



The autoregressive filter problem for multivariable degree one symmetric polynomials

Jeffrey S. Geronimo, Hugo J. Woerdeman  and Chung Y. Wong

Abstract. The multivariable autoregressive filter problem asks for a polynomial $p(z) = p(z_1, \dots, z_d)$ without roots in the closed d -disk based on prescribed Fourier coefficients of its spectral density function $1/|p(z)|^2$. The conditions derived in this paper for the construction of a degree one symmetric polynomial reveal a major divide between the case of at most two variables vs. the the case of three or more variables. The latter involves multivariable elliptic functions, while the former (due to [Geronimo and Woerdeman (Ann Math 160(3):839-906, 2004)]) only involve polynomials. The three variable case is treated with more detail, and entails hypergeometric functions. Along the way, we identify a seemingly new relation between ${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; z\right)$ and ${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \tilde{z}\right)$.

Mathematics Subject Classification. 33C05, 33C20, 42C05, 41A60, 47A57.

Keywords. Stable polynomial, Spectral density function, Fourier coefficients, Hypergeometric functions, Elliptic functions, Autoregressive filter, Multivariable Toeplitz matrices.

1. Introduction

The identification problem for wide sense stationary autoregressive stochastic processes is a classical signal processing problem. We consider (wide sense) stationary processes $X_m = X_{(m_1, \dots, m_d)}$ depending on d discrete variables defined on a fixed probability space (Ω, \mathcal{A}, P) . We shall assume that the random variables X_m are *centered*, i.e., their means $E(X_m)$ equal zero. Recall that the space $L^2(\Omega, \mathcal{A}, P)$ of square integrable random variables endowed with the *inner product of centered random variables*

$$\langle X, Y \rangle := E(Y^* X)$$

is a Hilbert space. A sequence $X = (X_m)_{m \in \mathbb{Z}^d}$ is called a *stationary process* on \mathbb{Z}^d if for $m, k \in \mathbb{Z}^d$ we have that

$$E(X_m^* X_k) = E(X_{m+p}^* X_{k+p}) =: R_X(m - k) \quad \text{for all } p \in \mathbb{Z}^d.$$

It is known that the function R_X , termed the *covariance function* of X , defines a positive semidefinite function, that is,

$$\sum_{i,j=1}^k \alpha_i \bar{\alpha}_j R_X(r_i - r_j) \geq 0$$

for all $k \in \mathbb{N}$, $\alpha_1, \dots, \alpha_k \in \mathbb{C}$, $r_1, \dots, r_k \in \mathbb{Z}^d$. Bochner's theorem [5, 6] on positive semidefinite functions states that for such a function R_X there is a positive measure μ_X defined for Borel sets on the torus $[0, 2\pi]^d$ such that

$$R_X(r) = \int e^{-i\langle r, u \rangle} d\mu_X(u)$$

for all d -tuples of integers $r \in \mathbb{Z}^d$. The measure μ_X is referred to as the *spectral distribution measure* of the process X .

For $n = (n_1, \dots, n_d) \in \mathbb{N}_0^d$ we let $\underline{n} = \prod_{j=1}^d \{0, \dots, n_j\}$. A centered stationary stochastic process X is said to be $\text{AR}(\underline{n})$ if there exist complex numbers a_k , $k \in \underline{n} \setminus \{0\}$, such that for every t ,

$$x_t + \sum_{\substack{k \in \underline{n} \\ k \neq 0}} a_k x_{t-k} = e_t, \quad t \in \mathbb{Z}^d, \quad (1.1)$$

where $\{e_k \mid k \in \mathbb{Z}^d\}$ is a white noise zero mean process with variance σ^2 . Here AR stands for auto-regressive. Let H be the standard half-space in \mathbb{Z}^d ; that is

$$H = \{(k_1, \dots, k_d) \in \mathbb{Z}^d : \text{there is } j \in \{1, \dots, d\} \text{ with } k_1 = \dots = k_{j-1} = 0 \text{ and } k_j > 0\}.$$

The $\text{AR}(\underline{n})$ process is said to be *causal* if there is a solution to (1.1) of the form

$$x_t = \sum_{k \in H \cup \{0\}} \phi_k e_{t-k}, \quad t \in \mathbb{Z}^d, \quad (1.2)$$

with $\sum_{k \in H \cup \{0\}} |\phi_k| < \infty$. Causality based on halfspaces and multivariable generalizations of the one variable case go back to the influential papers by Helson and Lowdenslager [16, 17]. It is not difficult to see that the $\text{AR}(\underline{n})$ process X is causal if and only if the polynomial

$$\tilde{p}(z) = 1 + \sum_{\substack{k \in \underline{n} \\ k \neq 0}} \bar{a}_k z^k$$

has no roots in the closed d -disk; we call such a polynomial *stable*. A causal $\text{AR}(\underline{n})$ process is in fact *positive orthant causal*, which by definition means that there is a solution to (1.1) of the form

$$x_t = \sum_{\substack{k \geq 0 \\ k \neq 0}} \phi_k e_{t-k}, \quad t \in \mathbb{Z}^d, \quad (1.3)$$

where $k = (k_1, \dots, k_d) \geq 0$ means that $k_j \geq 0$ for $j = 1, \dots, d$.

The *multivariate autoregressive filter design problem* is the following.
“Given are covariances

$$c_k = E(X_0^* X_k), \quad k \in \underline{n}.$$

What conditions must the covariances satisfy in order that these are the covariances of a causal AR(\underline{n}) process? And in that case, how does one compute the filter coefficients a_k , $k \in \underline{n} \setminus \{0\}$ and σ^2 ?”

The following characterization for the two variable autoregressive filter design problem appeared in [10].

Theorem 1.1. [10] *Let $n, m \in \mathbb{N}$ and c_{kl} , $(k, l) \in \{0, \dots, n\} \times \{0, \dots, m\}$, be given complex numbers. There exists a causal autoregressive process with the given covariances c_{kl} if and only if there exist complex numbers c_{kl} , $(k, l) \in \{1, \dots, n\} \times \{-m, \dots, -1\}$, such that*

1. *the $(n+1)(m+1) \times (n+1)(m+1)$ doubly indexed Toeplitz matrix $\Gamma = (c_{t-s})_{s, t \in \{0, \dots, n\} \times \{0, \dots, m\}}$ is positive definite;*
2. *the matrix $(c_{s-t})_{s \in \{1, \dots, n\} \times \{0, \dots, m\}, t \in \{0, \dots, n\} \times \{1, \dots, m\}}$ has rank equal to nm .*

In this case one finds the vector

$$\frac{1}{\sigma^2} [a_{nm} \cdots a_{n0} \cdots a_{0m} \cdots a_{01} \ 1]$$

as the last row of the inverse of Γ .

If we consider the polynomial $p(z) = \frac{1}{\sigma} \tilde{p}(z)$, then the Fourier coefficients of $\frac{1}{|p|^2}$ coincide exactly with the covariances c_k . In other words,

$$\widehat{\frac{1}{|p|^2}}(k) = c_k, \quad k \in \underline{n},$$

where $\widehat{f}(k)$ denotes the k th Fourier coefficient of the function f . There are actually two main ingredients in the solution to this problem:

- (i) one needs to find a trigonometric polynomial $q(z) = q(z_1, z_2)$ with positive values on the bitorus so that its reciprocal has the required Fourier coefficients c_{kl} , and
- (ii) this trigonometric polynomial needs to factor as $q(z) = |p(z)|^2$, where $p(z)$ is stable.

This factorization is referred to as *Fejér-Riesz factorization*, and while its existence is guaranteed in one variable, this is not the case in two or more variables. This close connection between the causal autoregressive filter problem and the Fejér-Riesz factorization problem led to necessary and sufficient conditions in [10] for the two-variable Fejér-Riesz factorization problem. These results are being used in, for instance, spectral estimation [32], image compression [26], multiwavelet design [22], frames [19], stability analysis [21], texture generation [27], radar applications [1], and digital watermarking [13].

Some of these publications go beyond the two-variable setting [19, 22, 26, 32], and indicate that a better understanding of the case of three or more variables is needed. One of the main take aways of this paper is that the three and more variable case require a substantially different approach than the case of one and two variables.

In this paper we will focus on the case where the polynomial $p(z)$ is a degree one symmetric polynomials in d variables, i.e.,

$$p(z_1, \dots, z_d) = p_0 + p_1(z_1 + \dots + z_d).$$

In general, a symmetric polynomial is a polynomial where a permutation of the variables does not change the polynomial. It is easy to see that p is stable if and only if $d|p_1| < |p_0|$. The corresponding autoregressive filter problem is as follows.

Problem. Given are complex numbers a and b . Find, if possible, a degree one stable symmetric polynomial in d variables so that

$$\widehat{\frac{1}{|p|^2}}(0, 0, \dots, 0) = a, \widehat{\frac{1}{|p|^2}}(1, 0, \dots, 0) = b.$$

Clearly, due to the symmetry, we have that

$$\widehat{\frac{1}{|p|^2}}(1, 0, \dots, 0) = \widehat{\frac{1}{|p|^2}}(0, 1, \dots, 0) = \dots = \widehat{\frac{1}{|p|^2}}(0, \dots, 0, 1),$$

so that it suffices to just require $\widehat{\frac{1}{|p|^2}}(1, 0, \dots, 0) = b$. Notice that $a > 0$ will be a necessary condition for the existence of a solution. If we apply a variation of Theorem 1.1 to this case (this variation is where c_{nm} is not pre-specified; see [3, Theorem 3.3.1]), we obtain the following.

