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Thomae's function and the space of ergodic measures

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ABSTRACT

We study the space of ergodic measures of the map

$$f : \mathbb{T}^2 \rightarrow \mathbb{T}^2, f(x, y) = (x, x + y) \pmod{1},$$

and show that its structure is similar to the graph of Thomae's function.

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1. Introduction

Thomae's function $\mathfrak{T} : (0, 1) \rightarrow \mathbb{R}$ is given by

$$\mathfrak{T}(x) = \begin{cases} 0 & \text{if } x \text{ is irrational;} \\ \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ for some coprime } p, q \in \mathbb{N} \end{cases}$$

and is also known as 'the popcorn function,' 'the raindrop function,' 'the countable cloud function,' the Riemann function, or even as 'the Stars over Babylon' (as was suggested by John Horton Conway). Some of these names are justified by the form of the graph of this function, see Figure 1:

Thomae's function is a standard example that is presented in most introductory real analysis classes. This function is continuous at irrational values of the argument and is discontinuous at rational values. This concept is a starting point for an interesting journey towards the Baire Category Theorem, where one can prove that there does not exist a function which is continuous on the rationals and discontinuous on the irrationals. At the same time, Thomae's function itself remains a mathematical curiosity and is something that doesn't usually appear in 'real life.' The purpose of this short note is to show how a structure similar to the graph of Thomae's function can appear in a natural way as a space of ergodic invariant measures of a very simple transformation.

Namely, define $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ via $f(x, y) = (x, x + y \pmod{1})$. We use additive notation for the unit circle $S^1 = \mathbb{R}/\mathbb{Z}$ so it is identified with $[0, 1)$. Thus, arithmetic on the 2-Torus $\mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ is understood to be done modulo 1, hence the notation given in the abstract. With this understanding, we omit this notation in the proofs. Let δ_ω be the atomic probability measure at the point $\omega \in \mathbb{T}^2$, and $\mathcal{M}(\mathbb{T}^2)$ be the space of all probability



Figure 1. The graph of Thomae's function.

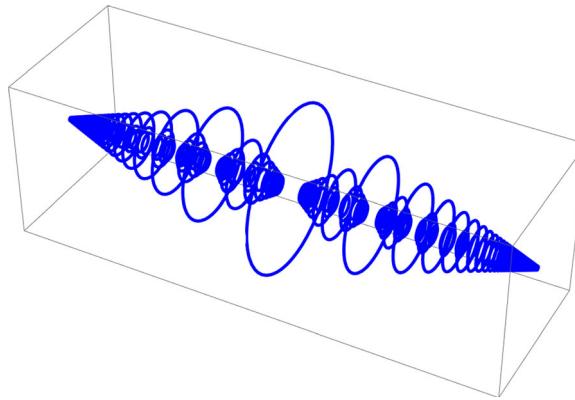


Figure 2. The set \mathcal{R} homeomorphic to the space of invariant ergodic measures of the map f .

measures on \mathbb{T}^2 . Consider the map $T : \mathbb{T}^2 \rightarrow \mathcal{M}(\mathbb{T}^2)$,

$$T(\omega) := \lim_{n \rightarrow \infty} \frac{\delta_\omega + \delta_{f(\omega)} + \cdots + \delta_{f^{n-1}(\omega)}}{n},$$

where the limit is taken in the weak-* topology. In this note, we proved that the function T is well-defined over \mathbb{T}^2 , describe the points at which T is continuous, and show that its image has a topological structure similar to the graph of the Thomae's function. Namely, define the set $\mathcal{R} \subset \mathbb{R}^3$ that can be obtained as a revolution of the graph of Thomae's function about the x -axis, see Figure 2:

$$\mathcal{R} = \left\{ (x, y, z) \in \mathbb{R}^3 \mid x \in (0, 1), \sqrt{y^2 + z^2} = \mathfrak{T}(x) \right\}.$$

Then we have the following:

Theorem 1.1: *The map T is well defined, and has the following properties:*

- For any $\omega \in \mathbb{T}^2$ the image $T(\omega)$ is an ergodic invariant measure of the map f .
- The map T is continuous at Lebesgue almost every point of \mathbb{T}^2 and discontinuous at a dense set of points. In particular, T is continuous at $\omega = (x_0, y_0)$ if and only if x_0 is irrational.
- The image $T(\mathbb{T}^2)$ is homeomorphic to \mathcal{R} .

