\,dt = \infty$, ϕ increases strictly from 0 at $\xi = 0$ to ∞ at $\xi = 1$, and in fact, $\phi'(\xi) \geq \alpha$ for all ξ.

As before, let ν_n denote the pushforward of the uniform probability measure on $[0, 1]$ under ϕ_n, and let ν be defined in the same way in terms of ϕ, so that $\nu = g(\xi)\,d\xi$ where $g(\xi) = 1/\phi'(\phi^{-1}(\xi))$. Thus we have:

Theorem 3.10. *Let w be a continuous probability density on $[0, 1]$. For each n and each $1 \leq j \leq n$, define $w_j = \int_{(j-1)/n}^{j/n} w(\xi)\,d\xi$. Let the energies E_n be chosen so that $\alpha_n \to \alpha$. Equip S_{n,ϵ_n} with the probability measure that is the pushforward under T_n (see Lemma 2.2) of the Dirichlet measure specified in (69) using these weights, averaged under permutations. Then this sequence is (α, g)-chaotic where g is given by (71).*

However, The chaotic sequences obtained in this manner do not satisfy detailed chaos. In this construction, for each n and $x_{j(n)}$, Each $\tilde{z}_{j(x)}$ has a Beta$(nw_{j(x)}, n - nw_{j(x)})$ distribution, and hence the probability density for $(n-1)\tilde{z}_{j(x)}$ converges to a non-exponential Gamma distribution unless $w_j \to 1/n$. To obtain sequence that satisfies detailed chaos, we must pushforward a different class of pre-chaotic measures on the standard simplices. In the next subsection, we describe one way of doing this.

Remark 3.11. If the concentration parameters in the Dirichlet distribution are multiplied by a common factor K, and hence equation (69) is replaced by

$$(72) \qquad h_w^K(\tilde{z}_1, \dots \tilde{z}_n) = \left(\prod_{j=1}^n \Gamma(Knw_j) \right)^{-1} \Gamma(Kn) \prod_{j=1}^n \tilde{z}_j^{Knw_j - 1},$$

we still have a Dirichlet distribution with the same expected values but with the variance covariance multiplied by the factor $1/K$. Equation (70) becomes

$$(73) \qquad \begin{aligned} \mathbb{E}[\tilde{z}_j] &= w_j, \\ \mathrm{Var}[\tilde{z}_i] &= \frac{w_j(1 - w_j)}{K(n+1)}, \\ \mathrm{Cov}[\tilde{z}_i, \tilde{z}_j] &= -\frac{w_i w_j}{K(n+1)}. \end{aligned}$$

Therefore all estimates leading to the proof of Theorem 3.10 are still valid, and therefore these measures are (α, g)-chaotic as well. But increasing K implies that the random variables $\tilde{z}_i$ become more concentrated around their mean w_i. And while this does not change the limiting density g, it changes the gap distribution for all finite n, and we will see in Section 4 that this is a fundamental difference for the limiting dynamics of the particle system.

3.4. *Detailed chaoticity via order statistics*

The construction that we now give uses another probability density $h(\eta)$ on $[0, 1]$ that has to do with the excess energy distribution.

The distribution of the random points $x_j \in \mathbb{R}_+$ is determined by the distribution of empty intervals $]0, x_1[, \ ]x_1 + \epsilon, x_2[$, or equivalently, as we have seen in Lemma 2.2, by the random variables z_j in (26). These specify a random partition

$$\big\{ [0, a_1], (a_1, a_2], \dots, (a_{n-1}, 1] \big\}$$

of $[0, 1]$ into n parts with $a_j - a_{j-1} = \tilde{z}_j$ and $a_0 = 0$. This random partition is closely related to a partition of the excess energy. Recall that the fraction of the total energy that is excess energy is $(1 - \alpha/2)$. Given a probability density $g(x)$ on $\mathbb{R}_+$ that satisfies (41), $(1 - \alpha g(x))$ represents the probability that the interval $[x, x + dx]$ is unoccupied. Opening up a gap in $[x, x + dx]$ would raise the energy of all the particles with energy higher than x by dx. Thus this would make a contribution of

$$\big(1 - \alpha g(x)\big)\big(1 - G(x)\big)$$

to the total excess energy, where G is defined as in Theorem 3.6. Therefore, the fraction of the excess energy that can be ascribed to gaps in $[x, x + dx]$ is

$$(74) \qquad h(x)\, dx = \frac{1}{1 - \alpha/2} \big(1 - \alpha g(x)\big)\big(1 - G(x)\big)\, dx.$$

One readily checks that $h(x)$ is indeed a probability density. Let $H(x)$ denote its cumulative distribution function. Out of G and H we define two maps from $[0, 1]$ to $[0, 1]$, namely $G \circ H^{-1}$ and $H \circ G^{-1}$. We may use these two maps to pushforward the uniform distribution on $[0, 1]$ onto $[0, 1]$ itself, producing two new probability measures on $[0, 1]$.

Define

$$(75) \qquad \psi(\eta) = \frac{g(H^{-1}(\eta))}{h(H^{-1}(\eta))},$$

and note that the cumulative distribution function of ψ is $\Psi(\eta) = G(H^{-1}(\eta))$. Likewise define

$$(76) \qquad w(\xi) = \frac{h(G^{-1}(\xi))}{g(G^{-1}(\xi))}$$

and note that the cumulative distribution function of w is $H(G^{-1}(\eta))$. Also, note that

$$(77) \qquad w(\xi) = \frac{1}{\psi(\Psi^{-1}(\eta))}.$$

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From (75), and the definition of h in terms of g,

$$(78) \qquad \psi(\eta) = \frac{1}{1 - \alpha/2}\left(\frac{1}{g(G^{-1}(\Psi(\eta)))} - \alpha\right)(1 - \Psi(\eta)).$$

When there is an $\epsilon > 0$ such that $g(x) \leq (\alpha + \epsilon)^{-1}$,

$$(79) \qquad \psi(\eta) \geq \frac{\epsilon}{1 - \alpha/2}(1 - \Psi(\eta)),$$

and this provides a lower bound on ψ on any interval $[0, \eta_*]$, for any $\eta_* < 1$.

Likewise, we may recover $w(\xi)$ as in equation (42) from the formula for h. When there is an $\epsilon > 0$ so that $1/g(x) \geq \alpha + \epsilon$ for all x,

$$w(x) \geq \epsilon(1 - \xi),$$

which provides a uniform lower bound on $w(\xi)$ on $[0, \xi_*]$ for any $\xi_* < 1$. We see that w is the probability density on $[0, 1]$ associated to g though Theorem 3.6.

Let Φ_j, $j = 1, 2$ be the cumulative distribution functions of two strictly positive probability densities ϕ_j, $j = 1, 2$ respectively, on intervals $[a_j, b_j]$, $j = 1, 2$. Then $\Phi_1^{-1} \circ \Phi_2 : [a_2, b_2] \to [a_1, b_1]$, and as is readily checked, $\Phi_1^{-1} \circ \Phi_2$ pushes $\phi_2(x)\,dx$ onto $\phi_1(x)\,dx$. In particular, if $\phi_2(x)\,dx$ is the uniform distribution on $[0, 1]$; i.e.; $\Phi_2(x) = x$ for all $x \in [0, 1]$, Φ_1^{-1} pushes forward the uniform distribution on $[0, 1]$ onto $\phi_1(x)\,dx$ on $[a_1, b_1]$. That is, for all continuous functions χ on $[a_1, b_1]$,

$$\int_{a_1}^{b_1} \chi(x)\phi_1(x)\,dx = \int_0^1 \chi(\Phi_1^{-1}(y))\,dy.$$

In particular if ξ is a random variable that is uniformly distributed on $[0, 1]$, $\Phi_1^{-1}(\xi)$ is a random variable with the law $\phi_1(x)\,dx$. Therefore, if $\xi_1, \ldots, \xi_{n-1}$ are the order statistics of $n - 1$ i.i.d uniformly distributed random variables, $\Phi_1^{-1}(\xi_1)$, $\ldots$, $\Phi_1^{-1}(\xi_{n-1})$ are the order statistics of $n - 1$ independent samples from the law $\phi_1(x)\,dx$.