Theorem 1.2. *The above problem has a solution in $d = 2$ variables if and only if $|b| < a$. In that case, the polynomial $p(z) = p_0 + p_1(z_1 + z_2)$ is given via*

$$\begin{bmatrix} a & \bar{b} & \bar{b} \\ b & a & \frac{|b|^2}{a} \\ b & \frac{|b|^2}{a} & a \end{bmatrix} \begin{bmatrix} |p_0|^2 \\ p_1 \bar{p}_0 \\ p_1 \bar{p}_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Indeed, in this case $c_{1,-1}$ is the only unknown in the matrix Γ , and item 2 in Theorem 1.1 requires

$$\begin{bmatrix} c_{1,-1} & c_{0,-1} \\ c_{1,0} & c_{0,0} \end{bmatrix} = \begin{bmatrix} c_{1,-1} & \bar{b} \\ b & a \end{bmatrix}$$

to be of rank 1, which leads to $c_{1,-1} = \frac{|b|^2}{a}$.

The main result in this paper addresses the case of d variables, which we will state in the next section. Recall that the hypergeometric function ${}_2F_1$ is defined for $|z| < 1$ via the power series

$${}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix}; z \right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

where c is not a negative integer. Here the Pochhammer function $(q)_n$ is defined by

$$(q)_n = \begin{cases} 1, & n = 0; \\ q(q+1) \cdots (q+n-1), & \text{otherwise.} \end{cases}$$

When we specify the result for $d = 3$ variables we obtain the following.

Theorem 1.3. *The above problem has a solution in $d = 3$ variables if and only if $|b| < a$. In that case, one finds the polynomial $p(z) = p_0 + p_1(z_1 + z_2 + z_3)$ by determining $c \geq 0$ so that*

$$\frac{a(a+2c)}{a^2+2ac-3|b|^2} = \frac{(a+2c)^2}{(a+2c)^2-3|b|^2} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{27|b|^4((a+2c)^2-|b|^2)}{((a+2c)^2-3|b|^2)^3}\right), \quad (1.4)$$

and

$$\begin{bmatrix} a & \bar{b} & \bar{b} & \bar{b} \\ b & a & c & c \\ b & c & a & c \\ b & c & c & a \end{bmatrix}$$

is positive definite. Next a solution $p(z)$ is found via the equation

$$\begin{bmatrix} a & \bar{b} & \bar{b} & \bar{b} \\ b & a & c & c \\ b & c & a & c \\ b & c & c & a \end{bmatrix} \begin{bmatrix} |p_0|^2 \\ p_1 \bar{p}_0 \\ p_1 \bar{p}_0 \\ p_1 \bar{p}_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

As one can see there is a significant difference between two and three variables. In two variables the unknown in the matrix is easily found by setting $c = \frac{|b|^2}{a}$, while in three variables one needs to solve the highly nontrivial equation (1.4) to find the unknown c in the matrix. The number c plays the role of

$$c = \widehat{\frac{1}{|p|^2}}(1, -1, 0, \dots, 0) = \widehat{\frac{1}{|p|^2}}(-1, 1, 0, \dots, 0) = \cdots = \widehat{\frac{1}{|p|^2}}(0, \dots, 0, 1, -1),$$

where again we used the symmetry of the polynomial. We will see that c is required to be nonnegative (see Proposition 2.2).

The paper is organized as follows. In Sect. 2 we present our main result giving necessary and sufficient condition for the existence of an autoregressive filter with a stable symmetric degree one polynomial in d variables, as well as a method how to find the polynomial. In Sect. 3 we further specify the results for the case of three variables and present a new relation between ${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; z\right)$ and ${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \tilde{z}\right)$. Finally, in Sect. 4 we explore formulas for other Fourier coefficients in the three variable case.

2. The main result

We will begin by determining some of the Fourier coefficients of $\frac{1}{|p(z)|^2}$, where $p(z) = p_0 + p_1(z_1 + \cdots + z_d)$, $z = (z_1, \dots, z_d)$. It will be convenient to do a

simple scaling and assume that $p_0 = 1$. Next we will write $p_1 = -s$. We will use the notation

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}, \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}, \overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}, \mathbb{N}_0 = \{0, 1, 2, \dots\}.$$

Lemma 2.1. *The polynomial $p(z) = 1 - s(z_1 + \dots + z_d)$ is stable if and only if $|s| < \frac{1}{d}$.*

Proof. Let $|s| < \frac{1}{d}$ and $(z_1, \dots, z_d) \in \overline{\mathbb{D}}^d$. Then we have that $|s(z_1 + \dots + z_d)| < 1$, and thus $p(z) \neq 0$. This gives that $p(z)$ is stable.

When $|s| \geq \frac{1}{d}$, then $z_1 = \dots = z_d = \frac{1}{sd}$ yields a root of $p(z)$ inside $\overline{\mathbb{D}}^d$. Thus $p(z)$ is not stable. \square

For $q \in \mathbb{Z}$ we let $q^+ = \max\{0, q\}$ and $q^- = \max\{0, -q\}$.

Proposition 2.2. *Let $p(z) = 1 - s(z_1 + \dots + z_d)$, $|s| < \frac{1}{d}$. Then for $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$,*

$$\widehat{\frac{1}{|p|^2}}(k) = \sum_{n=0}^{\infty} \sum_{\sum n_i = n} \binom{n + k_1^+ + \dots + k_d^+}{n_1 + k_1^+, \dots, n_d + k_d^+} \binom{n + k_1^- + \dots + k_d^-}{n_1 + k_1^-, \dots, n_d + k_d^-} |s|^{2n} s^{\sum_j k_j^+} \overline{s}^{\sum_j k_j^-}.$$

Here $n_1, \dots, n_d \geq 0$ range over all nonnegative numbers that sum up to n . In particular,

$$\widehat{\frac{1}{|p|^2}}(0, \dots, 0) > 0, \widehat{\frac{1}{|p|^2}}(1, -1, 0, \dots, 0) \geq 0. \quad (2.1)$$

Proof. For $(z_1, \dots, z_d) \in \mathbb{T}^d$ we have

$$\begin{aligned} \frac{1}{p(z)} &= \sum_{n=0}^{\infty} s^n (z_1 + \dots + z_d)^n = \sum_{n=0}^{\infty} \sum_{\sum n_i = n} \binom{n}{n_1, \dots, n_d} s^n z_1^{n_1} \dots z_d^{n_d}, \\ \frac{1}{\overline{p(z)}} &= \sum_{n=0}^{\infty} \overline{s}^n (z_1^{-1} + \dots + z_d^{-1})^n = \sum_{n=0}^{\infty} \sum_{\sum n_i = n} \binom{n}{n_1, \dots, n_d} \overline{s}^n z_1^{-n_1} \dots z_d^{-n_d}. \end{aligned}$$

Multiplying the two and extracting the coefficient of $z_1^{k_1} \dots z_d^{k_d}$ gives the stated formula for $\widehat{\frac{1}{|p|^2}}(k)$.

Finally, when $k = (0, \dots, 0)$ the number s only appears in $|s|^{2n}$ which is always ≥ 0 , and > 0 when $n = 0$, and when $k = (1, -1, 0, \dots, 0)$ the number s only appears in $|s|^{2n+2}$ which is always ≥ 0 . Clearly, all the multinomial coefficients are nonnegative, and thus (2.1) follows. \square

Proposition 2.3. *Let $p(z) = 1 - s(z_1 + \dots + z_d)$, $|s| < \frac{1}{d}$. Put*

$$a = \widehat{\frac{1}{|p|^2}}(0, 0, \dots, 0), b = \widehat{\frac{1}{|p|^2}}(1, 0, \dots, 0), c = \widehat{\frac{1}{|p|^2}}(1, -1, 0, \dots, 0).$$

Then $a > 0, c \geq 0$, and the $(d+1) \times (d+1)$ matrix

$$A = \begin{bmatrix} a & \bar{b} & \bar{b} & \cdots & \bar{b} \\ b & a & c & \cdots & c \\ b & c & a & \cdots & c \\ \vdots & \vdots & & \ddots & \vdots \\ b & c & c & \cdots & a \end{bmatrix}, \quad (2.2)$$

is positive definite. Furthermore

$$A \begin{bmatrix} 1 \\ -s \\ \vdots \\ -s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (2.3)$$

Proof. Let

$$\frac{1}{|p(z)|^2} = \sum_{k \in \mathbb{Z}^d} c_k z^k$$

denote its Fourier series. Thus $\widehat{\frac{1}{|p(z)|^2}}(k) = c_k, k \in \mathbb{Z}^d$. Since $\frac{1}{|p(z)|^2}$ is positive, the multiplication operator on $L_2(\mathbb{T}^d)$ with symbol $\frac{1}{|p(z)|^2}$ is positive definite. Its matrix representation with respect to the standard monomial basis is $(c_{k-\ell})_{k, \ell \in \mathbb{Z}^d}$. Consequently, any principal submatrix $(c_{k-\ell})_{k, \ell \in \Lambda}, \Lambda \subseteq \mathbb{Z}^d$, is positive definite. If we let $\Lambda = \{0, e_1, \dots, e_d\}$, where e_j is the j th standard basis vector of \mathbb{C}^d , we obtain

$$(c_{k-\ell})_{k, \ell \in \Lambda} = \begin{bmatrix} a & \bar{b} & \bar{b} & \cdots & \bar{b} \\ b & a & c & \cdots & c \\ b & c & a & \cdots & c \\ \vdots & \vdots & & \ddots & \vdots \\ b & c & c & \cdots & a \end{bmatrix}, \quad (2.4)$$

where

$$a = \widehat{\frac{1}{|p|^2}}(0, 0, \dots, 0), b = \widehat{\frac{1}{|p|^2}}(1, 0, \dots, 0), c = \widehat{\frac{1}{|p|^2}}(1, -1, 0, \dots, 0).$$

Thus (2.4) is positive definite.

