PAPER

On general-*n* coefficients in series expansions for row spin–spin correlation functions in the two-dimensional Ising model

IOP Publishing Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **55** (2022) 425001 (18pp) https://doi.org/10.1088/1751-8121/ac9654

On general-*n* coefficients in series expansions for row spin–spin correlation functions in the two-dimensional Ising model

Robert Shrock*

C N Yang Institute for Theoretical Physics and Department of Physics and Astronomy, Stony Brook University, Stony Brook, NY 11794, United States of America

E-mail: robert.shrock@stonybrook.edu

Received 18 July 2022, revised 16 September 2022
Accepted for publication 29 September 2022
Published 19 October 2022

Abstract

We consider spin–spin correlation functions for spins along a row, $R_n = \langle \sigma_{0,0}\sigma_{n,0}\rangle$, in the two-dimensional Ising model. We discuss a method for calculating general-*n* expressions for coefficients in high-temperature and low-temperature series expansions of R_n and apply it to obtain such expressions for several higher-order coefficients. In addition to their intrinsic interest, these results could be useful in the continuing quest for a nonlinear ordinary differential equation whose solution would determine R_n, analogous to the known nonlinear ordinary differential equation whose solution determines the diagonal correlation function $\langle \sigma_{0,0}\sigma_{n,n}\rangle$ in this model.

Keywords: correlation, functions, Ising, series, expansions

1. Introduction

Spin–spin correlation functions contain information about the degree of magnetic ordering in a spin model. The two-dimensional Ising model provides a valuable context in which one can obtain exact closed-form analytic expressions for these correlation functions. In thermal equilibrium at temperature T on the square lattice, the partition function of the (zero-field, isotropic, spin-1/2, nearest-neighbor (*nn*)) Ising model is given by

$$Z = \sum_{\{\sigma\}} e^{-\beta \mathcal{H}}, \tag{1.1}$$

*Author to whom any correspondence should be addressed.

where the Hamiltonian is

$$\mathcal{H} = -J\sum_{nn} \sigma_{\vec{r}}\sigma_{\vec{r}'}. \tag{1.2}$$

In equation (1.2), $\beta = 1/(k_{\rm B}T)$; the sum is over *nn* sites on the lattice; and $\sigma_{\vec{r}} = \pm 1$ is a classical spin variable defined on each lattice site. Given the well-known mapping on a bipartite lattice between the ferromagnetic ($J > 0$) and antiferromagnetic ($J < 0$) spin–spin couplings [5], one can, with no loss of generality, take $J > 0$, and we will do this. In the following, we assume the thermodynamic limit. This model has a global $\mathbb{Z}_2$ symmetry, which is spontaneously broken with the onset of a nonzero spontaneous magnetization M as the temperature decreases below the critical temperature, T_c. The system undergoes a continuous, second-order phase transition at this critical temperature.

The two-dimensional Ising model has the appeal that many of its properties are known exactly. The free energy was calculated by Onsager [1], and an expression for the magnetization was first published by Yang [2]. A method for calculating spin–spin correlation functions in terms of Toeplitz determinants was developed by Kaufman and Onsager [3] and later extended by Montroll *et al* [4]. Some reviews of the Ising model include [5, 6]. The critical behavior is known exactly; the thermal and magnetic exponents are $y_t = 1$ and $y_h = 15/8$, and hence the critical exponents ν, α, β, γ, etc, are known for models in the $d = 2$, $\mathbb{Z}_2$ universality class of second-order phase transitions, of which the *nn* spin-1/2 Ising model is arguably the simplest example. Generalizations to anisotropic couplings, spin $s \geqslant 1$, and non-*nn* two-spin and multi-spin interactions preserving the $\mathbb{Z}_2$ symmetry that do not cause frustration are in the same universality class. The critical behavior was elucidated in the context of continuous spatial dimensionality (above the lower critical dimensionality, $d = 1$) via the momentum-space renormalization group and ϵ expansion, where $\epsilon = 4 - d$ [7–9]. Further insight into the critical behavior of the $d = 2$, $\mathbb{Z}_2$ universality class was obtained by means of conformal algebra methods [10], which showed that the critical behavior is described by a rational conformal field theory with central charge $c = 1/2$ (reviewed, e.g. in [11]). Although the 2D Ising model is classical, it can be related to a 1D quantum spin chain [12]. In addition to these works, some other studies relevant to spin–spin correlation functions in the two-dimensional Ising model include [13–59].

While much is thus known about the two-dimensional Ising model, there are still interesting aspects to study. Among these are various properties of the spin–spin correlation functions. We denote the spin–spin correlation function as $\langle \sigma_{\vec{r}}\sigma_{\vec{r}'}\rangle$, where $\vec{r}$ and $\vec{r}'$ are sites on the square lattice and $\langle \mathcal{O}\rangle$ denotes the thermal average of an operator $\mathcal{O}$. Given the homogeneity of the square lattice, one can, with no loss of generality, take one spin to be located at the origin and thus consider

$$\langle \sigma_{\vec{0}}\sigma_{\vec{r}}\rangle \equiv C(\vec{r}). \tag{1.3}$$

We write $\vec{r} = (m, n)$ so that $C(\vec{r}) \equiv C(m, n)$. The spin–spin correlation function for two spins along a row is

$$R_n \equiv C(n, 0) \equiv \langle \sigma_{0,0}\sigma_{n,0}\rangle. \tag{1.4}$$

From the isotropy of the spin–spin couplings in (1.2), it follows that the correlation functions for equidistantly separated spins along a row and column are equal: $C(n, 0) = C(0, n)$. We denote the correlation function for spins along a diagonal of the lattice as

$$D_n \equiv C(n,n) \equiv \langle \sigma_{0,0}\sigma_{n,n}\rangle. \tag{1.5}$$

Note the symmetry relation $C(n,n) = C(n,-n)$. In studying spin–spin correlation functions of the two-dimensional Ising model, one acknowledges that these are not universal in the sense of the renormalization group; that is, modifications of the model (1.1) and (1.2) such as the generalization to spin $s \geqslant 1$ and/or addition of (nonfrustrating) non-*nn* spin–spin or multispin interactions preserving the $\mathbb{Z}_2$ symmetry would change $\langle \sigma_{0,0}\sigma_{m,n}\rangle$ without changing the universality class of the phase transition. Nevertheless, these correlation functions contain useful information about the behavior of the model. We define the following notation:

$$K = \beta J, \qquad v = \tanh K, \qquad x = v^2, \qquad z = \mathrm{e}^{-2K}, \qquad u = z^2 = \mathrm{e}^{-4K}. \tag{1.6}$$

Correlation functions are commonly expressed as functions of the variables

$$k_> = \sinh^2(2K), \qquad k_< = \frac{1}{k_>} = \frac{1}{\sinh^2(2K)}. \tag{1.7}$$

Recall that (as follows from duality) the critical point occurs at $v_c = z_c = \sqrt{2}-1$, i.e. $K_c = J/(k_B T_c) = (1/2)\ln(\sqrt{2}+1)$, at which point $k_> = k_< = 1$. The high-temperature (HT) series expansions of spin–spin correlation functions are commonly expressed as series in powers of v, while the low-temperature (LT) expansions on a bipartite lattice such as the square lattice considered here, are series in powers of u.