Lemma 3.12. *Let $F(\eta)$ be the cumulative distribution function of a continuous probability density $f(\eta)$ on $[0, 1]$ that is uniformly positive on $[0, \eta_*]$ for all $0 < \eta_* < 1$; i.e., for some $a > 0$, $f(\eta) \geq a$ for $0 \leq \eta \leq \eta_*$. Let $\xi_1, \ldots, \xi_{n-1}$ be the order statistics of $n - 1$ i.i.d uniformly distributed random variables, and set $\xi_0 = 0$, $\xi_n = 1$ and $\lambda_j = \xi_j - \xi_{j-1}$. Moreover let $\eta_j = F^{-1}(\xi_j)$ and set $\tilde{z}_j = \eta_j - \eta_{j-1}$. Then for $\eta_j, \eta_k \leq \eta_*$*

$$(80) \qquad \mathbb{E}[\tilde{z}_j] = \mathbb{E}\left[\frac{\lambda_j}{f(\eta_j)}\right] + \frac{1}{n}r_1,$$

$$(81) \qquad \mathrm{Var}[\tilde{z}_j] = \mathbb{E}\left[\left(\frac{\lambda_j}{f(\eta_j)}\right)^2\right] + \frac{1}{n^2}r_2,$$

$$(82) \qquad \mathrm{Cov}[\tilde{z}_j, \tilde{z}_k] = \frac{1}{n^2}r_3,$$

where r_1, r_2, and r_3 converge to zero as $n \to \infty$ with a rate depending on the modulus of continuity of f and the lower bound on f on $[0, \eta_*]$. If f is Lipschitz continuous and uniformly bounded below on all of $[0, 1]$, then $|r_i| < C/n$, with C depending on f.

Proof. Related results for order statistics can be found e.g. in [1]. The present result is preciely adapted for our situation. We know that $\lambda_j = \xi_j - \xi_{j-1}$ are distributed as a flat Dirichlet distribution, and hence that $\mathbb{E}[\lambda_i] = 1/n$, and that $\mathrm{Var}[\lambda_j] = 1/n^2 + \mathcal{O}(1/n^3)$ and $\mathrm{Cov}[\lambda_j, \lambda_k] = -1/n^3 + \mathcal{O}(1/n^4)$.

First, since $(F^{-1})'(t) = \frac{1}{f(F^{-1}(t))}$,

$$(83) \qquad \tilde{z}_j = \int_{\xi_{j-1}}^{\xi_j} \frac{1}{f(F^{-1}(s))}\,ds = \lambda_j \frac{1}{f(\eta_j)} + u_j,$$

where

$$(84) \qquad u_j := \int_{\xi_{j-1}}^{\xi_j} \left(\frac{1}{f(F^{-1}(s))} - \frac{1}{f(F^{-1}(\xi_j))}\right)ds.$$

If $f(s)$ is Lipschitz continuous and bounded below by $a > 0$ for $F^{-1}(\xi_j) \le \eta_*$, then

$$\left| \frac{1}{f(F^{-1}(s))} - \frac{1}{f(F^{-1}(\xi_j))} \right| \le \frac{\lambda_j^2}{a^2}.$$

Therefore $\mathbb{E}[u_j] \le C\mathbb{E}[\lambda_j^2] \le \frac{1}{a^2 n^2}$. Otherwise, if f is only continuous, there is a function $\omega(\delta) > 0$, the modulus of continuity, with $\lim_{\delta \to 0} \omega(\delta) = 0$ such that

$$\sup_{|t_1 - t_2| < \delta} \left| f(t_1) - f(t_2) \right| \le \omega(\delta).$$

Then, for δ small enough, and $\eta_j \le \eta_*$ and $f \ge a > 0$ on $[0, \eta_*]$,

$$\mathbb{E}\big[|u_j|\big] = \mathbb{E}\big[|u_j| \mathbb{1}_{\lambda_j < \delta}\big] + \mathbb{E}\big[|u_j| \mathbb{1}_{\lambda_j \ge \delta}\big]$$

$$\le \mathbb{E}\big[\lambda_j \omega(\delta) \mathbb{1}_{\lambda_j < \delta}\big] + \frac{1}{a} \mathbb{P}[\lambda_j > \delta]$$

$$\le \frac{1}{n} \omega(\delta) + \frac{1}{a} \frac{\mathbb{E}[\lambda_j^2]}{\delta^2}$$

$$\le \frac{1}{n} \omega(\delta) + \frac{1}{a} \frac{1}{n^2 \delta^2}.$$

Choosing $\delta = n^{-1/3}$, we find,

$$r_1 \le \frac{1}{n}\left(\omega(n^{-1/3}) + \frac{1}{a} n^{-1/3} \right).$$

The proof of (81), which we omit, is very similar. To estimate the covariance we write

(85) $$\text{Cov}[\tilde{z}_j, \tilde{z}_k] = \mathbb{E}[\tilde{z}_j \tilde{z}_k] - \mathbb{E}[\tilde{z}_j]\mathbb{E}[\tilde{z}_k],$$

and

(86)
$$\tilde{z}_j \tilde{z}_k = \int_{\xi_{j-1}}^{\xi_j} \int_{\xi_{k-1}}^{\xi_k} \frac{1}{f(F^{-1}(s))} \frac{1}{f(F^{-1}(t))} \, ds \, dt$$

$$= \lambda_j \lambda_k \frac{1}{f(\eta_j) f(\eta_k)}$$

$$+ \int_{\xi_{j-1}}^{\xi_j} \left(\frac{1}{f(F^{-1}(s))} - \frac{1}{f(\eta_j)} \right) ds \int_{\xi_{k-1}}^{\xi_k} \frac{1}{f(F^{-1}(t))} \, dt$$

$$+ \lambda_j \frac{1}{f(\eta_j)} \int_{\xi_{k-1}}^{\xi_k} \left(\frac{1}{f(F^{-1}(t))} - \frac{1}{f(\eta_k)} \right) dt.$$

For the first term we note that $\mathbb{E}[\lambda_j \lambda_k] = n^{-2} + \text{Cov}[\lambda_j, \lambda_k]$, and by estimates like the ones used to estimate r_1, we find that the remaining terms are $o(n^{-2})$, or even $\mathcal{O}(n^{-3})$ if $f(s)$ is Lipschitz. Hence computing the covariance in (85) by taking the expectation of (86) and using (80) yields (82). □

For the following theorem, recall the constraints for g as stated in equation (41), and the definition of h in (74), and that G and H are their cumulative distribution functions.

