

# Non-Invertible Symmetries from Holography and Branes

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We propose a systematic approach to deriving symmetry generators of Quantum Field Theories in holography. Central to this analysis are the Gauss law constraints in the Hamiltonian quantization of Symmetry Topological Field Theories (SymTFTs), which are obtained from supergravity. In turn, we realize the symmetry generators from world-volume theories of D-branes in holography. Our main focus is on non-invertible symmetries, which have emerged in the past year as a new type of symmetry in  $d \geq 4$  QFTs. We exemplify our proposal in the holographic confinement setup, dual to 4d  $\mathcal{N} = 1$  Super-Yang Mills. In the brane-picture, the fusion of non-invertible symmetries naturally arises from the Myers effect on D-branes. In turn, their action on line defects is modeled by the Hanany-Witten effect.

**Introduction.** The study of quantum dynamics is at the heart of uncovering any fundamental principles of nature. From various points of view, in condensed matter physics, mathematical physics and quantum field theory, such explorations have established the study of symmetries as an essential “backbone” of quantum systems. It thus comes as a genuine surprise in the past year where a dramatic extension to symmetries in 4d quantum field theories (QFTs) were uncovered, which unlike ordinary ones that form groups, obey fusion-like composition laws. These *non-invertible symmetries* are well-established in  $d = 2, 3$ , however, they are unexpected in  $d \geq 4$ . Within the past year various systematic approaches to the construction of non-invertible symmetries have appeared in [1–7]. Physical implications include characterization of de/confining vacua and constraints on pion decays [8, 9], and other applications appeared in [8–14].

All constructions thus far rely on field theory methods. Here we provide the holographic perspective from symmetry inflow, supergravity and branes. Other preliminary aspects of holography and non-invertible symmetries have been recently studied in [14–16]. Fundamental for the holographic construction is the *Symmetry Topological Field Theory (SymTFT)* [17–20], which naturally arises in brane/holographic setups from the anomaly polynomial and inflow [21–25]. The SymTFT on  $W_{d+1}$  encodes the full symmetry structure – the background fields for global symmetries and their ’t Hooft anomalies – of a QFT on  $W_d = \partial W_{d+1}$ . When placed on a slab with boundaries  $W_d$  and  $M_d$  and gapped boundary condition on  $M_d$ , the SymTFT reduces to the anomaly theory of the QFT.

In this paper we propose a holographic derivation of the SymTFT, as well as the study of the resulting symmetries – including non-invertible ones – that depend on said boundary conditions. We derive the SymTFT by descent from the anomaly polynomial in  $d + 2$  dimensions, which is encoded in the supergravity. Motivated by the work on BF-type theories in [26], the Hamiltonian quantization of the SymTFT on  $W_{d+1}$  allows us to extract the Gauss law constraints that generate gauge symmetry transformations. Under inflow, the bulk gauge symmetry

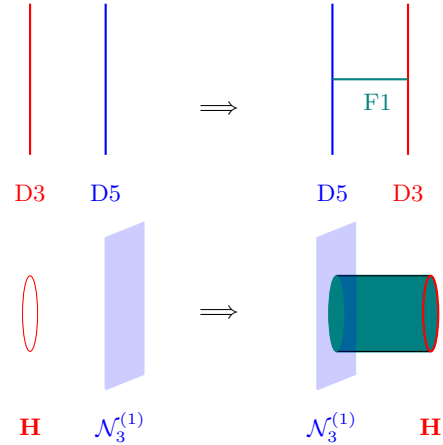


FIG. 1. Top: Hanany-Witten transition, where the  $(x_0, x_3)$ -plane is displayed to show the equivalence with the field theory transition. Bottom: ’t Hooft loop passing through the non-invertible defect  $\mathcal{N}_3^{(1)}$  becomes attached to a topological surface operator.

restricts to the global symmetries of the boundary theory and the bulk generators flow to the desired symmetry operators.

This is complemented by a realization of the symmetry generators in terms of D-branes and their worldvolume theories. The bulk supergravity fields, which define the symmetries, pull back on the brane worldvolume theories. In addition, the D-branes also contribute topological sectors that dress the symmetry defect, while the kinetic terms of the brane action drop out at the boundary. These defects become non-invertible depending on the boundary conditions for the bulk fields. The brane setup and its dynamics towards the boundary provide a compelling holographic interpretation for the non-invertible fusion via the Myers effect of  $Dp$ -branes into a single  $D(p + 2)$ -brane, which in turn implements the non-invertible fusion.

We demonstrate our proposal in the Klebanov-Strassler solution, that is dual to a flow to confining pure

$\mathcal{N} = 1$   $SU(M)$  Super-Yang Mills (SYM) [27]. Global properties of the gauge group can be identified in holography as in [28] and the study of holographic confinement using the 't Hooft anomalies of higher-form symmetries was carried out in [29]. In this paper, we determine a framework to construct all symmetries in this setup, in particular the non-invertible symmetries in the  $PSU(M) = SU(M)/\mathbb{Z}_M$  theory, which map between de/confining vacua when spontaneously broken. In the brane-picture the non-invertible fusion is naturally encoded in the Myers effect on D-branes [30]. Furthermore, the de/confining transition is beautifully modelled by the Hanany-Witten brane-transition [31], figure 1. Though we focus on holographic confinement, the methods are general and can be used to study the symmetry generators of any QFT from its SymTFT.

**Field Theory.** Non-Invertible symmetries in QFTs in spacetime dimensions  $d \geq 4$  have recently been constructed using various approaches. One is based on the presence of global symmetries that enjoy a mixed 't Hooft anomaly [1]. For concreteness we consider a 4d gauge theory (on a spin manifold) with 0-form symmetry  $\Gamma^{(0)} = \mathbb{Z}_{2M}$ , whose background field is  $A_1$ , and 1-form symmetry  $\Gamma^{(1)} = \mathbb{Z}_M$  with background field  $B_2$ . Consider the anomaly

$$\mathcal{A} = -2\pi \frac{1}{M} \int A_1 \cup \frac{\mathfrak{P}(B_2)}{2}, \quad (1)$$

where  $\mathfrak{P}$  is the Pontryagin square. This anomaly arises in 4d  $\mathcal{N} = 1$  supersymmetric Yang-Mills theories for instance, where  $\Gamma^{(0)}$  is the chiral symmetry. There is a non-anomalous  $\mathbb{Z}_2^{(0)} \subset \mathbb{Z}_{2M}^{(0)}$ .

The generalized 0- and 1-form symmetries [32] are generated by 3d and 2d topological defects  $D_3^g(M_3)$  and  $D_2^b(M_2)$ , respectively, which have group composition  $D_p^{g_1}(M_p) \otimes D_p^{g_2}(M_p) = D_p^{g_1 g_2}(M_p)$ . Due to the anomaly, the generators for  $\Gamma^{(0)}$  transform non-trivially in presence of background fields for  $\Gamma^{(1)}$

$$D_3^g(M_3) \rightarrow D_3^g(M_3) \exp\left(\int_{M_4} -\frac{2\pi i}{M} \frac{\mathfrak{P}(B_2)}{2}\right), \quad (2)$$

where  $\partial M_4 = M_3$ . Gauging the 1-form symmetry makes this defect inconsistent. The proposal in [1] is to dress the defect  $D_3^g(M_3)$  with a minimal TQFT  $\mathcal{A}^{M,p}$ , which has 1-form symmetry  $\mathbb{Z}_M$  and cancels the anomaly [33].

For  $\mathbb{Z}_M$  this is the minimal (spin) TQFT  $\mathcal{A}^{M,1} = U(1)_M$ . The dressed defects are

$$\mathcal{N}_3^{(1)} = D_3^{(1)} \otimes \mathcal{A}^{M,1}, \quad (3)$$

where the superscript labels the generator of the 0-form symmetry. This defect has non-invertible fusion [4, 8]. For  $M$  odd the TQFTs obey  $\mathcal{A}^{M,1} \otimes \mathcal{A}^{M,1} = \mathcal{A}^{M,2} \otimes \mathcal{A}^{M,2}$ . This results in the non-invertible fusion of the 3d defects in the  $PSU(M)$  theory

$$\mathcal{N}_3^{(1)} \otimes \mathcal{N}_3^{(1)} = \mathcal{A}^{M,2} \mathcal{N}_3^{(2)}. \quad (4)$$

Defining the conjugate  $\mathcal{N}_3^{(1)\dagger} = D_3^{-1} \otimes \mathcal{A}^{M,-1}$  results in

$$\mathcal{N}_3^{(1)} \otimes \mathcal{N}_3^{(1)\dagger} = \sum_{M_2 \in H_2(M_3, \mathbb{Z}_M)} \frac{(-1)^{Q(M_2)} D_2(M_2)}{|H^0(M_3, \mathbb{Z}_M)|}, \quad (5)$$

which is the condensation defect of the 1-form symmetry on  $M_3$  with  $D_2(M_2) = e^{i2\pi \int_{M_2} b_2/M}$ , where  $b_2$  is the gauge field for the 1-form symmetry. We will now turn to supergravity/branes and show how these non-invertible symmetries are naturally implemented in this framework.