Remark 1.2: If X is a measurable space and $f : X \rightarrow X$, denote the set of f -invariant probability measures by $\mathcal{M}(X, f)$. One can define a set-valued function $V : X \rightarrow 2^{\mathcal{M}(X, f)}$ where

$V(\omega)$ is the limit set of the sequence of measures $(\frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(\omega)})_n$. When $X = \mathbb{T}^2$ and f is given as above, Theorem 1.1 implies that each $V(\omega)$ is a singleton and hence can be thought of as a function into $\mathcal{M}(X, f)$, or that every point in X is generic for some measure. Systems of this kind have been studied in [6, 14] and such systems with the property that V is continuous have been characterized in [4, 8]. Besides, there is always a residual subset of X for which V is upper semi-continuous (see Proposition 3 in Chapter 8 of [1]), and in our case when $X = \mathbb{T}^2$ and f is as above, V is actually continuous on a residual subset of X . We are grateful to the anonymous referee for this remark.

Remark 1.3: One should expect structures similar to the graph of Thomae's function to appear in other contexts in dynamics whenever 'natural' semi-continuous functions on a parameter space appear; for another example of this kind, see Remark 4.3.2 from [9]. We are grateful to Simion Filip for this remark.

2. Spaces of invariant measures

Theorem 1.1 gives an example of a very simple transformation with a non-trivial topology of the space of ergodic measures. So, another way to incorporate this example into a larger picture is to ask a question about the structure of the space of invariant measures of a dynamical system.

If $f : M \rightarrow M$ is a diffeomorphism of a closed smooth manifold M , it is known that f must have some Borel invariant measures, and the space of all invariant probability measures is compact in weak-* topology. Moreover, it forms a convex subset in the space $\mathcal{M}(M)$ of all probability measures on M ; indeed, if ν_0 and ν_1 are invariant measures, then

$$\nu_t = t\nu_1 + (1-t)\nu_0 \tag{1}$$

is also a probability invariant measure for any $t \in [0, 1]$. Ergodic measures are exactly the extremal points (points that cannot be represented as ν_t from (1) for some $t \in (0, 1)$ and $\nu_0 \neq \nu_1$), of this convex set. So, one can think about ergodic measures as vertices of the simplex of all invariant measures. Moreover, the simplex is a Choquet simplex, i.e. in this case, a closed, convex subset of $\mathcal{M}(M)$ in which every point is a barycenter of a unique probability measure supported on the set of extremal points. Such a unique representation of an invariant measure as a weighted average of ergodic measures is known as *an ergodic decomposition*.

Thinking of invariant measures, it is tempting to think, say, about a tetrahedron, and to think about ergodic measures as its vertices. But this picture can be quite misleading. For example, for hyperbolic dynamical systems, the simplex of invariant measures typically forms the Poulsen simplex, a unique nontrivial compact Choquet simplex with a dense set of extreme points, see [17].

More specific questions on the structure of spaces of invariant measures were studied in the theory of dynamical systems at least since the 1970s. K. Sigmund in [19] showed that the space of ergodic measures for a transitive subshifts of finite type is path-connected; the result was recently extended by G. Iommi and A. Velozo [12] to the case of countable Markov shifts. Denseness of atomic ergodic measures in the space of all invariant measures for hyperbolic systems was studied by many authors, including K. Sigmund [18] and