In reference [29], Jimbo and Miwa showed that D_n can be calculated in terms of solutions to a (nonlinear, second-order) ordinary differential equation (ODE) of Painlevé VI type (see appendix A). Subsequently, there has been a quest to find an analogous nonlinear ODE whose solutions would yield the general spin–spin correlation function $C(m,n)$ in this model. However, as emphasized recently in [59], this is still an open problem. Even for R_n, to our knowledge, such a generalization of the Jimbo–Miwa ODE has not been found. Indeed, in the absence of an existence proof, it is not clear if such a (nonlinear, second-order) ODE whose solutions would yield the R_n, analogous to the Jimbo–Miwa Painlevé VI ODE for D_n (see appendix A), exists. Investigations into this can make use of exact calculations of correlation functions. The D_n and R_n can be expressed as Toeplitz determinants, and this method was used in [38, 39] to calculate these correlation functions for n up to 6 and to present exact expressions for n up to 5. Exact calculations of some other $C(m,n)$ were given in [41]. It was shown in [38] that D_n is a homogeneous polynomial of degree n in the complete elliptic integrals $K(k)$ and $E(k)$, where $k = k_>$ for $T \geqslant T_c$ and $k = k_<$ for $T \leqslant T_c$. The general structure of R_n for the model of equations (1.1) and (1.2) was determined in [39] and is substantially more complicated, as will be reviewed below. As shown in [46], the $C(m,n)$ for this model can be efficiently calculated recursively using certain quadratic relations [31] together with some initial inputs. Both of these methods yield specific correlation functions, e.g. R_6, R_7, etc for higher n. In searching for a nonlinear ODE for R_n analogous to the Jimbo–Miwa ODE for D_n, it would be convenient to use inputs that are general functions of n, rather than having to recursively compute R_n for successive fixed values of n. For this purpose, HT and LT series expansions can be useful, if one knows general-n expressions for the coefficients. However, standard procedures for calculating these series expansions are based on enumeration of graphs for a given correlation function $C(m,n)$ and, except for the first or second leading terms, do not normally yield expressions that are general functions of (m,n). Here we focus on R_n. The leading term in the HT series expansion of R_n is v^n, and an elementary graphical enumeration yields the

first higher-order term as $n(n+1)v^{n+2}$, but we are not aware of general-n expressions for still higher-order terms in the literature. Similar comments apply for the LT series expansion of this correlation function.

In this paper we shall discuss an approach that can yield general-n coefficients of higher-order terms in HT and LT expansions of the row correlation functions R_n for the two-dimensional Ising model defined by equations (1.1) and (1.2) on the square lattice. Our procedure makes use of exact calculations of individual R_n. We illustrate the approach by computing general-n coefficients of several higher-order terms in HT expansions of R_n and LT expansions of $(R_n)_{\rm conn.}$. In addition to their intrinsic interest, this method and these results should be useful in the continuing endeavor to find a nonlinear ODE for R_n analogous to the one derived for D_n by Jimbo and Miwa in [29]. Our work here is complementary to studies of form factor expansions for Ising correlation functions (e.g. [28, 50–52, 55, 56]). It is also complementary to studies of properties of the Ising model susceptibility χ (e.g. [26, 44, 45, 47, 54, 56]), since the latter involves a sum over all connected spin–spin correlation functions, not just R_n, via the relation $\beta^{-1}\chi = \sum_{\vec{r}} C(\vec{r})_{\rm conn.}$.

This paper is organized as follows. In section 2 we review the general structural form for R_n obtained in [39]. In sections 3 and 4 we use the exactly calculated R_n from [39] to infer general-n expressions for several coefficients of higher-order terms in HT series for R_n and LTs series for $(R_n)_{\rm conn.}$. Our conclusions are given in section 5. Some related results are included in appendices.

2. Structure of row correlation functions

From our analysis in [39], we inferred the following general structural form for the row correlation functions R_n. These have different analytic forms $R_{n,+}$ and $R_{n,-}$ for $T > T_c$ and $T < T_c$, respectively (which are equal at T_c):

$$n \text{ even}: \quad R_{n,\pm} = B_n k^{-p_n} \sum_{\ell=0}^{n/2} \pi^{-2\ell} \sum_{s=0}^{2\ell} \mathcal{R}^{(n,\pm)}_{2\ell-s,s}(k) E(k)^{2\ell-s} \bar{K}(k)^s, \tag{2.1}$$

where $k = k_>$ for $T \geqslant T_c$ and $k = k_<$ for $T \leqslant T_c$ and

n odd, $T \geqslant T_c$:

$$R_{n,+} = B_n k_>^{-p_n} (1+k_>^{-1})^{1/2} \sum_{\ell=0}^{n} \pi^{-\ell} \sum_{s=0}^{\ell} \mathcal{R}^{(n)}_{\ell-s,s}(k_>)(k_>-1)^{(1/2)[1-(-1)^\ell]\delta_{s,0}} E(k_>)^{\ell-s} \bar{K}(k_>)^s \tag{2.2}$$

n odd, $T \leqslant T_c$:

$$R_{n,-} = B_n k_<^{-p_n} (1+k_<)^{1/2} \sum_{\ell=0}^{n} (-\pi)^{-\ell} \sum_{s=0}^{\ell} \mathcal{R}^{(n)}_{\ell-s,s}(k_<)(k_<-1)^{(1/2)[1-(-1)^\ell]\delta_{s,0}} E(k_<)^{\ell-s} \bar{K}(k_<)^s. \tag{2.3}$$

In equations (2.1)–(2.3), B_n is a numerical prefactor; p_n is an integer power[1] and we define the compact notation

$$\bar{K}(k) \equiv (k-1)K(k). \tag{2.4}$$

[1] The quantities B_n and p_n and the dummy index s were denoted D_n, q_n, and r in [39]; here we relabel these to avoid confusion with our notation D_n for $C(n,n)$ and $r = |\vec{r}|$.

The first five B_n were given in [39], viz, $B_1 = B_2 = B_3 = 1, B_4 = 1/(3^2)$, and $B_5 = 1/(3^4)$ [39]. The sixth is $B_6 = 1/(3^6 \times 5^2)$. The values of the power p_n in equations (2.1)–(2.3) were listed (denoted as q_n) for n up to 5 in [39]. Here we observe that for the known p_n with $1 \leqslant n \leqslant 6$, the values are consistent with the general formula

$$p_n = \left[\frac{n^2}{4}\right]_{\rm floor}, \tag{2.5}$$

where for $\nu \in \mathbb{R}$, $[\nu]_{\rm floor}$ is the greatest integer $\leqslant \nu$. These powers p_n are the same as the powers that occur in the general structural form for the diagonal correlation function D_n that we found in [38]. In the remainder of the paper we will sometimes suppress the subscripts $\pm$ in the notation, with it being understood implicitly that $R_n \equiv R_{n,+}$ for $T \geqslant T_{\rm c}$ and $R_n \equiv R_{n,-}$ for $T \leqslant T_{\rm c}$. Note that although $K(k)$ is logarithmically divergent as $k \nearrow 1$, this divergence is removed by the prefactor $(k-1)$ in $\bar{K}(k)$. Indeed,

$$\lim_{k\to 1} \bar{K}(k) = 0, \tag{2.6}$$

although the derivative $({\rm d}/{\rm d}k)\bar{K}(k)$ is logarithmically divergent as $k \to 1$.

For a given n, the terms in R_n can be divided into sets such that all of the terms in each set are homogeneous polynomials in $E(k)$ and $\bar{K}(k)$ of a given degree. We label this degree as the 'level' of the set. For $R_{n,\pm}$ with odd n, as is evident from equations (2.2) and (2.3), these terms are explicitly of the form $E(k)^{\ell-s}\bar{K}(k)^s$ with ℓ in the range $0 \leqslant \ell \leqslant n$ and, for a given ℓ, with s in the range $0 \leqslant s \leqslant \ell$, where $k = k_>$ for $T \geqslant T_{\rm c}$ and $k = k_<$ for $T \leqslant T_{\rm c}$. The corresponding coefficients $\mathcal{R}^{(n)}_{\ell-s,s}$ in $R_{n,\pm}$ are polynomials in the respective k elliptic modulus variables. For even n, only even-degree levels occur, running over $2\ell = 0, 2, \ldots, n$, as is evident in equation (2.1). Another difference between the R_n with even and odd n is that for odd n, the same coefficient polynomial $\mathcal{R}^{(n)}_{\ell-s,s}(k)$ occurs for $T \geqslant T_{\rm c}$ and $T \leqslant T_{\rm c}$ with the respective assignments $k = k_>$ and $k = k_<$, whereas for even n, the $\mathcal{R}^{(n,+)}_{2\ell-s,s}(k_>)$ and $\mathcal{R}^{(n,-)}_{2\ell-s,s}(k_<)$ are different functions of their respective arguments, $k_>$ and $k_<$. A third difference between the R_n for even and odd n is that the R_n for odd n contain a square-root prefactor, $(1+k_>^{-1})^{1/2} = (1+k_<)^{1/2}$, whereas the R_n for even n do not contain such square-root prefactors[2].

