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Measures of spin ordering in the Potts model with a generalized external magnetic field

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ABSTRACT

We formulate measures of spin ordering in the q-state ferromagnetic Potts model in a generalized external magnetic field that favors or disfavors spin values in a subset $I_s = \{1, \ldots, s\}$ of the total set of q values. The results are contrasted with the corresponding measures of spin ordering in the case of a conventional external magnetic field that favors or disfavors a single spin value out of total set of q values. Some illustrative calculations are included.

1. Introduction

The q-state Potts model [1] has long been of interest as a classical spin model in which each spin can take on any of q values in the interval $I_q = \{1, 2, \ldots, q\}$, with a Kronecker delta function spin–spin interaction between spins on adjacent sites [2,3]. In contrast to the $q = 2$ case, which is equivalent to the Ising model, for $q \geq 3$, there are several different ways that one can incorporate the effect of a symmetry-breaking (uniform) external magnetic field. The conventional way is to define this field as favoring one particular spin value out of the q possible values in the set I_q, e.g., [4]. In [5–8] we defined and studied properties of the q-state Potts model with a generalized external magnetic field that favors or disfavors a subset consisting of more than just one value in I_q. By convention, with no loss of generality, we take this subset to consist of the first s values, denoted as the interval $I_s = \{1, \ldots, s\}$. The orthogonal subset in I_q is denoted $I_s^{\perp} = \{s + 1, \ldots, q\}$. In the case that we considered, the value of the magnetic field is a constant, consistent with its property as being applied externally. More general models with magnetic-like variables whose field values depend on the vertices have also been discussed [9–11], but we will not need this generality here.

In the present paper we continue the study of the q-state Potts model in this generalized uniform external magnetic field. We discuss measures of spin ordering in the presence of the external field and formulate an order parameter for this model. The results are contrasted with the corresponding measures of spin ordering in the case of a conventional external magnetic field that favors or disfavors a single spin value in I_q.

2. Definition and basic properties of the potts model in a generalized magnetic field

In this section we review the definition and basic properties of the model that we study. We will consider the Potts model on a graph $G(V, E)$ defined by its set V of vertices (site) and its set E of edges (bonds). For many physical

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applications, one usually takes G to be a regular d-dimensional lattice, but we retain the general formalism of graph theory here for later use. In thermal equilibrium at temperature T, the partition function for the q-state Potts model on the graph G in a generalized magnetic field is given by

$$Z = \sum_{\{\sigma_i\}} e^{-\beta \mathcal{H}} \ , \tag{2.1}$$

with the Hamiltonian

$$\mathcal{H} = -J \sum_{e_{ij}} \delta_{\sigma_i,\sigma_j} - \sum_{p=1}^{q} H_p \sum_{\ell} \delta_{\sigma_\ell,p} \ , \tag{2.2}$$

where $i,\ j,\ \ell$ label vertices of G; σ_i are classical spin variables on these vertices, taking values in the set $I_q = \{1, \dots, q\}$; $\beta = (k_B T)^{-1}$; e_{ij} is the edge (bonds) joining vertices i and j; J is the spin–spin interaction constant; and

$$H_p = \begin{cases} H & \text{if} \quad p \quad \in I_s \\ 0 & \text{if} \quad p \quad \in I_s^{\perp} \end{cases} \ . \tag{2.3}$$

Unless otherwise stated, we restrict our discussion to the ferromagnetic ($J > 0$) version of the model, since the antiferromagnetic model in a (uniform) external field entails complications due to competing interactions and frustration. If $H > 0$, the external field favorably weights spin values in the interval I_s, while if $H < 0$, this field favorably weights spin values in the orthogonal interval $I_s^{\perp}$. This model thus generalizes a conventional magnetic field, which would favor or disfavor one particular spin value. The zero-field Potts model Hamiltonian $\mathcal{H}$ and partition function Z are invariant under the global transformation in which $\sigma_i \to g\sigma_i \ \ \forall \ \ i \in V$, with $g \in \mathcal{S}_q$, where $\mathcal{S}_q$ is the symmetric (= permutation) group on q objects. In the presence of the generalized external field defined in Eq. (2.3), this symmetry group of $\mathcal{H}$ and Z is reduced from S_q at $H = 0$ to the tensor product

$$\mathcal{S}_s \otimes \mathcal{S}_{q-s} \ . \tag{2.4}$$

This simplifies to the conventional situation in which the external magnetic field favors or disfavors only a single spin value if $s = 1$ or $s = q - 1$, in which case the right-hand side of Eq. (2.4) is $\mathcal{S}_{q-1}$.

We use the notation

$$K = \beta J \ , \quad h = \beta H \ , \quad y = e^K \ , \quad v = y - 1 \ , \quad w = e^h \ . \tag{2.5}$$

The physical ranges of v are $v \ge 0$ for the Potts ferromagnet, and $-1 \le v \le 0$ for the Potts antiferromagnet. For fixed J and H, as $T \to \infty$, $v \to 0$ and $w \to 1$, while for $T \to 0$ (with our ferromagnetic choice $J > 0$), $v \to \infty$; and $w \to \infty$ if $H > 0$ while $w \to 0$ if $H < 0$. Recall that for $q = 2$, the equivalence with the Ising model with standard Hamiltonian (denoted with Is)

$$\mathcal{H}_{Is} = -J_{Is} \sum_{e_{ij}} \sigma_i^{(Is)} \sigma_j^{(Is)} - H_{Is} \sum_i \sigma_i^{(Is)} \ , \tag{2.6}$$

where $\sigma_i^{(Is)} = \pm 1$ makes use of the relations $J = 2J_{Is}$ and $H = 2H_{Is}$.