**Symmetries from Holography.** We illustrate the systematic approach by realizing it in the holographic confinement setup in type IIB supergravity introduced by Klebanov-Strassler [27]. It describes the near-horizon geometry of  $N$  D3-branes probing the conifold (*i.e.* the Calabi-Yau cone over the Sasaki-Einstein 5-manifold  $T^{1,1} \sim S^3 \times S^2$ ) with  $M$  D5-branes on  $S^2 \subset T^{1,1}$ . The near-horizon geometry is  $W_5 \times T^{1,1}$ , where the 4d spacetime where the QFT lives is  $W_4 = \partial W_5$ , see (C4). We assume integral  $N/M$ , so that the duality cascade in field theory ends on 4d  $\mathcal{N} = 1$   $SU(M)$  SYM. The 5d effective action is written in terms of  $p$ -form field strengths  $f_1, \mathcal{F}_2, g_2, h_3, f_3$  with Bianchi identities

$$df_1 = 2M\mathcal{F}_2, \quad dg_2 = Mh_3, \quad d\mathcal{F}_2 = dh_3 = df_3 = 0. \quad (6)$$

We solve the Bianchi identities (6) in terms of

$$f_1 = f_1^b + dc_0 + 2MA_1, \quad \mathcal{F}_2 = \mathcal{F}_2^b + dA_1, \quad (7)$$

$$g_2 = g_2^b + d\beta_1 + Mb_2, \quad h_3 = h_3^b + db_2, \quad f_3 = f_3^b + dc_2,$$

where  $A_1, c_0, b_2, \beta_1, c_2$  are globally defined  $p$ -form gauge potentials and  $f_1^b, \mathcal{F}_2^b, g_2^b, h_3^b, f_3^b$  are closed forms with integral periods, representing topologically non-trivial base-points. From (6) it follows that  $Mh_3$  and  $2M\mathcal{F}_2$  are cohomologically trivial. Assuming  $W_5$  has no torsion,  $h_3^b = 0 = \mathcal{F}_2^b$ . The base-points  $f_1^b, g_2^b$  represent integral lifts of classes in  $H^1(W_5; \mathbb{Z}_{2M}), H^2(W_5; \mathbb{Z}_M)$  describing discrete gauge fields for a  $\mathbb{Z}_{2M}$  0-form symmetry and  $\mathbb{Z}_M$  1-form symmetry.

The relevant terms in the 5d bulk action consist of standard kinetic terms and non-trivial topological terms. The latter can be extracted [29] from the consistent truncation of [34] or via anomaly inflow [24], as reported in appendix A. In order to construct the symmetry generators, it is convenient to dualize the 0-form potential  $c_0$  into a (globally defined) 3-form gauge potential  $c_3$ . Our task, carried out in appendix A, is then to write the 5d bulk action in terms of  $A_1, b_2, \beta_1, c_2, c_3$  and the base-point fluxes  $f_1^b, g_2^b, f_3^b$ . The final action consists of standard kinetic terms and

$$S_{\text{top}} = 2\pi \int_{W_5} \left[ \frac{1}{2} N(b_2 dc_2 - c_2 db_2) + M(A_1 dc_3 + c_3 dA_1) + Nb_2 f_3^b + A_1 (g_2^b)^2 \right]. \quad (8)$$

**Symmetry Generators.** We now analyze the 5d bulk action in the Hamiltonian formalism, treating the radial

direction of  $W_5$  as Euclidean time similarly to the AdS<sub>5</sub> cases [26, 28]. Crucially, the action does not depend on the time derivatives of the time components of the gauge potentials. As a result, the associated canonical momenta are identically zero. Varying the action with respect to the time components of the gauge potentials implements the (classical) Gauss constraints. We denote the variation of the action with respect to the time component of  $A_1$  as  $\mathcal{G}_{A_1}$ , and so on. We find that  $\mathcal{G}_{\beta_1} = \tilde{\mathcal{G}}_{\beta_1}$  and

$$\begin{aligned} \mathcal{G}_{b_2} &= \tilde{\mathcal{G}}_{b_2} - Nd_4 c_2 - Nf_3^b, \quad \mathcal{G}_{c_2} = \tilde{\mathcal{G}}_{c_2} + Nd_4 b_2 \\ \mathcal{G}_{A_1} &= \tilde{\mathcal{G}}_{A_1} + 2Md_4 c_3 + (g_2^b)^2, \quad \mathcal{G}_{c_3} = \tilde{\mathcal{G}}_{c_3} + 2Md_4 A_1. \end{aligned} \quad (9)$$

Here,  $d_4$  denotes external derivative along the spatial slice  $W_4$ , tilde the kinetic term contributions, and all fields are understood as restricted to  $W_4$ . The contributions  $\tilde{\mathcal{G}}$  of the bulk kinetic terms are suppressed near the boundary [26, 28].

We provide a detailed derivation of symmetry generators from the Gauss law constraints in appendix A. For concreteness, let us illustrate the general analysis here by considering  $e^{2\pi i \int_{M_4} (2Md_4 c_3 + (g_2^b)^2)}$  as it is brought to the boundary. Our task is to define a genuine operator on a 3-cycle  $M_3$  such that, when raised to the  $2M$ -th power and with  $M_3 = \partial M_4$ , it reproduces the operator  $e^{2\pi i \int_{M_4} (2Md_4 c_3 + (g_2^b)^2)}$  from the Gauss constraint. We consider two options. In Option (i), we fix  $g_2^b$  at the boundary as a classical background. This corresponds to 4d  $\mathcal{N} = 1$   $SU(M)$  SYM, with a global electric  $\mathbb{Z}_M$  1-form symmetry coupled to a non-dynamical discrete 2-form field. The genuine operator on  $M_3$  in this case is simply the standard holonomy  $e^{2\pi i \int_{M_3} c_3}$  accompanied by the  $c$ -number phase  $e^{\frac{2\pi i}{2M} \int_{M_4} (g_2^b)^2}$ . This operator obeys group-like fusion rules. In Option (ii) we sum over  $g_2^b$  at the boundary. In field theory, we gauge the electric 1-form symmetry of 4d  $\mathcal{N} = 1$   $SU(M)$  SYM, thereby getting the  $PSU(M)$  theory. Casting the phase  $e^{\frac{2\pi i}{2M} \int_{M_4} (g_2^b)^2}$  as a genuine operator on  $M_3$  we can rewrite  $\frac{1}{2M} (g_2^b)^2$  using a 3d auxiliary theory (this is a type of inflow from the bulk operator on  $M_4$  to  $M_3$ ), which we detail in appendix A. The symmetry generator on  $M_3$  is thus

$$\mathcal{N}_3^{(1)}(M_3) = \int \mathcal{D}a e^{2\pi i \int_{M_3} (c_3 + \frac{1}{2} M a da + a g_2^b)}, \quad (10)$$

which has the non-invertible fusion rule (4).