Theorem 3.13. *Let $\psi(\eta)$ be defined by* (75), *and let η_j, $j = 1, \ldots, n - 1$ be the order statistics of $n - 1$ i.i.d. random variables with distribution $\psi(\eta) \, d\eta$, and let $\eta_0 = 0$, $\eta_n = 1$. With $\tilde{z}_i = \eta_i - \eta_{i-1}$, $i = 1, \ldots, n$ this induces a measure on the standard $n - 1$ dimensional simplex S_1, whose pushforward to S_{n,ϵ_n} is (α, g)-chaotic. Let $x > 0$, and let $]x_{(j)}, x_{(j+1)}[$ be the random interval that contains x. If the density $\psi(\eta)$ is continuous, then the gap length $\zeta_{x,n} = x_{(j+1)} - x_{(j)} - \alpha/(n-1)$ satisfies*

(87) $$\lim_{n \to \infty} \mathbb{P}\big[(n-1)\zeta_{x,n}/\alpha > r\big] \to e^{-\frac{\alpha g(x)}{1 - \alpha g(x)} r}.$$
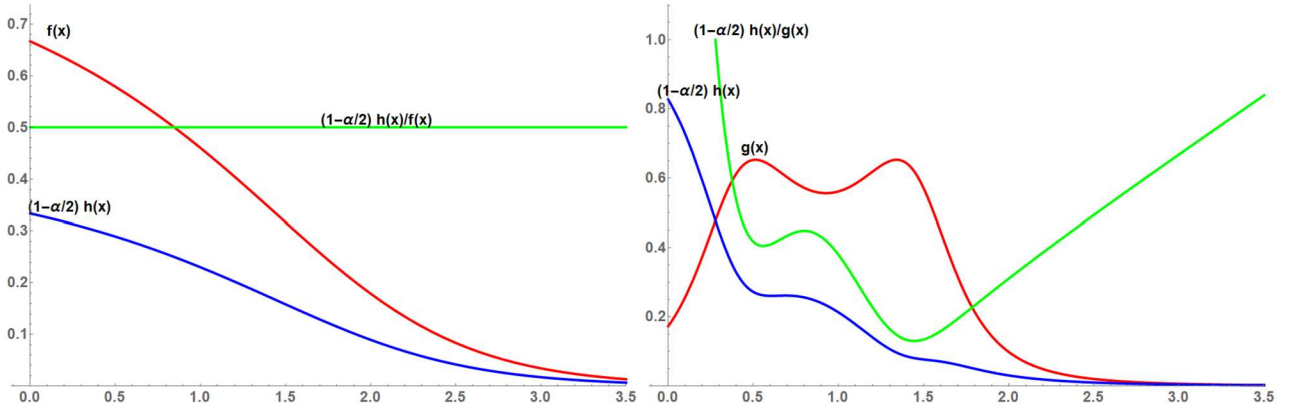
Fig. 2. The graphs show the particle density $g(x)$ (red), the corresponding excess energy distribution $h(x)$ corresponding to $\alpha = 1$ (blue), and the excess energy per particle, $h(x)/g(x)$ (green). To the left, the density is the equilibrium density $f_\alpha(x)$ as derived in Section 2, and to the right $g(x) = c_1/((1 + (c_2 x - 1)^2)(1 + (c_2 x - 4)^2))$ with the constants c_1 and c_2 chosen to make $g(x)$ a probability density with mean 1. We see that the density $f_\alpha(x)$ is equivalent to distributing the excess energy uniformly among the particles.

Proof. By Lemma 3.12 applied with $f = \psi$, together with (77), which implies that

$$\frac{1}{\psi(\eta_j)} = w(\xi_j).$$

This permits us to rewrite (80), (81) and (82) in terms of w, we see that the sequence of laws of $(\tilde z_1, \ldots, \tilde z_n)$, averaged over permutations, are w pre-chaotic family on S_1. Here w is given as in (42). Then by Theorem 3.5, if we fix a sequence of energies $\{E_n\}$ with $\alpha_n = \epsilon n(n-1)/E_n \to \alpha$, and define the maps T_n in terms of α_n as in Lemma (22), the sequence of their pushforwards onto S_{n,ϵ_n}, averaged over permutations, is (α, g)-chaotic. The statement about the gap distributions then follows from (83) with $f(\eta) = \frac{g(H^{-1}(\eta))}{h(H^{-1}(\eta))}$ so that

$$f(\eta_j) = \frac{g(G^{-1}(\xi_j))}{h(G^{-1}(\xi_j))}. \qquad \square$$

It follows from (76) that $w(\xi)\,d\xi$ results from pushing the excess energy distribution forward onto $[0, 1]$ using the distribution function G. That is $h(x) = w(G(x))g(x)$. In equilibrium, this excess energy density is uniform; i.e., $w(\xi) = 1$, and the approach to equilibrium for our process can be thought of as the approach of the excess energy distribution to uniform. This is illustrated in Figure 2, which shows the cumulative excess energy for a couple of different densities $g(x)$, and the excess energy per particle as a function of the position x of a particle, $(1 - \alpha g(x))G(x)/g(x)$.

Lemma 3.12 and Theorem 3.13 provide a means of sampling the empirical distributions μ_n: Let $\xi_{(i)}, \ldots, \xi_{(n-1)}$ be the order statistics of $n-1$ independent samples from the uniform distribution on $[0, 1]$ and then form $\eta_1, \ldots, \eta_n$ through $\eta_j = \Psi^{-1}(\xi_{(j)})$, which then gives us the order statistics of $n-1$ independent samples from $\psi(\eta)\,d\eta$. Then with $\eta_0 = 0$ and $\eta_n = 1$, we define $\tilde z_j = \eta_j - \eta_{j-1}$, from which we recover a sample of $(x_1, \ldots, x_n)$. We illustrate this way of sampling the empirical distribution in Figure 3, where g is as in Figure 2, but with $\alpha = 1.5$. Here the density is close to the maximal density $2/3$, which leads to slow convergence of the empirical measures.

Remark 3.14. The construction of the uniform measure on the simplex S_1 by independent i.i.d uniform random variables on the unit interval is not new here, and can be found for example in [17], where the authors study the asymptotics of partitions of a number n into a sum of m integers.

Remark 3.15. This construction also illustrates the difference between (α, g)-chaoticity and detailed (α, g)-chaoticity, and why detailed (α, g) chaos is difficult to express in terms of marginal distributions in the way Kac defined chaos (see equation (4)). Assuming that g is continuous as before, we have a one to one map between a point $(x_1, \ldots, x_n) \in S^*_{E_n, \epsilon}$ and points $\xi_j = \Psi(\eta_j)$ where $\eta_j = \sum_{i=1}^j \tilde z_i \in [0, 1]$, with $\eta_n = 1$. After symmetrization the pushforward of a measure $\sigma_{n,\epsilon}$ on $S_{E_n, \epsilon}$ by this map gives rise to a symmetric measure λ_n on $[0, 1]^n$, and one could compute the marginal distributions of σ_n and λ_n and see that if the sequence $\{\sigma_n\}$ is (α, g) chaotic, then $\{\lambda_n\}$ is chaotic with respect to the uniform measure on $[0, 1]$, and the other way around, and the fact that (α, g)-chaoticity of the sequence $\{\sigma_{n,\epsilon}\}$ corresponds to the usual notion of chaos $\{\lambda_n\}$ is encoded in the maps T_n. Detailed chaos can be expressed as saying that at the scale $1/n$, the points x_j
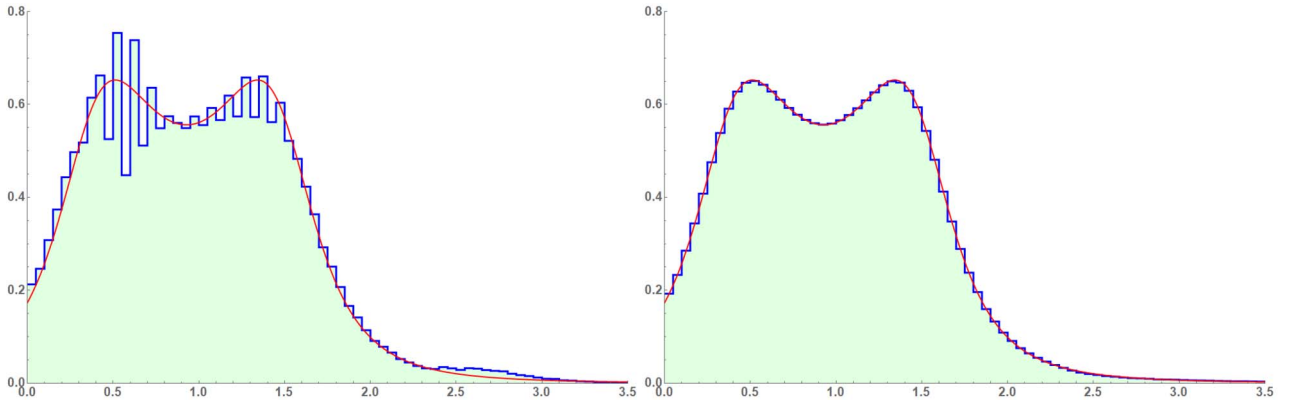
Fig. 3. The graphs show the particle density $g(x)$ (thin, red), and empirical histograms of samples (blue, thick) as defined in Theorem 3.13. To the left $n = 50$, and to the right $n = 1000$. The number of independent samples is 5000 and the bin width in the histograms is 0.05.