One can express the Potts model partition function on a graph G in a form that does not make any explicit reference to the spins σ_i or the summation over spin configurations in (2.1), but instead is expressed in a purely graph-theoretic manner, as a sum of terms arising from the spanning subgraphs $G' \subseteq G$, where $G' = (V, E')$ with $E' \subseteq E$. In zero field, this was done in [12], with the result

$$Z(G, q, v) = \sum_{G' \subseteq G} v^{e(G')} \, q^{k(G')} \ . \tag{2.7}$$

For the model with a conventional external magnetic field that favors or disfavors a single spin value in the set I_q, a spanning-subgraph formula for the partition function was given in [4]. For the model with a generalized magnetic field that favors or disfavors a larger set I_s consisting of two or more spin values in the set I_q and has a value $H_{i,p} = H_p$ that is the same for all vertices $i \in V$, a spanning-subgraph formula for the partition function was presented in Ref. [5] (see also [6]) and is as follows. Given a graph $G = (V, E)$, the numbers of vertices, edges, and connected components of G are denoted, respectively, by $n(G) \equiv n$, $e(G)$, and $k(G)$. The purely graph-theoretic expression of the partition function of the Potts model in a generalized magnetic field in this case is [5]

$$Z(G, q, s, v, w) = \sum_{G' \subseteq G} v^{e(G')} \prod_{i=1}^{k(G')} (q - s + s w^{n(G'_i)}) \ , \tag{2.8}$$

where G'_i, $1 \le i \le k(G')$ denotes one of the $k(G')$ connected components in a spanning subgraph G' of G. The formula (2.8) shows that Z is a polynomial in the variables q, s, v, and w, hence our notation $Z(G, q, s, v, w)$. For the case where the magnetic field favors (or disfavors) only a single spin value, i.e., for the case $s = 1$, the formula (2.8) reduces to the spanning subgraph formula for Z given in [4] (see also [13]). Parenthetically, we mention further generalizations that are different from the one we study here. First, one can let the spin–spin exchange constants J be edge-dependent, denoted as J_{ij} on the edge e_{ij} joining vertices i and j. Second, one can let the value of the magnetic-field-type variable be different for different vertices $\ell \in V$, so H_p is replaced by $H_{\ell,p}$. With these generalizations, a spanning-subgraph formula for the partition function was given in [10] and studied further in [11].

Focusing on the term $w^{n(G'_i)}$ in (2.8) and letting $\ell = n(G'_i)$ for compact notation, one can use the factorization relation

$$w^{\ell} - 1 = (w-1)\sum_{j=0}^{\ell-1} w^j \tag{2.9}$$

to deduce that the variable s enters in $Z(G, q, s, v, w)$, only in the combination

$$t = s(w-1) \ . \tag{2.10}$$

Hence, the special case of zero external field, $H = 0$, i.e., $w = 1$, is equivalent to the formal value $s = 0$ (outside the interval I_s).

Several relevant identities were derived in [5,6], including

$$Z(G, q, s, v, 1) = Z(G, q, v) \ , \tag{2.11}$$

$$Z(G, q, s, v, 0) = Z(G, q-s, v) \ , \tag{2.12}$$

$$Z(G, q, q, v, w) = w^n Z(G, q, v) \ , \tag{2.13}$$

and

$$Z(G, q, s, v, w) = w^n Z(G, q, q-s, v, w^{-1}) \ . \tag{2.14}$$

The identity (2.14) establishes a relation between the model with $H > 0$ and hence $w > 1$, and the model with $H < 0$ and hence $0 \le w < 1$. Given this identity, one may, with no loss of generality, restrict to $H \ge 0$, i.e., $w \ge 1$, and we will do this below, unless otherwise indicated.

In the limit $n(G) \to \infty$, the reduced, dimensionless free energy per vertex is

$$f(\{G\}, q, s, v, w) = \lim_{n(G)\to\infty} \frac{1}{n(G)} \ln[Z(G, q, s, v, w)] \ , \tag{2.15}$$

where the symbol $\{G\}$ denotes the formal $n \to \infty$ limit of a given family of graphs, such as a regular lattice with some specified boundary conditions. The actual Gibbs free energy per site is $F(T,H) = -k_B T f(T,H)$. For technical simplicity, unless otherwise indicated, we will restrict to the ferromagnetic case $J > 0$ here; in [5–8] we have also discussed the antiferromagnetic case. The zero-temperature limit of the antiferromagnetic version defines a weighted-set chromatic polynomial that counts the number of assignments from q colors to the vertices of G subject to the condition that no two adjacent vertices have the same color, with preferred (dispreferred) weighting given to colors in I_s for $H > 0$ ($H < 0$, respectively). Here and below, in order to avoid cumbersome notation, we will use the same symbol Z with different sets of arguments to refer to the full model, as $Z(G, q, s, v, w)$ and the zero-field special case, $Z(G, q, v)$.

The partition function of the zero-field Potts model is equivalent to an important function in mathematical graph theory, namely the Tutte (also called Tutte-Whitney) polynomial $T(G, x, y)$ [14]. This is defined as

$$T(G, x, y) = \sum_{G' \subseteq G} (x-1)^{k(G')-k(G)} (y-1)^{c(G')} \ , \tag{2.16}$$

where $c(G')$ denotes the number of linearly independent cycles on G'. Note that $c(G') = e(G') + k(G') - n(G') = e(G') + k(G') - n(G)$. The equivalence relation is

$$Z(G, q, v) = (x-1)^{k(G)} (y-1)^{n(G)} T(G, x, y) \ , \tag{2.17}$$

with

$$x = 1 + \frac{q}{v} \ , \quad y = v + 1 \ , \tag{2.18}$$

so that $q = (x-1)(y-1)$. Reviews of the Tutte polynomial and generalizations include [9–11,15–18].

3. Magnetic order parameter in the Ising, O(N), and Potts model with a conventional magnetic field

3.1. Ising and O(N) models

In the Ising model (2.6) the magnetization per site is given by

$$M(H) = -\frac{\partial F}{\partial H} = \frac{\partial f}{\partial h} \ , \tag{3.1}$$

and the spontaneous magnetization is $M \equiv \lim_{H\to 0} M(H)$. This M is (i) identically zero in the high-temperature phase where the theory is invariant under the global $\mathbb{Z}_2 \approx S_2$ symmetry and, (ii) for a regular lattice of dimensionality above the lower critical dimensionality $d_\ell = 1$, M is nonzero in the low-temperature phase where there is spontaneous breaking of this global $\mathcal{Z}_2$ symmetry, increasing from 0 to a maximum of 1 as T decreases from the critical temperature T_c to $T = 0$. Alternatively, in the ferromagnetic Ising model (2.6) on a regular lattice, the (square of the) magnetization can be calculated in the absence of an external field as the limit

$$M^2 = \lim_{r\to\infty} \langle \sigma_i^{(Is)} \sigma_j^{(Is)} \rangle \ , \tag{3.2}$$

where $\langle \sigma_i^{(Is)} \sigma_j^{(Is)} \rangle$ is the two-spin correlation function and r denotes the Euclidean distance between the lattice sites i and j on the lattice. For $T > T_c$, this correlation function $\langle \sigma_i^{(Is)} \sigma_j^{(Is)} \rangle \to 0$ as $r \to \infty$, so that $M = 0$. Regarding the Ising model as the $N = 1$ special case of an O(N) spin model, similar comments hold. Thus, consider the partition function