**The far IR for  $PSU(M)$ .** The  $\mathbb{Z}_{2M}^{(0)}$  global symmetry of 4d  $\mathcal{N} = 1$   $SU(M)$  SYM is spontaneously broken to  $\mathbb{Z}_2^{(0)}$  in the far IR and the theory has  $M$  confining vacua. The mixed anomaly (1) is matched by a non-trivial 4d Symmetry Enhanced Topological Phase (SET) [32]

$$\mathcal{L}_{4d} = M\phi dc_3 + \frac{1}{2}\phi db_1 db_1 + \Lambda_2 (db_1 + Mb_2), \quad (11)$$

where  $\phi$  is a compact scalar of period 1,  $c_3$ ,  $b_2$ ,  $b_1$  are gauge potentials and  $\Lambda_2$  a Lagrange multiplier. The  $b_1$ ,  $b_2$  fields are non-dynamical. The possible VEVs  $\langle e^{2\pi i \phi} \rangle =$

$e^{2\pi i p/M}$  ( $p = 0, 1, \dots, M-1$ ) label the  $M$  vacua, while  $e^{2\pi i \int_{c_3} c_3}$  describes a domain wall between vacua. The action (11) is invariant under the gauge transformations  $b_1' = b_1 - M\lambda_1$ ,  $c_3' = c_3 + db_1 \lambda_1 - \frac{1}{2} M \lambda_1 d\lambda_1$ . Thus  $e^{2\pi i \int_{M_3} c_3}$  has a 't Hooft anomaly, consistently with the fact that the domain walls in the  $SU(M)$  theory support a 3d TQFT  $\mathcal{A}^{N,-1}$  [35].

In [29] it is demonstrated how the SET (11) emerges from the 5d bulk couplings in the IR geometry  $T^*S^3$  (deformed conifold). In contrast to the UV analysis above, the IR analysis receives contributions from both topological and kinetic terms. The Lagrange multiplier  $\Lambda_2$  is an imprint of the Stückelberg pairing between  $b_1$ ,  $b_2$  in the 5d action. The scalar  $\phi$  is identified as  $c_0/M$ .

Let us now turn to the  $PSU(M)$  theory. The far IR is still described by (11), but now  $b_1$ ,  $b_2$  are local dynamical fields. Using  $db_1 = -Mb_2$ , we see that the vacuum with  $\langle e^{2\pi i \phi} \rangle = e^{2\pi i p/M}$  exhibits a discrete 2-form gauge theory  $\int_{M_4} \frac{pM}{2} b_2^2$ . The domain walls are no longer realized as  $e^{2\pi i \int_{M_3} c_3}$ , which is not gauge invariant, but precisely by (10). Indeed, this operator raised to the  $2M$ -th power with  $M_3 = \partial M_4$  reduces to the manifestly gauge invariant quantity  $e^{2\pi i \int_{M_4} (2Md_4 c_3 + db_1 db_1)}$  (where  $g_2^b$  is locally modeled by  $db_1$ ). On the domain wall, both  $a$  and  $b_1$  are dynamical and summed over. The total 3d theory is then an Abelian CS theory with levels encoded in the matrix  $\begin{pmatrix} M & 1 \\ 1 & 0 \end{pmatrix}$ . This is a Dijkgraaf-Witten theory with gauge group  $\mathbb{Z}_1$ , hence trivial, as anticipated in [33].

**D-branes as Symmetry Generators.** The topological defects also arise as boundary limits of probe branes in the bulk that are parallel to the boundary. Both in AdS in hyper-polar coordinates and in the  $W_5$  geometry of the KS solution, where the boundary sits at  $r \rightarrow \infty$ , the tension  $T_{Dp} \sim r^p$  ( $p > 0$ ), such that the DBI part of the action is subleading and the Wess-Zumino term dominates [36]. In addition, we stress that these D5-branes are not BPS but they can be stable in the sense of [37] in  $r \rightarrow \infty$ . The topological terms for a D5-brane wrapping the  $S^3$  contain the bulk forms  $c_3$  (from  $C_6$  on  $S^3$ ) and  $b_1$  (from  $C_4$  on  $S^3$ ), as well as the  $U(1)$  gauge field  $a$  on the brane. We derive the action on the defect by reducing the D5-brane Wess-Zumino action in appendix B. The result reads

$$S_{D5} = 2\pi \int_{M_3} \left( c_3 + \frac{M}{2} a da + a db_1 \right). \quad (12)$$

Here  $b_1$  is a local gauge field. The cohomology class of  $db_1$  is identified with  $g_2^b$  and is part of the data of the  $b_2$  configuration in (7). It is interesting to understand what happens as (12) is pushed to the boundary. We always perform a path integral over  $a$ , which is a localized mode on the D5-brane. We may or may not integrate over the topologically trivial part of  $b_1$ , which is a bulk mode, depending on the boundary conditions. If we do not integrate over it, the holonomy of  $c_3$  is dressed with the non-trivial TQFT  $U(1)_M$ . If we integrate over it,

it becomes a trivial theory just as in the supergravity derivation. The D-branes therefore precisely give rise to the minimal TQFT stacking.

**Non-invertible Fusion and Myers Effect.** To see the non-invertible fusion, we can either repeat the field theory analysis, given the explicit form of (12). There is a much more elegant way to obtain the fusion directly in string theory. The fusion is computed by stacking two D5-branes, which gives rise to a non-abelian gauge theory. However, a non-abelian brane configuration with an orthogonal  $S^2$  geometry and a non-trivial B-field undergoes the Myers effect [30] in reaching the configuration with minimal energy (see appendix C). The end point configuration is given by a single D7-brane with two units of worldvolume gauge flux on  $S^2$ . We then write  $f_2 = f_2^{S^2} + da$ , with  $\int_{S^2} f_2^{S^2} = 2$ . From the expansion of the Wess-Zumino action of the D7, integrating on  $S^2$  and  $S^3$ , the terms are

$$S_{D7/2D5} = 2\pi \int_{M_3} (2c_3 + Mada + 2adb_1). \quad (13)$$

Note that this argument is applicable for any integral value of  $M$ . From the brane we thus obtain the following perspective on the fusion. Each single D5-brane results in topological defects that are dressed with  $U(1)_M = \mathcal{A}^{M,1}$  CS theories – thus string theory construction automatically results in the minimal TQFT dressing of the defects. The “brane-fusion” predicts the action (13), which is  $U(1)_{2M}$  CS-theory coupled to  $b_2$ . This is obtained also by fusing two  $U(1)_M$  theories, [38] and therefore realizes the field theory fusion rule in (4).

It is also tempting to conjecture that the fusion of  $\mathcal{N}_3^{(1)}$  with its conjugate  $\mathcal{N}_3^{(1)\dagger}$  (5) is the fusion between defects created by brane and anti-brane, with a non-trivial field configuration. This result in the condensation defect, which is the lower-dimensional brane that couples to  $db_1$ . This is in fact expected from tachyon condensation of the D-Dbar system [39], which needs to preserve the charge under  $db_1$ , and thus is expected to give rise to a non-trivial condensate. This will be discussed elsewhere, and shown to correspond to a mesh of D3-branes.

**Action on ’t Hooft lines and Hanany-Witten.** The brane perspective makes the interaction between the ’t Hooft line  $\mathbf{H}$  and the non-invertible symmetry defect,  $\mathcal{N}_3^{(1)}$ , manifest. Field-theoretically, when such a line crosses the non-invertible topological defect, a topological surface operator is created, which connects  $\mathcal{N}$  and  $\mathbf{H}$ , see figure 1. This effect can be derived by observing that  $4d \mathcal{N} = 1 PSU(M)$  SYM is self-dual under  $\tau^{-1}\sigma\tau^{-1}$ , where  $\tau$  is stacking with  $e^{2\pi i \frac{1}{M} \int \mathfrak{P}(B_2)/2}$ ,  $B_2$  is the background field for the magnetic 1-form symmetry, and  $\sigma$  is gauging this 1-form symmetry (cfr. [1] section B). We may then realize the non-invertible topological defect  $\mathcal{N}_3^{(1)}$  via a half-space gauging argument, and use

this picture to derive the aforementioned action on ’t Hooft lines, along the lines of [4, 8].

In order to see this effect in supergravity we need to define a surface operator, which extends in the radial direction,  $r$ , and ends on the boundary,  $\mathcal{O}_2(M_2)$ . The 5d bulk EOMs select a natural candidate for  $\mathcal{O}_2(M_2)$ : the  $b_2$  EOM imply

$$-k_{f_3} d * f_3 = N f_3 + f_1 g_2 - M k_{g_2} * g_2 =: M \mathcal{F}_3, \quad (14)$$

where the  $k$ ’s are constants from the kinetic terms, and  $N f_3$  and  $f_1 g_2 - M k_{g_2} * g_2$  are separately closed. The latter combination encodes the 3-form field strength of the 2-form potential dual to  $b_1$ . On shell,  $\mathcal{F}_3 = d\hat{a}_2$  for some globally defined 2-form potential. The operator  $\mathcal{O}_2(M_2)$  is identified with a Wilson surface for  $\hat{a}_2$ ,

$$\mathcal{O}_2(M_2) = e^{2\pi i \int_{M_2} \hat{a}_2}, \quad \hat{a}_2 = a_2 + \kappa c_2, \quad (15)$$

where  $\kappa = N/M$  and  $a_2$  is the dual of  $b_1$  (in type IIB  $a_2$  comes from  $C_4$  on  $S^2$ , and the duality is a consequence of the self-dual  $F_5$  flux).