Next, we have that

$$\frac{1}{|p(z_1, \dots, z_d)|^2} p(z_1, \dots, z_d) = \frac{1}{p(1/\bar{z}_1, \dots, 1/\bar{z}_d)} = \sum_{k \in \mathbb{N}_0^d} \phi_k z^{-k}, z \in \mathbb{T}^d,$$

where $\phi_0 = 1$. Comparing the coefficients of $1, z_1, \dots, z_d$ on both sides we get the equality (2.3). \square

Proposition 2.4. Let $p_s(z) = 1 - s(z_1 + \dots + z_d)$, $|s| < \frac{1}{d}$. Put

$$a(s) = \widehat{\frac{1}{|p_s|^2}}(0, 0, \dots, 0), b(s) = \widehat{\frac{1}{|p_s|^2}}(1, 0, \dots, 0).$$

Then $a(s)$ is a function of $|s|$ that is strictly increasing on $[0, \frac{1}{d})$. In addition,

$$\{a(s) : |s| \in [0, \frac{1}{d})\} = [1, \gamma_d),$$

where

$$\gamma_d = \sum_{n=0}^{\infty} \sum_{\sum n_i=n} \binom{n}{n_1, \dots, n_d}^2 d^{-2n}. \quad (2.5)$$

We have $\gamma_1 = \gamma_2 = \gamma_3 = \infty$ and $\gamma_d < \infty$ for $d \geq 4$. Finally,

$$\left\{ \frac{|b(s)|}{a(s)} : |s| \in \left[0, \frac{1}{d}\right) \right\} = \left[0, 1 - \frac{1}{\gamma_d}\right),$$

where $\frac{1}{\infty} = 0$.

Remark 2.5. Writing (2.5) as $\gamma_d = \sum_{n=0}^{\infty} a_{n,d} d^{-2n}$ it seems that $\sum_{n=0}^{\infty} \frac{a_{n,d}}{(n!)^2} x^n = (I_0(2\sqrt{x}))^d = (\sum_{k=0}^{\infty} \frac{x^k}{(k!)^2})^d$, where $I_0(\cdot)$ is the modified Bessel function of the first kind. For $d = 4, 5, 6$ this statement is supported by items A002895, A169714, A169715, respectively, on the On-Line Encyclopedia of Integer Sequences (oeis.org).

Proof of Proposition 2.4. By the established asymptotic that was first ascertained in [25] and later generalized by [28, Theorem 4] and [7, Theorem 5.1], we have

$$\sum_{\sum n_i=n} \binom{n}{n_1, \dots, n_d}^2 d^{-2n} \approx d^{d/2} (4\pi n)^{(1-d)/2} \quad \text{as } n \rightarrow \infty.$$

Thus $\gamma_d = \infty$ for $d \leq 3$, and $\gamma_d < \infty$ for $d > 3$ follows. By Proposition 2.2 we have that

$$a(s) = \sum_{n=0}^{\infty} \sum_{\sum n_i=n} \binom{n}{n_1, \dots, n_d}^2 |s|^{2n},$$

thus $a(s)$ is a continuous function and is increasing as $|s|$ increases. Further, $a(0) = 1$ and $\lim_{|s| \rightarrow \frac{1}{d}-} a(s) = \gamma_d$, yielding that the range of $a(s)$ is the interval $[1, \gamma_d)$. Similarly,

$$|b(s)| = \sum_{n=0}^{\infty} \sum_{\sum n_i=n} \binom{n+1}{n_1+1, n_2, \dots, n_d} \binom{n}{n_1, \dots, n_d} |s|^{2n+1}$$

is a continuous function and is increasing as $|s|$ increases. Also, note that $b(0) = 0$. By (2.3) we have that

$$a(s) - d\overline{sb(s)} = 1,$$

and thus

$$\frac{|b(s)|}{a(s)} = \frac{1}{d|s|} \frac{a(s) - 1}{a(s)} = \frac{1}{d|s|} \left(1 - \frac{1}{a(s)}\right).$$

Since $\frac{|b(0)|}{a(0)} = 0$ and $\lim_{|s| \rightarrow \frac{1}{d}-} \frac{|b(s)|}{a(s)} = 1 - \frac{1}{\gamma_d}$, the last statement follows. \square

The main result is the following.

Theorem 2.6. Let $d \geq 3$ and define γ_d via (2.5). Given are $a > 0$ and $b \in \mathbb{C}$. Then there exists a stable degree one symmetric polynomial $p(z_1, \dots, z_d)$ so that

$$\widehat{\frac{1}{|p|^2}}(0, 0, \dots, 0) = a, \widehat{\frac{1}{|p|^2}}(1, 0, \dots, 0) = b,$$

if and only if $|b| < (1 - \frac{1}{\gamma_d})a$. In that case, the polynomial $p(z)$ may be found by finding $c \geq 0$ so that

$$\frac{a(a + (d-1)c)}{a^2 + (d-1)ac - d|b|^2} = \frac{1}{(2\pi)^{d-2}} \int_{[0, 2\pi]^{d-2}} \frac{1}{\sqrt{g(t_3, \dots, t_d)}} dt_3 \cdots dt_d, \quad (2.6)$$

where

$$g(t_3, \dots, t_d) = \left(1 - \frac{2|b|}{a + (d-1)c} \sum_{3 \leq j \leq d} \cos t_j + \frac{|b|^2}{(a + (d-1)c)^2} \sum_{3 \leq j, k \leq d} \cos(t_j - t_k) \right) \times \left(1 - \frac{2|b|}{a + (d-1)c} \sum_{3 \leq j \leq d} \cos t_j + \frac{|b|^2}{(a + (d-1)c)^2} \left(-4 + \sum_{3 \leq j, k \leq d} \cos(t_j - t_k) \right) \right),$$

and the matrix

$$\begin{bmatrix} a & \bar{b} & \bar{b} & \cdots & \bar{b} \\ b & a & c & \cdots & c \\ b & c & a & \cdots & c \\ \vdots & \vdots & & \ddots & \vdots \\ b & c & c & \cdots & a \end{bmatrix}$$

is positive definite. Subsequently, $p(z) = p_0 + p_1(z_1 + \cdots + z_d)$ is found via the equation

$$\begin{bmatrix} a & \bar{b} & \bar{b} & \cdots & \bar{b} \\ b & a & c & \cdots & c \\ b & c & a & \cdots & c \\ \vdots & \vdots & & \ddots & \vdots \\ b & c & c & \cdots & a \end{bmatrix} \begin{bmatrix} |p_0|^2 \\ p_1 \bar{p}_0 \\ \vdots \\ p_1 \bar{p}_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where one may choose $p_0 > 0$.

Remark 2.7. When we put $s = \frac{b}{a^2 + (d-1)ac - d|b|^2}$, the right hand side of (2.6) may be rewritten as

$$\frac{1}{(2\pi i)^{d-2}} \int_{\mathbb{T}^{d-2}} \frac{1}{|1 - s(z_3 + \cdots + z_d)|} \frac{1}{\sqrt{|1 - s(z_3 + \cdots + z_d)|^2 - 4|s|^2}} \frac{dz_3}{z_3} \cdots \frac{dz_d}{z_d}.$$

In determining the Fourier coefficients of $\frac{1}{|p(z)|^2}$, where

$$p(z) = 1 - s(z_1 + \cdots + z_d), \quad |s| < \frac{1}{d},$$

we let $w = z_3 + \cdots + z_d$, which we will treat as a parameter, and write

$$p(z) = p(z_1, z_2, w) = p_0(w) - s(z_1 + z_2),$$

where $p_0(w) = 1 - sw$. We write $f(z) = \frac{1}{|p(z)|^2}$ in Fourier series with w as a parameter

$$f(z) = \sum_{k,l \in \mathbb{Z}} c_{kl}(w) z_1^k z_2^l.$$

Proposition 2.8. *Let $p(z) = p(z_1, z_2, w) = p_0(w) - s(z_1 + z_2)$, $p_0(w) = 1 - sw$, $|s| < \frac{1}{d}$, and write $f(z) = \frac{1}{|p(z)|^2}$ in Fourier series as*

$$f(z) = \sum_{k,l \in \mathbb{Z}} c_{kl}(w) z_1^k z_2^l.$$

Then

$$\begin{bmatrix} c_{00}(w) & c_{0,-1}(w) & c_{-1,0}(w) \\ c_{01}(w) & c_{00}(w) & c_{-1,1}(w) \\ c_{10}(w) & c_{1,-1}(w) & c_{00}(w) \end{bmatrix}^{-1} = \begin{bmatrix} |1-sw|^2 & -\bar{s}(1-sw) & -\bar{s}(1-sw) \\ -s(1-\bar{s}\bar{w}) & \frac{1}{2}(|1-sw|^2 + \sqrt{|1-sw|^4 - 4|s|^2|1-sw|^2}) & 0 \\ -s(1-\bar{s}\bar{w}) & 0 & \frac{1}{2}(|1-sw|^2 + \sqrt{|1-sw|^4 - 4|s|^2|1-sw|^2}) \end{bmatrix} \quad (2.7)$$

and

$$\begin{bmatrix} c_{00}(w) & c_{0,-1}(w) & c_{-1,0}(w) & c_{-1,-1}(w) \\ c_{01}(w) & c_{00}(w) & c_{-1,1}(w) & c_{-1,0}(w) \\ c_{10}(w) & c_{1,-1}(w) & c_{00}(w) & c_{0,-1}(w) \\ c_{11}(w) & c_{10}(w) & c_{01}(w) & c_{00}(w) \end{bmatrix}^{-1} = \begin{bmatrix} |1-sw|^2 & -\bar{s}(1-sw) & -\bar{s}(1-sw) & 0 \\ -s(1-\bar{s}\bar{w}) & s^2 + \frac{1}{2}(|1-sw|^2 + \sqrt{|1-sw|^4 - 4|s|^2|1-sw|^2}) & s^2 & -\bar{s}(1-sw) \\ -s(1-\bar{s}\bar{w}) & \frac{s^2}{s^2} & s^2 + \frac{1}{2}(|1-sw|^2 + \sqrt{|1-sw|^4 - 4|s|^2|1-sw|^2}) & -\bar{s}(1-sw) \\ 0 & -s(1-\bar{s}\bar{w}) & -s(1-\bar{s}\bar{w}) & |1-sw|^2 \end{bmatrix}.$$