$$Z_{O(N)} = \int \prod_i d\Omega_i e^{-\beta \mathcal{H}_{O(N)}} \ , \tag{3.3}$$

with

$$\mathcal{H}_{O(N)} = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j - \vec{H} \cdot \sum_i \vec{S}_i \ , \tag{3.4}$$

where $\vec{S}_i$ is an N-component unit-normalized classical spin at site i on a given lattice and $d\Omega_i$ denotes the O(N) integration measure. For zero external field, this model has a global O(N) invariance. The presence of an external magnetic field $\vec{H}$ explicitly breaks the O(N) symmetry down to O($N-1$). For general $\vec{H}$, one has, for the thermal average of $\vec{M} = \vec{M}(\vec{H})$,

$$\vec{M}(\vec{H}) = -\frac{\partial F}{\partial \vec{H}} \ , \tag{3.5}$$

and the relation

$$|\vec{M}|^2 = \lim_{r\to\infty} \langle \vec{S}_i \cdot \vec{S}_j \rangle \ . \tag{3.6}$$

As usual, the spontaneous magnetization for the Ising and O(N) models is defined as $M_0 = \lim_{H\to 0} M(H)$ and $\vec{M}_0 = \lim_{|\vec{H}|\to 0} \vec{M}(\vec{H})$, respectively.

3.2. Measures of spin ordering in Potts model with conventional magnetic field

The situation is different in the Potts model, even with a conventional external magnetic field that favors only one spin value. In our formalism, this means that I_s consists of the single value $s = 1$. Before proceeding to derive our new results, we review this situation for this conventional case. For this purpose, it is useful to analyze the properties of the spin–spin correlation function. It will be sufficient here and below to assume that the graph G is a regular d-dimensional lattice. Let us denote as $P_{aa}(i,j)$ the probability (in the thermodynamic limit, in thermal equilibrium at temperature T) that the spins σ_i and σ_j at the sites i and j in the lattice have the value $a \in I_q$. At $T = \infty$, all spin configurations occur with equal probability, so the probability that σ_i has a particular value a is just $1/q$, and similarly with σ_j, so $P_{aa}(i,j) = 1/q^2$ at $T = \infty$. To define a correlation function with the usual property that in the high-temperature phase, as the distance r between the spins goes to infinity, they should be completely uncorrelated, one must therefore subtract this $1/q^2$ term. That is, in the Potts model, one defines the spin–spin correlation function as (e.g., [2])

$$\Gamma_{aa}(i,j) = P_{aa}(i,j) - \frac{1}{q^2} \ . \tag{3.7}$$

Thus, by construction, $\Gamma_{aa}(i,j) = 0$ at $T = \infty$. At $T = 0$, in the ferromagnetic Potts model, all of the spins take on the same value in the set I_q. Let us say that an infinitesimally small external field has been applied to favor the value $a \in I_q$, so then $P_{aa}(i,j) = 1/q$ and hence, under these conditions,

$$\Gamma_{aa}(i,j) = \frac{1}{q} - \frac{1}{q^2} = \frac{q-1}{q^2} \quad \text{at } T = 0 \ . \tag{3.8}$$

In the ferromagnetic Potts model with a conventional external (uniform) magnetic field favoring a single spin model, the magnetic order parameter, $\mathcal{M}$, normalized so that it is unity at $T = 0$, is then related to this spin–spin correlation function $\Gamma_{aa}(i,j)$ according to

$$\mathcal{M} = \left(\frac{q^2}{q-1} \right) \lim_{r\to\infty} \Gamma_{aa}(i,j) \ . \tag{3.9}$$

Although the quantity $-\partial F/\partial H = \partial f/\partial h$ yields one measure of magnetic ordering, it is not the order parameter itself, in contrast to the situation with both the Ising and O(N) spin models. Instead, as is evident from its definition, this partial derivative is equal to the fraction of the total number of sites in the (thermodynamic limit of the) lattice with spins taking one particular value out of the set of q values. We denote this as

$$M = -\frac{\partial F}{\partial H} = \frac{\partial f}{\partial h} = w\frac{\partial f}{\partial w} \ . \tag{3.10}$$

At $T = \infty$, since all spin values are weighted equally, it follows that the fraction of the spins taking on any particular value is $1/q$, i.e.,

$$M = \frac{1}{q} \quad \text{at } T = \infty \ . \tag{3.11}$$

In the opposite limit of zero temperature, given that $J > 0$, the spin–spin interaction forces all of the spins to have the same value. There is then a dichotomy in the behavior of the system, depending on whether H is positive or negative. If $H > 0$, then the external field forces this spin value to be the single value in I_s favored by this field, so

$$M = 1 \quad \text{at } T = 0 \text{ if } H > 0 \ , \tag{3.12}$$

If, on the other hand, $H < 0$, then the single spin value that all the spins are forced to have by the spin–spin interaction to lie in the orthogonal complement, which, for this case is the set $I_s^{\perp} = \{2, \ldots, q\}$, so the fraction in I_s is zero. Hence,

$$M = 0 \quad \text{at } T = 0 \text{ if } H < 0 \ . \tag{3.13}$$

The zero-field values of M at a given temperature for the respective cases $H > 0$ and $H < 0$ are $M_{0^+} = \lim_{H\to 0^+} M$ and $M_{0^-} = \lim_{H\to 0^-} M$.

Because the zero-field value of M does not vanish in the high-temperatures, S_q-symmetric phase, it cannot be the order parameter of the model, but instead is an auxiliary measure of the magnetic ordering per site. In the literature, studies focused on the case $H > 0$ in formulating an appropriate order parameter. In this case, to define an order parameter, one subtracts the $T = \infty$ value of M from the value for general T and normalizes the result so that the order parameter saturates at unity at $T = 0$. This yields a measure of spin ordering that we denote as $\mathcal{M}$:

$$\mathcal{M} = \frac{M - M_{T=\infty}}{M_{T=0} - M_{T=\infty}} = \frac{M - \frac{1}{q}}{1 - \frac{1}{q}} = \frac{qM - 1}{q - 1} \ . \tag{3.14}$$

The zero-field value of this order parameter, i.e., the spontaneous magnetization, is then

$$\mathcal{M}_0 = \lim_{H\to 0} \mathcal{M} \ . \tag{3.15}$$

This construction was given for the specific case $q = 3$ in [19] and for general q in [20] (see also [2], where $\mathcal{M}_0$ was denoted as m_0). Series expansions [19] and Monte-Carlo simulations [20] for the two-dimensional Potts model were consistent with the behavior expected of an order parameter, namely $\mathcal{M}_0 = 0$ in the high-temperature S_q-symmetric phase and $\mathcal{M} > 0$ in the low-temperature phase with spontaneous symmetry breaking of the global S_q symmetry to S_{q-1}.