This bulk picture fits with the D-brane picture, in which  $\mathcal{O}_2(M_2)$  is realized by a D1-D3-brane bound state on  $S^2 \subset T^{1,1}$ , or alternatively D3s with  $\kappa = N/M$  units of flux supported on  $S^2$  [29]. In brane engineering, a Hanany-Witten transition [31] can occur when two branes link non-trivially in spacetime and are passed through each other, thereby creating a new extended object stretching between them. In our setup this can happen for D3s wrapping  $S^2$  and extending along the radial direction  $r$  and D5s on  $S^3$  localized at the boundary:

Brane	$x_0$	$x_1$	$x_2$	$x_3$	$r$	$z_1$	$z_2$	$w_1$	$w_2$	$w_3$
D3	X				X	X	X			
D5	X	X	X					X	X	X
F1	X			X						

(16)

Here  $z_{1,2}$ ,  $w_{1,2,3}$  are local coordinates on  $S^2$  and  $S^3$ , respectively. The relevant brane linking in our system is measured by the following quantity  $L$  defined modulo  $M$ ,

$$L = \int_{M_2 \times S^3} F_5 = - \int_{M_1 \times S^2} F_3 = \int_{M_2} db_1 = - \int_{M_1} dc_0, \quad (17)$$

where  $M_2 = \mathbb{R}_{x_1} \times \mathbb{R}_{x_2}$  and  $M_1 = \mathbb{R}_r$ . On the worldvolume of  $\mathcal{N}_3^{(1)}$ , the EOM for  $a$  implies  $db_1 = -Mda$ . Thus,  $db_1$  is exact modulo  $M$ . As a result, the linking  $L$  must be conserved modulo  $M$ . When the D3 crosses the D5, this changes to  $db_1 = -Mda + \delta(\text{pt} \subset M_2)$ . The localized source is the effect of a new object (an F1-string) that is created, which intersects both  $M_2$  and  $M_3$  and extends along  $t = x_0, x_3$ , figure 1. The system in (16) is related to the original Hanany-Witten setup NS5-D5-D3 by S- and T-dualities. The D3- and D5-branes link in the direction  $x_3$ , this means that an F1 is created when the D3 crosses the D5 (16). F1 strings are indeed electrically charged under  $e^{-2\pi i \oint b_2}$ , which was precisely the dressing for  $\mathcal{O}_2(M_2)$ . This also matches the physics

of the action on the 't Hooft loop in  $PSU(M)$  through a non-invertible domain wall between de-/confining vacua, that mimicks closely the order/disorder transition in the Ising model.

**Outlook.** We provide a bottom up approach – via Gauss law constraints in supergravity – and top down one – via branes in string theory – for constructing symmetry operators in holography. Our methods are crucial for a systematic extraction of symmetry defects, whenever SymTFTs are available. It deserves further study. Future applications include theories that have similar type of mixed anomalies in the SymTFT, such as  $\mathcal{N} = 4$  SYM theories holographically dual to  $AdS_5 \times S^5$  with

non-invertible duality defects [2, 13]. A similar realization of these topological defects in terms of M5-branes at the boundary of conical in  $G_2$ -holonomy spaces is also tempting, and show similar features to (16). Finally we also briefly comment on the holographic realization of the (self-) duality and triality of non-invertible topological defects [4] for  $\mathcal{N} = 4$  SYM in  $AdS_5 \times S^5$ . Duality and Triality are all subgroup of  $SL(2, \mathbb{Z})$  symmetries, therefore is very tempting to conjecture that the topological defects in this case are engineered by 7-branes wrapping  $S^5$ . These are just example of possible applications of this approach, which we plan to come back in the future, but very importantly they show the broader scope of the holographic supergravity and brane approach, which are meant to address questions about symmetries of the QFTs living at the boundary.

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- [1] J. Kaidi, K. Ohmori, and Y. Zheng, Phys. Rev. Lett. **128**, 111601 (2022), arXiv:2111.01141 [hep-th].
- [2] Y. Choi, C. Cordova, P.-S. Hsin, H. T. Lam, and S.-H. Shao, Phys. Rev. D **105**, 125016 (2022), arXiv:2111.01139 [hep-th].
- [3] L. Bhardwaj, L. Bottini, S. Schafer-Nameki, and A. Tiwari, (2022), arXiv:2204.06564 [hep-th].
- [4] Y. Choi, C. Cordova, P.-S. Hsin, H. T. Lam, and S.-H. Shao, (2022), arXiv:2204.09025 [hep-th].
- [5] L. Bhardwaj, S. Schafer-Nameki, and J. Wu, (2022), arXiv:2208.05973 [hep-th].
- [6] L. Lin, D. Robbins, and E. Sharpe, (2022), arXiv:2208.05982 [hep-th].
- [7] T. Bartsch, M. Bullimore, A. E. V. Ferrari, and J. Pearson, (2022), arXiv:2208.05993 [hep-th].
- [8] Y. Choi, H. T. Lam, and S.-H. Shao, (2022), arXiv:2205.05086 [hep-th].
- [9] C. Cordova and K. Ohmori, (2022), arXiv:2205.06243 [hep-th].
- [10] B. Heidenreich, J. McNamara, M. Montero, M. Reece, T. Rudelius, and I. Valenzuela, JHEP **09**, 203 (2021), arXiv:2104.07036 [hep-th].
- [11] Y. Choi, H. T. Lam, and S.-H. Shao, (2022), arXiv:2208.04331 [hep-th].
- [12] V. Bashmakov, M. Del Zotto, and A. Hasan, (2022), arXiv:2206.07073 [hep-th].
- [13] J. Kaidi, G. Zafirir, and Y. Zheng, JHEP **08**, 053 (2022), arXiv:2205.01104 [hep-th].
- [14] J. A. Damia, R. Argurio, and E. Garcia-Valdecasas, (2022), arXiv:2207.02831 [hep-th].
- [15] J. A. Damia, R. Argurio, and L. Tizzano, (2022), arXiv:2206.14093 [hep-th].
- [16] F. Benini, C. Copetti, and L. Di Pietro, (2022), arXiv:2203.09537 [hep-th].
- [17] D. S. Freed and C. Teleman, Commun. Math. Phys. **326**, 459 (2014), arXiv:1212.1692 [hep-th].
- [18] D. Gaiotto and J. Kulp, JHEP **02**, 132 (2021), arXiv:2008.05960 [hep-th].
- [19] F. Apruzzi, F. Bonetti, I. n. G. Etxebarria, S. S. Hosseini, and S. Schafer-Nameki, (2021), arXiv:2112.02092 [hep-th].
- [20] F. Apruzzi, (2022), arXiv:2203.10063 [hep-th].
- [21] J. A. Harvey, R. Minasian, and G. W. Moore, JHEP **09**, 004 (1998), arXiv:hep-th/9808060.
- [22] D. Freed, J. A. Harvey, R. Minasian, and G. W. Moore, Adv. Theor. Math. Phys. **2**, 601 (1998), arXiv:hep-th/9803205.
- [23] I. Bah, F. Bonetti, R. Minasian, and E. Nardoni, JHEP **01**, 125 (2020), arXiv:1910.04166 [hep-th].
- [24] I. Bah, F. Bonetti, R. Minasian, and P. Weck, JHEP **02**, 116 (2021), arXiv:2002.10466 [hep-th].
- [25] I. Bah, F. Bonetti, and R. Minasian, JHEP **03**, 196 (2021), arXiv:2007.15003 [hep-th].
- [26] D. Belov and G. W. Moore, (2004), arXiv:hep-th/0412167.
- [27] I. R. Klebanov and M. J. Strassler, JHEP **08**, 052 (2000), arXiv:hep-th/0007191.
- [28] E. Witten, JHEP **12**, 012 (1998), arXiv:hep-th/9812012.
- [29] F. Apruzzi, M. van Beest, D. S. W. Gould, and S. Schäfer-Nameki, Phys. Rev. D **104**, 066005 (2021), arXiv:2104.12764 [hep-th].
- [30] R. C. Myers, JHEP **12**, 022 (1999), arXiv:hep-th/9910053.
- [31] A. Hanany and E. Witten, Nucl. Phys. B **492**, 152 (1997), arXiv:hep-th/9611230.
- [32] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, JHEP **02**, 172 (2015), arXiv:1412.5148 [hep-th].
- [33] P.-S. Hsin, H. T. Lam, and N. Seiberg, SciPost Phys. **6**, 039 (2019), arXiv:1812.04716 [hep-th].
- [34] D. Cassani and A. F. Faedo, Nucl. Phys. B **843**, 455 (2011), arXiv:1008.0883 [hep-th].
- [35] D. Gaiotto, A. Kapustin, Z. Komargodski, and N. Seiberg, JHEP **05**, 091 (2017), arXiv:1703.00501 [hep-th].
- [36] In addition, as suggested by [28], when compactifying on  $S^3$  the low-energy brane action, we can consider the truncation to the (topological) sector which describes flat fields only, and ignore any irrelevant 3d kinetic terms. These will only describe the dynamics of modes with non-zero flux.
- [37] D. Arean, D. E. Crooks, and A. V. Ramallo, JHEP **11**, 035 (2004), arXiv:hep-th/0408210.
- [38] Two  $U(1)_M$  theories correspond to  $\int DaDa'e^{2\pi i \int_{M_3} [\frac{M}{2}(ada+a'da')+(a+a')db_1]}$ . Define