Proof. The first inverse follows from [23, Theorem 1.1]. With $p(z_1, z_2) = p_{00} + p_{01}z_2 + p_{10}z_1 + p_{11}z_1z_2$ and using the notation from [23, Theorem 1.1] we have

$$A = \begin{bmatrix} p_{00} & 0 & 0 \\ p_{01} & p_{00} & 0 \\ p_{10} & 0 & p_{00} \end{bmatrix}, B = \begin{bmatrix} p_{11} & p_{10} & p_{01} \\ 0 & p_{11} & 0 \\ 0 & 0 & p_{11} \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 0 & p_{10}\overline{p_{00}} - \overline{p_{01}}p_{11} \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \end{bmatrix}, C_2 = \begin{bmatrix} 0 & p_{01}\overline{p_{00}} - \overline{p_{10}}p_{11} & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \end{bmatrix},$$

$$D_1 = \begin{bmatrix} |p_{00}|^2 + |p_{10}|^2 - |p_{01}|^2 & p_{00}\overline{p_{10}} & 0 & 0 & \cdots \\ p_{10}\overline{p_{00}} & |p_{00}|^2 + |p_{10}|^2 - |p_{01}|^2 & p_{00}\overline{p_{10}} & 0 & \cdots \\ 0 & p_{10}\overline{p_{00}} & |p_{00}|^2 + |p_{10}|^2 - |p_{01}|^2 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

$$D_2 = \begin{bmatrix} |p_{00}|^2 + |p_{01}|^2 - |p_{10}|^2 & p_{00}\overline{p_{01}} & 0 & 0 & \cdots \\ p_{01}\overline{p_{00}} & |p_{00}|^2 + |p_{01}|^2 - |p_{10}|^2 & p_{00}\overline{p_{01}} & 0 & \cdots \\ 0 & p_{01}\overline{p_{00}} & |p_{00}|^2 + |p_{01}|^2 - |p_{10}|^2 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

To invert D_1 we write $D_1 = K_1 K_1^*$, where K_1 is an upper triangular bidiagonal Toeplitz operator with α on the main diagonal and β on the superdiagonal, where $\alpha > 0$ and β are so that

$$\alpha^2 + |\beta|^2 = (|p_{00}|^2 + |p_{10}|^2 - |p_{01}|^2)^2 + |p_{00}\overline{p_{10}}|^2, \alpha\beta = (p_{00}\overline{p_{10}})(|p_{00}|^2 + |p_{10}|^2 - |p_{01}|^2).$$

Similarly for D_2 . Now we use the formula

$$\begin{bmatrix} c_{00}(w) & c_{0,-1}(w) & c_{-1,0}(w) \\ c_{01}(w) & c_{00}(w) & c_{-1,1}(w) \\ c_{10}(w) & c_{1,-1}(w) & c_{00}(w) \end{bmatrix}^{-1} = AA^* - B^*B - C_1^*D_1^{-1}C_1 - C_2^*D_2^{-1}C_2$$

to obtain (2.7).

For the second inverse, we use that the (4,1) entry in the inverse is 0 as $p(z)$ does not have a $p_{11}z_1z_2$ term. It now follows from the inverse block matrix formula

$$\begin{bmatrix} P & H_1 & H_3 \\ H_1^* & Q & H_2 \\ H_3^* & H_2^* & R \end{bmatrix}^{-1} = \begin{bmatrix} [P \ H_1]^{-1} & 0 \\ [H_1^* \ Q]^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & [Q \ H_2]^{-1} \\ 0 & [H_2^* \ R]^{-1} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & Q^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (2.8)$$

which holds if there is a zero in the (3,1) block of the inverse; see, for instance, [9, Formula (10)].¹ \square

Proposition 2.9. For $p(z) = 1 - s(z_1 + \cdots + z_d)$, $|s| < \frac{1}{d}$, we have

$$\widehat{\frac{1}{|p|^2}}(0, \dots, 0) = \frac{1}{(2\pi i)^{d-2}} \int_{\mathbb{T}^{d-2}} \frac{1}{|1 - s(z_3 + \cdots + z_d)|} \frac{1}{\sqrt{|1 - s(z_3 + \cdots + z_d)|^2 - 4|s|^2}} \frac{dz_3}{z_3} \cdots \frac{dz_d}{z_d}.$$

Proof. In general we have that

$$\begin{bmatrix} x & \bar{y} & \bar{y} \\ y & v & 0 \\ y & 0 & v \end{bmatrix}^{-1} = \frac{1}{xv - 2|y|^2} \begin{bmatrix} v & -\bar{y} & -\bar{y} \\ -y & x - \frac{|y|^2}{v} & \frac{|y|^2}{v} \\ -y & \frac{|y|^2}{v} & x - \frac{|y|^2}{v} \end{bmatrix}.$$

¹Formula (2.8) goes back to the 1980s and easily generalizes to $n \times n$ block matrices with a block tridiagonal inverse, but a reference for it is illusive.

Combining this with Proposition 2.8 we find

$$c_{00}(w) = \frac{v}{xv - 2|y|^2},$$

where

$$\begin{aligned} v &= \frac{1}{2}(|1 - sw|^2 + \sqrt{|1 - sw|^4 - 4|s|^2|1 - sw|^2}), \\ x &= |1 - sw|^2, y = -s(1 - \bar{s}\bar{w}). \end{aligned}$$

We have

$$\begin{aligned} xv - 2|y|^2 &= \frac{1}{2}(|1 - sw|^4 - 4|s|^2|1 - sw|^2 \\ &\quad + |1 - sw|^2\sqrt{|1 - sw|^4 - 4|s|^2|1 - sw|^2}) = \\ &= v\sqrt{|1 - sw|^4 - 4|s|^2|1 - sw|^2} = v|1 - sw|\sqrt{|1 - sw|^2 - 4|s|^2}. \end{aligned}$$

Thus

$$\begin{aligned} c_{00}(w) &= \frac{1}{|1 - sw|\sqrt{|1 - sw|^2 - 4|s|^2}} = \frac{1}{|1 - s(z_3 + \cdots + z_d)|} \\ &\quad \frac{1}{\sqrt{|1 - s(z_3 + \cdots + z_d)|^2 - 4|s|^2}}. \end{aligned}$$

To find the 0th Fourier coefficient of $\frac{1}{|p(z)|^2}$ we need to compute

$$\frac{1}{(2\pi i)^{d-2}} \int_{\mathbb{T}^{d-2}} c_{00}(z_3 + \cdots + z_d) \frac{dz_3}{z_3} \cdots \frac{dz_d}{z_d},$$

which yields the stated formula. \square

It is easy to check the following lemma.

Lemma 2.10. *Suppose that the $(d+1) \times (d+1)$ matrix*

$$A = \begin{bmatrix} a & \bar{b} & \bar{b} & \cdots & \bar{b} \\ b & a & c & \cdots & c \\ b & c & a & \cdots & c \\ \vdots & \vdots & & \ddots & \vdots \\ b & c & c & \cdots & a \end{bmatrix} \quad (2.9)$$

is invertible. Then the first column of the inverse equals

$$\frac{1}{a^2 + (d-1)ac - d|b|^2} \begin{bmatrix} a + (d-1)c \\ -b \\ \vdots \\ -b \end{bmatrix}.$$

Proof. Simply multiply A by the vector to obtain the first standard basis vector. \square

Proof of Theorem 2.6. By the last statement in Proposition 2.4 we see that $\frac{|b|}{a} \in [0, 1 - \frac{1}{\gamma_d})$ is necessary and sufficient.

Next, the polynomial $p(z)$ after normalization so that $p(0) = 1$ will satisfy (2.3). Starting with A as in (2.9) we can, by Lemma 2.10, rescale the matrix as $\frac{a+(d-1)c}{a^2+(d-1)ac-d|b|^2}A$ so that the (1,1) entry of its inverse is 1, which corresponds to the situation where $p(0) = 1$. Then, again using Lemma 2.10, we find that $s = -\frac{dp}{dz_1}|_{z=0}$ corresponds to the value $s = \frac{b}{a^2+(d-1)ac-d|b|^2}$. Using this value for s as well as $\widehat{\frac{1}{|p|^2}}(0, \dots, 0) = a \frac{a+(d-1)c}{a^2+(d-1)ac-d|b|^2}$, we find that Proposition 2.9 yields equality (2.6). \square

3. The three variable case

In this section we provide further details when $d = 3$. To be consistent with earlier results in [11] and [31], we consider the polynomial

$$p(z_1, z_2, z_3) = 1 - \frac{z_1 + z_2 + z_3}{r}, r > 3.$$

Comparing this with the previous section, we make the conversion $s = \frac{1}{r}$ and require $s > 0$. This is not a significant restriction as a phase appearing in s can always be absorbed in the variables via $(z_1, z_2, z_3) \rightarrow e^{i\theta}(z_1, z_2, z_3)$.