A parenthetical remark is in order concerning the trivial case $q = 1$ where the spins are all frozen to have the same value and hence are nondynamical. In this $q = 1$ case, up to a prefactor w^n, the partition function is equal to the free-field result, $w^n Z(G, q, v)$. As a consequence, for any temperature, $M = 1$, so $\mathcal{M}$ has an indeterminate form $0/0$. Thus, in using the formula (3.14), one restricts to the range $q \geq 2$.

4. Measures of spin ordering in the Potts model with generalized magnetic field

In this section we present our new results on measures of spin ordering in the Potts model with a generalized magnetic field, including, in particular, the order parameter for this model. In the limit $T \to \infty$, $P_{aa}(i,j) = 1/q^2$, independent of s. This is a consequence of the fact that the Boltzmann weight $e^{-\beta\mathcal{H}}$ in the expression for Z reduces to 1 for $\beta = 0$, and so the spins are completely random. However, the auxiliary measure of spin ordering, M, behaves differently in the Potts model with a generalized versus conventional external magnetic field. From the basic definition, calculating M from Eq. (3.10) and then letting $T \to \infty$, we find the following general behavior:

$$M = \frac{s}{q} \quad \text{at } T = \infty \text{ and any finite } H \ . \tag{4.1}$$

In the opposite limit, $T \to 0$, the value of M again depends on the sign of H. If $H > 0$, then (given that $J > 0$), the spin–spin interaction forces all spins to have the same value, and the presence of the external field forces this value to lie in the set I_s, so

$$\lim_{T\to 0} M = 1 \text{ for } H > 0 \ . \tag{4.2}$$

In this $T \to 0$ limit (again, given that $J > 0$), if $H < 0$, then the spin–spin interaction forces all spins to have the same value and this value lies in the orthogonal complement $I_s^{\perp}$, so the fraction of spins in I_s is 0:

$$\lim_{T\to 0} M = 0 \text{ for } H < 0 \ . \tag{4.3}$$

Finally, we record the behavior of M in the limits $H \to \pm\infty$ at fixed finite nonzero temperature. In terms of the Boltzmann weights, these two limits are $w \to \infty$ and $w \to 0$ with v finite. As $H \to \infty$ in this limit, all of the spins must take on values in I_s, so

$$\lim_{H\to\infty} M = 1 \text{ for any finite nonzero } T \ . \tag{4.4}$$

If $H \to -\infty$, then all spins must take on values in $I_s^{\perp}$, so the fraction in I_s is zero:

$$\lim_{H\to-\infty} M = 0 \quad \text{for any finite nonzero } T \ . \tag{4.5}$$

In order to obtain the zero-field value of M at a given temperature, as in the case of a conventional magnetic field, one would calculate $Z(G, q, s, v, w)$ on a given lattice graph G, take the thermodynamic limit, then calculate M and take the limit $H \to 0^+$ or $H \to 0^-$.

We now construct a magnetic order parameter $\mathcal{M}$ for the Potts model in a generalized magnetic field. As noted above, given the identity (2.14), we can, without loss of generality, restrict to $H > 0$ and we shall do so henceforth. We obtain

$$\mathcal{M} = \frac{M - M_{T=\infty}}{M_{T=0} - M_{T=\infty}} = \frac{M - \frac{s}{q}}{1 - \frac{s}{q}} = \frac{qM - s}{q - s} \ . \tag{4.6}$$

The spontaneous magnetization is then

$$\mathcal{M}_0 = \lim_{H\to 0^+} \mathcal{M} \ . \tag{4.7}$$

A word is in order concerning the apparent pole at $s = q$. If $s = q$, then the presence of the external field simply adds a constant term $-Hn$ to $\mathcal{H}$, or equivalently, i.e., the partition function is equivalent to the product of the factor w^n times the zero-field Z, as specified in the identity (2.13), so that

$$M = 1 \quad \text{if } s = q \ , \tag{4.8}$$

independent of temperature. Hence, just as was the case with the expression (3.14) for a conventional magnetic field favoring just one spin, so also here, the expression (4.6) takes the indeterminate form 0/0 in this case. Hence, in using (4.6), we restrict s to the interval $1 \le s \le q - 1$.

5. Some explicit examples

Some explicit examples illustrate the use of (4.6) for the order parameter. Although a Peierls-type argument shows that there is no spontaneous symmetry breaking of the $S_s \otimes S_{q-s}$ symmetry (2.4) on (the $n \to \infty$ limit of a) one-dimensional lattice or quasi-one-dimensional lattice strip, these types of lattices are, nevertheless, useful to illustrate some features of Eq. (4.6).

5.1. 1D lattice

As a first example, we use the exact expression for $Z(G, q, s, v, w)$ on a one-dimension lattice derived in [6]. This yields the reduced dimensionless free energy per site (in the thermodynamic limit, in the notation of [6])

$$f(1D, q, s, v, w) = \ln(\lambda_{Z,1,0,1}) \ , \tag{5.1}$$

where

$$\lambda_{Z,1,0,1} = \frac{1}{2}\left(A + \sqrt{R}\right) \ , \tag{5.2}$$

with

$$A = q + s(w - 1) + v(w + 1) \tag{5.3}$$

and

$$R = A^2 - 4v(q + v)w \ . \tag{5.4}$$

(As expected, in the thermodynamic limit, this result applies independent of the boundary conditions.) The resultant auxiliary measure of spin ordering, M, is

$$M = \frac{w}{\sqrt{R}} \left[s + v - \frac{2v(q + v)}{A + \sqrt{R}} \right] . \tag{5.5}$$

It is straightforward to confirm that Eq. (5.5) satisfies the general relations (4.1)–(4.5). As $T \to \infty$, i.e., $\beta \to 0$, the variables v and w (for finite J and H) approach the limits $v \to 0$ and $w \to 1$, i.e., $K \to 0$ and $h \to 0$. In this limit, we calculate a two-variable series expansion of M in K and h and find

$$M = \frac{s}{q}\Big[1 + \Big(1-\frac{s}{q}\Big)h\Big\{1 + \frac{2}{q}K + \frac{1}{2}\Big(1-\frac{2s}{q}\Big)h + \frac{1}{q}K^2 + \frac{3}{q}\Big(1-\frac{2s}{q}\Big)Kh + \Big(\frac{1}{6} - \frac{s}{q}\Big(1-\frac{s}{q}\Big)\Big)h^2 + O(K^3, K^2h, Kh^2, h^3)\Big\}\Big] \quad \text{as } T \to \infty \ . \tag{5.6}$$