$a_+ = a + a'$ , so that integrating out sets  $a_+ = 2a$ , [41] I. R. Klebanov and A. A. Tseytlin, Nucl. Phys. B **578**, 123 (2000), arXiv:hep-th/0002159.  
which gives precisely the brane-action (13).

[39] A. Sen, JHEP **08**, 012 (1998), arXiv:hep-th/9805170.

[40] I. n. García Etxebarria, arXiv:2208.07508 [hep-th].

**Acknowledgements.** We thank Lakshya Bhardwaj, Lea Bottini, Antoine Bourget, Inaki Garcia-Etxebarria, Zohar Komargodski, Carlos Nuñez, Ashoke Sen, Thomas Waddeleton for discussions. FB and SSN are supported by the ERC grant 682608. SSN also acknowledges support through the Simons Foundation Collaboration on “Special Holonomy in Geometry, Analysis, and Physics”, Award ID: 724073, Schafer-Nameki. IB is supported in part by NSF grant PHY-2112699 and by the Simons Collaboration on Global Categorical Symmetries. FA, IB and FB are grateful for the hospitality provided by The Lewiner Institute for Theoretical Physics at the Technion - Israel Institute of Technology. **Note.** A related paper [40] will appear at the same time and we thank the author for coordinating submission.

## Appendix A: Supergravity Analysis and (Non-Invertible) Symmetry Generators

The Klebanov-Strassler setup [27] provides a holographic realization of 4d  $\mathcal{N} = 1$  SYM theory in type IIB string theory. It consists of two main ingredients. First, a stack of  $N$  D3-branes, extending along  $\mathbb{R}^{1,3}$  and situated at the tip of the conifold, *i.e.* the non-compact Calabi-Yau metric cone over  $T^{1,1}$ , a Sasaki-Einstein 5-manifold with topology  $S^2 \times S^3$ . The near-horizon geometry at this stage is  $AdS_5 \times T^{1,1}$ , supported by  $N$  units of  $F_5$  flux. The second ingredient is a stack of  $M$  D5-branes wrapping the  $S^2$  inside  $T^{1,1}$  and extending on  $\mathbb{R}^{1,3}$ . The backreaction of the  $M$  D5-branes has two important effects: the  $AdS_5$  metric is deformed (see  $ds_{W_5}^2$  below (C4)); the  $F_5$  flux on  $S^5$  is no longer constant, see (C5). This appendix focuses on the low-energy 5d effective action on  $W_5$  obtained from reduction of the 10d couplings of type IIB supergravity on the horizon  $T^{1,1}$ , threaded by the fluxes in (C5).

*a. Bulk topological couplings from anomaly inflow and consistent truncation.* The relevant terms in the 5d bulk effective action take the form  $S = S_{\text{kin}} + S_{\text{top}}$ . In our conventions, the action enters the path integral as  $e^{iS}$ . The kinetic terms are standard, *i.e.* a sum of terms of the form  $f_1 \wedge *f_1$  and similar for the other field strengths, with constant coefficients. The topological terms  $S_{\text{top}}$  are written as an integral over a 6-manifold  $W_6$  such that  $\partial W_6 = W_5$ ,

$$S_{\text{top}} = 2\pi \int_{W_6} \left[ Nh_3 f_3 + \mathcal{F}_2 g_2^2 - f_1 g_2 h_3 - \frac{1}{4} N^2 \mathcal{F}_2^3 \right]. \quad (\text{A1})$$

These couplings can be derived via anomaly inflow in type IIB supergravity. The starting point is the 11-form [24]

$$\mathcal{I}_{11} = \frac{1}{2} \mathcal{F}_5 d\mathcal{F}_5 - \mathcal{F}_5 H_3 F_3. \quad (\text{A2})$$

This object captures both the Chern-Simons coupling in the 10d type IIB supergravity action, and the effect of the selfduality of  $F_5$ . The quantity  $\mathcal{F}_5$  is a non-closed 5-form, related to  $F_5$  as  $F_5 = (1 + *_{10})\mathcal{F}_5$ . The task at hand is to expand  $F_3$ ,  $H_3$ ,  $\mathcal{F}_5$  onto cohomologically non-trivial forms of the horizon  $T^{1,1}$ . The latter is topologically  $S^2 \times S^3$ . Its Einstein metric reads  $ds^2(T^{1,1}) = \frac{4}{9} D\psi^2 + \frac{1}{6} ds^2(S_1^2) + \frac{1}{6} ds^2(S_2^2)$ , where  $ds^2(S_i^2) = d\theta_i^2 + \sin^2 \theta_i d\phi_i$  ( $i = 1, 2$ ) are round metric on unit-radius  $S^2$ 's, the angle  $\psi$  has period  $2\pi$ , and  $D\psi = d\psi + \frac{1}{2} \cos \theta_1 d\phi_1 + \frac{1}{2} \cos \theta_2 d\phi_2$ . We turn on a Kaluza-Klein 1-form gauge field  $A_1$  associated to the isometry  $\partial_\psi$ , by means of the replacement  $\frac{d\psi}{2\pi} \rightarrow \frac{d\psi}{2\pi} + A_1$ . Let us define  $V_i = \frac{1}{4\pi} \sin \theta_i d\theta_i d\phi_i$  ( $i = 1, 2$ ). The expansion of  $H_3$ ,  $F_3$ ,  $\mathcal{F}_5$  reads

$$H_3 = h_3, \quad F_3 = M\omega_3 + f_1\omega_2 + f_3, \quad \mathcal{F}_5 = N\mathcal{V}_5 + g_2\omega_3. \quad (\text{A3})$$

The quantities  $h_3$ ,  $f_1$ ,  $f_3$ ,  $g_2$  are the field strengths of external gauge potentials. The integers  $M$ ,  $N$  capture the D5- and D3-brane charges in the Klebanov-Strassler setup. We have introduced  $\omega_3 = (V_1 - V_2) \frac{D\psi}{2\pi}$ ,  $\omega_2 = -\frac{1}{2}(V_1 - V_2)$ ,  $\mathcal{V}_5 = \omega_2\omega_3 + \frac{1}{2}\mathcal{F}_2(V_1 + V_2) \frac{D\psi}{2\pi}$ , where  $\mathcal{F}_2 = dA_1$ . The 3-form  $\omega_3$  is dual to the non-trivial 3-cycle in  $T^{1,1}$ , which can be represented by the  $S_\psi^1$  fibration over  $S_1^2$  at a generic point on  $S_2^2$ . The 2-form  $\omega_2$  is dual to the non-trivial 2-cycle in  $T^{1,1}$  and is normalized by requiring  $\int_{T^{1,1}} \omega_2\omega_3 = 1$ , which encodes the fact that the 2-cycle and 3-cycle in  $T^{1,1}$  have intersection number 1. We have  $d\omega_2 = 0$ ,  $d\omega_3 = -2\omega_2\mathcal{F}_2$ ,  $d\mathcal{V}_5 = \frac{1}{2}(V_1 + V_2)\mathcal{F}_2^2$ . In particular, the definition of  $\mathcal{V}_5$  is engineered in such a way that  $d\mathcal{V}_5$  contains no terms linear in  $\mathcal{F}_2$ . This is necessary to ensure the compatibility of (A3) with the type IIB Bianchi identities  $dF_5 = H_3 F_3$ ,  $dH_3 = 0$ ,  $dF_3 = 0$  (we have  $F_1 \equiv 0$ ). The 10d Bianchi identities also imply the Bianchi identities (6) for the external field strengths. The 6-form integrand in (A1) can now be readily reproduced by plugging (A3) into (A2) and fiber-integrating along  $T^{1,1}$ .