We will use the complete elliptic integral of the first kind, which is

$$\begin{aligned} K(m) &= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - m \sin^2(t)}} dt \\ &= \int_0^1 \frac{1}{\sqrt{1 - t^2} \sqrt{1 - mt^2}} dt = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; m\right). \end{aligned}$$

Theorem 3.1. Let $p(z_1, z_2, z_3) = 1 - \frac{z_1+z_2+z_3}{r}$, $r > 3$, and $f(z) = \frac{1}{|p(z)|^2}$, $z = (z_1, z_2, z_3)$. Write

$$f(z) = \sum_{k,l,m \in \mathbb{Z}} c_{klm} z_1^k z_2^l z_3^m, \quad (z_1, z_2, z_3) \in \mathbb{T}^3.$$

Then

$$\begin{aligned} c_{000} &= \frac{r^2}{2\pi} \int_0^{2\pi} \frac{1}{\sqrt{r^2+1-2r \cos t} \sqrt{r^2-3-2r \cos t}} dt \\ &= \frac{2r^2}{\pi(r-1)^{\frac{3}{2}}(r+3)^{\frac{1}{2}}} K\left(\frac{16r}{(r-1)^3(r+3)}\right) \\ &= \frac{r^2}{(r-1)^{\frac{3}{2}}(r+3)^{\frac{1}{2}}} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{16r}{(r-1)^3(r+3)}\right). \end{aligned} \quad (3.1)$$

Proof of Theorem 3.1. From Proposition 2.9 with $s = \frac{1}{r}$ and $z_3 = e^{it}$ we obtain

$$c_{000} = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1 - \frac{e^{it}}{r}|} \frac{1}{\sqrt{|1 - \frac{e^{it}}{r}|^2 - \frac{4}{r^2}}} dt.$$

Using that $|1 - \frac{e^{it}}{r}|^2 = (1 - \frac{\cos t}{r})^2 + \frac{\sin^2 t}{r^2} = \frac{1}{r^2}(r^2 - 2r \cos t + 1)$, formula (3.1) follows.

Next, use $\cos t = 2 \cos^2 \frac{t}{2} - 1 = 2 \sin^2(\frac{\pi}{2} - \frac{t}{2}) - 1$, do a change of variable $t \rightarrow \frac{\pi}{2} - \frac{t}{2}$, use the symmetry of the integrand, and (3.1) becomes

$$\frac{2r^2}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{(r+1)^2 - 4r \sin^2 t} \sqrt{(r+3)(r-1) - 4r \sin^2 t}} dt. \quad (3.2)$$

Now we let $p^2 = \frac{4r}{(r+1)^2}$ and $q^2 = \frac{4r}{(r+3)(r-1)}$, and use the first formula in Section 2.616 of [12], which is the equality²

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{(1-p^2 \sin^2 x)(1-q^2 \sin^2 x)}} = \frac{1}{\sqrt{1-p^2}} \int_0^{\frac{\pi}{2}} \frac{d\alpha}{\sqrt{1 - \frac{q^2-p^2}{1-p^2} \sin^2 \alpha}}.$$

This transforms (3.2) into

$$\begin{aligned} & \frac{2r^2}{\pi} \frac{1}{\sqrt{(r-1)^3(r+3)}} \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - \frac{16r}{(r-1)^3(r+3)} \sin^2 t}} dt \\ &= \frac{2r^2}{\pi(r-1)^{\frac{3}{2}}(r+3)^{\frac{1}{2}}} K\left(\frac{16r}{(r-1)^3(r+3)}\right). \end{aligned}$$

□

The following result is inspired by a generating function entry by Paul D. Hanna [14] regarding sequence A002893 on the On-Line Encyclopedia of Integer Sequences (oeis.org). Hanna arrived at this entry as a variation of the generating function for the triangle of cubed binomial coefficients (sequence A181543 on oeis.org) and numerically verified it for hundreds of terms [15].

Theorem 3.2. *Using the same notation as in Theorem 3.1, we have*

$$c_{000} = \frac{r^2}{r^2-3} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{27(r^2-1)}{(r^2-3)^3}\right), r > 3.$$

Proof. By Proposition 2.2, we have $c_{000}(r) = \sum_{n=0}^{\infty} \sum_{n_1+n_2+n_3=n} \binom{n}{n_1, n_2, n_3}^2 r^{-2n}$, $r > 3$. Letting $x = r^{-2}$, and

$$\begin{aligned} g(x) &= \sum_{n=0}^{\infty} \sum_{n_1+n_2+n_3=n} \binom{n}{n_1, n_2, n_3}^2 x^n, \\ h(x) &= \frac{1}{1-3x} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{27x^2(1-x)}{(1-3x)^3}\right), \end{aligned}$$

the stated equality now comes down to proving that $g(x) = h(x)$, $|x| < \frac{1}{9}$. We will show that both $g(x)$ and $h(x)$ satisfy the Heun differential equation (see [18]) with initial values

$$x(1-x)(1-9x)y'' + (1-20x+27x^2)y' + (9x-3)y = 0, y(0) = 1, y'(0) = 3. \quad (3.3)$$

²due to a change of variables $\sin \alpha = \frac{\sqrt{1-p^2} \sin x}{\sqrt{1-p^2 \sin^2 x}}$.

If we write $g(x) = \sum_{n=0}^{\infty} g_n x^n$, $|x| < \frac{1}{9}$, then it follows from [29, Theorem 1; see also Table 1] that

$$n^2 g_n - (10n^2 - 10n + 3)g_{n-1} + 9(n-1)^2 g_{n-2} = 0, n \geq 2, g_0 = 1, g_1 = 3. \quad (3.4)$$

But then it is a straightforward computation that $g(x)$ satisfies (3.3). Indeed, plugging $y = g(x) = \sum_{n=0}^{\infty} g_n x^n$ in the left hand side of (3.3) and extracting the coefficient of x^{n-1} we obtain

$$\begin{aligned} & n(n-1)g_n - 10(n-1)(n-2)g_{n-1} + 9(n-2)(n-3)g_{n-2} \\ & + ng_n - 20(n-1)g_{n-1} + 27(n-2)g_{n-2} + 9g_{n-2} - 3g_{n-1} \\ & = n^2 g_n + (-10n^2 + 30n - 20 - 20n + 20 - 3)g_{n-1} \\ & + (9n^2 - 45n + 54 + 27n - 54 + 9)g_{n-2} \\ & = n^2 g_n - (10n^2 - 10n + 3)g_{n-1} + 9(n-1)^2 g_{n-2} = 0, \end{aligned}$$

where in the last step we use (3.4).

Next, let us turn to $h(x)$. Introduce $z(x) = \frac{27x^2(1-x)}{(1-3x)^3}$ and $w(z) = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; z\right)$. Then (see, for instance, [12, Section 9.15])

$$(1-z)zw''(z) + (1-2z)w'(z) - \frac{2}{9}w(z) = 0.$$

We have that $h(x) = \frac{1}{1-3x}w(z(x))$, $h'(x) = \frac{3}{(1-3x)^2}w(z(x)) + \frac{54x}{(1-3x)^5}w'(z(x))$, and

$$h''(x) = \frac{18}{(1-3x)^3}w(z(x)) + \frac{54(15x+1)}{(1-3x)^6}w'(z(x)) + \frac{4(27x)^2}{(1-3x)^9}w''(z(x)).$$

Plugging $y = h(x)$ in the left hand side of (3.3) yields

$$\begin{aligned} & x(1-x)(1-9x)h''(x) + (1-20x+27x^2)h'(x) + (9x-3)h(x) = \\ & \frac{108x}{(1-3x)^3}((1-z(x))z(x)w''(z(x)) + (1-2z(x))w'(z(x)) \\ & - \frac{2}{9}w(z(x))) = 0. \end{aligned}$$

In addition, it is easy to check that $h(0) = 1, h'(0) = 3$.

Thus both $g(x)$ and $h(x)$ satisfy (3.3), and thus by uniqueness we find that $h(x) = g(x)$. \square

Remark 3.3. Using the Birkhoff-Trjitzinsky method (see [4], and [20] for complete proofs; see also [30] and [24]) one can obtain that the asymptotics of $g_n = h_n$ is $0.41349667 \cdot \frac{9^n}{n}(1 + O(n^{-1}))$. From this one can deduce that $g(x)$ is transcendental over $\mathbb{Q}(x)$; see [24, Corollary 2.1]. This implies that the autoregressive filter problem in three and more variables is of a different nature than the case of one or two variables in the sense that one can no longer expect necessary and sufficient conditions via polynomial expressions with rational coefficients, such as the low rank requirement in two variables.

Corollary 3.4.

$$\frac{1}{r^2-3} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{27(r^2-1)}{(r^2-3)^3}\right)$$

$$= \frac{1}{(r-1)^{\frac{3}{2}}(r+3)^{\frac{1}{2}}} {}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}; \frac{16r}{(r-1)^3(r+3)}\right).$$

Proof. Combine Theorems 3.1 and 3.2. \square

There are formulas that relate ${}_2F_1\left(\begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1 \end{matrix}; z\right)$ and ${}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}; \tilde{z}\right)$ (see, for instance, [2, page 112]), but the above equality seems to be of a different nature than those already known.

We end this section by providing a proof for Theorem 1.3.

Proof of Theorem 1.3. By Theorem 2.6 we see that $\frac{|b|}{a} < 1$ is necessary and sufficient. The proof is the same as the proof of Theorem 2.6, except that we will use the expression of c_{000} from Theorem 3.2. Let $d = 3$. As before, the polynomial $p(z)$ after normalization so that $p(0) = 1$ will satisfy (2.3). Starting with A as in (2.9) we can, by Lemma 2.10, rescale the matrix as $\frac{a+(d-1)c}{a^2+(d-1)ac-d|b|^2}A$ so that the (1,1) entry of its inverse is 1, which corresponds to the situation where $p(0) = 1$. Then, again using Lemma 2.10, we find that $\frac{1}{r} = -\frac{dp}{dz_1}|_{z=0}$ corresponds to the value $\frac{1}{r} = \frac{b}{a^2+(d-1)ac-d|b|^2}$. Using this value for r as well as $c_{000} = \widehat{\frac{1}{|p|^2}}(0, \dots, 0) = \frac{a(a+(d-1)c)}{a^2+(d-1)ac-d|b|^2}$, we find that Proposition 2.9 yields equality (1.4). \square

sing the same notation as in T

4. The three variable case: other Fourier coefficients

In [11] the current authors considered the two variable analog, and obtained the following expression for the Fourier coefficients of $f(z_1, z_2) = |1 - \frac{z_1+z_2}{r}|^{-2}$, $r > 2$.