Setting $\beta = 0$, one sees that this expansion satisfies the identity (4.1). Substituting Eq. (5.6) into our general expression for the order parameter, we obtain

$$\mathcal{M} = \frac{sh}{q}\Big[\ 1 + \frac{2}{q}K + \frac{1}{2}\Big(1-\frac{2s}{q}\Big)h + \frac{1}{q}K^2 + \frac{3}{q}\Big(1-\frac{2s}{q}\Big)Kh + \Big(\frac{1}{6} - \frac{s}{q}\Big(1-\frac{s}{q}\Big)\Big)h^2 + O(K^3, K^2h, Kh^2, h^3)\ \Big] \quad \text{as } T \to \infty \ , \tag{5.7}$$

where the notation $O(K^3, K^2h, Kh^2, h^3)$ refers to terms of order K^3, K^2h, Kh^2, or h^3. The proportionality of $\mathcal{M}$ to $(s/q)h = (s/q)\beta H$ as $\beta \to 0$ is the expression of the Curie–Weiss relation for the induced magnetization for this model.

Given Eq. (4.6) connecting M and $\mathcal{M}$, the susceptibilities defined via M and $\mathcal{M}$ are simply related to each other. Defining $\chi_M = \partial M/\partial H$ and $\chi_{\mathcal{M}} = \partial \mathcal{M}/\partial H$, we have

$$\chi_M = \Big(1 - \frac{s}{q}\Big)\chi_{\mathcal{M}} \ . \tag{5.8}$$

From Eq. (5.7), it follows that the two-variable high-temperature Taylor series expansion of $\chi_{\mathcal{M}}$ in powers of K and h is given by

$$\beta^{-1}\chi_{\mathcal{M}} = \frac{s}{q}\Big[\ 1 + \frac{2}{q}K + \Big(1-\frac{2s}{q}\Big)h + \frac{1}{q}K^2 + \frac{6}{q}\Big(1-\frac{2s}{q}\Big)Kh + \Big(\frac{1}{2} - \frac{3s}{q}\Big(1-\frac{s}{q}\Big)\Big)h^2 + O(K^3, K^2h, Kh^2, h^3)\ \Big] \quad \text{as } T \to \infty \ . \tag{5.9}$$

For $H \to \infty$ at finite T (equivalently, $w \to \infty$ with finite v), we calculate the Taylor series expansion

$$M = 1 - \frac{s(q-s)}{(s+v)^2 w} + \frac{s(q-s)[s(q-s) - 2v(q+v)]}{(s+v)^4 w^2} + O\Big(\frac{1}{w^3}\Big) \tag{5.10}$$

and hence

$$\mathcal{M} = 1 - \frac{sq}{(s+v)^2 w} + \frac{sq[s(q-s) - 2v(q+v)]}{(s+v)^4 w^2} + O\Big(\frac{1}{w^3}\Big) \ . \tag{5.11}$$

To show these results numerically for a typical case, we take the illustrative values $q = 5$ and $v = 2$. In Figs. 1 and 2 we plot $\mathcal{M}$ for this 1D lattice as a function of w in the intervals $1 \le w \le 8$ and $8 \le w \le 40$. Fixing the value of v corresponds most simply to fixing the values of J and T, so that the variation in w then amounts to a variation in H at fixed T. The results show that, as expected, $\mathcal{M}$ increases monotonically with increasing w and thus H, for fixed T. For small h, i.e., $w - 1 \to 0^+$, the values of $\mathcal{M}$ satisfy the relation $\mathcal{M} = (s/q)h$, in accord with (5.7) and hence are larger for larger s. However, as is evident from the series expansion (5.11) and from Fig. 2, this monotonicity is not preserved in $\mathcal{M}$ for this 1D lattice at large w.

5.2. $L_y = 2$ lattice strips

In [8] we calculated $Z(G, q, s, v, w)$ for the width L_y strips of the square and triangular lattice. This work generalized our previous calculations of $Z(G, q, v)$ in zero external field on these strips [21,22]. In the infinite-length limit (independent of longitudinal boundary conditions), the reduced free energy is given, respectively, by $f_{sq,L_y=2} = (1/2)\ln\lambda_{sq,L_y=2}$ and $f_{tri,L_y=2} = (1/2)\ln\lambda_{tri,L_y=2}$, where $\lambda_{sq,L_y=2}$ and $\lambda_{tri,L_y=2}$ are roots of respective degree-5 and degree-6 algebraic equations. Hence, it is not possible to calculate the derivatives $M = w\partial f/\partial w$ analytically to give explicit expressions for M and $\mathcal{M}$ for these infinite-length lattice strips. However, using numerical differentiation, it is still possible to obtain values for these quantities, given input values for v, q, and s. Using this method and again taking the illustrative values $q = 5$ and $v = 2$, we show plots of $\mathcal{M}$ for the infinite-length strips of the square and triangular lattices with width $L_y = 2$ in Figs. 3–6. As was the case with the 1D lattice, for small h, the values of $\mathcal{M}$ satisfy the relation $\mathcal{M} = (s/q)h$, and thus are larger for larger s, but this monotonicity relation does not apply for large w.