The general structural form that we inferred for R_n is considerably more complicated than the form that we had found in [38] for D_n. These have different expressions $D_{n,+}$ and $D_{n,-}$ for $T > T_{\rm c}$ and $T < T_{\rm c}$ (which are equal at $T = T_{\rm c}$):

$$D_{n,\pm} = A_n \pi^{-n} k^{-2p_n - [1-(-1)^n]\Theta(T-T_{\rm c})/2} \sum_{s=0}^{n} \mathcal{P}^{(n,\pm)}_{n-s,s}(k)(k^2-1)^{\delta_{s,1}\Theta(T_{\rm c}-T)} E(k)^{n-s}[(k^2-1)K(k)]^s, \tag{2.7}$$

where again $k = k_>$ if $T \geqslant T_{\rm c}$ and $k = k_<$ if $T \leqslant T_{\rm c}$; and $\Theta(x)$ is the Heaviside step function, defined as $\Theta(x) = 1$ if $x > 0$ and $\Theta(x) = 0$ if $x \leqslant 0$. One of the most striking differences is that D_n is a homogeneous polynomial of degree n in $E(k)$ and $K(k)$, while R_n has the multi-'level' structure of equations (2.1)–(2.3). Furthermore, calculations for D_n and the structural form presented in [38] apply for the general anisotropic case $J_1 \neq J_2$, with $k_> = k_<^{-1} = \sinh(2K_1)\sinh(2K_2)$ and $K_i = \beta J_i$, whereas in the anisotropic case, other correlation functions such as R_n would involve not just complete elliptic integrals of the first and

[2] We note some misprints in [4, 38, 39]. In equation (A19) of [4], the expression $-\frac{1}{2}\gamma_2[F_{m_1,m_2} + F_{m_1,m_2-1}]$ should read $-\frac{1}{2}\gamma_2[F_{m_1,m_2+1} + F_{m_1,m_2-1}]$. In [38] there was a misprint in the overall sign of $\mathcal{P}^{(5,-)}_{2,3}$, which should be reversed. In [39], the coefficient $\mathcal{R}^{(4,-)}_{3,1}$ should be multipled by $(k_< - 1)$.

second kinds, but also those of the third kind, as was already evident for R_1 [6, 58]. Our methods could be applied to this case in future work, although the series expansions would depend on two variables, e.g. $v_i = \tanh K_i$ where $K_i = \beta J_i$, $i = 1, 2$, or equivalently, $s_i = \sinh(2K_i)$ with $i = 1, 2$ for the HT expansions, and similarly on $1/s_i$ with $i = 1, 2$ for the LT expansions.

3. General-*n* coefficients in the high-temperature series expansion of R_n

Here we report our new results on general-n coefficients of higher-order terms in the HT Taylor series expansion of R_n. This is analogous to the calculation of general-n coefficients in the HT expansion of D_n in [37] (see appendix A), with the crucial difference that for D_n we were able to make use of the fact that the D_n can be determined in terms of solutions of the Painlevé VI ODE [29], whereas here no analogous (nonlinear) ODE for R_n is known. Hence, we make use of the R_n calculated in [39].

The standard procedure for calculating the HT Taylor series expansion for $R_n = \langle \sigma_{0,0}\sigma_{n,0}\rangle$ enumerates the contributions from paths on the bonds of the lattice of minimal length and progressively greater lengths joining the points $(0, 0)$ and $(n, 0)$ (e.g. [5]). The number of bonds in the path is then the power of v in a given term in the series expansion, and the coefficient of each term is a positive integer. If one factors out an overall factor of v^n in the small-v expansion of R_n, the rest of the series is a series in powers of v^2. This is an elementary consequence of the fact that if one reverses the sign of the spin–spin coupling J, and hence the sign of K and of $v = \tanh K$, then $R_n \to (-1)^n R_n$. The lowest-order term, v^n, in the small-v series expansion of R_n, arises from the unique graph consisting of a straight path from $(0, 0)$ to $(n, 0)$, of length n. Thus, the HT expansion for R_n has the general form

$$R_n = v^n \left[1 + \sum_{j=1}^{\infty} r_{n,2j} v^{2j} \right], \tag{3.1}$$

where the $r_{n,2j}$ are positive integers. Aside from the leading v^n term, the paths contributing to all higher-order terms include bonds above the direct, horizontal path, and corresponding paths that are related to these by reflection about the horizontal axis. That is, for each path including bonds above the direct route along the horizontal axis joining the sites $(0, 0)$ to $(n, 0)$, there is a path that is obtained by this reflection process. Therefore, the $r_{n,2j}$ are even integers.

To begin, we discuss the graphical derivation of the first subleading term in equation (3.1), namely, $r_{n,2}v^{n+2}$. This term arises from paths of length $n + 2$ bonds connecting the sites $(0, 0)$ and $(n, 0)$. An elementary enumeration counts these paths. The sites $(0, \ell)$ with $0 \leqslant \ell \leqslant n$ comprise $n + 1$ vertices on the square lattice. One set of paths of length $n + 2$ connecting $(0, 0)$ and $(n, 0)$ involves a 90° turn upwards at one of these $n + 1$ sites followed by a continuation along horizontal bonds, and then a 90° turn downward and final continuation to the point $(n, 0)$. For each such path, there is also a corresponding path obtained by reflecting about the horizontal axis, so that the first right-hand turn is downward instead of upward. There are $\binom{n+1}{2}$ paths in the first set, and hence $2\binom{n+1}{2} = n(n + 1)$ paths of length $n + 2$ joining the points $(0, 0)$ and $(n, 0)$. This simple combinatoric argument yields the coefficient, $r_{n,2}$, of the v^{n+2} term in the HT expansion of R_n, namely

$$r_{n,2} = n(n + 1). \tag{3.2}$$

This is manifestly even, since either n or $n + 1$ is even.

Now from HT series expansions of our calculations of R_n for n, we determine the following general-n expression for the next-to-next-leading-order coefficient, $r_{n,4}$. For reference, we list

these expansions for n up to 6 in appendix B. Our method is motivated by the structural form as polynomials in n that we obtained for coefficients of higher-order terms in the HT series for D_n in [37] (reviewed in appendix A). We thus fit the respective $O(v^{n+4})$ terms in the HT series expansions of the exact expressions for R_n to a polynomial. This is an overconstrained fit, and we obtain the result

$$r_{n,4} = \frac{n}{4}(n^3 + 2n^2 + 3n + 10). \tag{3.3}$$

Although there is an extensive literature on series expansions of quantities in the two-dimensional Ising model, we are not aware of this expression for $r_{n,4}$ having appeared in this literature. An alternate approach to determining $r_{n,4}$ would make use of an enumeration of all graphs that contribute to the $O(v^{n+4})$ term in the HT expansion of R_n for arbitrarily great n. This result is given to illustrate the method; clearly, one could proceed to calculate coefficients of more higher-order terms, $r_{n,6}$, etc. Since the $r_{n,2j}$ with $2j \geqslant 6$ are higher-degree polynomials in n, the procedure for calculating these polynomials via overconstrained fits requires the input of a larger number of row correlation functions. However, as emphasized, the value of this method is that the resultant coefficient applies for general n and hence is directly applicable to the search for a nonlinear differential equation whose solution would yield R_n.

Despite the prefactor of 1/4, it is easy to show that the expression for $r_{n,4}$ in equation (3.3) is an integer, and, furthermore, is even. This is proved by induction, starting from any of the known $r_{n,4}$ values for $1 \leqslant n \leqslant 6$, each of which is even. Given that there exists an n such that $r_{n,4}$ is even, to carry out the inductive proof, one must prove that $r_{n+1,4}$ is also even. This can be done by showing that the difference, $r_{n+1,4} - r_{n,4}$, is even, i.e. $r_{n+1,4} - r_{n,4} = 2p$ for some (positive) integer p. We calculate

$$r_{n+1,4} - r_{n,4} = (n+2)(n^2 + n + 2). \tag{3.4}$$

Since the factor $(n+2)$ can be even or odd, we thus need to show that $n^2 + n + 2$ is always even. This follows directly by observing that $n^2 + n + 2 = n(n+1) + 2$. Now $n(n+1)$ is manifestly even, since either n or $n+1$ is even, and hence $n(n+1) + 2$ is even. This completes the proof that the expression for $r_{n,4}$ in equation (3.3) is an even (positive) integer.