after symmetrization behave as a Poisson point process: take any point $\bar{\xi} \in \,]0, 1[$, and $a, b > 0$. Then number of points $\xi_j \in \,]\bar{\xi} - a/n, \bar{\xi} + b/n[$ will converge to a Poisson distribution with parameter $b - a$ when $n \to \infty$. Hence detailed chaos says in a sense that the set of points $x_1, \dots, x_n$ in the limit are as random as possible, given the constraint that their laws $\sigma_{n,\epsilon}$ are (α, g)-chaotic.

4. The Kac process

4.1. *Specification of the Master Equation*

Kac's result [9] for the original Kac model is that propagation of chaos as described in the introduction is sufficient to identify an evolution equation for the limiting densities, the Kac–Boltzmann equation. When the jumps are constrained by the exclusion principle the situation is more subtle, and propagation of chaos according to the definition 1.1 is not enough to identify a limiting equation. In this section we will present the Kac-process, and derive a limiting Kac–Boltzmann equation that is valid under the assumption of chaos according to Definition 1.2, with an exponential gap distribution as in Theorem 3.13.

The jump process is then as follows: With $x = (x_1, \dots, x_n) \in S_{n,\epsilon_n}$,

(1) pick a random waiting time t, exponentially distributed with rate n
(2) pick $1 \le j < k \le n$ uniformly among possible pairs, and let $\bar{x}_{j,k} = \frac{x_j + x_k}{2}$.
(3) Let $(x_j^*, x_k^*) = (\bar{x}_{j,k}(1 - \xi), \bar{x}_{j,k}(1 + \xi))$, where ξ is chosen uniformly in the $[-1, 1]$.
(4) If $(x_1, \dots, x_j^*, \dots, x_k^*, \dots, x_n) \in S_{n,\epsilon_n}$, then let $x^* = (x_1, \dots, x_j^*, \dots, x_k^*, \dots, x_n)$, else do nothing, i.e. let $x^* = x$

Note that the distribution of two particle energies after a collision would be exactly the same if the step (3) were replaced by

(3b) Pick ξ randomly from $[-1, 1]$, and let

$$x_j^* = x_j + \xi \bar{x}_{j,k} \quad (\mathrm{mod}\, x_j + x_k),$$
$$x_k^* = x_k - \xi \bar{x}_{j,k} \quad (\mathrm{mod}\, x_j + x_k),$$

where mod here simply means that if $x_j + \xi \bar{x}_{j,k} > x_j + x_k$, then $(x_j + x_k)$ is added or subtracted to map x_j^* back into the interval $0 \le x_j^* < (x_j + x_k)$.

This collision process is reversible for any fixed ξ, and can also be naturally generalized to collision models which favor small energy exchanges in the collision, or for "grazing collision limits", which are interesting in the classical setting.

However, for the purpose of writing down the generator of the process, the version as originally described is simplest. Let $\mathcal{L}$ denote the generator. Then for any continuous function F on S_{n,ϵ_n}

$$(88) \qquad \mathcal{L}F(x) = \frac{1}{n-1} \sum_{j<k} \int_{-1}^{1} 1_{S_{n,\epsilon_n}}\left(x_{i,k,\xi}^*\right)\left[F\left(x_{j,k,\xi}^*\right) - F(x)\right] \mathrm{d}\xi,$$

where $x = (x_1, \ldots, x_n)$, and $x_{j,k,\xi}^* = (x_1, \ldots, \bar{x}_{j,k}(1 - \xi), \ldots, \bar{x}_{j,k}(1 + \xi), \ldots, x_n)$. Recall that the process satisfies detailed balance, and is reversible, so that the operator $\mathcal{L}$ on the L^2 space given by the invariant measure is self-adjoint. The Kolmogorov forward equation, or what is the same thing, the Master Equation, of the process is then

$$(89) \qquad \frac{\partial}{\partial t} F(x, t) = \mathcal{L} F(x, t).$$

Let $\mathcal{P}_t$ denote the semigroup associated to (89), so that if $F(x, t)$ denotes the solutions with initial data $F(x, 0)$, $F(x, t) = \mathcal{P}_t F(x, 0)$.

4.2. *The exclusion factor*

To compute exactly the probability that the outcome from step (3) results in a jump as defined in (4) is difficult, but it is possible to derive formula for the limit as $n \to \infty$ under the assumption that the limiting gap distribution is known and that the events that x_j^* and x_k^* are admissible positions for particles are independent. We also assume here that the density $g(x)$ is continuous.

First, to see why propagation of chaos in the sense of Kac is not enough to identify a limiting equation we compare two different chaotic sequences that are (α, f_α)-chaotic, where f_α is the equilibrium density as found in Theorem 2.1. We take the empirical measures with the x_j defined as in (22), and $\alpha_n = \alpha$ for simplicity. On the other hand taking $\tilde{z}_j = 1/n$, for $j = 1, \ldots, n$ provides another chaotic sequence. In this latter sequence the gaps between particles are deterministic, $x_{j+1} - x_j = (1 - \alpha/2)/(n - j)$, and this means that to fit a new particle of size $\alpha/(n - 1)$ into an interval we must have $\frac{j}{n} > 1 - \frac{2-\alpha}{2\alpha}$, which is positive when $\alpha > 2/3$, and therefore for all x smaller than

$$x_j \geq \left(1 - \frac{\alpha}{2}\right) \sum_{k=1}^{\lfloor 1 - \frac{2-\alpha}{2\alpha} \rfloor} \frac{1}{n + 1 - k} + \frac{3 - \alpha}{2} - \frac{\alpha}{n}$$

which converges to

$$\bar{x}_\alpha = \log\left(\frac{2\alpha}{2 - \alpha}\right) + \frac{3\alpha - 2}{2},$$

when $n \to \infty$. So if $\alpha > 2/3$ this (α, f_α)-chaotic sequence does not allow any jump into an interval $[0, \bar{x}_\alpha]$. On the other hand, for the sequence constructed in Section 2, where the $(\tilde{z}_1, \ldots, \tilde{z}_n)$ taken from the flat Dirichlet distribution, the $\tilde{z}_j$ are close to being exponentially distributed with mean $1/n$. Hence for all j there is a positive probability that the j-th gap is bigger than $\alpha/(n - 1)$, and therefore jumps are possible to any point in the interval $[0, \infty[$, although the probability will be very small in intervals near the origin if α is large.