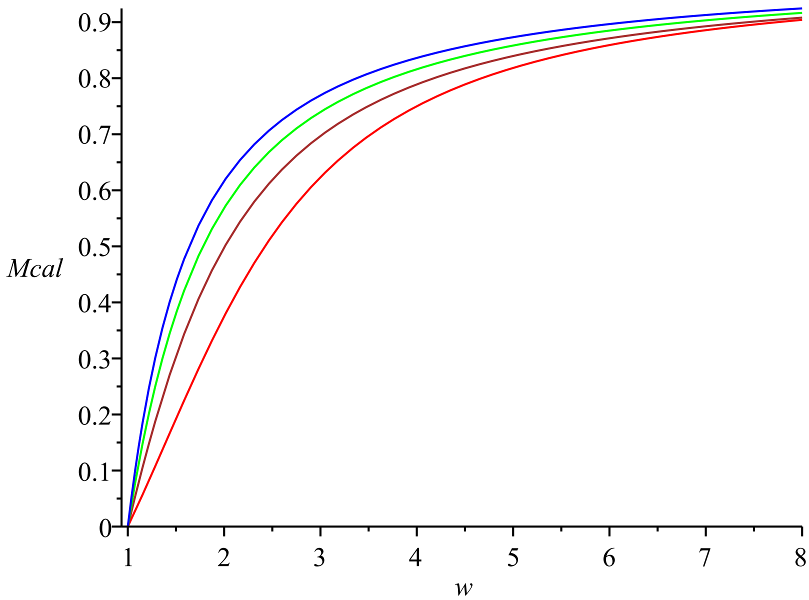

Fig. 1. Plot of $\mathcal{M}$ (denoted Mcal in this and other figures)for the 1D lattice as a function of w in the range $1 \le w \le 8$ for the illustrative values $q = 5$ and $v = 2$. For $w \gtrsim 1$, the curves, going from bottom to top, refer to $s = 1,\ 2,\ 3,\ 4$. The colors are $s = 1$ (red), $s = 2$ (brown), $s = 3$ (green), and $s = 4$ (blue).

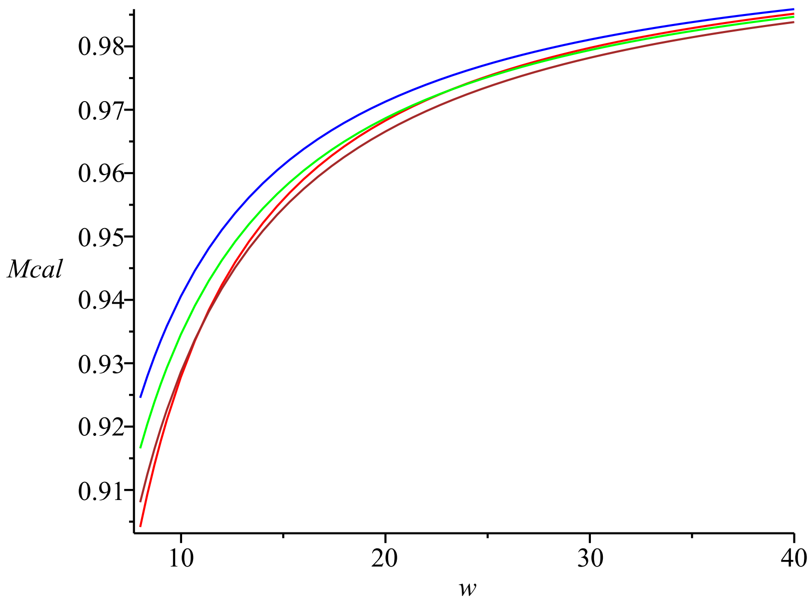

Fig. 2. Plot of $\mathcal{M}$ for the 1D lattice as a function of w in the range $8 \le w \le 40$ for the illustrative values $q = 5$ and $v = 2$. For $w = 8$, the curves, going from bottom to top, refer to $s = 1,\ 2,\ 3,\ 4$; crossings of curves are evident for larger w values. The color coding is the same as in Fig. 1.

6. Thermodynamic properties and critical behavior

As discussed in Section 2 above, in the presence of the generalized external magnetic field defined in Eq. (2.3), the symmetry group of $\mathcal{H}$ and Z is reduced from $\mathcal{S}_q$ at $H = 0$ to the tensor product in Eq. (2.4), and this further simplifies to $\mathcal{S}_{q-1}$ if $s = 1$ in which case, the external field favors or disfavors only a single spin value. From the identity (2.14), the case $s = q - 1$ is effectively equivalent to the conventional case $s = 1$. However, if s is in the interval

$$2 \le s \le q - 2 \ , \tag{6.1}$$

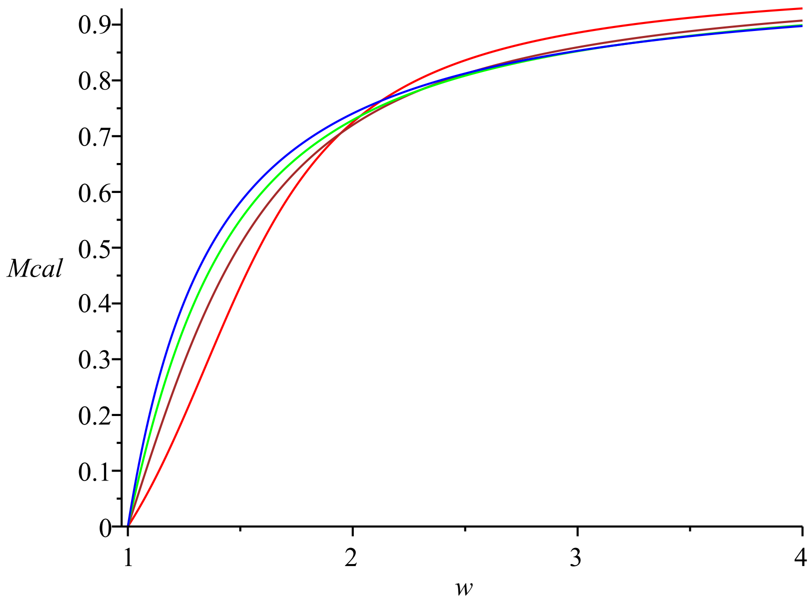

Fig. 3. Plot of $\mathcal{M}$ for the infinite-length strip of the square lattice of width $L_y = 2$, as a function of w in the range $1 \le w \le 4$ for the illustrative values $q = 5$ and $v = 2$. For $w \gtrsim 1$, the curves, going from bottom to top, refer to $s = 1,\ 2,\ 3,\ 4$. The color coding is the same as in Fig. 1.

