

Power corrections to electroweak boson production from threshold resummation

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We study the power corrections for electroweak boson production that are implied by threshold resummation, which we have extended to massive particles produced at measured transverse momentum, p_T , and rapidity. Power corrections in the resulting expressions arise from ambiguities in the low-scale behavior of the perturbative running coupling. Arguing for the relevance of the eikonal approximation, we show that such power corrections begin at order $1/p_T^2$ in full QCD, consistent with fixed-order, massive-gluon analysis. For large- N Mellin moments, the leading behavior is N^2/p_T^2 , which exponentiates along with the logarithms of threshold resummation.

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I. INTRODUCTION

Leading-power, factorized cross sections are central to the experimental program at high-energy accelerators, but the need for precision control over Standard Model calculations motivates the study of power-suppressed corrections in these cross sections. In this paper, we study the production of electroweak bosons, $\beta = \gamma, \gamma^*, W^\pm, Z$, with large transverse momenta, p_T , in hadronic collisions. For an observed color-singlet particle of momentum p produced in collisions between hadrons A and B , the generic, leading-power, factorized form is

$$p^0 \frac{d\sigma_{AB \rightarrow \beta+X}}{d^3p} = \sum_{ab} \int dx_a dx_b \phi_{a/A}(x_a, \mu^2) \phi_{b/B}(x_b, \mu^2) \times \omega_{ab \rightarrow \beta+X}(x_a P_A, x_b P_B, p, \mu, \alpha_s(\mu^2)) + \text{power corrections}, \quad (1)$$

where the sum is over parton flavors with distributions $\phi_{a/A}$ and $\phi_{b/B}$, and where μ denotes a factorization/renormalization scale. For real photons, we neglect fragmentation contributions, and for heavy bosons, we take $p_T \sim m$, with m the mass of the produced particle, to avoid the presence of large ratios. Corrections to Eq. (1) are suppressed relative

to the leading term by powers in p_T , or, generically, a hard scale Q , and it is these corrections that we study below.

The nature of power corrections to factorized cross sections of the form of Eq. (1) was discussed in Refs. [1,2], where arguments were presented that at fixed orders in perturbation theory, power corrections up to $1/Q^2$ in the hard scale Q take the form of factorized twist-four matrix elements. This leaves, however, the possibility of power corrections associated with resummations of higher-order leading-power corrections, especially near “partonic threshold,” defined as the integration region where

$$\hat{s} = x_a x_b S \rightarrow Q^2 \quad (2)$$

in Eq. (1). (We note that for single-particle cross sections, the hard scale Q is in general a function of the four-vectors $x_a P_A$, $x_b P_B$ and p .) Near partonic threshold, real radiation is suppressed, and although the hard scattering function $\omega_{ab \rightarrow \beta+X}$ remains infrared finite order-by-order in perturbation theory, the cancellation holds only up to scales that vanish when Eq. (2) becomes an equality. The threshold-resummed cross section generates a divergent series in this limit, which should be reinterpreted as implying the presence of power corrections to the factorized cross section [3–7]. As was shown in Refs. [8,9], this approach also predicts that power corrections to the inclusive Drell-Yan cross section start at $1/Q^2$ in full QCD. Extensions of these ideas to the case of high- p_T direct-photon production for integrated rapidity were carried out in Ref. [10], where leading power corrections of order $1/p_T^2$ to the cross section were anticipated, relying on the formalism of

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“joint resummation” [11], which organizes threshold and transverse-momentum resummations. The analysis we give below generalizes and extends the treatment of inclusive Drell-Yan in [8,9] and that of prompt photons in [10], including systematic treatments of final-state soft radiation at measured rapidity for the observed particle, and of the role of the recoiling colored parton in $AB \rightarrow \beta X$.

Our discussion is also inspired by recent arguments for the absence of nonperturbative $1/Q$ corrections in the hard scale for a phenomenologically relevant class of event shapes [12,13] in lepton pair annihilation and for the hadronic production of color neutral bosons at large transverse momentum [14,15]. The extensive arguments given to support this conclusion rely in part on a much older treatment of Drell-Yan cross sections in hadronic collisions, based on renormalon analysis in Abelian theories with large numbers of flavor in Refs. [4,5]. Related analyses rely on identifying sensitivity to a finite gluon mass with a nonperturbative representation of the running coupling [16–18], in the “dispersive” representation of the QCD running coupling at low scales [19].

Again, hard-scattering functions $\omega_{ab \rightarrow \beta+X}$ in Drell-Yan are singular at “partonic threshold,” where the incoming partons have just enough energy to produce the observed final state. At each order in the strong coupling α_s , the functions $\omega_{ab \rightarrow \beta+X}$ are characterized by singular plus distributions whose arguments vanish at partonic threshold. For example, in the inclusive Drell-Yan cross section the relevant argument is $1 - z$, with $z = Q^2/(x_a x_b S)$ and the leading distributions are of the form

$$\alpha_s^n \left[\frac{\ln^{2n-1}(1-z)}{1-z} \right]_+ . \quad (3)$$

These singular distributions, in turn, can be controlled by threshold resummation, which organizes all such logarithms, including those associated with the running of the coupling. The argument of the running coupling vanishes as z approaches 1. We will encounter similar behavior below, extending the analysis for photons in Ref. [20] to massive electroweak bosons produced at high p_T .

The relevant transform for the inclusive production of color-neutral particles involves, as we review below, the Mellin or Laplace factor with moment variable, N ,

$$(1 - \beta_r \cdot k/Q)^N \sim \exp[-N\beta_r \cdot k/Q](1 + \mathcal{O}(1/N)), \quad (4)$$

where k^μ is the sum of all radiation from a set of two incoming and one outgoing Wilson line, the latter characterized by velocity β_r , which is in the direction of recoil to the observed photon or color neutral boson at partonic threshold. The denominator that sets the scale of the exponent, Q , is the hard scale of the scattering, of order p_T . As N increases, real radiation in the moments is

restricted to energies of order Q/N , which can be of the order of Λ_{QCD} .

We will carry out our analysis using the eikonal approximation, which we will argue is appropriate for both real and virtual radiation at these low scales (see, for example, Refs. [21,22]). In the discussion that follows, we show that in full QCD, it is possible to infer the presence of specific power corrections to color-neutral boson production at large $p_T \sim Q$. These corrections appear through (exponentiating) terms of the form

$$\left(\frac{N}{Q}\right)^n \int_0^\kappa d\mu \mu^{n-1} A_i(\alpha_s(\mu^2)), \quad (5)$$

for an integer n , with $A_i(\alpha_s)$ a flavor-dependent function of the strong coupling, and κ an arbitrary cutoff. Such integrals are ambiguous in perturbation theory because of the behavior of the perturbative running coupling. If reexpressed as an inverse Borel transform, the singularity of the running coupling is mapped onto a singularity on the positive real axis in the Borel plane [3]. In particular, we show that threshold resummation implies the presence of such terms beginning at $n = 2$ for these cross sections, consistent with the results of Ref. [14]. We do not, however, rely on a Borel transform to identify the nature of the power correction. In Eq. (5) and the expressions we derive, the Q -dependence is manifest, appearing with a coefficient that is independent of Q , but that includes an integral over the Landau pole. We assume that the integral that includes the Landau pole of the perturbative coupling is replaced in the full theory by a physical coefficient that depends on both perturbative and nonperturbative dynamics. Also, because these contributions are found in integrals that exponentiate in moment space, we regard them as implying the presence of nonperturbative shape functions, very much as in resummed thrust and related cross sections [23–26].

We also note here that our analysis is not based directly on the evolution equations that characterize resummed cross sections in direct QCD or SCET resummations [27,28], although it is consistent with them. In the eikonal approximation used in our analysis, we will derive a form for the cross section in which all logarithmic corrections are organized into integrals over the QCD coupling evaluated at a well-defined scale. The resulting expressions generate the same logarithms as in threshold resummation (in eikonal approximation), but also provide a unique extension to the far infrared, well defined to any order in perturbation theory with the coupling reexpanded in terms of its value at a fixed scale, but otherwise encountering a Landau pole. In moment space, these integrals are finite except for the behavior of the running coupling, precisely as in Eq. (5).

In the following section, we review the double moment method of [20] for particle production at high p_T and

measured rapidity,¹ extending the formalism to the massive case. Of particular interest is the expression for the cross section as an inverse Mellin (or Laplace) transform, from which we infer the exponential suppression of nonperturbative behavior in the large-moment region. Section III reviews the construction of the cross section for color-singlet production in the eikonal approximation, found from matrix elements of Wilson lines, which will generate all corrections that behave as powers of N/p_T , with N the Mellin moment. We go on to show that final-state interactions cancel in these cross sections. Section IV reviews salient features of an explicit all-orders resummation for these processes, in terms of webs [32–36]. We present results generated by the lowest-order web, verifying the absence of linear power corrections at this level. We show, in Sec. V, how general features of exponentiated webs derived in Sec. IV lead to an expression for the resummed exponent, and we verify the absence of linear powers of N/p_T to all orders in the web expansion. We briefly discuss further applications and conclude.

II. DOUBLE-MOMENT CROSS SECTIONS

To set the stage for our analysis of logarithms associated with partonic threshold, we discuss the factorized cross section for the production of an electroweak boson as a function of transverse momentum and rapidity. In particular, we will present the moment-space formulation of the cross section, which is an essential ingredient of our method for organizing the threshold logarithms and will subsequently allow us to use the eikonal approximation to identify power corrections associated with renormalons. The following derivations follow those in our analysis of prompt photon production [20]; however, we extend them here to the case of a massive boson. Although we have mostly in mind the Drell-Yan process and the W and Z electroweak bosons, the application to Higgs production is immediate.

At measured transverse momentum p_T and rapidity η , the factorized form for the cross section for $AB \rightarrow \beta X$, as in Eq. (1), is given by

$$\begin{aligned} \frac{p_T^3 d\sigma_{AB \rightarrow \beta X}}{dp_T d\eta} &= \sum_{a,b} \int_{\frac{U}{S+T-m^2}}^1 dx_a \phi_{a/A}(x_a, \mu^2) \int_{\frac{-x_a(T-m^2)-m^2}{x_a S+U-m^2}}^1 dx_b \phi_{b/B}(x_b, \mu^2) \frac{p_T^3 d\hat{\sigma}_{ab}}{dp_T d\eta} \\ &\equiv \sum_{a,b} \int_{\frac{U}{S+T-m^2}}^1 dx_a \phi_{a/A}(x_a, \mu^2) \int_{\frac{-x_a(T-m^2)-m^2}{x_a S+U-m^2}}^1 dx_b \phi_{b/B}(x_b, \mu^2) \omega_{ab} \left(\hat{x}_T, \hat{\eta}, r, \frac{\mu^2}{\hat{s}} \right), \end{aligned} \quad (6)$$

where the ω_{ab} are dimensionless hard-scattering functions. We have introduced a set of kinematic variables: The mass of the produced boson β is denoted by m . The hadronic Mandelstam variables are $S = (P_A + P_B)^2$, $T = (P_A - p)^2$, $U = (P_B - p)^2$, with p the boson four-momentum. In terms of the boson's transverse momentum, mass and rapidity we have

$$\begin{aligned} T &= m^2 - \sqrt{S} \sqrt{m^2 + p_T^2} e^{-\eta}, \\ U &= m^2 - \sqrt{S} \sqrt{m^2 + p_T^2} e^{\eta}. \end{aligned} \quad (7)$$

For future reference we also define

$$x_T \equiv \frac{p_T + \sqrt{m^2 + p_T^2}}{\sqrt{S}} \equiv \frac{p_T + m_T}{\sqrt{S}}. \quad (8)$$

As indicated, the partonic hard-scattering functions may be written as functions of

$$\begin{aligned} \hat{x}_T &\equiv \frac{x_T}{\sqrt{x_a x_b}}, \\ \hat{\eta} &\equiv \eta - \frac{1}{2} \ln \frac{x_a}{x_b}, \\ r &\equiv \frac{p_T}{m_T}, \end{aligned} \quad (9)$$

and $\mu^2/\hat{s}$, with $\hat{s} = x_a x_b S$.