Theorem 4.1. [11, Theorem 1] *Let $p(z_1, z_2) = 1 - \frac{z_1+z_2}{r}$ with $r > 2$, and let c_{k_1, k_2} denote the Fourier coefficients of its spectral density function $f(z_1, z_2) = |1 - \frac{z_1+z_2}{r}|^{-2}$. Then we have*

$$c_{k_1, k_2} = \frac{1}{\sqrt{1 - \frac{4}{r^2}}} \left(\frac{r}{2} - \sqrt{\frac{r^2}{4} - 1} \right)^{|k_1|+|k_2|}, \quad k_1 k_2 \leq 0,$$

and

$$c_{k_1, k_2} = \frac{\binom{|k_1|+|k_2|}{|k_1|}}{r^{|k_1|+|k_2|}} {}_3F_2\left(1, \frac{|k_1|+|k_2|}{2} + 1, \frac{|k_1|+|k_2|+1}{2}; \frac{4}{r^2}\right), \quad k_1 k_2 > 0.$$

In an attempt to obtain a three variable generalization of the above result, we have found following expressions for the Fourier coefficients c_J , $J \in \{-1, 0, 1\}^3$ of $f(z_1, z_2, z_3) = |1 - \frac{z_1+z_2+z_3}{r}|^{-2}$, $r > 3$.

Theorem 4.2. *Using the same notation as in Theorem 3.1, we have*

$$c_{100} = \frac{r^2}{4\pi} \int_0^{2\pi} \frac{1}{r - e^{it}} \left(\sqrt{\frac{r^2 - 2r \cos t + 1}{r^2 - 2r \cos t - 3}} - 1 \right) dt =$$

$$-\frac{r}{2} + \frac{r^2}{4\pi} \int_0^{2\pi} \frac{r - \cos t}{\sqrt{r^2 - 2r \cos t + 1} \sqrt{r^2 - 2r \cos t - 3}} dt,$$

$$c_{-1,1,0} = \frac{r^2}{8\pi} \int_0^{2\pi} \sqrt{\frac{r^2 - 2r \cos t - 3}{r^2 - 2r \cos t + 1}} - 2 + \sqrt{\frac{r^2 - 2r \cos t + 1}{r^2 - 2r \cos t - 3}} dt,$$

and

$$\begin{aligned} c_{011} &= \frac{r^2}{\pi} \int_0^{2\pi} \frac{1}{e^{it}(r - e^{it})(\sqrt{(r^2 + 1 - 2r \cos t)(r^2 - 3 - 2r \cos t)} + r^2 - 3 - 2r \cos t)} dt = \\ &= -\frac{1}{2} + \frac{r^2}{4\pi} \int_0^{2\pi} \frac{r \cos t - \cos 2t}{\sqrt{r^2 + 1 - 2r \cos t} \sqrt{r^2 - 3 - 2r \cos t}} dt = \\ &= \frac{r}{4\pi} \int_0^{2\pi} \cos t \sqrt{\frac{r^2 - 2r \cos t + 1}{r^2 - 2r \cos t - 3}} dt \\ &+ \frac{r}{4\pi} \int_0^{2\pi} \frac{r - \cos t}{\sqrt{r^2 - 2r \cos t + 1} \sqrt{r^2 - 2r \cos t - 3}} dt - \frac{1}{2} = \\ &= \frac{r}{4\pi} \int_0^{2\pi} \cos t \sqrt{\frac{r^2 - 2r \cos t + 1}{r^2 - 2r \cos t - 3}} dt + \frac{c_{100}}{r}. \end{aligned} \quad (4.1)$$

Proof. From Proposition 2.8 we get

$$\begin{aligned} c_{01}(e^{it}) &= \frac{2r^2}{r - e^{it}} \left(\sqrt{(r^2 + 1 - 2r \cos t)(r^2 - 3 - 2r \cos t)} + r^2 - 3 - 2r \cos t \right)^{-1} \end{aligned}$$

Using $c_{011} = \frac{1}{2\pi} \int_0^{2\pi} c_{01}(e^{it}) e^{-it} dt$ we consequently obtain

$$c_{011} = \frac{r^2}{\pi} \int_0^{2\pi} \frac{1}{e^{it}(e^{it} - r) \sqrt{r^2 - 3 - 2r \cos t} (\sqrt{r^2 + 1 - 2r \cos t} + \sqrt{r^2 - 3 - 2r \cos t})} dt.$$

Multiplying numerator and denominator in the integrand with $\sqrt{r^2 + 1 - 2r \cos t} - \sqrt{r^2 - 3 - 2r \cos t}$, we obtain

$$\frac{r^2}{4\pi} \int_0^{2\pi} \frac{\sqrt{r^2 + 1 - 2r \cos t}}{e^{it}(r - e^{it}) \sqrt{r^2 - 3 - 2r \cos t}} dt - \frac{r^2}{4\pi} \int_0^{2\pi} \frac{1}{e^{it}(r - e^{it})} dt.$$

The second term equals $\frac{1}{2}$, and for the first term we can take its real part (since we know that c_{011} is real). This gives

$$\begin{aligned} c_{011} &= -\frac{1}{2} + \frac{r^2}{4\pi} \int_0^{2\pi} \frac{r \cos t - \cos 2t}{r^2 + 1 - 2r \cos t} \frac{\sqrt{r^2 + 1 - 2r \cos t}}{\sqrt{r^2 - 3 - 2r \cos t}} dt \\ &= -\frac{1}{2} + \frac{r^2}{4\pi} \int_0^{2\pi} \frac{r \cos t - \cos 2t}{\sqrt{r^2 + 1 - 2r \cos t} \sqrt{r^2 - 3 - 2r \cos t}} dt. \end{aligned}$$

The last equality for c_{011} is obtained by using $\frac{1}{z(r-z)} = \frac{1}{r} \left(\frac{1}{z} + \frac{1}{r-z} \right)$ and applying it to the first expression for c_{011} .

Next, from Proposition 2.8 we find

$$\begin{aligned} c_{-1,1}(e^{it}) &= \frac{4r^2}{\sqrt{r^2 + 1 - 2r \cos t} \sqrt{r^2 - 3 - 2r \cos t} (\sqrt{r^2 + 1 - 2r \cos t} + \sqrt{r^2 - 3 - 2r \cos t})^2}. \end{aligned}$$

Multiplying numerator and denominator with $(\sqrt{r^2 + 1 - 2r \cos t} - \sqrt{r^2 - 3 - 2r \cos t})^2$ we obtain

$$c_{-1,1}(e^{it}) = \frac{r^2}{4} \left(\frac{(\sqrt{r^2 + 1 - 2r \cos t} - \sqrt{r^2 - 3 - 2r \cos t})^2}{\sqrt{r^2 + 1 - 2r \cos t} \sqrt{r^2 - 3 - 2r \cos t}} \right)$$

$$= \frac{r^2}{4} \left(\sqrt{\frac{r^2 - 2r \cos t - 3}{r^2 - 2r \cos t + 1}} - 2 + \sqrt{\frac{r^2 - 2r \cos t + 1}{r^2 - 2r \cos t - 3}} \right).$$

Use now $c_{-1,1,0} = \frac{1}{2\pi} \int_0^{2\pi} c_{-1,1}(e^{it}) dt$ to obtain the result.

The proof for c_{100} is similar. \square

In Theorem 3.1 we have expressed c_{000} in terms of the complete elliptic integral of the first kind. We can express the other Fourier coefficients above in terms of the complete elliptic integral of the first, second and third kind, which are $K(m)$, $E(m)$ and $\Pi(n, m)$, respectively, where

$$E(m) = \int_0^{\frac{\pi}{2}} \sqrt{1 - m \sin^2 t} dt = \frac{\pi}{2} {}_2F_1 \left(-\frac{1}{2}, \frac{1}{2}; m \right),$$

and

$$\Pi(n, m) = \int_0^{\frac{\pi}{2}} \frac{1}{(1 - n \sin^2(t)) \sqrt{1 - m \sin^2(t)}} dt.$$

Proposition 4.3. *Using the same notation as in Theorem 3.1, we have*

$$\begin{aligned} c_{100} &= \frac{r}{3} (c_{000} - 1), \\ c_{011} &= \frac{1}{3} (c_{000} - 1) + \\ &\quad \frac{(r^4 - 2r^2 - 15)K\left(\frac{16r}{(r+3)(r-1)^2}\right) - (r+3)(r-1)^3 E\left(\frac{16r}{(r+3)(r-1)^2}\right) - 4(r-3)(r+1)\Pi\left(\frac{4r}{(r+3)(r-1)}, \frac{16r}{(r+3)(r-1)^2}\right)}{4\pi(r-1)\sqrt{(r+3)(r-1)}} \\ &= \frac{1}{3} (c_{000} - 1) - \frac{1}{2} + \\ &\quad \frac{(r+3)(r-1)^3 K\left(\frac{16r}{(r+3)(r-1)^2}\right) - (r+3)(r-1)^3 E\left(\frac{16r}{(r+3)(r-1)^2}\right) + 4(r-3)(r+1)\Pi\left(\frac{4}{(r-1)^2}, \frac{16r}{(r+3)(r-1)^2}\right)}{4\pi(r-1)\sqrt{(r+3)(r-1)}}, \\ c_{111} &= \frac{3}{r} c_{011}, \\ c_{0,1,-1} &= \frac{1}{2} (rc_{001} - c_{000}), \\ c_{1,1,-1} &= rc_{011} - 2c_{001}. \end{aligned} \tag{4.2}$$

Other Fourier coefficients c_J , $J \in \{-1, 0, 1\}^3$, are obtained via $c_J = c_{\sigma(J)} = c_{-J}$, where σ is a permutation.