In the following calculation we neglect the probability that x_j^* belongs to one of the gaps created when x_j and x_k are lifted out, i.e. when the particles fall back into nearly the same point as where they started. The probability that this happens converges to zero at the order $1/n$, and can be neglected unless the excess energy is very small. Also in this case, the effect of such a jump on the density will be very small, and therefore to see an effect of this one would need to consider the process over a very long time scale. It could be interesting to study this situation in a diffusive scaling, and to analyze as certain models for competing particle systems and rank based interacting diffusions [11,13,14].

For any x, consider an interval $[x - \delta/2, x + \delta/2]$, where δ is assumed to be small and eventually converging to 0. We will call a point x_* in this interval admissible if it satisfies the exclusion constraint, given the particles that are already present in the interval. The expected number of points x_j belonging to this interval will be $m + 1 = n \int_{x-\delta/2}^{x+\delta/2} g(y) \, dy \sim n\delta g(x)$ due to the assumption that g is continuous. Now let $x - \delta/2 < x_{(0)} < x_{(2)}, \ldots, < x_{(m)} < x + \delta/2$ be the positions of the m particles belonging to this interval, renumbered for convenience here, and let $\zeta_{j+1} = x_{(j)} - x_{(j-1)} - \alpha/(n - 1)$ be the gaps between particles. For fixed δ we have that $x_{(0)} \to x - \delta/2$ and $x_{(m)} \to x + \delta/2$ in probability, and therefore the error in considering only the interval $[x_{(0)}, x_{(m)}]$ will vanish in the limit as $n \to \infty$. For a given gap ζ_i, the interval available for putting a new particle x_* is $(\zeta_i - \alpha/(n - 1))\mathbb{1}_{\zeta_i > \alpha/(n-1)}$. In a jump, x_* is chosen uniformly over any interval, and therefore

$$(90) \qquad \mathbb{P}\big[x_* \text{ is admissible} \mid x_* \in]x_{(0)}, x_{(m)}[\big] = \frac{\frac{1}{m} \sum_{i=1}^{m} (\zeta_i - \frac{\alpha}{n-1}) \mathbb{1}_{\zeta_i > \alpha/(n-1)}}{\frac{1}{m}(x_{(m)} - x_{(0)})}$$

which holds for any particle configuration, if the interval $[x_{(0)}, x_{(m)}]$ does not contain the two particles that are selected for collision. The probability that x_* is admissible can now be computed by taking the expectation of the right hand side of equation (90) with respect to the other particles. To continue we make the following assumption:

Assumption 4.1. For $n \to \infty$, one may take $\delta = \delta_n \to 0$ such that $m \to \infty$ in probability, and such that $(n-1)\zeta_i/\alpha$ are asymptotically i.i.d with a density ρ_x.

This holds for the two constructions of chaotic sequences given in Theorem 3.10 and Theorem 3.13 if the density g is continuous. By the law of large numbers, the denominator of the right hand side of equation (90) is asymptotically $\mathbb{E}[\zeta_j] + \frac{\alpha}{n-1} \sim \frac{1}{(n-1)g(x)}$, and the enumerator is asymptotic to

$$(91) \qquad \frac{\alpha}{n-1}\mathbb{E}\left[\frac{n-1}{\alpha}\zeta_i - 1\right] = \frac{\alpha}{n-1}\int_1^\infty (s-1)\rho_x(s)\,\mathrm{d}s,$$

and therefore, for any interval not containing x_j or x_k we have

$$(92) \qquad \lim_{\delta \to 0}\lim_{n \to \infty} \mathbb{P}\big[x_* \text{ is admissible} \mid x_* \in [x - \delta/2, x + \delta/2]\big] = \alpha g(x)\int_1^\infty (s-1)\rho_x(s)\,\mathrm{d}s.$$

For example, with the chaotic sequence from Theorem 3.13, $\rho_x(s) = \frac{\alpha g(x)}{1 - \alpha g(x)}\exp(-\frac{\alpha g(x)}{1 - \alpha g(x)}s)$, and we find in the limit that

$$
\begin{aligned}
(93) \qquad \mathbb{P}[x_* \text{ is admissible}] &= \frac{\alpha^2 g(x_*)^2}{1 - \alpha g(x_*)}\int_1^\infty (s-1)e^{-\frac{\alpha g(x_*)}{1 - \alpha g(x_*)}s}\,\mathrm{d}s \\
&= \big(1 - \alpha g(x_*)\big)\exp\left(-\frac{\alpha g(x_*)}{1 - \alpha g(x_*)}\right).
\end{aligned}
$$

Here we recognize the first factor $1 - \alpha g(x_*)$ as the exclusion factor in the Uehling–Uhlenbeck equation for discrete energy levels, and the second exponential factor reflects the fact that the continuous spacing of gaps is a less efficient use of the available excess energy.

With the chaotic sequences constructed through the Dirichlet distribution as in Theorem 3.10 the distribution of the gaps are Beta-distributions, as shown in equation (73), which gives the density for the distribution of a gap in the partition of excess energy as

$$(94) \qquad \frac{\Gamma(Kn)}{\Gamma(Knw_j)\Gamma(Kn(1 - w_j))}\tilde{z}^{Knw_j - 1}(1 - \tilde{z})^{Kn(1-w_j)-1},$$

and hence, because $\zeta_j = (1 - \alpha/2))\frac{\tilde{z}_j}{1 - \frac{j-1}{n}}$ that $s = (n-1)\zeta_j/\alpha$ has density

$$(95) \qquad \rho_{x,n}(s) = cs^{Knw_j}\left(1 - \frac{s}{\lambda_j}\right)^{Kn(1-w_j)-1},$$

where c is a normalizing constant and $\lambda_j = \frac{(n-1)2\alpha}{2-\alpha}\frac{1}{1 - G(x)}$. Because $\frac{j}{n} \sim G(x)$ we have asymptotically

$$(96) \qquad nw_j = n\int_{(j-1)/n}^{j/n} w(\xi)\,\mathrm{d}x \sim w\big(G(x)\big) = \frac{2\alpha}{2-\alpha}\frac{1 - \alpha g(x)}{\alpha g(x)}\big(1 - G(x)\big) \equiv w_{g(x),G(x)},$$

and $Kn/\lambda_n \sim K\frac{2-\alpha}{2\alpha}(1 - G(x)))$. It follows that

$$(97) \qquad \rho_{x,n}(s) \to \rho_{g(x),G(x)}(s) = s^{Kw_{g(x),G(x)}-1}\exp\left(-K\frac{2-\alpha}{2\alpha}\big(1 - G(x)\big)s\right).$$

One can now obtain a formula similar to equation (93) corresponding to the density $\rho_{g(x),G(x)}$. The notable difference is that with this density the probability that a point x_* is admissible asymptotically does not only depend on the limiting density $g(x)$ but also on the cumulative distribution function $G(x)$.