We now take combined Mellin/Fourier moments of the hadronic cross section at fixed p_T ,

$$\begin{aligned} \Sigma_{AB \rightarrow \beta X}(N, M, p_T) &\equiv \int_{-\infty}^{\infty} d\eta e^{iM\eta} \int_0^{x_{T,\max}^2} dx_T^2 (x_T^2)^{N-1} \frac{p_T^3 d\sigma_{AB \rightarrow \beta X}}{dp_T d\eta}. \end{aligned} \quad (10)$$

Here $x_{T,\max}^2$ is the kinematic upper limit on x_T^2 , given at fixed rapidity by

$$x_{T,\max}^2 = \frac{\cosh^2 \eta}{(1-r)^2} \left(1 - \sqrt{1 - \frac{1-r^2}{\cosh^2 \eta}} \right)^2. \quad (11)$$

Applying the moment integrals to the expression in the second line of Eq. (6) we find, after some straightforward algebra,

¹The use of double moments in threshold resummation was introduced in [29]. For a related treatment of resummation at measured rapidity, see Refs. [30,31].

$$\begin{aligned}
 \Sigma_{AB \rightarrow \beta X}(N, M, p_T) &= \sum_{a,b} \int_0^1 dx_a x_a^{N+iM/2} \phi_{a/A}(x_a, \mu^2) \int_0^1 dx_b x_b^{N-iM/2} \phi_{b/B}(x_b, \mu^2) \\
 &\quad \times \int_{-\infty}^{\infty} d\hat{\eta} e^{iM\hat{\eta}} \int_0^{\hat{x}_T^2, \max} d\hat{x}_T^2 (\hat{x}_T^2)^{N-1} \omega_{ab} \left(\hat{x}_T, \hat{\eta}, r, \frac{\mu^2}{\hat{s}} \right) \\
 &\equiv \sum_{a,b} \tilde{\phi}_{a/A}^{N+1+\frac{iM}{2}}(\mu^2) \tilde{\phi}_{b/B}^{N+1-\frac{iM}{2}}(\mu^2) \tilde{\omega}_{ab} \left(N, M, r, \frac{\mu^2}{p_T^2} \right). \quad (12)
 \end{aligned}$$

Here the Mellin moments of the parton distribution functions are defined as usual by

$$\tilde{\phi}_{i/H}^n(\mu^2) \equiv \int_0^1 dx x^{n-1} \phi_{i/H}(x, \mu^2), \quad (13)$$

and we have introduced

$$\begin{aligned}
 \tilde{\omega}_{ab} \left(N, M, r, \frac{\mu^2}{p_T^2} \right) \\
 \equiv \int_{-\infty}^{\infty} d\hat{\eta} e^{iM\hat{\eta}} \int_0^{\hat{x}_T^2, \max} d\hat{x}_T^2 (\hat{x}_T^2)^{N-1} \omega_{ab} \left(\hat{x}_T, \hat{\eta}, r, \frac{\mu^2}{\hat{s}} \right), \quad (14)
 \end{aligned}$$

with

$$\hat{x}_T^2, \max = \frac{\cosh^2 \hat{\eta}}{(1-r)^2} \left(1 - \sqrt{1 - \frac{1-r^2}{\cosh^2 \hat{\eta}}} \right)^2. \quad (15)$$

As seen in Eq. (11), in moment space the contribution to the cross section for a given partonic channel factorizes into a product of the moments of the parton distributions and a double moment of the partonic hard-scattering function. We note that in the limit $m \rightarrow 0$ (or $r \rightarrow 1$) Eq. (12) reverts to the corresponding expression for the case of prompt photons discussed in Ref. [20].

In threshold resummation, we organize terms that become singular at partonic threshold, when the active

partons a and b have just enough energy to produce a boson at fixed $\hat{x}_T$ and $\hat{\eta}$. In order to investigate the moment-space expression near threshold, it is convenient to introduce the variable

$$\zeta \equiv \frac{s_4}{\hat{s}} \equiv \frac{\hat{s} + \hat{t} + \hat{u} - m^2}{\hat{s}}, \quad (16)$$

where

$$\begin{aligned}
 \hat{t} &= m^2 + x_a(T - m^2), \\
 \hat{u} &= m^2 + x_b(U - m^2). \quad (17)
 \end{aligned}$$

The invariant s_4 provides a natural measure of the distance from threshold. In terms of $\hat{x}_T, \hat{\eta}$ and r we have

$$\zeta = 1 + \hat{x}_T^2 \frac{1-r}{1+r} - \frac{2\hat{x}_T}{1+r} \cosh \hat{\eta}, \quad (18)$$

which may be inverted to give

$$\hat{x}_T(\zeta) = \frac{\cosh \hat{\eta}}{1-r} \left(1 - \sqrt{1 - \frac{(1-r^2)(1-\zeta)}{\cosh^2 \hat{\eta}}} \right). \quad (19)$$

Using this to replace the integration over $\hat{x}_T^2$ in (14) we have for our double moments

$$\begin{aligned}
 \tilde{\omega}_{ab} \left(N, M, r, \frac{\mu^2}{p_T^2} \right) &= \int_{-\infty}^{\infty} d\hat{\eta} e^{iM\hat{\eta}} \frac{1+r}{(1-r)^{2N-1}} (\cosh \hat{\eta})^{2N-2} \int_0^1 d\zeta \frac{\left(1 - \sqrt{1 - \frac{(1-r^2)(1-\zeta)}{\cosh^2 \hat{\eta}}} \right)^{2N-1}}{\sqrt{1 - \frac{(1-r^2)(1-\zeta)}{\cosh^2 \hat{\eta}}}} \omega_{ab} \left(\hat{x}_T, \hat{\eta}, r, \frac{\mu^2}{\hat{s}} \right) \\
 &= \int_{-\infty}^{\infty} d\hat{\eta} e^{iM\hat{\eta}} \frac{1+r}{\cosh \hat{\eta}} \left(\frac{\cosh \hat{\eta}}{(1-r)} \left(1 - \sqrt{1 - \frac{1-r^2}{\cosh^2 \hat{\eta}}} \right) \right)^{2N-1} \\
 &\quad \times \int_0^1 \frac{d\zeta}{\sqrt{1 - \frac{(1-r^2)(1-\zeta)}{\cosh^2 \hat{\eta}}}} \left(\frac{1 - \sqrt{1 - \frac{(1-r^2)(1-\zeta)}{\cosh^2 \hat{\eta}}}}{1 - \sqrt{1 - \frac{1-r^2}{\cosh^2 \hat{\eta}}}} \right)^{2N-1} \omega_{ab} \left(\hat{x}_T, \hat{\eta}, r, \frac{\mu^2}{\hat{s}} \right). \quad (20)
 \end{aligned}$$

The form given by the second equality shows that the ζ integrand is exponentially suppressed away from $\zeta = 0$ at large N . We can therefore expand the fraction raised to the $(2N-1)$ st power in the ζ integral to find

$$\begin{aligned} \tilde{\omega}_{ab}\left(N, M, r, \frac{\mu^2}{p_T^2}\right) &= \int_{-\infty}^{\infty} d\hat{\eta} e^{iM\hat{\eta}} \frac{1+r}{\cosh \hat{\eta}} \left(\frac{\cosh \hat{\eta}}{(1-r)} \left(1 - \sqrt{1 - \frac{1-r^2}{\cosh^2 \hat{\eta}}} \right) \right)^{2N-1} \\ &\times \int_0^1 \frac{d\zeta}{\sqrt{1 - \frac{(1-r^2)(1-\zeta)}{\cosh^2 \hat{\eta}}}} \exp\left[-(2N-1) \frac{\zeta}{2} \left(1 + \frac{\cosh \hat{\eta}}{\sqrt{r^2 + \sinh^2 \hat{\eta}}} \right)\right] \omega_{ab}\left(\hat{x}_T, \hat{\eta}, r, \frac{\mu^2}{\hat{s}}\right) + \dots, \end{aligned} \quad (21)$$

with corrections that are suppressed by powers of $1/N$. In this form, we can carry out a “standard” resummation of logarithms of N for $\tilde{\omega}_{ab}$, although for nonzero mass m the effective value of N is linked in a mild way to $\hat{\eta}$. The $\hat{\eta}$ integrand still decreases rapidly for large $\hat{\eta}$, in the same manner as seen in the massless case [20].

Given any determination of the large- N behavior of $\tilde{\omega}_{ab}$, the behavior of the physical cross section is given in turn by the inverse of the double transform,

$$\begin{aligned} \frac{p_T^3 d\sigma_{AB \rightarrow \beta X}}{dp_T d\eta} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dM e^{-iM\eta} \\ &\times \frac{1}{2\pi i} \int_{\mathcal{C}} dN (x_T^2)^{-N} \Sigma_{AB \rightarrow \beta X}(N, M, p_T), \end{aligned} \quad (22)$$

where $\mathcal{C}$ denotes a suitable contour in the complex-Mellin plane [20], extending into the left half-plane where the integrand is exponentially suppressed. For large values of the moment variable, that is, when $N \geq Q/\Lambda_{\text{QCD}}$, the integrand in this expression is suppressed at least as $\exp[\ln x_T(Q/\Lambda_{\text{QCD}})]$. In our arguments below, we will identify nonperturbative corrections proportional to $(N/Q)^2$ in an exponentiated, resummed hard-scattering function. Our arguments, however, apply only when $N\Lambda_{\text{QCD}}/Q$ remains order unity or smaller, so that the large- N behavior of the physical cross section converges as long as $x_T < 1 - \Lambda_{\text{QCD}}/Q$. This is the case, for example, in the minimal resummation prescription [6].

The outcome of this analysis is that in the physical cross section the inverse transform in $\hat{x}_T$ will remain exponentially suppressed at large $|N|$ along our inverse contour, as $\exp[-N \ln(1/x_T)]$. This enables us to argue that the very large N behavior of the resummed exponent does not influence the result. That is, we only need to follow N to order p_T/Λ .

III. EIKONAL CROSS SECTIONS

Near partonic threshold, the limit $\zeta \rightarrow 0$ in Eq. (16), the underlying processes for electroweak boson production are quark-antiquark annihilation to gluon plus boson, and (anti) quark-gluon to (anti)quark plus boson, accompanied by soft gluons, both of the form

$$a + b \rightarrow \beta + r, \quad (23)$$

where again, β represents the color-singlet particle, and where we refer to r as the recoil parton, which is taken here to be a massless quark, antiquark or gluon.

The eikonal cross section is appropriate for soft radiation, and is naturally relevant for partonic threshold. As we recall below, it is generated from cross sections involving products of Wilson lines, which act as sources for gluons. We begin by discussing the kinematics at partonic threshold. This enables us to show that to leading power in N , the transform of the hard-scattering function in Eq. (21) (in the general massive case) is determined by an eikonal cross section with a Laplace transform that depends on the soft radiation. Here, and in the remainder of the paper, for brevity of notation, we will replace the value $2N - 1$, which appears in Eq. (21), by simply N .

For large enough N , the moment-space hard scattering functions, $\tilde{\omega}_{ab}(N, M)$, are well approximated by their eikonal approximations, in which terms quadratic in soft radiation are neglected compared to linear powers that are contracted with hard vectors, $k^2 \ll 2p \cdot k$. The contributions of corrections to the eikonal approximation start with terms that include relative factors of the general form $\beta_k^\mu |k_0|/Q$, with k_0 the energy of the radiation and β_k the velocity vector for k . Contributions of such terms to the variables s_4 in Eq. (16) and ζ in Eq. (18), which measure the distance to partonic threshold, are suppressed by a factor of $\beta_k \cdot \beta_r/N$ relative to the leading eikonal behavior. In the special case of radiation in the recoil direction ($\beta_k = \beta_r$), the radiation of momentum k^μ will factor into a partonic jet function. Such jets contribute to the Laplace transform in Eq. (21) through a factor $\exp[-Nm_{\text{jet}}^2/Q^2]$, with nominal contributions like N/Q^2 , rather than N^2/Q^2 . Taking as an assumption that N^2/Q^2 contributions will be larger than those of order N/Q^2 , the eikonal approximation will give a good first approximation to the pattern of $1/Q^2$ power corrections. This is the assumption we shall entertain, as motivation to work in eikonal approximation.