The topological couplings (A1) can also be inferred from the consistent truncation of [34]. With reference to the full set of topological couplings in the 5d action of [34] (and using their notation) we freeze  $C_0$  to a constant, we gauge fix the scalars  $b^J$ ,  $c^J$  to zero, we set to zero the massive 1-form fields  $b_1$ ,  $c_1$  and the massive scalar  $b^\Phi$ , we

set  $p = 0$ ,  $q = M$ ,  $k = N$ . The couplings (A1) are reproduced with the identifications  $(A_1, g_2, f_1, h_3, f_3)_{\text{here}} = (-\frac{1}{2}A, f_2^\Phi, Dc^\Phi, db_2, dc_2)_{\text{there}}$ . The term  $\mathcal{F}_2^3$  in (A1) originates from  $dA(d\tilde{a}_1^J)^2$  in [34], due to the fact that  $A + \tilde{a}_1^J$  is a massive vector. Since we are interested in massless modes, we can set effectively  $\tilde{a}_1^J = -A$ .

*b. Dualization of  $c_0$ .* After solving the Bianchi identities (6) in terms of base-point fluxes and globally defined gauge potentials, we can write the topological action (A1) as an integral of a globally defined 5-form on  $W_5$ ,

$$S_{\text{top}} = 2\pi \int_{W_5} \left[ Nb_2(dc_2 + f_3^b) + A_1(g_2^b)^2 + b_2g_2^b f_1^b + \frac{1}{2M}\tilde{f}_1(\tilde{g}_2^2 + 2\tilde{g}_2g_2^b - \frac{1}{4}N^2(dA_1)^2) + \frac{1}{2M}f_1^b\tilde{g}_2^2 \right], \quad (\text{A4})$$

where we have introduced the shorthand notation  $\tilde{f}_1 = dc_0 + 2MA_1$ ,  $\tilde{g}_2 = d\beta_1 + Mb_2$ , and suppressed wedge products for brevity. We add a Lagrange multiplier to the action, implementing the Bianchi identity for  $\tilde{f}_1$ ,

$$S_{\text{mult}} = 2\pi \int_{W_5} (2MdA_1 - d\tilde{f}_1)c_3, \quad (\text{A5})$$

where  $c_3$  is a globally defined 3-form potential. After integrating by parts the term  $d\tilde{f}_1c_3$ , the total action  $S_{\text{kin}} + S_{\text{top}} + S_{\text{mult}}$  depends algebraically on  $\tilde{f}_1$ , which can be integrated out using its classical equation of motion  $\tilde{f}_1 \propto *f_4$  (recall that  $f_1 = f_1^b + \tilde{f}_1$ ), with  $f_4$  given as

$$f_4 = dc_3 - \frac{1}{2M}(\tilde{g}_2^2 + 2\tilde{g}_2g_2^b - \frac{1}{4}N^2(dA_1)^2). \quad (\text{A6})$$

The new action after eliminating  $\tilde{f}_1$  has standard kinetic terms and a new set of topological couplings as in (8). More precisely, to reproduce (8) we also have to add some total derivatives to the 5d action, constructed with the globally-defined gauge potentials and the closed base-point fluxes.

*c. Canonical momenta and Gauss constraints.* We consider a 5d spacetime of the form  $W_5 = M_4 \times \mathbb{R}_t$  with product metric  $ds^2(W_5) = ds^2(M_4) - dt^2$ . We write the 5d exterior derivative as  $d = d_4 + dt\partial_t$ . We decompose the gauge potentials in components without and with a leg along the time direction  $t$ , (where we denote the spatial slice part with a bar)

$$A_1 = \bar{A}_1 + dt A_0^0, \quad \beta_1 = \bar{\beta}_1 + dt \beta_0^0, \quad b_2 = \bar{b}_2 + dt b_1^0, \quad c_2 = \bar{c}_2 + dt c_1^0, \quad c_3 = \bar{c}_3 + dt c_2^0. \quad (\text{A7})$$

The Lagrangian  $\mathcal{L}$  is defined by  $S = 2\pi \int dt\mathcal{L}$ , and it is the integral over  $M_4$  of a 4-form Lagrangian density. The canonical momenta associated to the spatial components  $\bar{A}_1, \bar{\beta}_1, \bar{b}_2, \bar{c}_2, \bar{c}_3$  are defined as variational derivatives of  $\mathcal{L}$  with respect to  $\partial_t\bar{A}_1, \partial_t\bar{\beta}_1$ , and so on. Similarly, the Gauss constraints are defined as the variational derivatives of  $\mathcal{L}$  with respect to the time components  $A_0^0, \beta_0^0$ , and so on,

$$\delta\mathcal{L} = \int_{M_4} \left[ \Pi_{A_1}\delta\partial_t\bar{A}_1 + \Pi_{\beta_1}\delta\partial_t\bar{\beta}_1 + \Pi_{b_2}\delta\partial_t\bar{b}_2 + \Pi_{c_2}\delta\partial_t\bar{c}_2 + \Pi_{c_3}\delta\partial_t\bar{c}_3 + \mathcal{G}_{A_1}\delta A_0^0 + \mathcal{G}_{\beta_1}\delta\beta_0^0 + \mathcal{G}_{b_2}\delta b_1^0 + \mathcal{G}_{c_2}\delta c_1^0 + \mathcal{G}_{c_3}\delta c_2^0 \right],$$

where we have denoted the momenta with  $\Pi$  and the Gauss constraints with  $\mathcal{G}$ . We compute

$$\Pi_{b_2} = \tilde{\Pi}_{b_2} - \frac{1}{2}N\bar{c}_2, \quad \Pi_{c_2} = \tilde{\Pi}_{c_2} + \frac{1}{2}N\bar{b}_2, \quad \Pi_{A_1} = \tilde{\Pi}_{A_1} - M\bar{c}_3, \quad \Pi_{c_3} = \tilde{\Pi}_{c_3} - M\bar{A}_1, \quad \Pi_{\beta_1} = \tilde{\Pi}_{\beta_1}, \quad (\text{A8})$$

where we used a tilde to denote the contributions originating from the kinetic terms. The Gauss constraints are reported in (9). The quantities  $\tilde{\Pi}, \tilde{\mathcal{G}}$  satisfy

$$\tilde{\mathcal{G}}_{A_1} = -d_4\tilde{\Pi}_{A_1}, \quad \tilde{\mathcal{G}}_{\beta_1} = -d_4\tilde{\Pi}_{\beta_1}, \quad \tilde{\mathcal{G}}_{b_2} = d_4\tilde{\Pi}_{b_2} + M\tilde{\Pi}_{\beta_1}, \quad \tilde{\mathcal{G}}_{c_2} = d_4\tilde{\Pi}_{c_2}, \quad \tilde{\mathcal{G}}_{c_3} = -d_4\tilde{\Pi}_{c_3}. \quad (\text{A9})$$