One can also express these results equivalently as series expansions in powers of the variable $\sqrt{k_>}$, using the relation (B2). It is convenient to introduce the variable $\hat{k}_> = (1/4)k_>$ as in equation (B4). Then equation (3.1) can be written as

$$R_n = \hat{k}_>^{n/2}\left[1 + \sum_{\ell=1}^{\infty} \tilde{r}_{n,\ell}\hat{k}_>^{\ell}\right], \tag{3.5}$$

where

$$\tilde{r}_{n,1} = n^2 \tag{3.6}$$

and

$$\tilde{r}_{n,2} = \frac{1}{4}n(n-1)(n^2 - n - 8). \tag{3.7}$$

While the coefficients $r_{n,2}$, $r_{n,4}$, and $\tilde{r}_{n,1}$ are positive and monotonically increasing as functions of n in the interval $n \geqslant 1$, the behavior of $\tilde{r}_{n,2}$ is more complicated. As a function of n, with n generalized from integral values to real values in this interval $n \geqslant 1$, $\tilde{r}_{n,2}$ decreases from zero at $n = 1$ through negative values, reaching a minimum of -4 at $n = (1/2)(1 + \sqrt{17}) = 2.561\,55$

and then increases monotonically for larger n, passing through zero again at $n = (1/2)(1 + \sqrt{33}) = 3.372\,28$. Thus, $\tilde{r}_{n,2}$ is negative for $n = 2$ and $n = 3$, taking the values $\tilde{r}_{2,2} = \tilde{r}_{3,2} = -3$.

4. General-*n* coefficients of higher-order terms in the low-temperature series expansion of $(R_n)_{\rm conn.}$

In addition to its intrinsic interest, the spin–spin correlation function $C(\vec{r})$ is important because its limit as $r \to \infty$ determines the (square of the) spontaneous magnetization:

$$\lim_{r\to\infty} C(\vec{r}) = M^2, \tag{4.1}$$

where $r \equiv |\vec{r}|$. The connected correlation function is then

$$C(\vec{r})_{\rm conn.} \equiv C(\vec{r}) - M^2. \tag{4.2}$$

For $T < T_{\rm c}$ where the spontaneous magnetization is nonzero, an interesting question concerns the approach to the limit (4.1), for a given $\vec{r}$, a quantitative measure of this approach is provided by the ratio

$$A_{C(\vec{r})} = \frac{C(\vec{r})}{M^2} = 1 + \frac{C(\vec{r})_{\rm conn.}}{M^2}. \tag{4.3}$$

In this section, we present our results on general-n coefficients of higher-order terms in the LT Taylor series expansions of $(R_n)_{\rm conn.} = R_n - M^2$ and $A_{R_n} = R_n/M^2$. In calculating $(R_n)_{\rm conn.}$, we make use of the result first published by Yang [2] for the spontaneous magnetization in the two-dimensional Ising model on the square lattice,

$$M = (1 - k_<^2)^{1/8} = \frac{(1+u)^{1/4}(1-6u+u^2)^{1/8}}{(1-u)^{1/2}}. \tag{4.4}$$

The quantity M^2 has the resultant LT Taylor series expansion

$$\begin{aligned} M^2 = {} & 1 - 4u^2 - 16u^3 - 64u^4 - 272u^5 - 1228u^6 - 5792u^7 - 28\,192u^8 \\ & - 140\,448u^9 - 712\,276u^{10} - 3663\,664u^{11} - O(u^{12}). \end{aligned} \tag{4.5}$$

The property (4.1), together with the property that M and R_n are continuous functions of u, implies that as n increases, the LT (i.e. small-u) Taylor series expansion of R_n must coincide with the small-u expansion of M^2 to an increasingly high order, and the order of the first term in the small-u expansion of $R_n - M^2$ must go to infinity as $n \to \infty$. From general arguments, for an Ising ferromagnet on (the thermodynamic limit of) a given lattice, R_n is a monotonically decreasing function of n for fixed temperature T, i.e. $R_n \geqslant R_{n+1}$, and hence $R_n \geqslant M^2$. (The two points at which this inequality is realized as an equality are (i) $T = 0$ for any n, where $R_n = M^2 = 1$, and (ii) $T = \infty$, where for $n \geqslant 1$, $R_n = M^2 = 0$.)

The LT Taylor series expansions of the R_n in powers of u or $k_<$ match the corresponding expansions of M^2 to $O(u^{n+1}) = O(k_<^{n+1})$ inclusive. Thus, the LT series expansion of R_n has the general form

$$(R_n)_{\rm conn.} = 4u^{n+2}\left[1 + \sum_{j=1}^{\infty} \rho_{n,j} u^j\right] = 4\hat{k}_<^{n+2}\left[1 + \sum_{j=1}^{\infty} \tilde{\rho}_{n,j}\hat{k}_<^j\right]. \tag{4.6}$$

Here it is convenient to use the rescaled variable $\hat{k}_< = (1/4)k_<$ (as defined in equation (B4)), since this yields integral coefficients $\tilde{\rho}_{n,j}$. Using LT expansions of the R_n that we have calculated exactly, we apply the same polynomial fitting procedure that we used for the HT expansions. For reference, we list these LT expansions in appendix C. We obtain the following general-n expressions for the u^{n+3} and u^{n+4} terms in equation (4.6):

$$\rho_{n,1} = n^2 + 2n + 4, \tag{4.7}$$

and

$$\rho_{n,2} = \frac{1}{2}(n^4 + 4n^3 + 13n^2 + 26n + 32). \tag{4.8}$$

Equivalently, for the expansion of $(R_n)_{\rm conn.}$ in terms of $\hat{k}_<$ in equation (4.6), we have

$$\tilde{\rho}_{n,1} = n^2 \tag{4.9}$$

and

$$\tilde{\rho}_{n,2} = \frac{1}{2}(n+2)(n^3 - 2n^2 + n + 6). \tag{4.10}$$

By combining the LT expansion for M^2 with these results, one can thus obtain the corresponding general-n LT expansion for R_n up to $O(u^{n+4}) = O(\hat{k}_<^{n+4})$. We are not aware of the expressions (4.7)–(4.10) having appeared before in the literature. These results are given to illustrate the method and could be extended to higher order using additional R_n correlation functions as input.

Note that, despite the prefactor of 1/2, the expression for $\rho_{n,2}$ is an integer. We give an inductive proof of this. First, this integral property holds for the LT series for R_1. Hence, it is necessary and sufficient to show that with $\rho_{n,2}$ being integral, so is $\rho_{n+1,2}$. To do this, we show that the difference $\rho_{n+1,2} - \rho_{n,2}$ is integral. This difference is

$$\rho_{n+1,2} - \rho_{n,2} = (n+1)(2n^2 + 5n + 11), \tag{4.11}$$

which is obviously integral. The same inductive method shows that $\tilde{\rho}_{n,2}$ is an integer.

Combining these results with the definition $A_{R_n} = R_n/M^2$ yields

$$\begin{aligned} A_{R_n} &= 1 + 4u^{n+2}\big[1 + \rho_{n,1}u + (\rho_{n,2}+4)u^2 + O(u^3)\big] \\ &= 1 + 4u^{n+2}\bigg[1 + (n^2+2n+4)u + \frac{1}{2}\big(n^4 + 4n^3 \\ &\quad + 13n^2 + 26n + 40\big)u^2 + O(u^3)\bigg]. \end{aligned} \tag{4.12}$$

Equivalently, in terms of the $\hat{k}_<$ variable,

$$\begin{aligned} A_{R_n} &= 1 + 4\hat{k}_<^{n+2}\Big[1 + \tilde{\rho}_{n,1}\hat{k}_< + (\tilde{\rho}_{n,2}+4)\hat{k}_<^2 + O(\hat{k}_<^3)\Big] \\ &= 1 + 4\hat{k}_<^{n+2}\bigg[1 + n^2\hat{k}_< + \frac{1}{2}(n^4 - 3n^2 + 8n + 20)\hat{k}_<^2 + O(\hat{k}_<^3)\bigg]. \end{aligned} \tag{4.13}$$

5. Conclusions

In this paper we have discussed a method for obtaining general-n expressions for coefficients of higher-order terms in the HT and LT series expansions of the spin–spin correlation function R_n in the two-dimensional Ising model on the square lattice and have applied it to obtain general-n coefficients of several higher-order terms in these series. This method is complementary to the standard method for calculating these coefficients, which is via enumeration of graphs that contribute in a given order of expansion. It is also complementary to another method that was used in [37] for the HT expansions of diagonal correlation functions D_n, which was based on the property that the D_n can be computed in terms of solutions to the Painlevé VI ODE of [29]. In addition to the intrinsic interest in the general-n coefficients discussed here, they provide further inputs to the continuing quest to find a nonlinear ODE whose solution would determine the R_n.