A. The large- N transform and threshold kinematics

At large values of N in Eq. (21), the integral will be restricted to a region where $\zeta = s_4/\hat{s}$ is of order $1/N$. We would like to relate the integral over ζ to an integral over the momenta of final states. In general, this is a nonlinear relation, because both s_4 and $\hat{s}$ depend on these momenta. We can, however, relate $\hat{s}$ and s_4 , and we find that for soft radiation,

$$\hat{s} = \hat{s}_{\min} + \mathcal{O}(s_4) \quad (24)$$

where $\hat{s}_{\min}$, the invariant corresponding to a final state consisting only of the color-singlet boson and the recoil parton, is

$$\hat{s}_{\min} = (p_T^2 + m^2) \cosh^2 \hat{\eta} \left[1 + \sqrt{1 - \frac{1 - r^2}{\cosh^2 \hat{\eta}}} \right]^2. \quad (25)$$

As defined in Eq. (9), $r = p_T/m_T$. For the region of interest, we can represent the approximations that connect the ζ integral to a Laplace transform in $s_4/\hat{s}_{\min}$ as,

$$\int_0 d\zeta (1 - \zeta)^N \rightarrow \int_0 d\zeta e^{-N\zeta} \rightarrow \int_0 \frac{ds_4}{\hat{s}_{\min}} \exp\left(-N \frac{s_4}{\hat{s}_{\min}}\right), \quad (26)$$

where corrections are suppressed by powers of $s_4/\hat{s} \sim 1/N$. As we have observed above, this is the level of accuracy of the eikonal cross section.

At partonic threshold, the final state consists of only the color-singlet boson and the lightlike recoil parton, r . In the partonic center of mass, the spatial momentum of p_r is back-to-back with the color-singlet particle, and depends on the rapidity $\hat{\eta}$ of that particle through

$$\begin{aligned} p_r^0 &= \sqrt{p_T^2 \cosh^2 \hat{\eta} + m^2 \sinh^2 \hat{\eta}}, \\ p_r^3 &= -\sqrt{p_T^2 + m^2} \sinh \hat{\eta}, \\ \mathbf{p}_{r,T} &= -\mathbf{p}_T. \end{aligned} \quad (27)$$

The explicit expression of s_4 for a state consisting of a recoil parton of momentum p_r and soft radiation k is

$$s_4 = (k + p_r)^2 = (k + p_r^0 \beta_r)^2 \sim 2p_r^0 \beta_r \cdot k, \quad (28)$$

where β_r is a lightlike velocity, with $\beta_r^0 = 1$, and where in the final relation we invoke the eikonal approximation. In these terms, we will denote the transform function as

$$\exp\left(-N \frac{s_4}{\hat{s}_{\min}}\right) = \exp\left(-N \frac{\beta_r \cdot k}{Q}\right) \quad (29)$$

where the hard scale Q is defined by

$$Q \equiv \frac{\hat{s}_{\min}}{2p_r^0}, \quad (30)$$

with $\hat{s}_{\min}$ given by (25) and p_r^0 by (27). We readily check that in the massless limit, $Q \rightarrow 2p_T \cosh \hat{\eta}$.

In summary, we have shown that for large moment variable N , the total momentum of radiation k in the hard-scattering function is limited to regions where $s_4 \sim p_r \cdot k < 1/N$. In this region, up to corrections suppressed by powers of $1/N$, the eikonal approximation holds

and the N -dependence in the transform is approximated by Eq. (29).

B. Eikonal cross sections and Wilson lines

We now turn to the construction of the eikonal approximation of the hard-scattering function $\tilde{\omega}_{ab}$ in Eq. (21). For a generic color-singlet boson production process $ab \rightarrow \beta r$ we have [35]

$$\begin{aligned} \tilde{\omega}_{abr}^{(\text{eik})}(N, Q, \hat{\eta}, \mu) &= H_{abr}(p_T, \hat{\eta}, \mu) \\ &\times \frac{\tilde{\sigma}_{abr}^{(\text{eik})}(N/Q, \hat{\eta}, \mu, \epsilon)}{\tilde{\phi}_{a/a}^{(\text{eik})}(N_a, \mu, \epsilon) \tilde{\phi}_{b/b}^{(\text{eik})}(N_b, \mu, \epsilon)}, \end{aligned} \quad (31)$$

where, again, N stands for $2N - 1$ in (21), and Q , which is of order p_T , is defined in Eq. (30). It will be convenient here and in the following to exhibit the label of the recoil parton, r , on the eikonal cross section and hard-scattering function. The purely virtual short-distance function H_{abr} is independent of N and begins with the partonic Born cross section, which for convenience we normalize to the zeroth-order eikonal cross section (see below),

$$H_{abr}(p_T, \hat{\eta}, \mu) = \frac{1}{\sigma_{abr}^{(\text{eik},0)}} \frac{p_T^3 d\hat{\sigma}_{ab \rightarrow \beta r}^{(\text{Born})}}{dp_T d\hat{\eta}} (1 + \mathcal{O}(\alpha_s(\mu^2))). \quad (32)$$

Dependence on N is all in the eikonal cross section, $\tilde{\sigma}_{abr}^{(\text{eik})}$ and eikonal parton distributions $\tilde{\phi}_{i/i}^{(\text{eik})}$. As is characteristic of direct photon and similar single-particle cross sections, the moment values of the distributions are scaled by functions of the scattering [35],

$$\begin{aligned} N_a &= N \frac{\beta_b \cdot \beta_r}{\beta_a \cdot \beta_b}, \\ N_b &= N \frac{\beta_a \cdot \beta_r}{\beta_a \cdot \beta_b}, \end{aligned} \quad (33)$$

where $\beta_{a,b}$ are the velocity four-vectors of the incoming particles, and β_r of the recoil particle in the final state.

In the same way as in the full partonic calculation, the eikonal cross sections have uncanceled collinear singularities associated with their incoming eikonal lines. As indicated in Eq. (31), we regulate collinear singularities by continuation to $D = 4 - 2\epsilon$ dimensions. The resulting collinear poles in ϵ exponentiate, and are cancelled by the collinear poles of the eikonal parton distributions in the denominator of Eq. (31), giving a finite ratio for each $\tilde{\omega}_{abr}$. We will encounter these cancellations below. The cancellation follows from standard arguments for factorization [37–39], and to leading power in N requires only a simple ratio in moment space, for each choice of incoming eikonal lines. This flavor-diagonal cancellation is a general

feature of threshold resummation at leading power in N [29,40,41].

Both the eikonal distributions and cross sections are defined as matrix elements of Wilson lines. For the latter, we use the notation,

$$\Phi_{\beta}^{(R)}(\lambda_2, \lambda_1; x) \equiv \text{P exp} \left[-ig \int_{\lambda_1}^{\lambda_2} d\lambda \beta \cdot A^{(R)}(\lambda \beta + x) \right] \quad (34)$$

in color representation R , where “P” denotes path ordering. For the annihilation channel, $q\bar{q} \rightarrow \beta g$, we use these to construct a three-Wilson line product, which represents the source of radiation near partonic threshold:

$$[U_{q\bar{q}g}(x)]_{d;ji} \equiv \text{T}([\Phi_{\beta_g}^{(g)}(\infty, 0; x)]_{d,e} u_{e,ji}^{(q\bar{q})}(x)), \quad (35)$$

where “T” denotes time ordering, and where

$$u_{e,ji}^{(q\bar{q})}(x) \equiv [\Phi_{\beta_{\bar{q}}}^{(\bar{q})}(0, -\infty; x)]_{jl} (T_e^{(q)})_{lk} [\Phi_{\beta_q}^{(q)}(0, -\infty; x)]_{ki}, \quad (36)$$

with $T_e^{(q)}$ the SU(3) generator in the fundamental representation. For the Compton channel, $qg \rightarrow \beta q$, the corresponding product is

$$\begin{aligned} [U_{qqg}(x)]_{j;di} &= \text{T}([\Phi_{\beta_q}^{(q)}(\infty, 0; x)]_{jk} u_{k,di}^{(gq)}(x)), \\ u_{k,di}^{(gq)}(x) &= (T_e^{(q)})_{kl} [\Phi_{\beta_g}^{(g)}(0, -\infty; x)]_{ed} [\Phi_{\beta_q}^{(q)}(0, -\infty; x)]_{li}, \end{aligned} \quad (37)$$

and analogously for the $\bar{q}g \rightarrow \beta \bar{q}$ channel.

In terms of the operators just defined and the transform of Eq. (29), the eikonal cross sections are given by

$$\begin{aligned} \tilde{\sigma}_{abr}^{(\text{eik})}(N/Q, \hat{n}, \mu, \epsilon) &= \sum_X e^{-N(\beta_r \cdot p_X/Q)} \langle 0 | U_{abr}^\dagger(0) | X \rangle \langle X | U_{abr}(0) | 0 \rangle \\ &= \sum_X \langle 0 | U_{abr}^\dagger(0) | X \rangle \langle X | e^{-N(i\beta_r \cdot \partial_x/Q)} U_{abr}(x) | 0 \rangle_{x=0} \\ &= \langle 0 | U_{abr}^\dagger(0) e^{-N(i\beta_r \cdot \partial_x/Q)} U_{abr}(x) | 0 \rangle_{x=0}. \end{aligned} \quad (38)$$

In computing this quantity, an average over initial-state colors and a sum over final-state colors (for example, i, j and d , respectively, in $[U_{q\bar{q}g}]_{d;ji}$) is assumed but suppressed. The derivative operator in this form is fixed by the velocity of the outgoing Wilson line in the U_{abr} operators. We will use this feature in the following arguments.

Before analyzing the eikonal cross section using a graphical approach, we observe that at fixed values of $\beta_r \cdot p_X$, the matrix elements encountered in Eq. (38) for the eikonal cross section are the same as those in the SCET soft function analyzed in Ref. [42]. As in that analysis, we make

use of the scaling behavior of the integrands of eikonal cross section in terms of the velocities, $\beta_i \rightarrow \lambda \beta_i$. In the following, we will show that it is possible to write the logarithm of this function in an integral form that is the sum of far-infrared integrals that include the Landau pole times an explicit power of Q , as in Eq. (5), plus perturbative integrals that generate logarithms in the moment variation N of threshold resummation.

C. Cancellation of final-state interactions

The outgoing Wilson line effectively changes the exponential of the derivative to the exponential of a covariant derivative,

$$e^{-\frac{iN\beta_r \cdot \partial_x}{Q}} \Phi_{\beta_r}^{(r)}(\infty, 0; x) = \Phi_{\beta_r}^{(r)}(\infty, 0; x) e^{-\frac{iN\beta_r \cdot D^{(r)}(A(x))}{Q}}, \quad (39)$$

in terms of the covariant derivative $\beta_r \cdot D^{(r)}(A(x)) = \beta_r \cdot (\partial_x + igA^{(r)}(x))$, in the color representation of the recoil parton. Substituting this basic relation into the expression in Eq. (38), we readily find the cancellation of the unitary final-state Wilson lines, giving a form in which all interactions associated with the recoil line are in an exponentiated operator localized at the origin,

$$\begin{aligned} \tilde{\sigma}_{abr}^{(\text{eik})}(N/Q, \hat{n}, \mu, \epsilon) &= \langle 0 | u^{ab\dagger}(0) [e^{-\frac{iN\beta_r \cdot D^{(r)}(A(0))}{Q}} u^{ab}(0)] | 0 \rangle \\ &= \langle 0 | u^{ab\dagger}(0) \left[\left(1 - \frac{iN\beta_r \cdot D^{(r)}(A)}{Q} + \dots \right) u^{ab}(0) \right] | 0 \rangle \\ &= \sigma_{abr}^{(\text{eik},0)} (1 + \mathcal{O}(\alpha_s)). \end{aligned} \quad (40)$$

In the second equality, we exhibit the expansion in N/Q that we will employ below. Any truncation to a finite power of N/Q corresponds to a local operator of dimension N . In the third equality, we define the normalization factor $\sigma_{abr}^{(\text{eik},0)}$ that appears in the definition of the factor H_{abr} in Eq. (32). Note that $\sigma_{abr}^{(\text{eik},0)}$ is a pure color factor.