A. Katok [13]. T. Downarowicz showed that for every Choquet simplex K there exists a minimal subshift with the space of ergodic invariant measures affinely homeomorphic to K [7]. Questions about structure of the space of hyperbolic ergodic measures on a locally maximal homoclinic class of a diffeomorphism were studied by A. Gorodetski and Y. Pesin [11]. Some of the results by C. Bonatti and J. Zhang [3], by L.J. Diaz, K. Gelfert, T. Marcarini, and M. Rams [5], and by D. Yang and J. Zhang [20] can be interpreted in terms of the structure of the space of ergodic invariant measures of partially hyperbolic diffeomorphisms under different assumptions. Recently, a notion of emergence have been developed by P. Berger and J. Bochi in [2]; it can be viewed as a way to quantify the ‘size’ or ‘dimension’ of the space of ergodic measures of a dynamical system. For other related results, see also [10,15,16].

3. Proof of Theorem 1.1

We first recall the following useful facts about sequences of real numbers:

Lemma 3.1: (1) *If $(\frac{p_n}{q_n})_n$ is a sequence of fully reduced rationals with an irrational limit, then $q_n \rightarrow \infty$ as $n \rightarrow \infty$.*
 (2) *If $(a_n)_n$ is a sequence such that there is some M satisfying $a_n = a_{n+M}$, then*

$$\lim_{n \rightarrow \infty} \frac{a_1 + \cdots + a_n}{n} = \frac{a_1 + \cdots + a_M}{M}.$$

Consider the ergodic invariant measure of \mathbb{T}^2 given by

$$\mu_{(p/q, y_0)} := \frac{1}{q} \sum_{i=0}^{q-1} \delta_{\left(\frac{p}{q}, y_0 + i\frac{p}{q}\right)}$$

where p/q is a fully reduced rational. Furthermore, denote the Lebesgue measure on S^1 by m_{S^1} .

Lemma 3.2: *For $\omega = (x_0, y_0) \in \mathbb{T}^2$,*

$$T(\omega) = \begin{cases} \delta_{x_0} \times m_{S^1} & \text{if } x_0 \notin \mathbb{Q} \\ \mu_{(p/q, y_0)} & \text{if } x_0 \in \mathbb{Q}, \text{ with } x_0 = \frac{p}{q} \text{ fully reduced} \end{cases}.$$

Proof: Fix $\omega = (x_0, y_0) \in \mathbb{T}^2$ and let $\phi \in C_0(\mathbb{T}^2)$. For simplicity, denote

$$\mu_n := \frac{\delta_\omega + \delta_{f(\omega)} + \cdots + \delta_{f^{n-1}(\omega)}}{n}.$$

First suppose that $x_0 \notin \mathbb{Q}$. Then, we note that y_0 has dense orbit in S^1 under the rotation function R_{x_0} . Now, by the continuity of ϕ and the Weyl Equidistribution Theorem, we have

$$\lim_{n \rightarrow \infty} \int \phi \, d\mu_n = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \phi(x_0, R_{x_0}^i(y_0)) = \int_0^1 \phi \, d(\delta_{x_0} \times m_{S^1}).$$

Now, suppose that $x_0 = p/q$ is a fully reduced rational number. Since the sequence $(\phi(\frac{p}{q}, y_0 + i\frac{p}{q}))_i$ satisfies the premises of Lemma 3.1 part (2), we have that μ_n converges

to $\mu_{(p/q, y_0)}$ since

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n-1} \phi \left(\frac{p}{q}, y_0 + i \frac{p}{q} \right) = \frac{1}{q} \sum_{i=0}^{q-1} \phi \left(\frac{p}{q}, y_0 + i \frac{p}{q} \right).$$

■

Proof of Theorem 1.1: That T is well-defined and the first statement, follow easily from Lemma 3.2 and construction of T . We now prove the sufficient and necessary conditions for the continuity of T at $\omega \in \mathbb{T}^2$.