Hence, when analyzing the limiting behavior of the Kac process for this n-particle system, it is important to take the gap distribution into account. We formulate this asymptotic result for the exponential gap distribution as a proposition:

Proposition 4.2. *Let $g(x)$ be a continuous probability density on $[0, \infty[$, and let $((x_1, \ldots, x_n))_{n=2}^\infty$ be a chaotic sequence constructed as in Theorem 3.13. There is a sequence $\delta_n \to 0$ such that if x_* is chosen uniformly in an interval $[x, x + \delta_n]$,*

then

(98)
$$\lim_{n\to\infty} \mathbb{P}[x_* \text{ is admissible}] = \left(1 - g(x)\right) \exp\left(-\frac{\alpha g(x)}{1 - \alpha g(x)}\right).$$

4.3. *The Boltzmann equation*

For the original Kac process, it is enough to prove propagation of chaos to identify an equation that describes the evolution of a density in the limit of infinitely many particles. Here the situation is more complicated, because the asymptotic gap distribution is important. We conjecture that the process defined here propagates chaos with exponential gap distribution, but we do not have a proof. The conjecture is supported by numerical simulations that are presented in Section 5. Under the assumption that detailed chaos is propagated, it is then possible to write down the corresponding kinetic equation, and compare this with the corresponding Kac and Uehling–Uhlenbeck equations; a formal proof would follow along the same lines as Kac's original derivation.

Theorem 4.3. *Suppose that the evolution specified by* (89) *propagates chaos in detail with parameter* α, *so that the asymptotic gap distribution is exponential as in Theorem* 3.13. *Then the limiting empirical distribution* g_t *evolves according to*

(99)
$$\frac{\partial}{\partial t} g(x, t) = Q[g](x, t),$$

where

(100)
$$Q[g](x) = \frac{1}{2} \int_0^\infty \int_{-1}^1 \big(g(x')g(y')\Pi(\alpha g(x))\Pi(\alpha g(y))$$
$$- g(x)g(y)\Pi(\alpha g(x'))\Pi(\alpha g(y'))\big)\, d\xi\, dy,$$

(101)
$$x' = (1 - \xi)(x + y)/2,$$
$$y' = (1 + \xi)(x + y)/2,$$

and

(102)
$$\Pi(u) = (1 - u)\exp\left(-\frac{u}{1 - u}\right).$$

The function $\Pi(u)$ specifies the effects of the exclusion constraint which slows down the evolution. The function is plotted here in Figure 4, together with the function $\Pi(u) = 1 - u$, which is the corresponding factor in the Uehling–Uhlenbeck equation. With the car parking analogy from the introduction, this factor quantifies how much less efficient it is to let cars park at will along a road compared to using fixed parking slots. In Kac's original paper there is no exclusion factor, and in that case $\Pi(u)$ is constant, equal to 1.
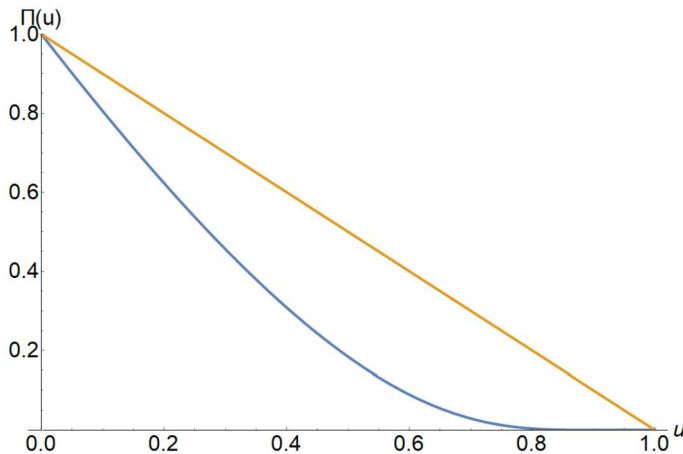

Fig. 4. The exclusion factor as a function of u (blue), compared with the fermionic factor $1 - u$.
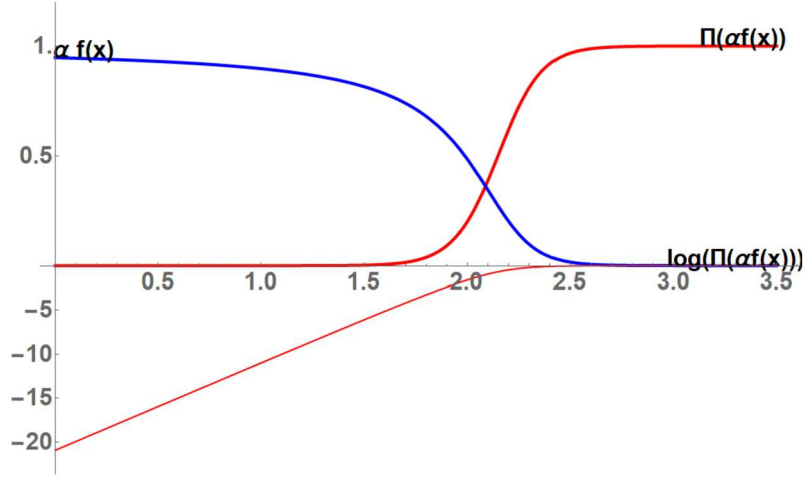
Fig. 5. The exclusion factor $\Pi(\alpha f_\alpha(x))$ for $\alpha = 1.8$, as a function of x. The thin red curve shows the logarithm of the same function, and the equilibrium distribution $f_\alpha(x)$ scaled by a factor α is given in blue.

Note that α^{-1} is the maximum density possible, and hence that $\alpha f(x) = 1$ implies that the particles are densely packed near x. The exclusion factor reduces the effective jump rate much more strongly than the usual factor $1 - u$ from the Boltzmann equation for Fermions, and is a significant difference between the continuous setting that we study here, and the discrete, quantized models. In Figure 5 we plot the function $\Pi(\alpha f(x))$, i.e. the exclusion factor evaluated at the equilibrium density as a function of the energy x, which indicates that particles will very seldom get a new energy close to $x = 0$, and therefore that the rate of convergence to equilibrium could be very low.

4.4. *Properties of the collision operator*

The collision operator $Q[g]$ as defined in equation (100) is amenable to very much the same manipulations as the ordinary collision operator for the Boltzmann equation, except that, in addition to the mass, there is only one conserved quantity, the energy.

Theorem 4.4. *Let $Q[g]$ be defined as in equation* (100). *Then the following holds*:
For any $a, b \in \mathbb{R}$, and any $g(x)$ satisfying $\int_0^\infty x g(x)\, dx = 1$

$$
(103) \qquad \int_0^\infty (a + bx) Q[g](x)\, dx = 0.
$$

Let $f_\alpha(x)$ defined by equation (18) *and* (19). *Then*

$$
(104) \qquad Q[f_\alpha](x) = 0.
$$

If $g(x, t)$ is a solution to equation (99), *then*

$$
(105) \qquad \frac{d}{dt} \int_0^\infty g(x, t) \log\left(\frac{\alpha g(x, t)}{1 - \alpha g(x, t)} \right) dx \leq 0.
$$

Proof. Let

$$
(106) \qquad R(x', y', x, y) = \big(g(x')g(y')\Pi(\alpha g(x))\Pi(\alpha g(y)) - g(x)g(y)\Pi(\alpha g(x'))\Pi(\alpha g(y'))\big).
$$

Here x' and y' depend on a parameter ξ as defined in equation (101). Formally, for any $h(x)$, a change of variables gives

$$
(107) \qquad
\begin{aligned}
&\frac{1}{2} \int_0^\infty \int_0^\infty \int_{-1}^1 R(x', y', x, y) h(x)\, d\xi\, dx\, dy \\
&\qquad = \int_0^\infty \frac{1}{u} \int_0^u \int_0^u R(z, u - z, v, u - v) h(v)\, dv\, dz\, du.
\end{aligned}
$$