In Eq. (40), the cancellation of IR singularities associated with the final state is manifest at any order in the expansion. The eikonal cross section is an infinite sum of insertions of the covariant derivative in the β_r direction on the incoming lines of u^{ab} at the origin. After an order-by-order sum over final states, there are no singularities at any point where lines are parallel to the outgoing eikonal β_r , simply because there are no eikonal propagators in the perturbative expansion of the right-hand side of Eq. (40). Collinear singularities associated with the incoming eikonals remain, of course. As discussed following Eq. (31) all collinear singularities will be eliminated in the ratios that define the eikonal hard functions $\tilde{w}_{abr}^{(\text{eik})}$ in that equation. We will see how this occurs below.

Logarithms of the moment variable, N in $\tilde{\sigma}_{abr}^{(\text{eik})}$, organized by threshold resummation, are generated in the usual fashion by an incomplete cancellation of real and virtual emissions. This mismatch between real and virtual contributions requires all orders in the expansion of the exponent in Eq. (40), and we will not base our analysis below on this particular form. The essential conclusion that we employ from this expression is the absence of singular behavior associated with the outgoing eikonal in an alternative expression for the eikonal cross section, which is fully equivalent to (40) point-by-point in momentum space. This is the web expansion, to which we now turn.

IV. GRAPHICAL EXPONENTIATION AND RESUMMATION

We now recall the representation of eikonal cross sections as exponentials of web functions, and go on to review some of the relevant properties of web functions, in particular their renormalization-scale independence and absence of subdivergences. We explain the relation between perturbative resummation and web exponentiation, and compute the exponent at lowest order in the running coupling. Using the renormalization group independence of webs, we see already at this order that power corrections induced by the running coupling, or finite gluon mass, are characterized by even powers of N/p_T . In the following section, we go on to extend this result to all orders.

A. Graphical exponentiation and web functions

The value of the expression for the cross section, Eq. (40) is that it shows that all infrared singularities associated with couplings to the outgoing line cancel in the eikonal cross section. This result follows because the moment variable is defined by the same velocity vector, β_r , that defines the single outgoing Wilson line for the eikonal cross sections. In this section, we will exploit this cancellation, using the exponentiated web representation of these eikonal cross sections based on three eikonal lines. A specific discussion of exponentiation for products of three Wilson lines at the amplitude level can be found in [36], and an eikonal cross section written as in Eq. (38) or (40) falls into the general arguments of [43] as long as the integral over phase space is symmetric for all final-state partons. This is the case for the cross section defined by Eq. (38). It may therefore be written in the form

$$\tilde{\sigma}_{abr}^{(\text{eik})}(N/Q, \hat{\eta}, \mu, \epsilon) = \tilde{\sigma}_{abr}^{(\text{eik},0)} e^{E_{abr}(N/Q, \hat{\eta}, \mu, \epsilon)}, \quad (41)$$

where

$$\begin{aligned} E_{abr}(N/Q, \hat{\eta}, \mu, \epsilon) &= \int \frac{d^D k}{(2\pi)^D} \left(e^{-N \frac{\theta_{rk}}{Q}} - 1 \right) \theta\left(\frac{Q}{\sqrt{2}} - k^+\right) \theta\left(\frac{Q}{\sqrt{2}} - k^-\right) \\ &\times w_{abr}\left(\left\{\frac{\beta_i \cdot k \beta_j \cdot k}{\beta_i \cdot \beta_j}\right\}, k^2, \mu^2, \alpha_s(\mu^2)\right). \end{aligned} \quad (42)$$

For this three-eikonal case, the web function w_{abr} is a scalar in group space, and the overall factor $\tilde{\sigma}_{abr}^{(\text{eik},0)}$ is the zeroth-order eikonal cross section. The dependence of the web function on momentum k is fixed by the invariance of straight Wilson lines under rescalings of their defining velocities β_i . The explicit Laplace transform corresponds to contributions of all diagrams with at least one parton in the final state, weighted as in Eqs. (29) and (38). The step functions cut off final momenta at the scale of the partonic threshold center-of-mass energy [9]. Dependence of this cutoff on Q is exponentially suppressed for real radiation. For brevity of notation, we will suppress these theta functions below. The contributions of purely virtual diagrams are summarized by the term with “1” that is subtracted in the integrand. The form of the virtual contributions is fixed by imposing that the eikonal cross section vanish at $N = 0$, that is, for the fully inclusive eikonal cross section.

The perturbative expansion of the cross section in Eq. (42) is, of course, collinear-singular, and requires collinear subtractions for its incoming eikonals, as in Eq. (31). In computing the ratio of eikonal cross sections to eikonal parton distributions in Eq. (31), we use the exponential form for the latter [9], given in $D = 4 - 2\epsilon$ dimensions by

$$\tilde{\phi}_{i/i}(N, \mu, \epsilon) = \exp \left[\int_0^{\mu^2} \frac{dk_T^2}{k_T^{2(1+\epsilon)}} A_i(\alpha_s(k_T^2)) \int_0^1 dz \frac{z^N - 1}{1 - z} \right], \quad (43)$$

in terms of the familiar anomalous dimension $A_i(\alpha_s) = C_i(\alpha_s/\pi) + \dots$ with $C_q = C_F = 4/3$ and $C_g = C_A = 3$. Here, we exhibit the N -dependence of the full anomalous dimension $[-A(\alpha_s) \ln(Ne^{\gamma_E}) + \mathcal{O}(1/N)$, with γ_E the Euler constant] as an integral over a variable z . This form will be convenient in Sec. IV D.

Substituted into Eq. (31), the eikonal cross section (41) and subtractions (43) give for the eikonal hard-scattering cross section,

$$\tilde{\omega}_{abr}^{(\text{eik})}(N, Q, \hat{\eta}, \mu) = H_{abr}(p_T, \hat{\eta}, \mu) \sigma_{abr}^{(\text{eik},0)} e^{\hat{E}_{abr}(N/Q, \hat{\eta}, \mu)}, \quad (44)$$

where

$$\begin{aligned}
\hat{E}_{abr}(N/Q, \hat{\eta}, \mu) &= \int \frac{d^D k}{(2\pi)^D} \left(e^{-N\beta_r \cdot k/Q} - 1 \right) w_{abr} \left(\left\{ \frac{\beta_i \cdot k \beta_j \cdot k}{\beta_i \cdot \beta_j} \right\}, k^2, \mu^2, \alpha_s(\mu^2) \right) \\
&\quad - \sum_{i=a,b} \int_0^{\mu^2} \frac{dk_T^2}{k_T^{2(1+\epsilon)}} A_i(\alpha_s(k_T^2)) \int_0^1 dz \frac{z^{N_i} - 1}{1-z}. \quad (45)
\end{aligned}$$

In this form, the cancellation of collinear singularities appears in the exponent, the logarithm of the cross section, which as we have seen is given at each order by a sum of web diagrams. The integrals of the web diagrams will thus automatically generate a single collinear-singular integral, exactly matching the collinear integrals of the eikonal parton distributions, specified by the anomalous dimensions A_a . We will organize these integrals below in a manner that exhibits collinear-finite power corrections in N/Q , but the cancellation of leading-power collinear singularities is guaranteed by factorization, as discussed below Eq. (31).

Leading powers in Q come from the range of k where $N\beta_r \cdot k/Q > 1$ and real-gluon emission is exponentially suppressed. Potential contributions at nonleading powers in N/Q arise from the region $N\beta_r \cdot k/Q < 1$, and can be isolated by expanding the exponential, as in Eq. (40), for the full eikonal cross section. We will analyze this expansion in Sec. V, after discussing relevant properties of the web functions.

B. Webs, their renormalization and absence of subdivergences

The web functions in Eq. (42) can be defined recursively, in perturbation theory, with a general form given by [43]

$$\begin{aligned}
w^{(N+1)} &= \sum_{D^{(N+1)}} D^{(N+1)} \\
&\quad - \left[\sum_{m=2}^{N+1} \frac{1}{m!} \sum_{i_m=1}^N \dots \sum_{i_1=1}^N w^{(i_m)} w^{(i_{m-1})} \dots w^{(i_1)} \right]^{(N+1)}, \quad (46)
\end{aligned}$$

where the $D^{(N+1)}$ make up the full set of eikonal diagrams at the $(N+1)$ st order in α_s , and the web function $w^{(M)}$ is the sum of web diagrams at M th order. The square bracket with superscript $(N+1)$ indicates that the $(N+1)$ st order only is kept in the multiple sum over lower-order webs. At first order, the webs are given by the sum of single-gluon exchanges between Wilson lines, and Eq. (46) determines all higher orders. We have suppressed momentum dependence, integrations and parton labels, but we emphasize that this form applies to any eikonal cross section with symmetric phase space, not simply to the three-eikonal case at hand. In the general case, with four or more eikonals, the web functions are group space matrices, which have been studied to high order [44–46].

The webs for electroweak boson production cross sections in our case are given by diagrams with three-eikonal irreducibility, which differ from their standard forms by the subtraction of certain color factors [17,36] (maximally non-Abelian configurations are left). Their propagators are the usual causal free Green functions for quarks and gluons, and the usual eikonal denominators for Wilson lines. Because the propagators are standard, web diagram integrals over the components opposite to β_r can be carried out in light-cone perturbation theory, and the cancellation of final states holds diagram-by-diagram in the inclusive cross section [38]. The scale invariance of eikonal diagrams implies the homogeneous dependence on the variables involving the Wilson line velocities, β_i^μ shown in Eq. (42).

The integral of the virtual web function over momentum k in Eq. (45) requires ultraviolet renormalization, which is additive, and corresponds to the renormalization of the composite three-eikonal vertex that defines the cross section. At fixed momentum k , however, the web functions are unrenormalized, that is, renormalization scale invariant,

$$\mu \frac{d}{d\mu} w_{abr} \left(\left\{ \frac{\beta_i \cdot k \beta_j \cdot k}{\beta_i \cdot \beta_j} \right\}, k^2, \mu^2, \alpha_s(\mu^2) \right) = 0. \quad (47)$$

This is a consequence of the lack of an analog of wave function renormalization for Wilson lines [47]. Equation (47) holds for both massive and massless Wilson lines.

In addition to their renormalization properties, webs also have the important property that they are free of subdivergences, aside from logarithms that can be absorbed into the running coupling through Eq. (47). This means that IR divergences in the integral over k arise only when both k^2 and one of the combinations $\beta_i \cdot k \beta_j \cdot k$ vanish. For the cross section at hand, these are soft and collinear limits of web diagrams.

To make the discussion specific, let us consider a subspace S_L of the loop momenta of web $w^{(N)}$ in an L th-order subdiagram, $L < N$. We denote by “ $S_L \rightarrow 0$ ” a configuration where all of the loop momenta of the subdiagram are soft or collinear. We want to show that

$$w^{(N)} \xrightarrow{S_L \rightarrow 0} 0, \quad (48)$$

where the zero on the right-hand side refers to the absence of an infrared subdivergence. The lack of subdivergences, Eq. (48), certainly holds at $N=1$, which is the case of a single gluon. We next assume that it applies for webs up to order N .

Extending Eq. (48) to order $N+1$ is readily seen as a consequence of the order-by-order factorization properties of cross sections in such limits. We consider the L -loop on-shell, soft or collinear limit of $w^{(N+1)}$ as given by Eq. (46). For the first term on the right-hand side of (46), general factorization properties give

$$\sum_{D^{(N+1)}} D^{(N+1)} \xrightarrow{S_L \rightarrow 0} \sum_{D^{(L)}} D^{(L)} \sum_{D^{(N+1-L)}} D^{(N+1-L)}, \quad (49)$$

where corrections are again nonsingular. To treat the second term, consisting of sums of lower-order webs, we use our assumption that up to order N , web functions only diverge

$$\begin{aligned} w^{(N+1)} &\xrightarrow{S_L \rightarrow 0} \sum_{D^{(L)}} D^{(L)} \sum_{D^{(N+1-L)}} D^{(N+1-L)} - \left\{ w^{(L)} + \left[\sum_{m'=2}^L \frac{1}{m'!} \sum_{j_m=1}^L \dots \sum_{j_1=1}^L w^{(j_{m'})} w^{(j_{m'-1})} \dots w^{(j_1)} \right]^{(L)} \right\} \times \sum_{D^{(N+1-L)}} D^{(N+1-L)} \\ &= 0, \end{aligned} \quad (50)$$

where we have also used Eq. (46) to identify the full set of $(N+1-L)$ th order diagrams as a sum of webs. Referring above to Eq. (46), the definition of the web function at $(N+1)$ st order, we see that the term in curly brackets cancels the sum over the $D^{(L)}$, by the definition of the web function $w^{(L)}$ for $L \leq N$. The subdivergences of $w^{(N+1)}$ thus also cancel in the sum over webs, which is what we set out to show.