Proof. First observe that

$$\frac{1}{|p(z_1, \dots, z_d)|^2} p(z_1, \dots, z_d) = \frac{1}{p(\frac{1}{z_1}, \dots, \frac{1}{z_d})} = \sum_{k \in \mathbb{N}_0^d} \phi_k z^{-k}, \quad z \in \mathbb{T}^d, \tag{4.3}$$

where $\phi_0 = 1$. If we extract the Fourier coefficients indexed by $\Lambda = \{0, 1\}^3$ on both sides, we obtain

$$\begin{bmatrix} c_{000} & c_{00,-1} & c_{0,-1,0} & c_{0,-1,-1} & c_{-100} & c_{-1,0,-1} & c_{-1,-1,0} & c_{-1,-1,-1} \\ c_{001} & c_{000} & c_{0,-1,1} & c_{0,-1,0} & c_{-101} & c_{-1,0,0} & c_{-1,-1,1} & c_{-1,-1,0} \\ c_{010} & c_{01,-1} & c_{000} & c_{0,0,-1} & c_{-110} & c_{-1,1,-1} & c_{-1,0,0} & c_{-1,0,-1} \\ c_{011} & c_{010} & c_{001} & c_{000} & c_{-111} & c_{-1,1,0} & c_{-1,0,1} & c_{-1,0,0} \\ c_{100} & c_{1,0,-1} & c_{1,-1,0} & c_{1,-1,-1} & c_{000} & c_{0,0,-1} & c_{0,-1,0} & c_{0,-1,-1} \\ c_{101} & c_{100} & c_{1,-1,1} & c_{1,-1,0} & c_{001} & c_{000} & c_{0,-1,1} & c_{0,-1,0} \\ c_{110} & c_{1,1,-1} & c_{1,0,0} & c_{1,0,-1} & c_{010} & c_{0,1,-1} & c_{000} & c_{0,0,-1} \\ c_{111} & c_{110} & c_{101} & c_{100} & c_{011} & c_{010} & c_{001} & c_{000} \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{1}{r} \\ -\frac{1}{r} \\ -\frac{1}{r} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Since p is a symmetric polynomial with real coefficients we have that $c_J = c_{\sigma(J)} = c_{-J}$, where σ is a permutation. Thus we obtain

$$\begin{aligned} c_{000} - \frac{3c_{001}}{r} &= 1, (1 - \frac{1}{r})c_{001} - \frac{c_{000}}{r} - \frac{c_{0,-1,1}}{r} = 0, \\ c_{100} - \frac{2c_{0,-1,1}}{r} - \frac{c_{1,-1,-1}}{r} &= 0, \\ c_{011} - \frac{c_{1,1,-1} + c_{100} + c_{0,-1,1}}{r} &= 0, c_{111} - \frac{2c_{011}}{r} - \frac{c_{100}}{r} = 0. \end{aligned}$$

This yields the stated relations between the different Fourier coefficients (see also [31, Proposition 3.1.1]).

Finally, we turn to c_{011} . To prove the first expression for c_{011} , by (4.1) it suffices to prove

$$\begin{aligned} \int_0^{2\pi} \cos t \sqrt{\frac{r^2 - 2r \cos t + 1}{r^2 - 2r \cos t - 3}} dt = \\ \frac{-(r+3)(r-1)^3 E(\frac{16r}{(r+3)(r-1)^3}) + (r^4 - 2r^2 - 15)K(\frac{16r}{(r+3)(r-1)^3}) - 4(r-3)(r+1)\Pi(\frac{4r}{(r+3)(r-1)}, \frac{16r}{(r+3)(r-1)^3})}{r(r-1)\sqrt{(r+3)(r-1)}}. \end{aligned} \quad (4.4)$$

The left hand side of (4.4) can be rewritten as

$$\begin{aligned} \int_0^{2\pi} \cos t \sqrt{\frac{r^2 - 2r \cos t + 1}{r^2 - 2r \cos t - 3}} dt = \int_0^{2\pi} \sqrt{\frac{r^2 - 2r \cos t + 1}{r^2 - 2r \cos t - 3}} dt \\ - \int_0^{2\pi} (1 - \cos t) \sqrt{\frac{r^2 - 2r \cos t + 1}{r^2 - 2r \cos t - 3}} dt. \end{aligned} \quad (4.5)$$

We will first show that

$$\begin{aligned} \int_0^{2\pi} \sqrt{\frac{r^2 - 2r \cos t + 1}{r^2 - 2r \cos t - 3}} dt \\ = \frac{4 \left(4K(\frac{16r}{(r+3)(r-1)^3}) + (r-3)(r+1)\Pi(\frac{4r}{(r+3)(r-1)}, \frac{16r}{(r+3)(r-1)^3}) \right)}{(r-1)\sqrt{(r+3)(r-1)}}. \end{aligned} \quad (4.6)$$

To prove (4.6) we need to show the equality

$$\int_0^{2\pi} \sqrt{\frac{r^2 - 2r \cos t + 1}{r^2 - 2r \cos t - 3}} dt = \int_0^{2\pi} \frac{\sqrt{(r+3)(r-1)^3 - 16r \sin^2 t}}{(r+3)(r-1) - 4r \sin^2 t} dt.$$

To prove the above equality we make some simplifications. In the second integral, because everything is in terms of $\sin^2 t$, the integral over $[0, 2\pi]$ is equal to 4 times the integral over $[0, \pi/2]$. For the first integral, make the change of variables $\cos t = 1 - 2\sin^2 t/2$ then $t \rightarrow t/2$, then put everything on $[0, \pi/2]$ and divide by 4, to obtain,

$$\frac{r-1}{\sqrt{(r-3)(r+1)}} \int_0^{\frac{\pi}{2}} \sqrt{\frac{1 + \frac{4r}{(r-1)^2} \sin^2 t}{1 + \frac{4r}{(r-3)(r+1)} \sin^2 t}} dt$$

$$= \sqrt{\frac{r-1}{r+3}} \int_0^{\frac{\pi}{2}} \frac{\sqrt{1 - \frac{16r}{(r+3)(r-1)^3} \sin^2 t}}{1 - \frac{4r}{(r+3)(r-1)} \sin^2 t} dt.$$

Now let $p^2 = -\frac{4r}{(r-1)^2}$ and $q^2 = -\frac{4r}{(r-3)(r+1)}$. Then $1 - q^2 = \frac{(r+3)(r-1)}{(r-3)(r+1)}$ and $\frac{q^2 - p^2}{1 - q^2} = \frac{16r}{(r+3)(r-1)^3}$. The integrals become

$$\int_0^{\frac{\pi}{2}} \sqrt{\frac{1 - p^2 \sin^2 t}{1 - q^2 \sin^2 t}} dt = \frac{1}{\sqrt{1 - q^2}} \int_0^{\frac{\pi}{2}} \frac{\sqrt{1 - \frac{q^2 - p^2}{1 - q^2} \sin^2 t}}{1 + \frac{q^2}{1 - q^2} \sin^2 t} dt. \quad (4.7)$$

On the right hand integral make the change of variable $\sin t = \frac{\sqrt{1 - q^2} \sin x}{\sqrt{1 - q^2 \sin^2 x}}$ ³, then $[0, \pi/2] \rightarrow [0, \pi/2]$ and the right hand integral goes to the left hand integral. Indeed, we have

$$\cos t dt = \frac{\sqrt{1 - q^2 \sin^2 x} \sqrt{1 - q^2} \cos x - \sqrt{1 - q^2} \sin x \frac{1}{2\sqrt{1 - q^2 \sin^2 x}} (-2q^2 \sin x \cos x)}{1 - q^2 \sin^2 x} dx.$$

Using $\cos t = \sqrt{1 - \frac{(1 - q^2) \sin^2 x}{1 - q^2 \sin^2 x}} = \frac{\cos x}{\sqrt{1 - q^2 \sin^2 x}}$, we find

$$dt = \frac{\sqrt{1 - q^2}}{1 - q^2 \sin^2 x} dx.$$

Now equality (4.7) (and thus (4.6)) follows after some manipulations.

Next, we deal with the second term of the right hand side of (4.5):

$$\int_0^{2\pi} (1 - \cos t) \sqrt{\frac{r^2 - 2r \cos t + 1}{r^2 - 2r \cos t - 3}} dt = 2 \int_0^{\pi} (1 - \cos t) \sqrt{\frac{\frac{r^2 - 1}{2r} - \cos t}{\frac{r^2 - 3}{2r} - \cos t}} dt. \quad (4.8)$$

By using the change of variables $u = \cos t$ (and thus $dt = -\frac{1}{\sqrt{1 - u^2}} du$), we can rewrite this as

$$2 \int_{-1}^1 \sqrt{\frac{(\frac{r^2 + 1}{2r} - u)(1 - u)}{(\frac{r^2 - 3}{2r} - u)(u - (-1))}} du.$$

Let

$$a = \frac{r^2 + 1}{2r}, b = \frac{r^2 - 3}{2r}, c = 1, d = -1,$$

and observe that $a > b > c \geq y > d$. We can now use [8, Equations 252.17 and 362.16], which yield

$$\begin{aligned} \int_d^y \sqrt{\frac{(c - u)(a - u)}{(b - u)(u - d)}} du = \\ \frac{(a - d)(c - d)g}{2\alpha^2(k^2 - \alpha^2)} (\alpha^2 E(k^2) + (k^2 - \alpha^2)K(k^2)) \\ + (2k^2\alpha^2 - \alpha^4 - k^2)\Pi(\alpha^2, k^2), \end{aligned}$$

³The reverse change of variables is $\sin x = \frac{\sin t}{\sqrt{1 - q^2 + q^2 \sin^2 t}}$ and we get $dx = \frac{\sqrt{1 - q^2}}{1 - q^2 + q^2 \sin^2 t} dt$.

where

$$g = \frac{2}{\sqrt{(a-c)(b-d)}}, \alpha^2 = \frac{d-c}{a-c}, k^2 = \frac{(a-b)(c-d)}{(a-c)(b-d)}.$$

We obtain that (4.8) equals

$$\frac{(r-1)^3(r+3)E\left(\frac{16r}{(r+3)(r-1)^3}\right) - (r-1)^2(r+1)^2K\left(\frac{16r}{(r+3)(r-1)^3}\right) + 4(r+1)^3\Pi\left(\frac{-4r}{(r-1)^2}, \frac{16r}{(r+3)(r-1)^3}\right)}{r\sqrt{(r-1)^3(r+3)}}. \quad (4.9)$$

Next we observe that [8, Equation 117.03], after multiplying with $\frac{(r-1)^5(r+3)}{4r(r+1)^2}$, gives

$$\begin{aligned} (r+1)^2\Pi\left(\frac{-4r}{(r-1)^2}, \frac{16r}{(r+3)(r-1)^3}\right) = \\ (r-3)(r+1)\Pi\left(\frac{4r}{(r+3)(r-1)}, \frac{16r}{(r+3)(r-1)^3}\right) + 4K\left(\frac{16r}{(r+3)(r-1)^3}\right). \end{aligned} \quad (4.10)$$

Putting these together with (4.6), yields (4.4).