We see, just as for the usual Boltzmann equation, that $R(x', y', x, y)$ is symmetric with respect to the changes $(x, y) \rightarrow (y, x)$ and anti symmetric with respect to changing $(x', y', x, y) \rightarrow (x, y, x', y')$, and therefore the righthand side of equation (107) is

$$
\begin{aligned}
(108) \qquad &\frac{1}{4} \int_0^\infty \frac{1}{u} \int_0^u \int_0^u R(z, u-z, v, u-v)\big(h(v) + h(u-v) - h(z) - h(u-z)\big)\,dv\,dz\,du \\
&= \frac{1}{8} \int_0^\infty \int_0^\infty \int_{-1}^1 R(x', y', x, y)(h(x) + h(y) - h(x') - h(y'))\,d\xi\,dx\,dy,
\end{aligned}
$$

which implies (103). To prove (104), we write $R(x', y', x, y)$ as

$$
(109) \qquad \alpha^2 g(x') g(y') g(x) g(y) \left(\frac{\Pi(\alpha g(x))}{\alpha g(x)} \frac{\Pi(\alpha g(y))}{\alpha g(y)} - \frac{\Pi(\alpha g(x'))}{\alpha g(x')} \frac{\Pi(\alpha g(y'))}{\alpha g(y')} \right).
$$

Next we take $g(x) = f_\alpha(x)$ and define

$$
(110) \qquad r(x) = \log \frac{\Pi(\alpha f_\alpha(x))}{\alpha f_\alpha(x)} = -\log(\alpha f_\alpha(x)) + \log(1 - \alpha f_\alpha(x)) - \frac{\alpha f_\alpha(x)}{1 - \alpha f_\alpha(x)}.
$$

Then

$$
\begin{aligned}
(111) \qquad r'(x) &= -f_\alpha'(x) \left(\frac{1}{f_\alpha(x)} + \frac{\alpha}{1 - \alpha f_\alpha(x)} + \frac{\alpha}{(1 - \alpha f_\alpha(x))^2} \right) \\
&= -f_\alpha'(x) \frac{1}{f_\alpha(x)(1 - \alpha f_\alpha(x))^2}.
\end{aligned}
$$

On the other hand $f_\alpha(x)$ satisfies

$$
(112) \qquad f_\alpha(x) = \frac{1}{\phi'(F(x))},
$$

where $F(x) = \int_0^x f(y)\,dy$ and

$$
(113) \qquad \phi(\xi) = \left(1 - \frac{\alpha}{2}\right) \log \frac{1}{1 - \xi} + \alpha \xi.
$$

Therefore

$$
\frac{1}{f_\alpha(x)} = \phi'(F(x)) = \left(1 - \frac{\alpha}{2}\right) \frac{1}{1 - F(x)} + \alpha \quad \text{and}
$$

$$
f_\alpha'(x) = -\frac{\phi''(F(x))}{\phi'(F(x))^2} f(x) = -\phi''(F(x)) f(x)^3,
$$

which when inserted into (111) gives

$$
(114) \qquad r'(x) = \left(1 - \frac{\alpha}{2}\right) \frac{1}{(1 - F(x))^2} \frac{1}{(\frac{1}{f_\alpha(x)} - \alpha)^2} = \left(1 - \frac{\alpha}{2}\right)^{-1}.
$$

Hence $r(x)$ is a linear function, and because the parenthesis in equation (109) is

$$
(115) \qquad \exp(r(x) + r(y)) - \exp(r(x') + r(y'))
$$

and $x + y = x' + y'$ we see that $R(x', y', x, y)$ vanishes when $g(x) = f_\alpha(x)$. Therefore not only does $Q[f_\alpha](x)$ vanish, but the whole integrand, which is to say that the collision process satisfies a detailed balance condition also after passing to the limit.

Finally, to prove (105) we write

$$
(116) \qquad \frac{\partial}{\partial t} \left(g \log \left(\frac{\alpha g}{1 - \alpha g} \right) \right) = \frac{\partial g}{\partial t} \left(\log \left(\frac{\alpha g}{1 - \alpha g} \right) + \frac{\alpha g}{1 - \alpha g} \right) = -Q[g](x) r_g(x),
$$

where $r_g(x)$ is the expression in (110) with f_α replaced by g. Using the expression in (108), we then find

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int_0^\infty g(x,t) \log\left(\frac{\alpha g(x,t)}{1-\alpha g(x,t)}\right) \mathrm{d}x
$$

$$
= -\frac{1}{8} \int_0^\infty \int_0^\infty \int_{-1}^1 R(x',y',x,y)\left(r_g(x)+r_g(y)-r_g(x')-r_g(y')\right) \mathrm{d}\xi \, \mathrm{d}x \, \mathrm{d}y
$$

$$
= -\frac{1}{8} \int_0^\infty \int_0^\infty \int_{-1}^1 g(x)g(y)g(x')g(y')\left(e^{r_g(x)+r_g(y)}-e^{r_g(x')+r_g(y')}\right)
$$

$$
\times \left(r_g(x)+r_g(y)-r_g(x')-r_g(y')\right) \mathrm{d}\xi \, \mathrm{d}x \, \mathrm{d}y \le 0,
$$

which proves (105). Therefore $\int_0^\infty g(x) \log\left(\frac{\alpha g(x)}{1-\alpha g(x)}\right) \mathrm{d}x$ is an entropy for the Boltzmann equation (100). □

5. Simulation results

We present here simulations to illustrate the results presented in the previous sections, and to provide support for the conjecture that the Kac process on S_{n,ϵ_n} propagates detailed chaos according to Definition 1.2, and moreover to investigate the long time behavior of solutions of different types.

The sampling of initial data has been done as described by Theorem 3.10 and Theorem 3.13. A very large number of random numbers have been used, and in particular for simulations with a large number of particles, it is necessary to use random numbers with high precision. We have generated random numbers with 64 bits precision, using routines from the GNU Scientific Library [7].

In order to avoid having to compute the distance between the new energy of a particle, x_j^* with the energies of all other particles, which would imply a computational cost of $\mathcal{O}(n)$ for each jump, the x_j are kept in an ordered list, which is implemented as minor modification of the AVL-tree as described by Ben Pfaff [12]. In this way the computational cost of one collision grows as $\mathcal{O}(\log(n))$.

In the first example the initial distributions are (α, f_α)-chaotic, i.e. chosen to converge to the equilibrium distribution with $\alpha = 1$. We compare sampling initial data that are equidistributed (which corresponds to taking samples as in Remark 3.11 with $K \to \infty$ in equation (72)), and a Dirichlet distributed initial data with $K = 0.02$. These are shown at $t = 0$ (Figure 6), $t = 0.1$ (Figure 7), and $t = 10.0$ (Figure 8), showing that although the initial distributions are equilibrium-chaotic, they are not at equilibrium for this jump process. Figure 9 shows the gap distribution at $t = 0$ and $t = 10$, and includes also the result when the initial distribution is a true equilibrium for this process, with asymptotically exponentially distributed energy gaps. For these cases, the gap distribution is very close to exponential at $t = 10$, which supports our conjecture that this property is propagated in time. More simulation results can be found in the supplementary material [3].
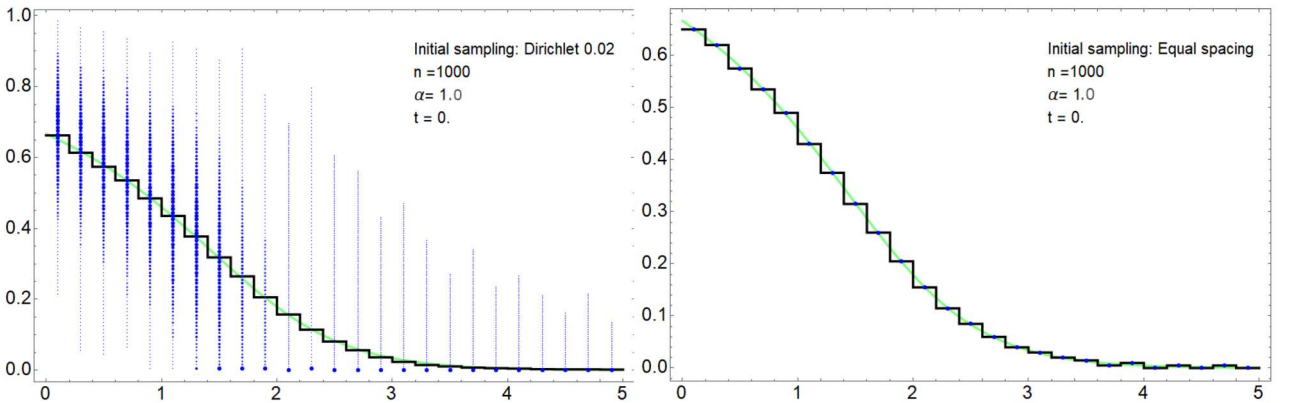