C. Perturbative resummations and eikonal power corrections

Before studying power corrections, we return to the full form of the eikonal cross section in Eq. (42) and the hard-scattering function (45), and remark that perturbative and nonperturbative contributions separate naturally in the integral over k . Perturbative resummation of logarithms in N can be derived from the range of momenta k for which $\beta_r \cdot k \geq Q/N$. In this region, the term due to real emission is exponentially suppressed, and logarithms in N arise entirely from the contributions of virtual diagrams, given by the function w_{abr} , with subtractions specified as in Eq. (45) in the range $1-z > 1/N$, where $z^N \ll 1$. The results for direct photons, developed from this point of view, are found in Ref. [9]. Alternative (and more familiar) approaches to resummation for single-particle production are based on refactorizations of the hard scattering function, leading to evolution equations [48–54], and on related analyses in soft collinear effective theory [55]. These analyses go beyond the eikonal approximation that we are discussing here. From a practical point of view, a purely eikonal analysis of resummation would miss contributions associated with partonic spin and kinematics starting at next-to-leading logarithms. For the reasons discussed at the beginning of Sec. III, however, we believe that the eikonal approximation captures the leading nonperturbative behavior that appears in powers of N/Q .

In what follows, we take this viewpoint, and concentrate on the range of k where $\beta_r \cdot k \leq Q/N$, and hence for which an expansion of the exponential $\exp[-N\beta_r \cdot k/Q]$ converges rapidly in Eqs. (42) and (45). In this range, which is not

when the entire web becomes collinear or soft altogether. Thus, the limit $S_L \rightarrow 0$ can only be realized when a single web in order L has loop momenta all in S_L , or when a set of web functions whose union is a diagram in the sum over $D^{(L)}$ have loop momenta that together are in S_L . Factoring these webs algebraically from the sum in (46), we find

normally included in perturbative resummation, there is an interplay between real and virtual corrections at low momentum scales. To study the integral in this region, the properties of the web functions derived above will be useful.

D. The lowest order web

We illustrate these considerations with the lowest-order web function for $q\bar{q} \rightarrow \beta + g$, given by the interference terms between gluon emission from each of the three Wilson lines. The C_F part of the calculation proceeds in much the same way as the one for the inclusive Drell-Yan cross section in Ref. [8]. It is natural to specialize to the frame where β_a^μ and β_b^μ are incoming along the 3 axis. We continue to normalize velocities as $\beta_{a,b}^0 = 1$, so that

$$\begin{aligned} \beta_a^\mu &= \sqrt{2}\delta_{\mu+}, \\ \beta_b^\mu &= \sqrt{2}\delta_{\mu-}, \end{aligned} \quad (51)$$

giving $\beta_a \cdot \beta_b = 2$ and

$$\frac{\beta_a \cdot k \beta_b \cdot k}{\beta_a \cdot \beta_b} = \frac{k^2 + k_T^2}{2}. \quad (52)$$

It is now convenient to project the recoil velocity in the following way:

$$\beta_r^\mu = \frac{\beta_r \cdot \beta_b}{\beta_a \cdot \beta_b} \beta_a^\mu + \frac{\beta_r \cdot \beta_a}{\beta_a \cdot \beta_b} \beta_b^\mu + \beta_{r,T}. \quad (53)$$

A bit of algebra then gives for the first-order web function in the $q\bar{q}$ channel:

$$\begin{aligned} w_{q\bar{q}g}^{(1)} &= 4\pi g^2 \mu^{2\epsilon} \delta(k^2) \\ &\times \left(C_F \frac{\beta_a \cdot \beta_b}{\beta_a \cdot k \beta_b \cdot k} + \frac{C_A}{2} \frac{\beta_a \cdot \beta_b}{\beta_a \cdot k \beta_b \cdot k} \frac{\beta_{r,T} \cdot k_T}{\beta_r \cdot k} \right) \\ &\equiv C_F u_{q\bar{q}g}^{(1)} + C_A v_{q\bar{q}g}^{(1)}. \end{aligned} \quad (54)$$

The term $u_{q\bar{q}g}^{(1)}$, multiplied by C_F contains all collinear singularities in the web contribution to Eq. (45). It is easy to

see that with $A(\alpha_s) = C_F \alpha_s / \pi + \dots$, these singularities are cancelled by the collinear subtraction, as they must be. The term $w_{q\bar{q}g}^{(1)}$, proportional to C_A is by itself collinear singular when k is collinear to β_r , but this singularity cancels between real and virtual corrections in Eq. (45). We may thus substitute the web function $w_{q\bar{q}g}^{(1)}$ of Eq. (54) in Eq. (45) and evaluate the resulting integral with $D = 4$. In the large- N limit, we can approximate the z integrals in the subtraction term of (45) by $\ln N_i e^{\gamma_E} \equiv \ln \bar{N}_i$. We then have, at lowest order in α_s ,

$$\begin{aligned} \hat{E}_{abr}^{(1)}(N/Q, \hat{\eta}, \mu) &= C_F \int \frac{d^4 k}{(2\pi)^4} \left(e^{-N \frac{\beta_r \cdot k}{Q}} - 1 \right) u_{abr}^{(1)}(k) \\ &\quad + \int_0^{\mu^2} \frac{dk_T^2}{k_T^2} C_F \frac{\alpha_s(k_T^2)}{\pi} \ln \left(\frac{\bar{N} \beta_{r,T}}{2} \right) \\ &\quad + C_A \int \frac{d^4 k}{(2\pi)^4} \left(e^{-N \frac{\beta_r \cdot k}{Q}} - 1 \right) v_{abr}^{(1)}(k) \\ &\equiv C_F U_{q\bar{q}g}^{(1)}(N/Q, \hat{\eta}, \mu) \\ &\quad + C_A V_{q\bar{q}g}^{(1)}(N/Q, \hat{\eta}, \mu), \end{aligned} \quad (55)$$

where we have used the choices of moment variables for single-particle cross sections given in Eq. (33) and that, in the normalization of Eq. (51), $\beta_{r,T}^2 = \beta_b \cdot \beta_r \beta_a \cdot \beta_r$.

It is straightforward to reduce the resulting expression for the C_F term in Eq. (55) to a single integral over k_T , in the limit that $N \gg 1$ for values of k_T small enough that $Nk_T/Q \leq 1$. We begin by expressing the mass-shell delta function in the web as $(1/2k^+) \delta(k^- - k_T^2/2k^+)$ to do the k^- integral, imposing the limits on the k^+ and k^- integrals shown explicitly in Eq. (42). In the large- N limit, we can then perform the k^+ integral to get a Bessel function, up to terms that are exponentially suppressed in N , using the relation

$$\begin{aligned} \int_{k_T^2/\sqrt{2}Q}^{Q/\sqrt{2}} \frac{dk^+}{2k^+} \left(e^{-\frac{N}{Q}(\beta_r k^+ + \beta_r^+ \frac{k_T^2}{2k^+})} - 1 \right) \\ = K_0 \left(\frac{N \beta_{r,T} k_T}{Q} \right) + \ln \left(\frac{k_T}{Q} \right) + \mathcal{O}(e^{-N}). \end{aligned} \quad (56)$$

We note that this approximation holds for any k_T less than Q , and that, in fact, the function $K_0(z)$ behaves as $z^{-1/2} e^{-z}$ for large z . For the function $U_{q\bar{q}g}^{(1)}(N/Q, \hat{\eta}, \mu)$, we then find

$$\begin{aligned} U_{q\bar{q}g}^{(1)}(N/Q, \hat{\eta}, \mu) \\ = 2 \int_0^{Q^2} \frac{dk_T^2}{k_T^2} \frac{\alpha_s(k_T^2)}{\pi} \left[I_0 \left(\frac{N \beta_{r,T} k_T}{Q} \right) K_0 \left(\frac{N \beta_{r,T} k_T}{Q} \right) \right. \\ \left. + \ln \left(\frac{\bar{N} \beta_{r,T} k_T}{2Q} \right) \right], \end{aligned} \quad (57)$$

where $I_0(x)$ is a standard Bessel function, for which $I_0(0) = 1$. In the range for which k_T is small enough that $Nk_T/Q < 1$, standard expansion formulas for the functions K_0 and I_0 show that power corrections involve only even powers of N/Q . As noted above, for large N and k_T , the Bessel function K_0 decreases exponentially. In Eq. (57), this leaves the exponentiating double-logarithmic corrections of threshold resummation. In summary, as anticipated in the discussion of the eikonal cross section at the end of Sec. III B, we have shown that at lowest order the logarithm of the cross section additively generates both the logarithms of threshold resummation and power corrections. We find the perturbative logarithms for large Nk_T , and inverse, even powers of Q^2 , times integrals with a Landau pole. In the next section, we extend this result to arbitrary order in the exponent.

For the second, C_A , term in Eq. (55), it is easier to compute the derivative with respect to N , which cancels the $\beta_r \cdot k$ denominator in Eq. (54). We find that

$$\begin{aligned} \frac{d}{dN} V_{q\bar{q}g}^{(1)}(N/Q, \hat{\eta}, \mu) &= \frac{2\beta_{r,T}}{Q} \int_0^{Q^2} \frac{dk_T^2}{k_T} \frac{\alpha_s(k_T^2)}{\pi} I_1 \left(\frac{N \beta_{r,T} k_T}{Q} \right) \\ &\quad \times K_0 \left(\frac{N \beta_{r,T} k_T}{Q} \right). \end{aligned} \quad (58)$$

Although N is large, in the region of interest we can still expand the Bessel functions in terms of their arguments, leading to an expression that begins at power N/Q times even powers of N/Q . Such an expansion is legitimate since the integrand in Eq. (55) vanishes at $N = 0$ for any k_T . This also means that there is no term constant in N in this expansion. Assuming $V^{(1)}$ has an expansion in the same variables, it must start at N^2/Q^2 , just as for $U^{(1)}$. In summary, the lowest-order behavior in N/Q for the functions $U^{(1)}$ and $V^{(1)}$ in the region $N \gg 1$, $Nk_T/Q \leq 1$ is

$$\begin{aligned} U_{q\bar{q}g}^{(1)}(N/Q, \hat{\eta}, \mu) &= \frac{1}{2} \left(\frac{N \beta_{r,T}}{Q} \right)^2 \int_0^{Q^2} dk_T^2 \frac{\alpha_s(k_T^2)}{\pi} \left[1 - 2 \ln \left(\frac{\bar{N} \beta_{r,T} k_T}{2Q} \right) \right] + \mathcal{O}(N^4/Q^4), \\ V_{q\bar{q}g}^{(1)}(N/Q, \hat{\eta}, \mu) &= -\frac{1}{2} \left(\frac{N \beta_{r,T}}{Q} \right)^2 \int_0^{Q^2} dk_T^2 \frac{\alpha_s(k_T^2)}{\pi} \ln \left(\frac{\bar{N} \beta_{r,T} k_T}{Q} \right) + \mathcal{O}(N^4/Q^4). \end{aligned} \quad (59)$$

Here we see explicitly the absence of linear powers. The part with $U_{q\bar{q}g}^{(1)}$ is very similar to the result for inclusive Drell-Yan obtained in [8]. As noted, the expansions of the relevant Bessel functions generate even powers to all orders in this expansion. Although we are not able of course to do the k_T integrals in (57) and (58) in closed form, this feature is completely general. We have checked that it equally applies in the case of qg scattering, despite the fact that the collinear subtraction here contains two separate terms, one with C_F and one with C_A .