This can be seen by noting that the part of the 4-form Lagrangian density that originates from the 5d kinetic terms depends on the various gauge potentials only via the following combinations:  $d_4\bar{A}_1, d_4\bar{b}_2, d_4\bar{\beta}_1 + M\bar{b}_2, d_4\bar{c}_2, d_4\bar{c}_3, \partial_t\bar{A}_1 - d_4A_0^0, \partial_t\bar{b}_2 - d_4b_1^0, \partial_t\bar{\beta}_1 - d_4\beta_0^0 + Mb_1^0, \partial_t\bar{c}_2 - d_4c_1^0, \partial_t\bar{c}_3 - d_4c_2^0$ . The relations (A9) then follow from the chain rule for functional derivatives. Combining (9), (A8), and (A9) we arrive at the expression for the Gauss constraints in terms of the canonical momenta,

$$\begin{aligned} \mathcal{G}_{b_2} &= d_4\Pi_{b_2} + M\Pi_{\beta_1} - \frac{1}{2}Nd_4\bar{c}_2 - Nf_3^b, & \mathcal{G}_{A_1} &= -d_4\Pi_{A_1} + Md_4\bar{c}_3 + (g_2^b)^2, & \mathcal{G}_{\beta_1} &= d_4\Pi_{\beta_1}. \\ \mathcal{G}_{c_2} &= d_4\Pi_{c_2} + \frac{1}{2}Nd_4\bar{b}_2, & \mathcal{G}_{c_3} &= -d_4\Pi_{c_3} + Md_4\bar{A}_1, \end{aligned} \quad (\text{A10})$$

*d. Symmetry Generators from Gauss law constraints.* We briefly review why the Gauss law constraint can be used to construct symmetry operators. For this we consider a  $p$ -form gauge field  $A_p$  with transformation  $A_p \rightarrow A_p + d\lambda_{p-1}$  on a  $d + 1$ -dimensional spacetime with constant time slices  $W_d$ . Let  $\mathcal{G}_A$  be the Gauss constraint, which is a closed  $(d - p + 1)$ -form. In Hamiltonian quantization, it generates a small gauge transformation as the operator

$$\exp\left(2\pi i \int_{W_d} \lambda_{p-1} \wedge \mathcal{G}_A\right). \quad (\text{A11})$$

Now we consider a singular gauge transformation that has support on a boundary surface  $M_{d-p} = \partial M_{d-p+1}$  as

$$d\lambda_{p-1} = \delta(M_{d-p}). \quad (\text{A12})$$

This allows us to associate a symmetry generator for gauge transformation on the pair  $(M_{d-p}, M_{d-p+1})$  as

$$\exp\left(2\pi i \int_{M_{d-p+1}} \mathcal{G}_A\right). \quad (\text{A13})$$

If  $\mathcal{G}_A = d\mathcal{O}_A$ , with  $\mathcal{O}_A$  a globally defined operator, we can integrate by parts to obtain a genuine dynamical operator defined on  $d - p$ -cycles as  $e^{2\pi i \int_{M_{d-p}} \mathcal{O}_A}$ . Now as we take the  $W_4$  near the boundary of  $W_5$ , the kinetic terms are suppressed and the operator is topological on  $M_{d-p}$  [26, 28].

If  $\mathcal{G}_a$  cannot be expressed as a derivative of a gauge invariant and globally defined operator, consider  $M_{d-p} = \partial M_{d-p+1}$  and define

$$\mathcal{S}_A(M_{d-p}) = e^{2\pi i \int_{M_{d-p+1}} \mathcal{G}_A}. \quad (\text{A14})$$

In this operator the contribution from the kinetic terms are derivatives of local operators. The main task is now to define a genuine operator on  $M_{d-p}$ .

Let us concretely carry this out in the case of  $A = A_1$  with Gauss law constraints

$$\mathcal{G}_{A_1} = \tilde{\mathcal{G}}_{A_1} + 2Md_4\bar{c}_3 + (\bar{g}_2^b)^2. \quad (\text{A15})$$

Here  $\tilde{\mathcal{G}}_{A_1} = d\tilde{\mathcal{O}}_{A_1}$  is exact and  $\bar{c}_3$  is globally defined. We wish to define a genuine operator on  $M_3 = \partial M_4$  such that, when raised to the  $2M$ -th power it reproduces the operator constructed from the Gauss law constraint  $\mathcal{S}_{A_1}(M_4)$ . At this stage, we will also consider the operator near the boundary of  $W_5$  and therefore can drop the contribution from the kinetic term. However we must now consider the different choices of boundary conditions. The first case we fix  $b_2$  and thus  $\bar{g}_2^b$  at the boundary as classical backgrounds. This corresponds to 4d  $\mathcal{N} = 1$   $SU(M)$  SYM, with a global electric  $\mathbb{Z}_M$  1-form symmetry coupled to a non-dynamical discrete 2-form field. The genuine operator on  $M_3$  in this case is simply the standard holonomy  $e^{2\pi i \int_{M_3} \bar{c}_3}$  accompanied by the  $c$ -number phase  $e^{2\pi i \frac{1}{2M} \int_{M_4} (\bar{g}_2^b)^2}$ . This operator obeys group-like fusion rules.

The more interesting scenario is when we sum over  $\bar{g}_2^b$  at the boundary. In field theory, we gauge the electric 1-form symmetry of 4d  $\mathcal{N} = 1$   $SU(M)$  SYM, thereby getting the  $PSU(M)$  theory. Here we cannot treat  $e^{2\pi i \frac{1}{2M} \int_{M_4} (\bar{g}_2^b)^2}$  as a  $c$ -number. However we observe that

$$e^{2\pi i \int_{M_4} (\bar{g}_2^b)^2} = \int \mathcal{D}a\mathcal{D}c e^{-2\pi i \int_{M_4} (M^2 da \wedge da + 2Mda \wedge \bar{g}_2^b + dc \wedge \bar{g}_2^b)}, \quad (\text{A16})$$

where  $\bar{g}_2^b$  is flat on the left hand side. Therefore integrating over  $a$  and  $c$  reproduces the right hand side. Now we are free to write

$$e^{2\pi i \int_{M_4} (\bar{g}_2^b)^2} = \int \mathcal{D}a\mathcal{D}c e^{-2\pi i \int_{M_4} (M^2 da \wedge da + 2Mda \wedge \bar{g}_2^b + dc \wedge \bar{g}_2^b)} = \int \mathcal{D}a e^{-2\pi i \int_{M_3} 2M \left(\frac{M}{2} a \wedge da + a \wedge \bar{g}_2^b\right)}. \quad (\text{A17})$$

In the middle term, the integral localizes to configurations that satisfy  $d(Mda + g_2^b) = 0$  and  $dg_2^b = 0$  thereby realizing the first equality. In the last expression we have integrated over  $c$  and integrated by part to  $M_3$ . Now we have rewritten the right hand side in a way that we can take the  $2M$ -th root. The genuine operator on  $M_3$  can then be determined using this prescription. For the anomaly in the solution at hand we find (10), which precisely has the non-invertible fusion rule (4).



## Appendix B: D5-brane action and reduction

The topological couplings in the D5-brane action are encoded in the 6-form

$$I_6^{\text{D5}(0)} = \left[ e^{da-B_2} (C_6 + C_4 + C_2 + C_0) \right]_{6\text{-form}} = C_6 + C_4(da - B_2) + \frac{1}{2}C_2(da - B_2)^2 + \frac{1}{6}C_0(da - B_2)^3. \quad (\text{B1})$$

We neglect contributions from the tangent and normal bundles. The associated 7-form anomaly polynomial is

$$I_7^{\text{D5}} = dI_6^{\text{D5}(0)} = F_7 + F_5(da - B_2) + \frac{1}{2}F_3(da - B_2)^2 + \frac{1}{6}F_1(da - B_2)^3, \quad (\text{B2})$$

where  $F_7 = dC_6 - H_3C_4$ ,  $F_5 = dC_4 - H_3C_2$ ,  $F_3 = dC_2 - H_3C_0$ ,  $F_1 = dC_0$ . In these conventions, the Bianchi identities read  $dF_p = H_3F_{p-2}$  ( $p = 3, 5, 7$ ) and  $dF_1 = 0$ . To reduce  $I_7^{\text{D5}}$  on  $S^3$ , we use

$$F_7 = f_4 \text{vol}_{S^3} + \dots, \quad F_5 = g_2 \text{vol}_{S^3} + \dots, \quad F_3 = M \text{vol}_{S^3} + \dots, \quad F_1 = 0 + \dots, \quad B_2 = b_2 + \dots \quad (\text{B3})$$

where the ellipses stand for contributions that are not relevant for the  $S^3$  reduction. The 10d Bianchi identities require in particular  $df_4 = h_3g_2$ ,  $dg_2 = Mh_3$ , where  $h_3 = db_2$ . We obtain

$$I_4^{\text{D5}} := \int_{S^3} I_7^{\text{D5}} = f_4 + g_2(da - b_2) + \frac{1}{2}M(da - b_2)^2, \quad (\text{B4})$$

with  $\int_{S^3} \text{vol}_{S^3} = 1$ . Let us collect powers of the D5-brane gauge field  $a$ ,

$$I_4^{\text{D5}} = \left( f_4 - g_2b_2 + \frac{1}{2}Mb_2^2 \right) + da(g_2 - Mb_2) + \frac{1}{2}Mdada. \quad (\text{B5})$$