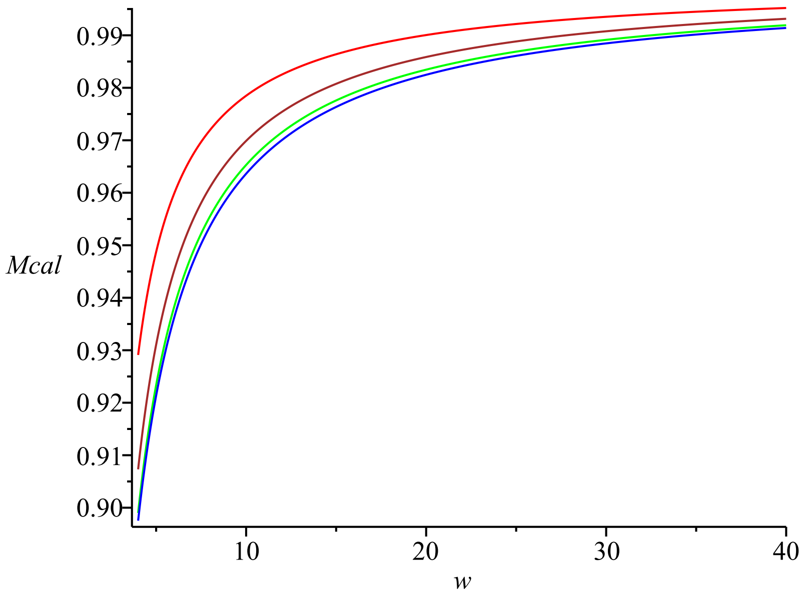

Fig. 4. Plot of $\mathcal{M}$ for the infinite-length strip of the square lattice of width $L_y = 2$, as a function of w in the range $4 \le w \le 40$ for the illustrative values $q = 5$ and $v = 2$. The curves, going from bottom to top, refer to $s = 4,\ 3,\ 2,\ 1$. The color coding is the same as in Fig. 1.

then the general model of Eqs. (2.2) and (2.3) exhibits properties that are interestingly different from those of a q-state Potts model in a conventional magnetic field. In this section we will consider both signs of H and J. With a conventional magnetic field, at a given temperature T, if $H \gg |J|$, the interaction with the external field dominates over the spin–spin interaction, and if $h = \beta H$ is sufficiently large, the spins are frozen to the single favored value. In contrast, in the model with a generalized magnetic field, if s lies in the interval (6.1), and if $|H| \gg |J|$, this effectively reduces the model to (i) an s-state Potts model if $H > 0$, or (ii) a $(q-s)$-state Potts model if $H < 0$. For given values of q and s, taking the thermodynamic limit of a given regular lattice, there are, in general, four types of possible models, depending on the sign of H and the sign of J. A discussion of these models, including the types of critical behavior, where present in the case of square lattice, was given in (Section 4 of) [7] with details for the illustrative case $q = 5$ and $s = 2$. We generalize this here to $q \ge 5$. For $H = 0$, the ferromagnetic version of the model has a first-order phase transition, with spontaneous breaking

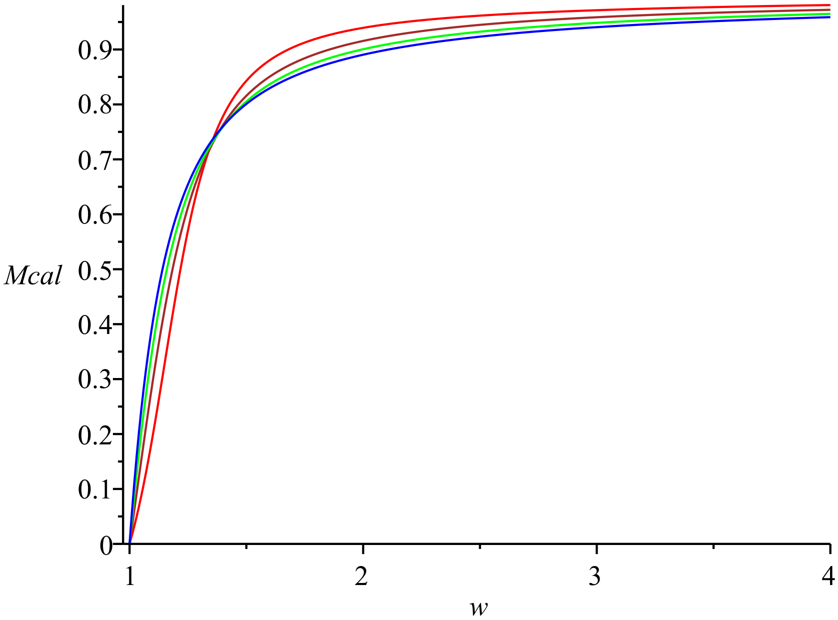

Fig. 5. Plot of $\mathcal{M}$ for the infinite-length strip of the triangular lattice of width $L_y = 2$, as a function of w in the range $1 \le w \le 4$ for the illustrative values $q = 5$ and $v = 2$. For $w \gtrsim 1$, the curves, going from bottom to top, refer to $s = 1,\ 2,\ 3,\ 4$. The color coding is the same as in Fig. 1.

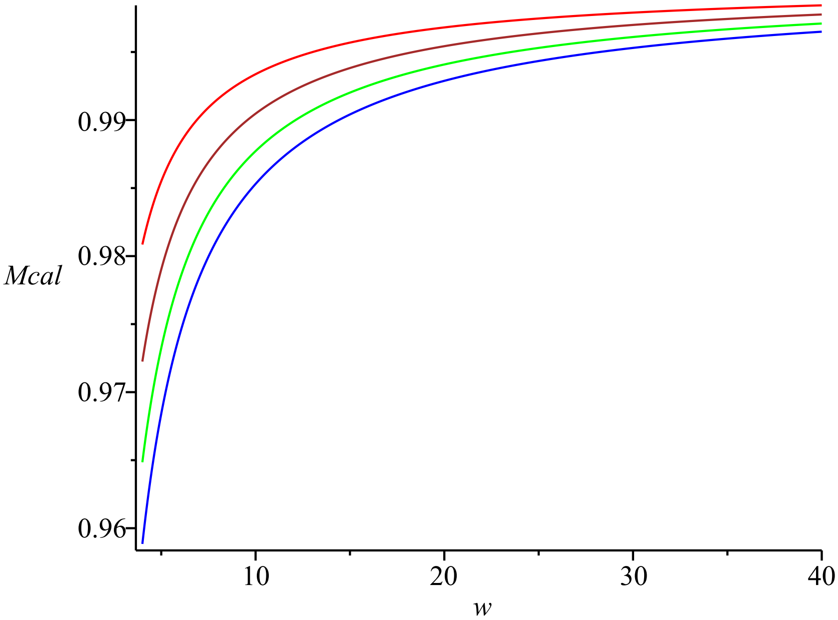

Fig. 6. Plot of $\mathcal{M}$ for the infinite-length strip of the triangular lattice of width $L_y = 2$, as a function of w in the range $4 \le w \le 40$ for the illustrative values $q = 5$ and $v = 2$. The curves, going from bottom to top, refer to $s = 4,\ 3,\ 2,\ 1$. The color coding is the same as in Fig. 1.