We have used k_T as the scale of the running coupling in all of these expressions. This is a consistent choice, given the scale invariance of web functions, Eq. (47) and the lack of subdivergences in webs discussed above. The ambiguity present in resummed perturbation theory is then evident in Eq. (59), due to the Landau singularity in $\alpha_s(k_T^2)$. These singularities may be organized, for example, in Borel form [3], by using the representation

$$\alpha_s(k_T^2) = \int_0^\infty d\sigma \left(\frac{k_T^2}{\Lambda^2} \right)^{-\sigma\beta_0/4\pi}, \quad (60)$$

in Eq. (59), where $\beta_0 = 11 - 2N_f/3$ (with N_f the number of flavors) is the lowest-order coefficient of the QCD β function. In any case, the absence of a linear N/Q correction is the main result, familiar from Ref. [4]. We extend this result to all orders in the following section.

In closing this section, we can check that analogous exponentiating power corrections appear at lowest order and fixed coupling if we provide the gluon with a mass, as in Ref. [15]. This can be done by making the simple replacement

$$\delta(k^2) \rightarrow \delta(k^2 - \lambda^2) \quad (61)$$

in the lowest-order web, Eq. (54), also modifying k_T^2 to $k_T^2 + \lambda^2$ in the subtraction term. The resulting corrections also involve only even powers of λ .

V. INFRARED MOMENTA AND POWER CORRECTIONS

It is our goal in this section to demonstrate that the lack of linear powers in N/Q extends to all orders in the web expansion. The web expansion includes information on power corrections, derived by expanding the exponent in the region $\beta_r \cdot k \leq Q/N$ in Eq. (42) for the eikonal cross section. We have seen in Eq. (40) that in the full moment integral all final-state interactions generated by the outgoing Wilson line $\Phi_{\beta_r}^{(r)}$ can be replaced by an expansion in terms of operators that are local at the three-eikonal vertex (the “origin”). These operators are generated from the exponential of the covariant derivative $\beta_r \cdot D^{(r)}(A(0))$ multiplied by N/Q . Hence, each inverse power of Q in

the expansion comes with a polynomial of the same order in the vector β_r^μ .

We will begin our general study of power corrections with contributions that are linear in the vector β_r^μ . We will see that despite its overall factor of N/Q the $N\beta_r \cdot k/Q$ term is actually leading power and collinear singular. These singular terms, however, are cancelled in the eikonal hard-scattering function $\tilde{w}_{abr}^{(\text{eik})}$, leaving a residue that is free of linear, $1/Q$ corrections, and which begins at order N^2/Q^2 . We will then go on to show that this is the case for any power in the expansion in $N\beta_r \cdot k/Q$. In the following arguments, we use our ability to identify possible sources of logarithmic singularities with well-characterized regions in web loop momenta. We will find power corrections beginning at N^2/Q^2 associated with the low-scale behavior of the QCD running coupling at all orders.

A. Web functions in the power expansion

Perturbative resummations that are sensitive to the Landau singularity at power $1/Q$ must come from momentum integrals that are pinched at on-shell configurations in Eq. (42). The web function itself is finite order-by-order at fixed nonvanishing values of its arguments. It is only when a subset of the invariants vanish that there is the possibility of generating logarithms of these invariants.

We begin our discussion of power corrections with those associated with the first power in the expansion of the exponential E_{abr} in Eq. (42) in $N\beta_r \cdot k/Q$, which we consider to all orders in the strong coupling. In view of our observation that dependence on the outgoing velocity β_r^μ is local at the three-eikonal vertex in our cross section to any fixed power in $1/Q$, all true final-state interactions, associated with the outgoing Wilson line, cancel in the web function just as in the full moment of the eikonal cross section. We denote this result as follows:

$$\begin{aligned} E_{abr}^{[1]}(N/Q, \hat{n}, \mu, \epsilon) &\equiv -N \int \frac{d^D k}{(2\pi)^D} \left(\frac{\beta_r \cdot k}{Q} \right) \\ &\times w_{abr} \left(\left\{ \frac{\beta_i \cdot k \beta_j \cdot k}{\beta_i \cdot \beta_j} \right\}, k^2, \mu^2, \alpha_s(\mu^2) \right) \\ &\equiv -N \int \frac{d^D k}{(2\pi)^D} \left(\frac{\beta_r \cdot k}{Q} \right) \\ &\times \rho_{abr}^{[1]} \left(\left\{ \frac{\beta_i \cdot k \beta_j \cdot k}{\beta_i \cdot \beta_j} \right\}, k^2, \mu^2, \alpha_s(\mu^2) \right), \end{aligned} \quad (62)$$

where in the second form, the function $\rho_{abr}^{[1]}$ represents the sum of contributions to the full web function w_{abr} that remain after the sum over final states, that is, when integrated over final-state momentum k , weighted by $\beta_r \cdot k$. The superscript [1] in square brackets for the web integrand and the exponent refers to first order in the power

expansion, and remains to all orders in the coupling expansion. We note that at this point, we are working with the exponent of the full cross section rather than the subtracted exponent $\hat{E}_{abr}$ of Eq. (45). We will return to the role of the initial-state collinear subtractions that relate the two below.

Given the expression for the cross section in (42), in which final-state interactions have cancelled, we can conclude that the function $\rho_{abr}^{[1]}$ in Eq. (62) is much simpler than the full web function. Indeed, because $E_{abr}^{[1]}$ can depend on β_r only through a single linear factor, we can write the residual web function $\rho_{abr}^{[1]}$ in the form

$$\begin{aligned} \beta_r \cdot k \rho_{abr}^{[1]} & \left(\left\{ \frac{\beta_i \cdot k \beta_j \cdot k}{\beta_i \cdot \beta_j} \right\}, k^2, \mu^2, \alpha_s(\mu^2) \right) \\ &= \beta_r \cdot k g_{ab} \left(\frac{\beta_a \cdot k \beta_b \cdot k}{\beta_a \cdot \beta_b}, k^2, \mu^2, \alpha_s(\mu^2) \right) \\ &+ \frac{\beta_r \cdot \beta_a \beta_b \cdot k}{\beta_a \cdot \beta_b} g_{ar} \left(\frac{\beta_a \cdot k \beta_b \cdot k}{\beta_a \cdot \beta_b}, k^2, \mu^2, \alpha_s(\mu^2) \right) \\ &+ \frac{\beta_r \cdot \beta_b \beta_a \cdot k}{\beta_a \cdot \beta_b} g_{br} \left(\frac{\beta_a \cdot k \beta_b \cdot k}{\beta_a \cdot \beta_b}, k^2, \mu^2, \alpha_s(\mu^2) \right), \end{aligned} \quad (63)$$

where the functions g_{ij} have the same mass dimension as the webs, $-D$ in D dimensions. Note that the renormalization

group invariance of the full web, Eq. (47), is inherited by these functions individually. This is because the local subtractions involved in renormalization are independent of the expansion in $N\beta_r \cdot k/Q$. We therefore have

$$\mu \frac{d}{d\mu} g_{ij} = 0, \quad (64)$$

which we will use below.

B. Web structure at lowest order and beyond

Before turning to the identification of power corrections, we would like to see how the decomposition of the web functions $\rho_{abr}^{[1]}$ in Eq. (63) happens in perturbation theory. To this end, we return to the lowest order addressed in Sec. IV D where the web is a single gluon, exchanged among the three Wilson lines. This time, we only keep the linear term in the expansion in $N\beta_r \cdot k/Q$.

To illustrate the lowest order, we consider $q + \bar{q} \rightarrow \beta + g$, represented here, of course, by eikonal lines in the appropriate color representations. There are six terms (two pairs of three) corresponding to the interference between gluon emission from pairs of different webs. Factoring out the zeroth-order color factor ($C_F N_c$), which is not part of the web, we find, in $D = 4 - 2\epsilon$ dimensions,

$$\begin{aligned} \beta_r \cdot k \rho_{q\bar{q}g}^{[1],(1)} &= 2(2\pi)g^2\mu^{2\epsilon}\delta_+(k^2)\beta_r \cdot k \left\{ \left[C_F - \frac{C_A}{2} \right] \frac{\beta_a \cdot \beta_b}{\beta_a \cdot k \beta_b \cdot k} + \frac{C_A}{2} \left[\frac{\beta_r \cdot \beta_a}{\beta_a \cdot k \beta_r \cdot k} + \frac{\beta_r \cdot \beta_b}{\beta_b \cdot k \beta_r \cdot k} \right] \right\} \\ &= 2(2\pi)g^2\mu^{2\epsilon}\delta_+(k^2) \frac{\beta_a \cdot \beta_b}{\beta_a \cdot k \beta_b \cdot k} \left\{ \beta_r \cdot k \left[C_F - \frac{C_A}{2} \right] + \frac{C_A}{2} \left(\frac{\beta_r \cdot \beta_a \beta_b \cdot k}{\beta_a \cdot \beta_b} + \frac{\beta_r \cdot \beta_b \beta_a \cdot k}{\beta_a \cdot \beta_b} \right) \right\}, \end{aligned} \quad (65)$$

where the superscript with square brackets, [1], refers to the power expansion in N/Q , and the parentheses in superscript (1) refer to the order in α_s . In the second equality in Eq. (65), we match the kinematic factors involving β_r to the form in Eq. (63). This enables us to read off the functions g_{ij} at lowest order. In this case, two functions are equal,

$$\begin{aligned} g_{q\bar{q}}^{(1)} &= 2 \left[C_F - \frac{C_A}{2} \right] (2\pi)g^2\mu^{2\epsilon}\delta_+(k^2) \frac{\beta_a \cdot \beta_b}{\beta_a \cdot k \beta_b \cdot k}, \\ g_{qg}^{(1)} &= g_{\bar{q}g}^{(1)} = 2 \frac{C_A}{2} (2\pi)g^2\mu^{2\epsilon}\delta_+(k^2) \frac{\beta_a \cdot \beta_b}{\beta_a \cdot k \beta_b \cdot k}. \end{aligned} \quad (66)$$

All of these functions show collinear singularities at $k_T = 0$, with k_T defined relative to the scattering axis of the initial state. After the cancellation of final states, however, collinear singularities at order α_s occur only with color factor C_F , corresponding to the incoming eikonals in the quark pair representations, which are present in $g_{q\bar{q}}^{(1)}$. The C_A terms must be free of collinear singularities, which

in fact they are, since we can readily check that the terms proportional to C_A in (65) cancel when k is in the a or b direction. These observations are consistent with what we found in Sec. IV D; see Eq. (55).

For completeness, we present the corresponding results for the quark-gluon channel,

$$\begin{aligned} g_{qg}^{(1)} &= 2 \left[C_F - \frac{C_A}{2} \right] (2\pi)g^2\mu^{2\epsilon}\delta_+(k^2) \frac{\beta_a \cdot \beta_b}{\beta_a \cdot k \beta_b \cdot k}, \\ g_{gq}^{(1)} &= g_{qg}^{(1)} = 2 \frac{C_A}{2} (2\pi)g^2\mu^{2\epsilon}\delta_+(k^2) \frac{\beta_a \cdot \beta_b}{\beta_a \cdot k \beta_b \cdot k}. \end{aligned} \quad (67)$$

Here, we readily verify that collinear singularities again appear in the a and b directions with color factors consistent with collinear factorization.

We now show how the same structure is preserved in the function $E_{abr}^{[1]}$ to arbitrary orders in the coupling. For any final state, the moment function can be thought of as a sum of two contributions,

$$\beta_r \cdot k = \beta_r \cdot k_{ab} + \beta_r \cdot k_r, \quad (68)$$

where k_{ab} is the total momentum flowing out of all vertices on the incoming eikonals, and k_r is the total momentum flowing out of the composite vertex at which the outgoing eikonal couples to the incoming eikonals. The sum over final states can be taken at fixed k_{ab} , and the resulting sum vanishes after integration. The $\beta_r \cdot k_r$ term on the right-hand side of Eq. (68) is the inverse of the first outgoing eikonal propagator, corresponding to the gauge field term in the covariant derivative in the expansion of Eq. (40). This new vertex, with field $\beta_r \cdot A^{(r)}$ associated with the three-eikonal vertex, appears for every remaining final state, for which the sum over final states again cancels.