The Bianchi identity for  $g_2$  is solved by setting  $g_2 = db_1 + Mb_2$ , hence  $g_2 - Mb_2 = db_1$ . (Here we do not separate the base-point for  $g_2$  explicitly; it is understood that  $b_1$  can be topologically non-trivial.) The Bianchi identities for  $f_4$  and  $g_2$  imply that  $d(f_4 - g_2b_2 + \frac{1}{2}Mb_2^2) = 0$ . We may therefore introduce a bulk 3-form gauge potential  $c_3$  satisfying

$$f_4 - g_2b_2 + \frac{1}{2}Mb_2^2 = dc_3. \quad (\text{B6})$$

The 4-form anomaly polynomial  $I_4$  then reads

$$I_4^{\text{D5}} = dc_3 + dadb_1 + \frac{1}{2}Mdada, \quad \text{hence} \quad I_3^{\text{D5}(0)} = c_3 + \frac{M}{2}ada + adb_1. \quad (\text{B7})$$

The bulk extended operator of interest is the holonomy of  $I_3^{(0)}$  with charge 1,  $e^{2\pi i \int_{M_3} I_3^{(0)}}$ .

The topological terms for a D7-brane wrapping  $T^{1,1} \cong S^3 \times S^2$  are derived analogously from the 9-form anomaly polynomial  $I_9^{\text{D7}} = F_9 + F_7(da - B_2) + \frac{1}{2}F_5(da - B_2)^2 + \frac{1}{3!}F_3(da - B_2)^3 + \frac{1}{4!}F_1(da - B_2)^4$ . We take into account the worldvolume gauge flux on  $S^2$  with the replacement  $da \rightarrow f_2^{S^2} + da$ ,  $\int_{S^2} f_2^{S^2} = 2$ . We may use  $\int_{S^2} (f_2^{S^2} + da - B_2)^p = 2p(da - B_2)^{p-1}$ . In total,

$$I_4^{\text{D7}} := \int_{T^{1,1}} I_9^{\text{D7}} = 2I_4^{\text{D5}}. \quad (\text{B8})$$

We have not included an  $F_5 \propto \text{vol}_{T^{1,1}}$  contribution because this flux is not quantised in the UV KS solution, see (C5).

## Appendix C: Review of Myers Effect and Application to D5-branes on $S^3$

In this appendix we briefly review the main aspects of the Myers effect [30], applying it to  $k$  D5-branes wrapping  $S^3$  in the Klebanov-Strassler geometry. Before proceeding with the details of the explicit construction that we consider, let us set the basis by providing the general expression for the non-Abelian DBI and WZ action of the brane. First of all the coordinate orthogonal to the Dp-brane stack are promoted to matrices in the adjoint of  $U(k)$ ,  $X^i$ . The action is then

$$S_{\text{D}p}^{\text{non-Abelian}} = -\mu_p \int d^{p+1}x \text{STr} \left( e^{-\phi} \sqrt{-\det g^{\mu\nu}} \sqrt{\det \Theta} - P \left[ e^{\frac{1}{2\pi} \iota_X \iota_X} (e^{-B} \wedge C) \right] \right), \quad (\text{C1})$$

where  $g^\parallel$  is the pullback of the metric onto the branes,  $S\text{Tr}$  is the symmetrized trace and  $P$  the pullback on the worldvolume. The matrix  $\Theta$  is

$$\Theta^i_j = \delta^i_j + \frac{1}{2\pi} [X^i, X^k] (g_{kj}^\perp - B_{kj}), \quad (\text{C2})$$

where  $g^\perp$  is metric perpendicular to the branes. In the WZ action we have the contraction  $\iota_X$ , which is given by the action of the operator  $X^i \partial_{x^i}$  on the potentials. For instance, on a two-form  $C^{(2)} = \frac{1}{2} C_{ij} dx^i \wedge dx^j$ :

$$\iota_X \iota_X C^{(2)} = X^j X^i C_{ij} = \frac{1}{2} C_{ij} [X^j, X^i]. \quad (\text{C3})$$

To perform explicit computation with the action (C1) we need to Taylor expand up to two-derivatives terms and at most quartic scalar interactions, which can be thought as first order correction given by the non-Abelian action with respect to the Abelian one.

Let us go back to the setup we are studying. The metric of the KS solution for large values of the radial coordinate is given by the Klebanov-Tseytlin metric [41],

$$ds_{10}^2 = ds_{W_5}^2 + \mathcal{R}^2(r) ds_{T^{1,1}}^2, \quad \mathcal{R}(r) \sim \ln(r/r_s)^{1/4}, \quad (\text{C4})$$

where  $ds_{W_5}^2 = \frac{r^2 d\vec{x}^2}{\mathcal{R}^2(r)} + \frac{\mathcal{R}^2(r) dr^2}{r^2}$ , and  $r_s = r_0 e^{-\frac{2\pi N}{3g_s M^2} - \frac{1}{4}}$ . The non-trivial fluxes in the background are

$$\int_{S^3} F_3 = M, \quad \int_{S^2} B_2 = \mathcal{L}(r), \quad \int_{T^{1,1}} F_5 = \mathcal{K}(r) = N + M\mathcal{L}, \quad \mathcal{L} := \frac{3g_s M}{2\pi} \ln(r/r_0). \quad (\text{C5})$$

The stack of  $k$  D5-branes wraps  $S^3$  engineering the topological defects, and we would like to study its dynamics close to the boundary at  $r \rightarrow r_0$ , with  $r_0 \gg 0$ , in order for the supergravity regime to be reliable. The orthogonal directions are spanned by  $\{x_3, r, \theta, \gamma\}$ , where the last two are the coordinates of  $S^2 \subset T^{1,1}$ . Let us put aside  $x_3$ , which will not affect the analysis of this section. Since we are in a regime where for  $r \rightarrow \infty$  the radial dynamics of the entire  $k$  D5-branes system is frozen and we focus on the dynamics of the relative positions of the  $k$  D5-branes, we also fix the warping factor  $\mathcal{R}(r = r_0)$ . The geometry spanned by the coordinates  $\{r, \theta, \gamma\}$  orthogonal to the D5 brane stack in the limit  $r \rightarrow r_0$  is  $\mathbb{R}^3$  where the metric is  $ds_{\mathbb{R}^3}^2 = \frac{1}{2r_0^2} (dr^2 + r^2 d\Omega_2)$ . The  $B$  components in  $\mathbb{R}^3$  which depend on  $X^i$  are then given by

$$B_{ij} = \frac{1}{r_0^3} \epsilon_{ijk} X^k. \quad (\text{C6})$$

The fluxes (C5) allow for no WZ action for the D5 stack wrapping  $S^3$  since the only background which pulls back to the brane is  $B_2$ . Hence we need to expand the DBI action, which up to quadratic order reads,

$$S_{D5}^{\text{non-Abelian}} = -\mu_5 \text{Tr} \int d^6 x \sqrt{g^\parallel} \left( 1 + \frac{1}{2} g^{\mu\nu} g_{ij}^\perp \partial_\mu X^i \partial_\nu X^j - \frac{1}{4\pi} B_{ij} [X^j, X^i] + \frac{1}{16\pi^2} [X^i, X^j] g_{jk}^\perp [X^k, X^l] g_{li}^\perp \right). \quad (\text{C7})$$

The static solutions are described by

$$[[X^i, X^j], X^j] - 2[X^k, X^j] \epsilon_{kj}^i = 0, \quad (\text{C8})$$

where we rescaled  $X^i \rightarrow r_0 X^i$ . Commuting matrices are always solutions, however the minimal energy solution is one where the fields are non-commutative ones such that  $X^i \sim \alpha^i$ , where  $\alpha^i$  are the realization as  $k \times k$  matrices of  $\mathfrak{su}_2$  in the  $k$ -dimensional representation. The dynamical consequence for the D5 branes is that they puff up and polarize into a D7 with  $k$  unit of  $f_2^{S^2}$  flux, or alternatively a bound state of D5/D7, in analogy with the D0/D2 bound state system studied in the original paper [30].