Acknowledgments

I would like to thank Prof. B M McCoy for questions and discussions that led me to revisit this subject. I am also grateful to Prof. J W Essam for an informative communication. This research was supported in part by the US National Science Foundation Grant NSF-PHY-1915093.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. General-n coefficients in the high-temperature series expansion of D_n

The HT series expansion of D_n in the Ising model on the square lattice has the form

$$D_n = \sum_{j=n}^{\infty} c_{n,j}^{(D)} x^j, \tag{A1}$$

where $x \equiv v^2$ (cf equation (1.6)) and the subscript $+$ in $D_{n,+}$ is understood implicitly. An elementary combinatoric argument determines the coefficient of the leading-order term as

$$c_{n,n}^{(D)} = \frac{(2n)!}{(n!)^2}. \tag{A2}$$

The series (A1) can equivalently be written as

$$D_n = c_{n,n}^{(D)} x^n \left[1 + \sum_{\ell=1}^{\infty} r_{n,\ell}^{(D)} x^\ell \right], \tag{A3}$$

where the ratio $r_{n,j}^{(D)}$ is given by

$$r_{n,\ell}^{(D)} = \frac{c_{n,n+\ell}^{(D)}}{c_{n,n}^{(D)}}. \tag{A4}$$

Let us define a variable t as $t = k_>^{-2}$ for $T \geqslant T_c$ and $t = k_<^{-2}$ for $T \leqslant T_c$, and auxiliary functions $\sigma_{n,\pm}$ as follows:

$$\sigma_{n,+} = t(t-1)\frac{\mathrm{d}\,\ln D_{n,+}}{\mathrm{d}t} - \frac{t}{4} \tag{A5}$$

and

$$\sigma_{n,-} = t(t-1)\frac{\mathrm{d}\,\ln D_{n,-}}{\mathrm{d}t} - \frac{1}{4}. \tag{A6}$$

In [29], Jimbo and Miwa showed that the σ_n functions are solutions to the following ODE of Painlevé VI type (where subscripts $\pm$ on σ_n are understood implicitly for $T \geqslant T_c$ and $T \leqslant T_c$ and $\sigma_n' \equiv \mathrm{d}\sigma_n/\mathrm{d}t$):

$$[t(t-1)\sigma_n'']^2 - n^2[(t-1)\sigma_n' - \sigma_n]^2 + 4\sigma_n'\left[(t-1)\sigma_n' - \sigma_n - \frac{1}{4}\right](t\sigma_n' - \sigma_n) = 0. \tag{A7}$$

The diagonal correlation functions $D_{n,\pm}$ are then determined in terms of the $\sigma_{n,\pm}$. In [37] with Ghosh, using the result from [29], we derived the general form of the nine terms beyond the leading term in the HT expansion of $D_n = D_{n,+}$. (In [37], the coefficients $c_{n,j}^{(D)}$ and the ratios $r_{n,\ell}^{(D)}$ were denoted as $c_{n,j}$ and $r_{n,n+j}$, respectively.) Our results in [37] included the following for the ratios $r_{n,\mathrm{ell}}^{(D)}$, in our present notation:

$$r_{n,1}^{(D)} = 2n \tag{A8}$$

$$r_{n,2}^{(D)} = \frac{n(2n^2+3n+5)}{n+1} \tag{A9}$$

$$r_{n,3}^{(D)} = \frac{2n(2n^3+5n^2+16n+25)}{3(n+1)} \tag{A10}$$

$$r_{n,4}^{(D)} = \frac{4n^6+24n^5+103n^4+372n^3+943n^2+726n-48}{6(n+1)(n+2)} \tag{A11}$$

$$r_{n,5}^{(D)} = \frac{4n^7+32n^6+183n^5+930n^4+4031n^3+10228n^2+6972n-960}{15(n+1)(n+2)}, \tag{A12}$$

and so forth for higher-order terms up to $r_{n,9}^{(D)}$. It is interesting to note that, in an analogous manner, it was possible to use the fact that correlation functions for the transverse Ising quantum spin chain at critical field and $T = 0$ satisfy a Painlevé V equation to derive a number of properties of the correlation functions for these functions [33–36, 42, 43].

Appendix B. High-temperature series for R_n

For reference, in this appendix we list the HT series that we calculate from our exact results for R_n with n up to 6. Note that R_1 was calculated in [3]. These series have the general form of equation (3.1). We first record some relations between the elliptic moduli $k_> = 1/k_<$ and the respective high- and LT expansion variables v and z. The latter two variables are dual to each other and satisfy

$$v = \frac{1-z}{1+z}, \quad \text{equivalently,} \quad z = \frac{1-v}{1+v}. \tag{B1}$$

Then

$$k_> = \left[\frac{2v}{1-v^2}\right]^2 \tag{B2}$$

and

$$k_< = \left[\frac{2z}{1-z^2}\right]^2 = \frac{4u}{(1-u)^2}. \tag{B3}$$

It convenient to introduce the rescaled quantities

$$\hat{k}_> = \frac{k_>}{4}, \qquad \hat{k}_< = \frac{k_<}{4}. \tag{B4}$$

The HT series expansions are

$$R_1 = v + 2v^3 + 4v^5 + 12v^7 + 42v^9 + 164v^{11} + 686v^{13} + 3012v^{15} + O(v^{17}) \tag{B5}$$

$$R_2 = v^2 + 6v^4 + 16v^6 + 46v^8 + 158v^{10} + 618v^{12} + 2618v^{14} + 11\,654v^{16} + O(v^{18}) \tag{B6}$$

$$R_3 = v^3 + 12v^5 + 48v^7 + 152v^9 + 506v^{11} + 1900v^{13} + 7902v^{15} + 35\,114v^{17} + O(v^{19}) \tag{B7}$$

$$R_4 = v^4 + 20v^6 + 118v^8 + 452v^{10} + 1564v^{12} + 5684v^{14} + 22\,726v^{16} + 98\,708v^{18} + O(v^{20}) \tag{B8}$$

$$R_5 = v^5 + 30v^7 + 250v^9 + 1200v^{11} + 4606v^{13} + 16\,920v^{15} + 65\,452v^{17} + 274\,422v^{19} + O(v^{21}) \tag{B9}$$

$$R_6 = v^6 + 42v^8 + 474v^{10} + 2862v^{12} + 12\,662v^{14} + 49\,282v^{16} + 189\,702v^{18} + 770\,190v^{20} + O(v^{22}). \tag{B10}$$

These series can be extended to higher n, but these are sufficient to illustrate our method.

These HT series for R_n can equivalently be expressed as series expansions in powers of the variable $\sqrt{k_>}$, using the relation (B2). However, in contrast to the series in v, the series in powers of $\sqrt{k_>}$ have coefficients that vary in sign, and do not increase monotonically in magnitude; indeed, some terms have zero coefficients. We list these equivalent expansions here. It is convenient to use the rescaled quantity $\hat{k}_> = (1/4)k_>$ as defined equation (B4), since this avoids fractional coefficients. We have

$$R_1 = \hat{k}_>^{1/2}\left[1 + \hat{k}_> + 5\hat{k}_>^3 - 4\hat{k}_>^4 + 44\hat{k}_>^5 - 60\hat{k}_>^6 + 469\hat{k}_>^7 - 820\hat{k}_>^8 + O\left(\hat{k}_>^9\right)\right] \tag{B11}$$

$$R_2 = \hat{k}_>\left[1 + 4\hat{k}_> - 3\hat{k}_>^2 + 20\hat{k}_>^3 - 24\hat{k}_>^4 + 160\hat{k}_>^5 - 235\hat{k}_>^6 + 1556\hat{k}_>^7 - 2568\hat{k}_>^8 + O\left(\hat{k}_>^9\right)\right] \tag{B12}$$

$$R_3 = \hat{k}_>^{3/2}\Big[1 + 9\hat{k}_> - 3\hat{k}_>^2 + 28\hat{k}_>^3 + 8\hat{k}_>^4 + 153\hat{k}_>^5 + 233\hat{k}_>^6 + 1008\hat{k}_>^7 + 3588\hat{k}_>^8 + O\Big(\hat{k}_>^9\Big)\Big] \tag{B13}$$

$$R_4 = \hat{k}_>^2\Big[1 + 16\hat{k}_> + 12\hat{k}_>^2 + 201\hat{k}_>^4 - 240\hat{k}_>^5 + 2332\hat{k}_>^6 - 3584\hat{k}_>^7 + 27\,280\hat{k}_>^8 + O\Big(\hat{k}_>^9\Big)\Big] \tag{B14}$$

$$R_5 = \hat{k}_>^{5/2}\Big[1 + 25\hat{k}_> + 60\hat{k}_>^2 - 75\hat{k}_>^3 + 561\hat{k}_>^4 - 699\hat{k}_>^5 + 4876\hat{k}_>^6 - 5420\hat{k}_>^7 + 45\,516\hat{k}_>^8 + O\Big(\hat{k}_>^9\Big)\Big] \tag{B15}$$

$$R_6 = \hat{k}_>^3\Big[1 + 36\hat{k}_> + 165\hat{k}_>^2 - 140\hat{k}_>^3 + 821\hat{k}_>^4 + 276\hat{k}_>^5 + 3092\hat{k}_>^6 + 15\,440\hat{k}_>^7 - 2484\hat{k}_>^8 + O\Big(\hat{k}_>^9\Big)\Big] \tag{B16}$$