The cancellation can be made manifest by integrating over loop and phase space momentum components projected by β_r , including $\beta_r \cdot k$, in light-cone ordered perturbation theory. This cancellation corresponds to the statement that inserting a complete set of states between the two local operators of Eq. (40), both of which are fixed at the origin, does not change the value of the matrix element. The exact cancellation assumes that we integrate over all values of the final-state quantity $\beta_r \cdot k$. In the expressions we consider below, this integration will only be carried out to the order of Q/N . Induced corrections will be suppressed by additional powers of N/Q , which we consider to be small. In this approximation, all final state interactions cancel in the resulting sum over final states, and the result is an expression in which the velocity β_r appears only in products with k and/or the incoming velocities, times functions that are boost invariant on the $\beta_a - \beta_b$ axis and invariant under scalings of β_a, β_b , just as in Eq. (63).

C. Organizing collinear singularities

We adopt the frame introduced in Sec. IV D, specifically Eqs. (51) and (52). In the following, we will relabel the arguments of the web functions as $g_{ab}(k_T^2, k^2, \mu^2, \alpha_s(\mu^2))$, and so forth, or for succinctness, simply $g_{ab}(k_T^2, k^2)$. With these choices, the first term in the power expansion, using (63) for $\rho_{abr}^{[1]}$ in Eq. (42) for the exponent, is given by

$$\begin{aligned} E_{abr}^{[1]}(N/Q, \hat{n}, \mu, \epsilon) = & -\frac{N}{Q} \int \frac{d^D k}{(2\pi)^D} \left[\beta_r \cdot k g_{ab}(k_T^2, k^2) \right. \\ & + \frac{\beta_r \cdot \beta_a \beta_b \cdot k}{\beta_a \cdot \beta_b} g_{ar}(k_T^2, k^2) \\ & \left. + \frac{\beta_r \cdot \beta_b \beta_a \cdot k}{\beta_a \cdot \beta_b} g_{br}(k_T^2, k^2) \right]. \quad (69) \end{aligned}$$

Because we are at the level of the cross section, the g_{ij} are collinear divergent when k is parallel to either β_a or β_b . Because they appear in the web function, however, these

singularities appear additively, and require that all lines associated with the web become collinear at the same time.

The collinear singularities of $E_{abr}^{[1]}$ associated with the incoming eikonals are to be removed by eikonal collinear subtractions, as in Eq. (45), and it is the subtracted form that must be analyzed for renormalon or finite-mass power corrections. The analysis is closely related to that for the Drell-Yan cross section [9].

We next use the expansion of β_r^μ given in Eq. (53) for the factor $\beta_r \cdot k$ in Eq. (69). The term proportional to $\beta_{r,T} \cdot k_T$ vanishes because the integrand is otherwise azimuthally symmetric. It is then natural to combine the terms proportional to $\beta_a \cdot k$, and those proportional to $\beta_b \cdot k$, defining

$$\begin{aligned} G_a & \equiv g_{ab} + g_{ar}, \\ G_b & \equiv g_{ab} + g_{br}, \end{aligned} \quad (70)$$

in terms of which (69) becomes

$$\begin{aligned} E_{abr}^{[1]}(N/Q, \hat{n}, \mu, \epsilon) & = -\frac{N}{Q} \int \frac{d^D k}{(2\pi)^D} \left[\frac{\beta_r \cdot \beta_a \beta_b \cdot k}{\beta_a \cdot \beta_b} G_a(k^2, k_T^2) \right. \\ & \quad \left. + \frac{\beta_r \cdot \beta_b \beta_a \cdot k}{\beta_a \cdot \beta_b} G_b(k^2, k_T^2) \right] \\ & \equiv \mathcal{E}_a^{[1]}(N/Q, \hat{n}, \mu, \epsilon) + \mathcal{E}_b^{[1]}(N/Q, \hat{n}, \mu, \epsilon), \quad (71) \end{aligned}$$

where we exhibit only the momentum arguments for G_a and G_b , and introduce a notation for their integrals. All singularities for k collinear to β_a are in the $\mathcal{E}_a^{[1]}$ term, and all singularities for k collinear to β_b are in the $\mathcal{E}_b^{[1]}$ term.

We are now ready to analyze the k integrals of $E_{abr}^{[1]}$, to all orders. Because we are interested in singularities in the running coupling at small μ , we limit ourselves to the region where $k_T^2 \leq \kappa^2 \sim \Lambda_{\text{QCD}}^2$. An expansion in $\beta_r \cdot k$ is then similar to an expansion in energy for gluons emitted at low transverse momentum relative to the incoming directions.

D. Infrared momenta and the absence of linear power corrections

We recall that to identify the power corrections that emerge in the cross section, we have expanded the Laplace transform exponential, $\exp(-N\beta_r \cdot k/Q)$, in the exponent E_{abr} of Eq. (42) in collinear and soft regions. This gives rise to the following conditions on the energy of soft radiation:

$$\begin{aligned} k_0 & \leq \frac{1}{\beta_a \cdot \beta_r} \frac{Q}{N}, \quad \text{for } k \text{ collinear to } \beta_a, \\ k_0 & \leq \frac{1}{\beta_b \cdot \beta_r} \frac{Q}{N}, \quad \text{for } k \text{ collinear to } \beta_b. \end{aligned} \quad (72)$$

In these terms, the eikonal cross section will depend on the overall scale Q and the invariants $\beta_a \cdot \beta_r$ and $\beta_b \cdot \beta_r$.

Correspondingly, in the subtraction terms in Eq. (45), we split the integral over z into two regions,

$$\int_0^1 dz \frac{z^N - 1}{1 - z} = \int_0^{1-\frac{1}{N}} dz \frac{z^N - 1}{1 - z} + \int_{1-\frac{1}{N}}^1 dz \frac{z^N - 1}{1 - z}. \quad (73)$$

The first region produces the familiar perturbative logarithm of N in the moments of the anomalous dimension for the eikonal parton distributions, whose role is to cancel a similar term in the eikonal hard-scattering functions $\tilde{\sigma}_{abr}^{(\text{eik})}$ [9], as in Eq. (31). We will not consider the integration over this region further. The second term, with the integration over $1 - 1/N \leq z \leq 1$, is relevant for our analysis of power corrections. Making the approximation $z^N \sim \exp[-N(1-z)]$, which has corrections suppressed by additional powers of N , and considering the region $k_T^2 \leq \kappa^2 \sim \Lambda_{\text{QCD}}^2$, we have

$$\begin{aligned} \sum_{i=a,b} \int_0^{\kappa^2} \frac{dk_T^2}{k_T^{2(1+\epsilon)}} A_i(\alpha_s(k_T^2)) \int_{1-\frac{1}{N}}^1 dz \frac{e^{-N(1-z)} - 1}{1 - z} \\ = - \sum_{i=a,b} \int_0^{\kappa^2} \frac{dk_T^2}{k_T^{2(1+\epsilon)}} A_i(\alpha_s(k_T^2)) \left[1 - \sum_{P=2}^{\infty} \frac{(-1)^P}{P!P} \right]. \quad (74) \end{aligned}$$

This is a leading-power and collinear-singular correction, sensitive to low momentum scales, which will precisely cancel the singularities of the eikonal cross section, for $1 - 1/N < z < 1$. We have isolated the first power, $P = 1$ in the sum, which will cancel collinear singularities in $E_{abr}^{[1]}$, Eq. (71), the first term in the eikonal cross section expanded in powers of $N\beta_r \cdot k/Q$. We will return to the higher values of P in the following subsection.

Equation (71) and the $P = 1$ term of (74) specify the full integral that includes the nonperturbative corrections that we are after. Identifying these terms as they appear in Eq. (45) we obtain

$$\begin{aligned} \hat{E}_{abr}^{[1]}(N/Q, \hat{\eta}, \mu) &= \mathcal{E}_a^{[1]}(N/Q, \hat{\eta}, \mu, \epsilon) \\ &+ \int_0^{\kappa^2} \frac{dk_T^2}{k_T^{2(1+\epsilon)}} A_a(\alpha_s(k_T^2)) \\ &+ \mathcal{E}_b^{[1]}(N/Q, \hat{\eta}, \mu, \epsilon) \\ &+ \int_0^{\kappa^2} \frac{dk_T^2}{k_T^{2(1+\epsilon)}} A_b(\alpha_s(k_T^2)). \quad (75) \end{aligned}$$

We look first at the terms from the cross section that contain the collinear singularities in the a direction, again restricting to the range $k_T^2 \leq \kappa^2$:

$$\begin{aligned} \mathcal{E}_a^{[1]}(N/Q, \hat{\eta}, \mu, \epsilon) &= - \frac{N \beta_a \cdot \beta_r}{Q \beta_a \cdot \beta_b} \Omega_{1-\epsilon} \int_0^{\kappa^2} \frac{dk_T^2 k_T^{-2\epsilon}}{2(2\pi)^D} \int_0^{Q^2/(\beta_a \cdot \beta_r N)^2 - k_T^2} dk^2 G_a(k^2, k_T^2) \\ &\times \int_{\sqrt{k^2 + k_T^2}}^{Q/(\beta_a \cdot \beta_r N)} dk_0 \int \frac{dk_3}{2|k_3|} \beta_b \cdot k \left[\delta(k_3 - \sqrt{k_0^2 - k_T^2 - k^2}) + \delta(k_3 + \sqrt{k_0^2 - k_T^2 - k^2}) \right], \quad (76) \end{aligned}$$

where $\Omega_{1-\epsilon} = 2\pi^{1-\epsilon}/\Gamma(1-\epsilon)$ is the result of the angular integral that generalizes the azimuthal integral in dimensional regularization. Because of the boost invariance of the web functions along the 3 axis, the energy integral can be carried out explicitly. With $\beta_b \cdot k = k_0 + k_3$ we find

$$\begin{aligned} \mathcal{E}_a^{[1]}(N/Q, \hat{\eta}, \mu, \epsilon) &= - \frac{N \beta_a \cdot \beta_r}{Q \beta_a \cdot \beta_b} \Omega_{1-\epsilon} \int_0^{\kappa^2} \frac{dk_T^2 k_T^{-2\epsilon}}{2(2\pi)^D} \int_0^{Q^2/(\beta_a \cdot \beta_r N)^2 - k_T^2} dk^2 G_a(k^2, k_T^2) \sqrt{\left(\frac{Q}{\beta_a \cdot \beta_r N}\right)^2 - k^2 - k_T^2} \\ &= - \frac{\Omega_{1-\epsilon}}{2\pi} \int_0^{\kappa^2} dk_T^2 k_T^{-2\epsilon} \int_0^{Q^2/(\beta_a \cdot \beta_r N)^2 - k_T^2} \frac{dk^2}{4(2\pi)^{D-1}} G_a(k^2, k_T^2) \left[1 - \frac{(k^2 + k_T^2)(\beta_a \cdot \beta_r)^2 N^2}{2Q^2} + \dots \right], \quad (77) \end{aligned}$$

where in the second equality, we have used $\beta_a \cdot \beta_b = 2$, in the normalization of Eq. (51). In the expansion, we identify a leading-power term, corrections beginning at the N^2/Q^2 level, as anticipated. As discussed after Eq. (45), factorization ensures that the singularities associated with the leading-power term from the web function integrals are cancelled by the subtraction in Eq. (75) from the eikonal parton distribution [9]. This implies that

$$\begin{aligned} \int_0^{Q^2/(\beta_a \cdot \beta_r N)^2 - k_T^2} \frac{dk^2}{4(2\pi)^{D-1}} G_a(k^2, k_T^2) \\ = \frac{A_a(\alpha_s(k_T^2))}{k_T^2} + \mathcal{A}_a(k_T^2, Q^2), \quad (78) \end{aligned}$$

where the function $\mathcal{A}_a$ is collinear finite. In fact, because $\mathcal{A}_a$ can depend only on k_T^2 and Q^2 , it must be suppressed by a power of Q/N . More importantly, by the properties of

web functions, it has no logarithmic scales except for $Q/(N\mu)$. The scale of the running coupling is thus naturally chosen as Q/N in this term. This is still a perturbative scale, and does not imply a nonperturbative power correction from the integral over k_T . We assume, therefore, that it can be neglected. A result completely analogous to Eq. (78) holds for collinear radiation in the b direction, with subscript b everywhere. Since collinear singularities now cancel, we can return our full expression in Eq. (75) to $D = 4$.