To prove the second equality for c_{011} from the first, we use (see [8, Formula 117.02])

$$\Pi(n, m) = K(m) - \Pi\left(\frac{m}{n}, m\right) + \frac{\pi}{2} \sqrt{\frac{n}{(1-n)(n-m)}},$$

with $n = \frac{4r}{(r+3)(r-1)}$ and $m = \frac{16r}{(r+3)(r-1)^3}$. The constant here works out to equal $\frac{\pi}{2} \frac{(r+3)^{\frac{1}{2}}(r-1)^{\frac{3}{2}}}{(r-3)(r+1)}$. Thus (4.2) follows. \square

Equation (4.3) yields the relations

$$c_{klm} - \frac{c_{k-1,l,m} + c_{k,l-1,m} + c_{k,l,m-1}}{r} = 0, (k, l, m) \notin -\mathbb{N}_0^3.$$

These equalities provide a partial picture of the Fourier coefficients of $|1 - \frac{z_1+z_2+z_3}{r}|^{-2}$, $r > 3$. Our method to determine other relations rely on the formulas obtained in Proposition 2.8. The inverses in this proposition are obtained via [23, Theorem 1.1] and the ability to find a formula for the inverse of a tridiagonal infinite Toeplitz matrix. If we want to use this method to obtain expressions for Fourier coefficients beyond the indices $\{-1, 0, 1\}^3$, we will need to be able to find manageable expressions for (part of) the inverse of more involved infinite (block) Toeplitz matrices, which is a challenge.

Acknowledgements

We would like to thank Professor Bruce C. Berndt for his feedback on Corollary 3.4. Also we thank Paul D. Hanna for giving us background on his entries on the On-Line Encyclopedia of Integer Sequences. Finally, we thank Professor Stephen Melczer for his help with Remark 3.3.

Funding Research of HJW is supported by Simons Foundation grant 355645 and National Science Foundation grant DMS 2000037.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

References

- [1] Abramovich, Y.I., Johnson, B.A., Spencer, N.K.: Two-dimensional multivariate parametric models for radar applications-part II: maximum-entropy extensions for hermitian-block matrices. *IEEE Trans. Signal Process.* **56**, 5527–5539 (2008)
- [2] Andrews, G.E., Berndt, B.C.: *Ramanujan's Lost Notebook Part V*. Springer, Cham (2018)
- [3] Bakonyi, Mihály, Woerdeman, Hugo J.: *Matrix Completions, Moments, and Sums of Hermitian Squares*. Princeton University Press, Princeton (2011)
- [4] Birkhoff, George D., Trjitzinsky, W. J.: Analytic theory of singular difference equations. *Acta Math.* **60**(1), 1–89 (1933)
- [5] Bochner, S.: *Vorlesungen über Fourierische Integral*. Akademische Verlagsgesellschaft, Leipzig, (1932)
- [6] Bochner, S.: Monotone Funktionen, Stieltjessche integrale und harmonische analyse. *Math. Anal.* **108**, 378–410 (1933)
- [7] Burnette, C., Wong, C.: Abelian Squares and Their Progenies. [arXiv:1609.05580](https://arxiv.org/abs/1609.05580) (2016)
- [8] Byrd, Paul F., Friedman, Morris D.: *Handbook of Elliptic Integrals for Engineers and Scientists*. Springer, Berlin (1971)
- [9] Dela Rosa, K.L., Woerdeman, H.J.: Completing an operator matrix and the free joint numerical radius. *Comp. Anal. Op. Theory* **16**(8), 114 (2022)
- [10] Geronimo, Jeffrey S., Woerdeman, Hugo J.: Positive extensions, Fejér-Riesz factorization and autoregressive filters in two variables. *Ann. Math.* **160**(3), 839–906 (2004)
- [11] Geronimo, Jeffrey S., Woerdeman, Hugo J., Wong, Chung Y.: Spectral density functions of bivariable stable polynomials. *Ramanujan J.* **56**(1), 265–295 (2021)
- [12] Gradshteyn, I.S., Ryzhik, I.M., Geronimus, Y.V., Tseytlin, M.Y., Jeffrey, A., Zwillinger, D., Moll, V.H.: *Table of Integrals, Series, and Products Translated by Scripta Technica Inc.* Academic Press, New York (1965)
- [13] Guiduo, Duan, Jianping, Li., Tianxi, Huang: A new construction of 2-dimensional orthonormal filter based on cosine function for improving the robustness of digital watermarking. *HKIE Trans.* **17**(1), 11–15 (2010)
- [14] Hanna, P.D.: contribution dated February 26, 2012, regarding the sequence A002893 on the On-Line Encyclopedia of Integer Sequences (oeis.org)
- [15] Hanna, P.D.: private communication, September 2020.

- [16] Helson, Henry, Lowdenslager, David: Prediction theory and Fourier series in several variables. *Acta Math.* **99**, 165–202 (1958)
- [17] Helson, Henry, Lowdenslager, David: Prediction theory and Fourier series in several variables. II. *Acta Math.* **106**, 175–213 (1961)
- [18] Heun, K.: Zur Theorie der Riemann'schen Functionen zweiter Ordnung mit vier Verzweigungspunkten. *Math. Ann.* **33**(2), 161–179 (1888)
- [19] Hur, Youngmi, Lubberts, Zachary, Okoudjou, Kasso A.: Multivariate tight wavelet frames with few generators and high vanishing moments. *Int. J. Wavelets Multiresolut. Inf. Process.* **20**(5), 2250009 (2022)
- [20] Immink, G.K.: Reduction to canonical forms and the stokes phenomenon in the theory of linear difference equations *SIAM. J. Math. Anal.* **22**(1), 238–259 (1991)
- [21] Kojima, C., Rapisarda, P., Takaba, K.: Lyapunov stability analysis of higher-order 2-D systems. *Multidim. Syst. Sign. Process.* **22**, 287–302 (2011)
- [22] Kolev, V., Cooklev, T., Keinert, F.: Design of a simple orthogonal multiwavelet filter by matrix spectral factorization. *Circuits Syst. Signal Process.* **39**, 2006–2041 (2020)
- [23] Koyuncu, S., Woerdeman, H.J.: The inverse of a two-level positive definite Toeplitz operator matrix. In: Dym, H., Kaashoek, M.A., Lancaster, P., Langer, H., Lerer, L. (eds.) *A Panorama of Modern Operator Theory and Related Topics*, vol. 218. Springer, Basel (2012)
- [24] Melczer, Stephen: *An Invitation to Analytic Combinatorics: From One to Several Variables*. Springer, Berlin (2020)
- [25] Richmond, L.B., Rousseau, C.: Comment on problem 87–2. *SIAM Rev.* **31**(1), 122–125 (1989)
- [26] Ringh, A., Karlsson, J., Lindquist, A.: Multidimensional rational covariance extension with applications to spectral estimation and image compression. *SIAM J. Control. Optim.* **54**, 1950–1982 (2016)
- [27] Ringh, A., Karlsson, J., Lindquist, A.: Further results on multidimensional rational covariance extension with application to texture generation. In: *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, Melbourne, VIC, Australia. pp. 4038–4045 (2017)
- [28] Richmond, L.B., Shallit, Jeffrey: Counting abelian squares. *Electron. J. Comb.* **16**(1), R72 (2009)
- [29] Verrill, H.A.: Sums of squares of binomial coefficients, with applications to Picard-Fuchs equations. [arXiv:math/0407327](https://arxiv.org/abs/math/0407327)
- [30] Wimp, Jet, Zeilberger, Doron: Resurrecting the asymptotics of linear recurrences. *J. Math. Anal. Appl.* **111**(1), 162–176 (1985)
- [31] Wong, C.Y.: *Spectral Density Functions and Their Applications*, Thesis (PhD). Drexel University. Philadelphia (2016)
- [32] Zhu, Bin, Ferrante, Augusto, Karlsson, Johan, Zorzi, Mattia: M^2 -spectral estimation: a relative entropy approach. *Automatica* **125**, 109404 (2021)

Jeffrey S. Geronimo
School of Mathematics
Georgia Institute of Technology
225 North Ave
Atlanta GA30332
USA

Hugo J. Woerdeman (✉)
Department of Mathematics
Drexel University
3141 Chestnut Street
Philadelphia PA19104
USA
e-mail: hjw27@drexel.edu

Chung Y. Wong
Department of Mathematics
County College of Morris
214 Center Grove Rd.
Randolph NJ07869
USA

Received: October 8, 2022.

Accepted: December 11, 2022.