Note that in the square bracket for R_1 in equation (B11) there is no $\hat{k}_>^2$ term and in the square bracket for R_4 in equation (B14) there is no $\hat{k}_>^3$ term.

For reference, we list numerical values of the R_n for $T \geqslant T_c$ in table 1. For comparison with the numerical values of R_n as $T \to T_c$, the analytic values of $(R_n)_{cr}$ with n up to 6 from [39] are as follows:

$$(R_1)_{cr} = 2^{-1/2} = 0.707\,107 \tag{B17}$$

$$(R_2)_{cr} = 1 - \frac{2^2}{\pi^2} = \left(1 - \frac{2}{\pi}\right)\left(1 + \frac{2}{\pi}\right) = 0.594\,715 \tag{B18}$$

$$(R_3)_{cr} = 2^{3/2}\left(1 - \frac{2^3}{\pi^2}\right) = 0.535\,790\,45 \tag{B19}$$

$$(R_4)_{cr} = 2^4\left(1 - \frac{2^4 \times 7}{3^2\pi^2} + \frac{2^8}{3^2\pi^4}\right) = 0.497\,989 \tag{B20}$$

$$(R_5)_{cr} = 2^{15/2}\left(1 - \frac{2^3 \times 19}{3^2\pi^2} + \frac{2^9 \times 11}{3^4\pi^4}\right) = 0.470\,724 \tag{B21}$$

$$(R_6)_{cr} = 2^{12}\left(1 - \frac{2^2 \times 13 \times 31}{3 \times 5^2\pi^2} + \frac{2^{10} \times 7 \times 13}{3^3 \times 5^2 \times \pi^4} - \frac{2^{22}}{3^6 \times 5^2\pi^6}\right) = 0.449\,637. \tag{B22}$$

Factorizations of these $(R_n)_{cr}$ for even n were given in [39]; we have only shown the first of these factorizations, for R_2, here. See also [40].

Appendix C Low-temperature series for $(R_n)_{conn.}$

For reference, we list here the LT series expansions of the connected correlation functions $(R_n)_{conn.}$ for n up to 6 here. These have the general form (4.6) and are as follows:

$$(R_1)_{conn.} = 4u^3 + 28u^4 + 152u^5 + 780u^6 + 3972u^7 + 20\,348u^8 + 105\,192u^9 + 548\,792u^{10} + O(u^{11}) \tag{C1}$$

$$(R_2)_{conn.} = 4u^4 + 48u^5 + 368u^6 + 2320u^7 + 13\,428u^8 + 74\,848u^9 + 410\,576u^{10} + 2238\,496u^{11} + O(u^{12}) \tag{C2}$$

Table 1. Numerical values of the R_n for $1 \leqslant n \leqslant 6$ and $T \geqslant T_c$ as functions of $k_>$. For reference, the values of v and T/T_c corresponding to each value of $k_>$ are also shown.

$k_>$	v	T/T_c	R_1	R_2	R_3	R_4	R_5	R_6
0	0	∞	0	0	0	0	0	0
0.1	0.1543	8.828	0.1621	0.027 46	0.4837×10^{-2}	0.8797×10^{-3}	1.642×10^{-4}	0.3127×10^{-4}
0.2	0.2134	4.435	0.2349	0.059 74	0.016 17	0.4578×10^{-2}	1.338×10^{-3}	0.4000×10^{-3}
0.3	0.2559	2.981	0.2950	0.096 84	0.034 31	0.012 79	0.4926×10^{-2}	1.939×10^{-3}
0.4	0.2897	2.260	0.3494	0.1389	0.060 11	0.027 40	0.012 90	0.6200×10^{-2}
0.5	0.3178	1.832	0.4013	0.1864	0.094 63	0.050 59	0.027 90	0.015 69
0.6	0.3420	1.549	0.4525	0.2400	0.139 35	0.085 07	0.053 48	0.034 26
0.7	0.3632	1.350	0.5045	0.3011	0.1965	0.1345	0.094 62	0.067 74
0.8	0.3820	1.203	0.5595	0.3723	0.2700	0.2046	0.1590	0.1256
0.9	0.3989	1.090	0.6210	0.4596	0.3684	0.3070	0.2617	0.2262
1	0.4142	1	0.7071	0.5947	0.5358	0.4980	0.4707	0.4496

$$(R_3)_{\rm conn.} = 4u^5 + 76u^6 + 832u^7 + 6648u^8 + 44\,852u^9 + 276\,456u^{10} + 1623\,704u^{11} + 9293\,292u^{12} + O(u^{13}) \tag{C3}$$

$$(R_4)_{\rm conn.} = 4u^6 + 112u^7 + 1712u^8 + 17\,584u^9 + 141\,756u^{10} + 988\,192u^{11} + 6317\,392u^{12} + 38\,365\,984u^{13} + O(u^{14}) \tag{C4}$$

$$(R_5)_{\rm conn.} = 4u^7 + 156u^8 + 3224u^9 + 42\,412u^{10} + 414\,228u^{11} + 3331\,068u^{12} + 23\,619\,120u^{13} + 154\,485\,248u^{14} + O(u^{15}) \tag{C5}$$

$$(R_6)_{\rm conn.} = 4u^8 + 208u^9 + 5632u^{10} + 93\,680u^{11} + 1111\,492u^{12} + 10\,437\,824u^{13} + 83\,409\,104u^{14} + 596\,805\,184u^{15} + O(u^{16}). \tag{C6}$$

These series can equivalently be expressed in terms of the variable $k_<$, using the relation (B3). As before, it is convenient to use the rescaled variable $\hat{k}_< = (1/4)k_<$ as defined in equation (B4), since this avoids fractional coefficients. We have

$$(R_1)_{\rm conn.} = 4\hat{k}_<^3 + 4\hat{k}_<^4 + 36\hat{k}_<^5 + 52\hat{k}_<^6 + 384\hat{k}_<^7 + 668\hat{k}_<^8 + 4500\hat{k}_<^9 + 8820\hat{k}_<^{10} + O(\hat{k}_<^{11}) \tag{C7}$$

$$(R_2)_{\rm conn.} = 4\hat{k}_<^4 + 16\hat{k}_<^5 + 64\hat{k}_<^6 + 192\hat{k}_<^7 + 908\hat{k}_<^8 + 2256\hat{k}_<^9 + 12\,704\hat{k}_<^{10} + O(\hat{k}_<^{11}) \tag{C8}$$

$$(R_3)_{\rm conn.} = 4\hat{k}_<^5 + 36\hat{k}_<^6 + 180\hat{k}_<^7 + 440\hat{k}_<^8 + 2948\hat{k}_<^9 + 5604\hat{k}_<^{10} + 42\,808\hat{k}_<^{11} + 74\,980\hat{k}_<^{12} + O(\hat{k}_<^{13}) \tag{C9}$$

Table 2. Numerical values of the R_n for $1 \leqslant n \leqslant 6$ and $T \leqslant T_c$ as functions of $k_<$ For reference, the values of z and T/T_c corresponding to each value of $k_<$ are also shown.