The leading correction that remains after the cancellation of collinear singularities is then obtained from (75) as [we recall from (70) that the flavor index r for the recoil parton is implicit in the functions $G_{a,b}$]

$$\begin{aligned} \hat{E}_{abr}^{[1]}(N/Q, \hat{\eta}, \mu) &= \frac{(\beta_a \cdot \beta_r)^2 N^2}{2Q^2} \int_0^{\kappa^2} dk_T^2 \int_0^{Q^2/(\beta_a \cdot \beta_r N)^2 - k_T^2} \frac{dk^2}{4(2\pi)^3} (k^2 + k_T^2) \\ &\times [G_a(k^2, k_T^2, \alpha_s(k_T^2)) + G_b(k^2, k_T^2, \alpha_s(k_T^2))]. \end{aligned} \quad (79)$$

In this expression, we have reintroduced dependence on the running coupling in the functions $G_{a,b}$, using their renormalization-scale independence, Eq. (64). We note that by the general properties of the web functions discussed in Sec. IV B, all logarithmic behavior is associated with k^2 and k_T^2 vanishing at the same time. Then, choosing $\mu^2 = k_T^2$ absorbs all perturbative logarithms into the running coupling. The ambiguities in the low-scale behavior of the coupling induce the power corrections to perturbation theory.

On the basis of this discussion, we see that the term nominally power-suppressed by N/Q includes a leading power contribution from values of k_0 of order Q/N , which, however, is cancelled by the collinear subtraction. Nevertheless, the next power corrections appear at order N^2/Q^2 , with no linear, N/Q contribution. To complete our argument on the absence of linear power corrections, we return now to higher orders in the expansion in $N\beta_r \cdot k/Q$.

E. Higher orders in the power expansion

We know that higher powers in the expansion of the exponent Eq. (45) contribute at leading and nonleading powers in N/Q , because they are necessary to match to the full collinear subtraction, beyond the first term in the expansion shown in Eq. (74).

The treatment of these terms closely follows that of the lowest order, N/Q , just described in detail. We consider order P in the expansion of the exponent, and note that the arguments of Sec. V B apply to arbitrary powers of the weight $\beta_r \cdot k$. The general form is [see Eq. (62)]

$$\begin{aligned} E_{abr}^{[P]}(N/Q, \hat{\eta}, \mu, \epsilon) &\equiv -\frac{1}{P!} \left(\frac{N}{Q}\right)^P \int \frac{d^D k}{(2\pi)^D} (\beta_r \cdot k)^P \\ &\times \rho_{abr}^{[P]} \left(\left\{ \frac{\beta_i \cdot k \beta_j \cdot k}{\beta_i \cdot \beta_j} \right\}, k^2, \mu^2, \alpha_s(\mu^2) \right), \end{aligned} \quad (80)$$

where $\rho_{abr}^{[P]}$ is the web function that remains after the cancellation of final state interactions. Generalizing the case of $P = 1$, the product $(\beta_k \cdot k_j)^P \rho_{abr}^{[P]}$ is proportional to the P th power of β_r and of dot products of k and the β_i , $i = a, b, r$, and is otherwise a function of boost-invariant factors of k^2 and k_T^2 , using the frame chosen above. We may thus write

$$\begin{aligned} (\beta_r \cdot k)^P \rho_{abr}^{[P]} &= \sum_{A=0}^P \sum_{B=0}^{P-A} (\beta_r \cdot k)^{P-A-B} \left(\frac{\beta_r \cdot \beta_a \beta_b \cdot k}{\beta_a \cdot \beta_b} \right)^A \\ &\times \left(\frac{\beta_r \cdot \beta_b \beta_a \cdot k}{\beta_a \cdot \beta_b} \right)^B g_{AB}^{(P)}(k^2, k_T^2), \end{aligned} \quad (81)$$

where by comparison to the $P = 1$ case, (63) we can identify

$$\begin{aligned} g_{00}^{(1)} &= g_{ab}, \\ g_{10}^{(1)} &= g_{ar}, \\ g_{01}^{(1)} &= g_{br}. \end{aligned} \quad (82)$$

Expanding factors of $\beta_r \cdot k$ as in Eq. (53), we derive a set of terms that are polynomials in $\beta_a \cdot k$, $\beta_b \cdot k$, and in even powers of $\beta_{r,T} \cdot k_T$.²

At P th order in the power expansion, we want to identify terms in the integral over k that can compensate for the overall power of $(N/Q)^P$ in Eq. (80), and to confirm that there are no corrections at linear, $1/Q$, order. As in the case of $P = 1$, contributions that grow with Q can come only from the k_0 integral. Although for large P there are many terms in Eq. (81), powers Q^0 and Q^{-1} can come about only from factors $(\beta_b \cdot k)^P$, which give collinear singularities in the β_a direction, or $(\beta_a \cdot k)^P$, which give collinear singularities in the β_b direction. In particular, any factor $\beta_a \cdot k \beta_b \cdot k = (k^2 + k_T^2)/2$ or $(\beta_{r,T} \cdot k_T)^2$ gives corrections that suppress the integral by $1/Q^2$.

In summary, the only possible sources of $1/Q$ corrections are integrals of the type

$$\begin{aligned} \mathcal{E}_a^{[P]}(N/Q, \hat{\eta}, \mu, \epsilon) &= -\frac{1}{P!} \left(\frac{N}{Q}\right)^P \int \frac{d^D k}{(2\pi)^D} \left(\frac{\beta_r \cdot \beta_a \beta_b \cdot k}{\beta_a \cdot \beta_b} \right)^P \\ &\times G_a^{[P]}(k^2, k_T^2), \end{aligned} \quad (83)$$

²We have simplified Eq. (81) somewhat by neglecting terms with invariants $\beta_r \cdot \beta_a \beta_r \cdot \beta_b / \beta_a \cdot \beta_b$ times k^2 or k_T^2 . In such terms, the leading contributions begin at order $1/Q^2$.

where $G_a^{[P]}$ generalizes G_a in Eq. (70),

$$G_a^{[P]} = \sum_{A=0}^P \sum_{B=0}^{P-A} g_{AB}^{[P]}(k^2, k_T^2). \quad (84)$$

Expanding the integrals, this is

$$\begin{aligned} \mathcal{E}_a^{[P]}(N/Q, \hat{\eta}, \mu, \epsilon) &= -\frac{1}{P!} \left(\frac{N}{Q}\right)^P \left(\frac{\beta_a \cdot \beta_r}{\beta_a \cdot \beta_b}\right)^P \Omega_{1-\epsilon} \int_0^{\kappa^2} \frac{dk_T^2 k_T^{-2\epsilon}}{2(2\pi)^D} \\ &\times \int_0^{Q^2/(\beta_a \cdot \beta_r N)^2 - k_T^2} dk^2 G_a^{[P]}(k^2, k_T^2) \int_{\sqrt{k^2 + k_T^2}}^{Q/(\beta_a \cdot \beta_r N)} dk_0 \\ &\times \int \frac{dk_3}{2|k_3|} (\beta_b \cdot k)^P \left[\delta\left(k_3 - \sqrt{k_0^2 - k_T^2 - k^2}\right) \right. \\ &\left. + \delta\left(k_3 + \sqrt{k_0^2 - k_T^2 - k^2}\right) \right]. \end{aligned} \quad (85)$$

$$\begin{aligned} I_{m,n}(Q/N) &= \left(\frac{N}{Q}\right)^{m+n+1} \left\{ \frac{1}{m+n+1} \left(\left(\frac{Q}{N}\right)^{m+n+1} - l_0^{m+n+1} \right) - l_0^{m+n+1} \int_1^{Q/(Nl_0)} dx x^{m+n} \left(1 - \left(1 - \frac{1}{x^2}\right)^{n/2} \right) \right\} \\ &= \frac{1}{m+n+1} + \mathcal{O}\left(\frac{N^2}{Q^2}\right). \end{aligned} \quad (87)$$

Substituting this result into Eq. (85), the Q^0 contribution leads to a collinear-singular integral over k_T^2 . This is just that part of the standard initial-state collinear singularity of the eikonal cross section that shows up in the $(N/Q)^P$ term of the expansion of the exponential in Eq. (45). As such, the collinear finiteness of the eikonal hard-scattering function $\tilde{\omega}_{abr}^{(\text{eik})}$ again ensures that the dimensional poles of this integral will match the corresponding term in the expansion of the eikonal collinear subtraction, Eq. (74). This implies a relation between the function $G_a^{[P]}$ and the anomalous dimension $A_a(\alpha_s)$, analogous to Eq. (78).

It will be of interest for future work to study the systematics of the full set of $(N/Q)^2$ nonperturbative contributions, in terms of their low-order expansions in the running coupling, using methods related to those in Refs. [3,19], and perhaps the all-order structure of these contributions.

VI. CONCLUSIONS AND OUTLOOK

We have shown in full QCD the absence of coupling-induced linear power corrections in the eikonal approximation, for electroweak boson production at fixed transverse momentum and rapidity. This result is consistent with the recent detailed analysis at order g^2 based on models with a massive gluon [15] and in particular the use of azimuthal symmetry for isolated terms proportional to a

single power of k_T . It is not difficult to show that although this integral produces exactly the leading power in the P th terms of the collinear subtraction Eq. (74), its corrections linear in N/Q vanish, leaving corrections that begin at $(N/Q)^2$. To verify this claim, we can consider the class of integrals,

$$I_{m,n}(Q/N) \equiv \left(\frac{N}{Q}\right)^{m+n+1} \int_{l_0}^{Q/N} dk_0 k_0^m (k_0^2 - l_0^2)^{n/2}, \quad (86)$$

with $l_0^2 = k^2 + k_T^2$. The case $P = 1$ is realized by $m = 1$, $n = -1$ or by $m = n = 0$, but only the former is relevant here since for $m = n = 0$ the k_3 integral in Eq. (85) vanishes by antisymmetry of the integrand. The $P > 1$ integrals provide $m + n \geq 1$. Adding and subtracting k_0^{m+n} in the integrand, and changing variables to $x = k_0/l_0$, we find

single power of k_T . It is worth noting that the set of integrals considered in Ref. [15] includes several that reflect the recoil of hard partons to soft radiation. These terms are not included in our analysis because they are suppressed by a power of the soft gluon energy, and hence by $1/N$ in the Laplace moment, before expansion of the exponential. Such terms are independent of the resummation of threshold logarithms.

Looking ahead, it may be possible to generalize our analysis beyond the eikonal approximation, relying on the “next to eikonal” structure of these cross sections investigated in recent years [56–59]. We also expect our results to have valuable ramifications for phenomenology, given the vast body of experimental measurements for single electroweak boson production that have been carried out. Prompt photons produced in fixed-target scattering have presented a long-standing challenge to theory [60–62], which makes an improved treatment of power corrections highly desirable. Similarly, recent studies [63,64] have found that fixed-order and even threshold-resummed perturbation theory for semi-inclusive deep-inelastic scattering and the Drell-Yan process at large transverse momentum do not describe the available experimental data well in the fixed-target regime. Equipped with a more robust understanding of power corrections, one may now attempt to resolve the discrepancies between data and theory for these reactions, perhaps using models of the strong coupling, as for example, in Ref. [19].

Finally, it will clearly be very interesting to extend our study to the case of “pure QCD” processes, that is, replacing the produced electroweak boson by a QCD parton that subsequently fragments into a hadron or produces a jet. In this way, one can address power corrections for a larger class of single-particle reactions, where the eikonal approximation involves the full complexity of QCD. Here the question as to whether power corrections start at order $1/p_T$ or $1/p_T^2$ is still open. Its resolution will likely impact the description of fixed-target scattering data for single-hadron production, as well as the fragmentation contribution to the prompt-photon cross

section. In this context, it is interesting to note that recent Phenix photon data [65] are in better agreement with NLO theory for the isolated case than for the nonisolated one at $p_T \lesssim 10$ GeV, the latter having a much larger fragmentation component.

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