of the $\mathcal{S}_5$ symmetry, at $K_c = \ln(1+\sqrt{q})$, while the antiferromagnetic version has no finite-temperature phase transition and is disordered even at $T = 0$ [2,3]. For $H > 0$ and $H \gg |J|$, the theory reduces effectively to a two-state Potts model, i.e., an Ising model. Owing to the bipartite property of the square lattice, there is an elementary mapping that relates the ferromagnetic and antiferromagnetic versions of the model, and, as is well known, both have a second-order phase transition, with spontaneous symmetry breaking of the $\mathcal{S}_2 \approx \mathbb{Z}_2$ symmetry, at $|K_c| = \ln(1+\sqrt{2}) \simeq 0.881$ (where $K = \beta J$), with thermal and magnetic critical exponents $y_t = 1$, $y_h = 15/8$, described by the rational conformal field theory (RCFT) with central charge $c = 1/2$. For $H < 0$ and $|H| \gg |J|$, the theory effectively reduces to a $(q-2)$-state Potts model. In the ferromagnetic case, $J > 0$, if (a) $q = 5$, then the resultant 3-state Potts ferromagnet has a well-understood second-order phase transition, with spontaneous symmetry breaking of the $\mathcal{S}_3$ symmetry, at $K_c = \ln(1+\sqrt{3}) \simeq 1.01$, with thermal and magnetic critical exponents $y_t = 6/5$, $y_h = 28/15$, described by a RCFT with central charge $c = 4/5$; (b) if $q = 6$, then the resultant 4-state Potts ferromagnet also has a second-order phase transition with thermal and magnetic critical exponents $y_t = 3/2$, $y_h = 15/8$, described by a RCFT with central charge $c = 1$ [2,23,24]; and (c) if $q \geq 7$, then the resultant Potts ferromagnet has a first-order transition [2,3]. In the antiferromagnetic case, $J < 0$, if $q = 5$, the resultant 3-state Potts antiferromagnet has no finite-temperature phase transition but is critical at $T = 0$ (without frustration), with nonzero ground-state entropy per site $S/k_B = (3/2)\ln(4/3) \simeq 0.432$ [2,25]. If $q \geq 6$, then the resultant $(q-2)$-state Potts antiferromagnet (on the square lattice) does not have any symmetry-breaking phase transition at any finite temperature and is disordered also at $T = 0$. Similar discussions can be given for other lattices.

7. Conclusions

In this paper we have discussed measures of spin ordering in the q-state Potts model in a generalized external magnetic field that favors or disfavors spin values in a subset $I_s = \{1, \ldots, s\}$ of the total set of q values. In particular, we have constructed an order parameter $\mathcal{M}$ (given in Eq. (4.6)) and have presented an illustrative evaluation of it, together with relevant series expansions, for the (thermodynamic limit of the) one-dimensional lattice, as well as quantitative plots of $\mathcal{M}$ for this 1D lattice and for strips of the square and triangular lattices.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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References

[1] R.B. Potts, Some generalized order–disorder transformations, Proc. Cambridge Philos. Soc. 48 (1952) 106–109.
[2] F.Y. Wu, The Potts model, Rev. Modern Phys. 54 (1982) 235–268.
[3] R.J. Baxter, Exactly Solved Models in Statistical Mechanics, Oxford University Press, London, 1982.
[4] F.Y. Wu, Percolation and the Potts model, J. Stat. Phys. 18 (1978) 115–123.
[5] S.-C. Chang, R. Shrock, Weighted graph colorings, J. Stat. Phys. 138 (2010) 496–542.
[6] R. Shrock, Y. Xu, Weighted-set graph colorings, J. Stat. Phys. 139 (2010) 27–61.
[7] R. Shrock, Y. Xu, Exact results on Potts model partition functions in a generalized external field and weighted-set graph colorings, J. Stat. Phys. 141 (2010) 909–939.
[8] S.-C. Chang, R. Shrock, Exact partition functions for the q-state Potts model with a generalized magnetic field on lattice strip graphs, J. Stat. Phys. 161 (2015) 915–932.
[9] S.D. Noble, D.J.A. Welsh, A weighted graph polynomial from chromatic invariants of knots, Ann. Inst. Fourier 49 (1999) 1057–1087.
[10] A.D. Sokal, Bounds on complex zeros of (di)chromatic polynomials and Potts model partition functions, Combin. Probab. Comput. 10 (2001) 41–77.
[11] J.A. Ellis-Monaghan, I.I. Moffatt, The Tutte-Potts connection in the presence of an external magnetic field, Adv. Appl. Math. 47 (2011) 772–782, [arXiv:1005.5470].
[12] C.M. Fortuin, P.W. Kasteleyn, On the random cluster model, Physica (Amsterdam) 57 (1972) 536–564.
[13] S.-C. Chang, R. Shrock, Some exact results on the Potts model in a magnetic field, J. Phys. A 42 (2009) 385004.
[14] W.T. Tutte, On dichromatic polynomials, J. Comb. Theory 2 (1967) 301–320.
[15] B. Bollobás, Modern Graph Theory, Springer, New York, 1998.
[16] A.D. Sokal, The multivariate Tutte polynomial (alias Potts model) for graphs and matroids, in: Surveys in Combinatorics, Cambridge Univ. Press, Cambridge, UK, 2005, pp. 173–226.
[17] J.A. Ellis-Monaghan, C. Merino, Graph polynomials and their applications I: The Tutte polynomial, in: M. Dehmer (Ed.), Structural Analysis of Complex Networks, Birkauser, Boston, 2011, pp. 219–255, [arXiv:0803.3079].
[18] J.A. Ellis-Monaghan, I. Moffatt (Eds.), Handbook of the Tutte Polynomial, CRC Press, Boca Raton, 2022.
[19] J.P. Straley, M.E. Fisher, Three-state Potts model and anomalous tricritical points, J. Phys. A 6 (1973) 1310–1326.
[20] K. Binder, Static and dynamic critical phenomena of the two-dimensional q-state Potts model, J. Stat. Phys. 24 (1981) 69–86.
[21] R. Shrock, Exact Potts model partition functions for ladder graphs, Physica A 283 (2000) 388–446.
[22] S.-C. Chang, R. Shrock, Exact Potts model partition functions on strips of the triangular lattice, Physica A 286 (2000) 189–238.
[23] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory, Nuclear Phys. B 241 (1984) 333–380.
[24] P. Di Francesco, P. Mathieu, D. Sénéchal, Conformal Field Theory, Springer, New York, 1997.
[25] E.H. Lieb, Residual entropy of square ice, Phys. Rev. 162 (1962) 162.