$k_<$	z	T/T_c	R_1	R_2	R_3	R_4	R_5	R_6
0	0	0	0	0	0	0	0	0
0.1	0.1543	0.2940	0.9976	0.9975	0.9975	0.9975	0.9975	0.9975
0.2	0.2134	0.3811	0.9904	0.9800	0.9899	0.9898	0.9898	0.9898
0.3	0.2559	0.4593	0.9786	0.9769	0.9767	0.9767	0.9767	0.9767
0.4	0.2897	0.5351	0.9622	0.9580	0.957 45	0.9574	0.957 35	0.957 35
0.5	0.3178	0.6105	0.9410	0.9325	0.9310	0.9307	0.9306	0.9306
0.6	0.3420	0.6865	0.9144	0.8992	0.8958	0.8949	0.8946	0.8945
0.7	0.3632	0.7634	0.8817	0.8562	0.8491	0.8467	0.8458	0.8454
0.8	0.3820	0.8413	0.8412	0.8003	0.7864	0.780 65	0.7779	0.7765
0.9	0.3989	0.9202	0.7893	0.7245	0.6978	0.6842	0.6764	0.6716
1	0.4142	1	0.7071	0.5947	0.5358	0.4980	0.4707	0.4496

$$(R_4)_{\rm conn.} = 4\hat{k}_<^6 + 64\hat{k}_<^7 + 504\hat{k}_<^8 + 1344\hat{k}_<^9 + 7720\hat{k}_<^{10} + 22\,912\hat{k}_<^{11} + 108\,608\hat{k}_<^{12} + 352\,256\hat{k}_<^{13} + O(\hat{k}_<^{14}) \tag{C10}$$

$$(R_5)_{\rm conn.} = 4\hat{k}_<^7 + 100\hat{k}_<^8 + 1204\hat{k}_<^9 + 4900\hat{k}_<^{10} + 19\,224\hat{k}_<^{11} + 84\,708\hat{k}_<^{12} + 311\,588\hat{k}_<^{13} + 1230\,068\hat{k}_<^{14} + O(\hat{k}_<^{15}) \tag{C11}$$

$$(R_6)_{\rm conn.} = 4\hat{k}_<^8 + 144\hat{k}_<^9 + 2496\hat{k}_<^{10} + 15\,872\hat{k}_<^{11} + 58\,484\hat{k}_<^{12} + 250\,896\hat{k}_<^{13} + 1104\,448\hat{k}_<^{14} + 3668\,416\hat{k}_<^{15} + O(\hat{k}_<^{16}). \tag{C12}$$

Combining equations (C1)–(C6) with the LT series expansion for M^2, one obtains the LT series expansions for the full R_n correlation functions:

$$R_1 = 1 - 4u^2 - 12u^3 - 36u^4 - 120u^5 - 448u^6 - 1820u^7 - 7844u^8 - 35\,256u^9 - 163\,484u^{10} - O(u^{11}) \tag{C13}$$

$$R_2 = 1 - 4u^2 - 16u^3 - 60u^4 - 224u^5 - 860u^6 - 3472u^7 - 14764u^8 - 65\,600u^9 - 301\,700u^{10} - O(u^{11}) \tag{C14}$$

$$R_3 = 1 - 4u^2 - 16u^3 - 64u^4 - 268u^5 - 1152u^6 - 4960u^7 - 21\,544u^8 - 95\,596u^9 - 435\,820u^{10} - O(u^{11}) \tag{C15}$$

$$R_4 = 1 - 4u^2 - 16u^3 - 64u^4 - 272u^5 - 1224u^6 - 5680u^7 - 26\,480u^8 - 122\,864u^9 - 570\,520u^{10} - O(u^{11}) \tag{C16}$$

$$R_5 = 1 - 4u^2 - 16u^3 - 64u^4 - 272u^5 - 1228u^6 - 5788u^7 - 28\,036u^8 - 137\,224u^9 - 669\,864u^{10} - O(u^{11}) \tag{C17}$$

$$R_6 = 1 - 4u^2 - 16u^3 - 64u^4 - 272u^5 - 1228u^6 - 5792u^7 - 28\,188u^8 - 140\,240u^9 - 706\,644u^{10} - O(u^{11}). \tag{C18}$$

The equivalent series expansions, expressed in terms of the variable $\hat{k}_<$, are

$$R_1 = 1 - 4\hat{k}_<^2 + 4\hat{k}_<^3 - 20\hat{k}_<^4 + 36\hat{k}_<^5 - 172\hat{k}_<^6 + 384\hat{k}_<^7 - 1796\hat{k}_<^8 + 4500\hat{k}_<^9 - 20\,748\hat{k}_<^{10} + O(\hat{k}_<^{11}) \tag{C19}$$

$$R_2 = 1 - 4\hat{k}_<^2 - 20\hat{k}_<^4 + 16\hat{k}_<^5 - 160\hat{k}_<^6 + 192\hat{k}_<^7 - 1556\hat{k}_<^8 + 2256\hat{k}_<^9 - 16\,864\hat{k}_<^{10} - 27\,392\hat{k}_<^{10} + O(\hat{k}_<^{11}) \tag{C20}$$

$$R_3 = 1 - 4\hat{k}_<^2 - 24\hat{k}_<^4 + 4\hat{k}_<^5 - 188\hat{k}_<^6 + 180\hat{k}_<^7 - 2024\hat{k}_<^8 + 2948\hat{k}_<^9 - 23\,964\hat{k}_<^{10} + O(\hat{k}_<^{11}) \tag{C21}$$

$$R_4 = 1 - 4\hat{k}_<^2 - 24\hat{k}_<^4 - 220\hat{k}_<^6 + 64\hat{k}_<^7 - 1960\hat{k}_<^8 + 1344\hat{k}_<^9 - 21\,848\hat{k}_<^{10} + O(\hat{k}_<^{11}) \tag{C22}$$

$$R_5 = 1 - 4\hat{k}_<^2 - 24\hat{k}_<^4 - 224\hat{k}_<^6 + 4\hat{k}_<^7 - 2364\hat{k}_<^8 + 1204\hat{k}_<^9 - 24\,668\hat{k}_<^{10} + O(\hat{k}_<^{11}) \tag{C23}$$

$$R_6 = 1 - 4\hat{k}_<^2 - 24\hat{k}_<^4 - 224\hat{k}_<^6 - 2460\hat{k}_<^8 + 144\hat{k}_<^9 - 27\,072\hat{k}_<^{10} + O(\hat{k}_<^{11}). \tag{C24}$$

We list numerical values of the R_n for $T \leqslant T_c$ in table 2.

ORCID iDs

Robert Shrock https://orcid.org/0000-0001-6541-0893

References

[1] Onsager L 1944 Crystal statistics: I. A two-dimensional model with an order-disorder transition *Phys. Rev.* **65** 117–49
[2] Yang C N 1952 The spontaneous magnetization of a two-dimensional Ising model *Phys. Rev.* **85** 808–16
[3] Kaufman B and Onsager L 1949 Crystal statistics: III. Short-range order in a binary Ising lattice *Phys. Rev.* **76** 1244–52
[4] Montroll E W, Potts R B and Ward J C 1963 Correlations and spontaneous magnetization of the two-dimensional Ising model *J. Math. Phys.* **4** 308–22
[5] Domb C 1960 On the theory of cooperative phenomena in crystals *Adv. Phys.* **9** 149–244
[6] McCoy B M and Wu T T 1973 *The Two-Dimensional Ising Model* (Cambridge, MA: Harvard University Press)
[7] Wilson K G and Kogut J B 1974 The renormalization group and the ε expansion *Phys. Rep.* **12** 75–199
[8] Fisher M E 1975 The renormalization group in the theory of critical behavior *Rev. Mod. Phys.* **46** 597–616
[9] Le Guillou J C and Zinn-Justin J 1980 Critical exponents from field theory *Phys. Rev.* B **21** 3976–98
[10] Belavin A A, Polyakov A M and Zamolodchikov A B 1984 Infinite conformal symmetry in two-dimensional quantum field theory *Nucl. Phys.* B **241** 333–80
[11] Di Francesco P, Mathieu P and Sénéchal D 1997 *Conformal Field Theory* (New York: Springer)
[12] Suzuki M 1971 Relationship among exactly soluble models of critical phenomena: I *Prog. Theor. Phys.* **46** 1337–59
[13] Stephenson J 1964 Ising-model spin correlations on the triangular lattice *J. Math. Phys.* **5** 1009–24
[14] Sykes M F, Essam J W and Gaunt D S 1965 Derivation of low-temperature expansions for the Ising model of a ferromagnet and an antiferromagnet *J. Math. Phys.* **6** 283–98
[15] Wu T T 1966 Theory of Toeplitz determinants and the spin correlations of the two-dimensional Ising model: I *Phys. Rev.* **149** 380–401