Fig. 6. The graphs show the equilibrium distribution (green) and the result from 5000 independent samples of the empirical distribution with $n = 1000$, counted in bins of width 0.2. The black step function shows the mean outcome, and the blue dots illustrate the distribution of counts in the bins, with the area of the dots proportional to the number of samples with the same count. The exclusion parameter $\alpha = 1.0$.
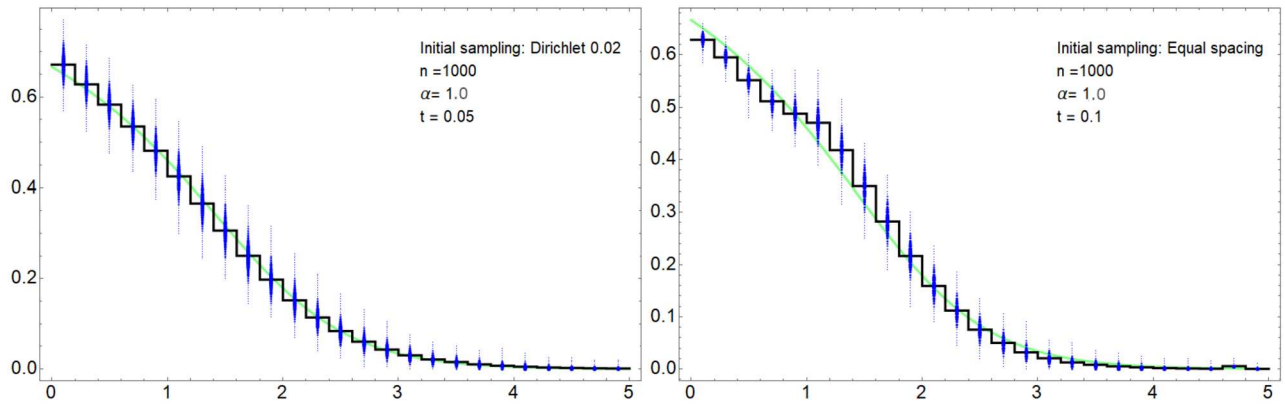
Fig. 7. The graphs show the outcome of the same simulation as in Figure 6 at time $t = 0.05$ and $t = 0.1$. This shows that although the initial data in both cases are equilibrium chaotic, the non-equilibrium state gives quite different behavior of the evolution.
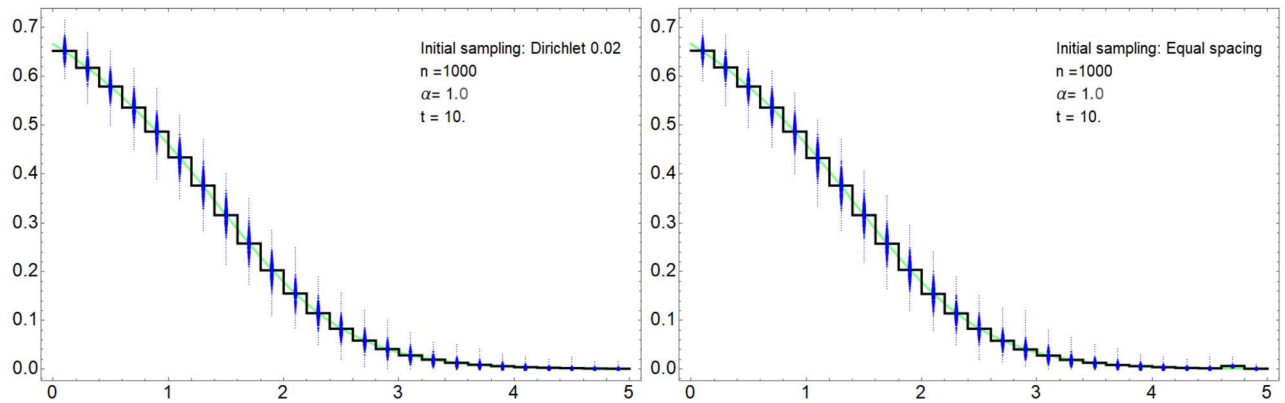

Fig. 8. These graphs represent the solutions of the same simulation as in Figure 6 at time $t = 10.0$. Here thee two simulations give the same result, a convergence to the true equilibrium.
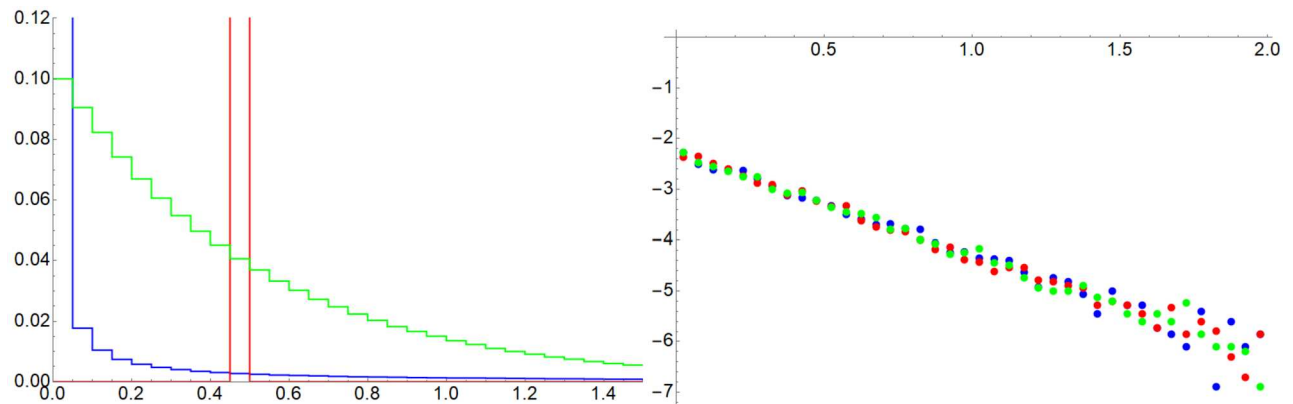

Fig. 9. The graphs show the gap distribution at time $t = 0$ (left) and $t = 2$ (right) for initial data with equal spacing of the excess energy (red) and the Dirichlet 0.01-distribution (blue). At $t = 2$ this is presented in logarithmic scale to show that the distribution becomes exponential as conjectured. The red and blue dots almost overlap here. At $t = 0$ the red curve represents a Dirac measure, and the blue curve shows that with the Dirichlet 0.01-distribution, most gaps are very close to zero, and the excess energy is essentially distributed to a few very large gaps.

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Supplementary Material

Supplement to "A Kac model with exclusion" (DOI: 10.1214/22-AIHP1276SUPP; .pdf). Additional numerical results.

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