[16] Kadanoff L P 1966 Spin–spin correlations in the two-dimensional Ising model *Nuovo Cimento* B **44** 276–305
[17] McCoy B M and Wu T T 1967 Theory of Toeplitz determinants and the spin correlations of the two-dimensional Ising model: II *Phys. Rev.* **155** 438–52
[18] Cheng H and Wu T T 1967 Theory of Toeplitz determinants and the spin correlations of the two-dimensional Ising model: III *Phys. Rev.* **164** 719–35
[19] McCoy B M and Wu T T 1967 Theory of Toeplitz determinants and the spin correlations of the two-dimensional Ising model: IV *Phys. Rev.* **162** 436–75
[20] Fisher M E and Burford R J 1967 Theory of critical-point scattering and correlations: I. The Ising model *Phys. Rev.* **156** 583–622
[21] Griffiths R B 1967 Correlations in Ising ferromagnets: I *J. Math. Phys.* **8** 478–83
[22] Griffiths R B 1967 Correlations in Ising ferromagnets: II. External magnetic fields *J. Math. Phys.* **8** 484–9
[23] Hartwig R E and Stephenson J 1968 Ising-model spin correlations *J. Math. Phys.* **9** 836–48
[24] Sykes M F, Gaunt D S, Essam J W and Hunter D L 1973 Derivation of low-temperature expansions for Ising model: II. General theory *J. Math. Phys.* **14** 1060–5
[25] Sykes M F, Gaunt D S, Mattingly S R, Essam J W and Elliott C J 1973 Derivation of low-temperature expansions for Ising model: III. Two-dimensional lattices-field grouping *J. Math. Phys.* **14** 1066–70
[26] Sykes M F, Gaunt D S, Martin J L, Mattingly S R and Essam J W 1973 Derivation of low-temperature expansions for Ising model: IV. Two-dimensional lattices-temperature grouping *J. Math. Phys.* **14** 1071–4
[27] Tarko H B and Fisher M E 1975 Theory of critical point scattering and correlations: III. The Ising model below T_c and in a field *Phys. Rev.* B **11** 1217–53
[28] Wu T T, Mccoy B M, Tracy C A and Barouch E 1978 Spin–spin correlation functions for the two-dimensional Ising model: exact theory in the scaling region *Phys. Rev.* B **13** 316–74
[29] Jimbo M and Miwa T 1980 Studies on holonomic quantum fields, XVII *Proc. Japan Acad.* A **56** 405
Jimbo M and Miwa T 1981 Studies on holonomic quantum fields, XVII *Proc. Japan Acad.* A **57** 347 (erratum)
[30] McCoy B M and Wu T T 1980 Nonlinear partial difference equations for the two-dimensional Ising model *Phys. Rev. Lett.* **45** 675–8
[31] Perk J H H 1980 Quadratic identities for Ising model correlations *Phys. Lett.* A **79** 3–5
[32] McCoy B M and Wu T T 1981 Non-linear partial difference equations for the two-spin correlation function of the two-dimensional Ising model *Nucl. Phys.* B **180** 89–115
[33] McCoy B M, Perk J H H and Shrock R E 1983 Time-dependent correlation functions of the transverse Ising chain at the critical magnetic field *Nucl. Phys.* B **220** 35–47
[34] McCoy B M, Perk J H H and Shrock R E 1983 Correlation functions of the transverse Ising chain at the critical field for large temporal and spatial separations *Nucl. Phys.* B **220** 269–82
[35] Müller G and Shrock R E 1983 Dynamic correlation functions for quantum spin chains *Phys. Rev. Lett.* **51** 219–22
[36] Müller G and Shrock R E 1984 Dynamic correlation functions for one-dimensional quantum-spin systems: new results based on a rigorous approach *Phys. Rev.* B **29** 288–301
[37] Ghosh R K and Shrock R E 1984 New method for calculating series expansions of correlation functions in the $d = 2$ Ising model *Phys. Rev.* B **30** 19–23
[38] Ghosh R K and Shrock R E 1984 Exact expressions for diagonal correlation functions in the $d = 2$ Ising model *Phys. Rev.* B **30** 3790–4
[39] Ghosh R K and Shrock R E 1985 Exact expressions for row correlation functions in the isotropic $d = 2$ Ising model *J. Stat. Phys.* **38** 473–82
[40] Au-Yang H and Perk J H H 1984 Ising correlations at the critical temperature *Phys. Lett.* A **104** 131–4
[41] Shrock R E and Ghosh R K 1985 Off-axis correlation functions in the isotropic $d = 2$ Ising model *Phys. Rev.* B **31** 1486–9
[42] Its A R, Izergin A G, Korepin and Slavnov N A 1990 Differential equations for quantum correlation functions *Int. J. Mod. Phys.* B **04** 1003–37
[43] Korepin V E, Bogoliubov N M and Korepin V 1993 *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge: Cambridge University Press)

[44] Briggs K M, Enting I G and Guttmann A 1994 Series studies of the Potts model: II. Bulk series for the square lattice *J. Phys. A: Math. Gen.* **27** 1503–23
[45] Matveev V and Shrock R 1995 Complex-temperature singularities of the susceptibility in the $D = 2$ Ising model: I. Square lattice *J. Phys. A: Math. Gen.* **28** 1557–83
[46] Au-Yang H and Perk J H H 2002 Correlation functions and susceptibility in the Z-invariant Ising model *MathPhys Odyssey 2001 Prog. Math. Phys. vol 23* (Boston, MA: Birkhäuser) pp 23–48
[47] Orrick W P, Nickel B, Guttmann A J and Perk J H H 2001 The susceptibility of the square lattice Ising model: new developments *J. Stat. Phys.* **102** 795–841
[48] Ghosh R K 2005 On the low temperature series expansion for the diagonal correlation functions in the two-dimensional Ising model (arXiv:cond-mat/0505166)
[49] Ghosh R K 2007 Duality and even number spin-correlation functions in the two dimensional square lattice Ising model (arXiv:cond-mat/0703602)
[50] Boukraa S, Hassani S, Maillard J-M, McCoy J-M, Weil J-A and Zenine N 2006 Painlevé versus fuchs *J. Phys. A: Math. Gen.* **39** 12245–63
[51] Boukraa S, Hassani S, Maillard J-M, McCoy J-M, Weil J-A and Zenine N 2007 Fuchs versus Painlevé *J. Phys. A: Math. Theor.* **40** 12589–605
[52] Lyberg I and McCoy B M 2007 Form factor expansion of the row and diagonal correlation functions of the two-dimensional Ising model *J. Phys. A: Math. Theor.* **40** 3329–46
[53] Perk J H H and Au-Yang H 2009 New Results for the correlation functions of the Ising model and the transverse Ising chain *J. Stat. Phys.* **135** 599–619
[54] Bostan A, Boukraa S, Hassani S, Maillard J-M, Weil J-A and Zenine N 2009 Globally nilpotent differential operators and the square Ising model *J. Phys. A: Math. Theor.* **42** 125206
[55] Mangazeev V and Guttmann A J 2010 Form factor expansions in the 2D Ising model and Painlevé VI *Nucl. Phys.* B **838** 391–412
[56] Boukraa S, Hassani S, Jensen I, Maillard J-M and Zenine N 2011 High order Fuchsian equations for the square lattice Ising model: $\chi(6)$ *J. Phys. A: Math. Theor.* **43** 115201
[57] Deift D, Its A and Krasovsky I 2013 Toeplitz matrices and Toeplitz determinants under the impetus of the Ising model: some history and some recent results *Commun. Pure Appl. Math.* **66** 1360–438
[58] McCoy B M and Maillard J-M 2016 The anisotropic Ising correlations as elliptic integrals: duality and differential equations *J. Phys. A: Math. Theor.* **49** 434004
[59] Boukraa S, Maillard J-M and McCoy B M 2020 Ising correlation $C(M, N)$ for $\nu = -k$ *J. Phys. A: Math. Theor.* **53** 465202