For sufficiency, it is enough to prove that if $\omega = (p/q, y_0)$, where p/q is fully reduced, then T is not continuous at ω . We consider a continuous function $h : S^1 \rightarrow S^1$ such that

$$\int_0^1 h(y) dy > 0 \quad \text{and} \quad h \left(y_0 + i \frac{p}{k} \right) = 0,$$

for $i = 0, 1, \dots, q-1$. Now, let $((x_k, y_k))_k$ be a sequence such that $x_k \notin \mathbb{Q}$ and $(x_k, y_k) \rightarrow (x_0, y_0)$, and consider the function $\phi \in C_0(\mathbb{T}^2)$ via $\phi(x, y) = h(y)$. Then, by Lemma 3.2, it is clear that

$$\lim_{k \rightarrow \infty} \int_0^1 \phi d(T((x_k, y_k))) \neq \int_0^1 \phi d(T(\omega)).$$

Therefore, T is not continuous at $(p/q, y_0)$.

Conversely, let $\omega = (x_0, y_0) \in \mathbb{T}^2$ where $x_0 \notin \mathbb{Q}$. We will show that if $\omega_k \rightarrow \omega$, then $T(\omega_k) \rightarrow T(\omega)$ in the weak-* topology. It suffices to consider the cases when $\omega_k = (x_k, y_k)$ with $x_k \notin \mathbb{Q}$ and $\omega_k = (p_k/q_k, y_k)$, where each p_k/q_k is a fully reduced rational value.

Let $\phi \in C_0(\mathbb{T}^2)$. The former case is almost immediate by the uniform continuity of ϕ and the Weyl Equidistribution Theorem. In the latter case, we have that

$$\int_0^1 \phi d(T(\omega_k)) = \frac{1}{q_k} \sum_{i=0}^{q_k-1} \phi \left(\frac{p_k}{q_k}, y_k + i \frac{p_k}{q_k} \right).$$

As ϕ is uniformly continuous, for sufficiently large k , we can make the quantity $\phi(\frac{p_k}{q_k}, y_k + i \frac{p_k}{q_k})$ arbitrarily close to $\phi(x_0, y_0 + i \frac{p_k}{q_k})$, so it suffices to show that

$$\lim_{k \rightarrow \infty} \frac{1}{q_k} \sum_{i=0}^{q_k-1} \phi \left(x_0, y_0 + i \frac{p_k}{q_k} \right) = \int \phi d(T(\omega)).$$

Notice that the partition of S^1 given by $P_k = \{R_{\frac{p_k}{q_k}}^i(y_0)\}_{i=0}^{q_k}$ splits S^1 into q_k many intervals of equal length. Thus, by Lemma 3.1 part (1), since $q_k \rightarrow \infty$, we must have

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{1}{q_k} \sum_{i=0}^{q_k-1} \phi \left(x_0, y_0 + i \frac{p_k}{q_k} \right) &= \lim_{k \rightarrow \infty} \sum_{i=0}^{q_k-1} \phi \left(x_0, R_{\frac{p_k}{q_k}}^i(y_0) \right) \cdot (\text{mesh}(P_k)) \\ &= \int_0^1 \phi(x_0, y) dy. \end{aligned}$$

Therefore, T is continuous at (x_0, y_0) if $x_0 \notin \mathbb{Q}$.

Finally, we want to show that $T(\mathbb{T}^2)$ is homeomorphic to \mathcal{R} . First consider the function

$$\tilde{\mathfrak{T}}(x) = \begin{cases} \mathfrak{T}(x) & 0 < x < 1 \\ 1 & x = 0 \end{cases}$$

and denote the revolution of its graph on $[0, 1)$ by $\tilde{\mathcal{R}}$. Now, consider

$\psi_1 : T(\mathbb{T}^2) \rightarrow \tilde{\mathcal{R}}$ defined by

$$\psi_1(T((x, y))) = \begin{cases} (x, 0, 0) & x \notin \mathbb{Q} \\ \left(\frac{p}{q}, \frac{1}{q} \cos(2\pi i q y), \frac{1}{q} \sin(2\pi i q y)\right) & x = \frac{p}{q} \text{ fully reduced} \\ (0, \cos(2\pi i y), \sin(2\pi i y)) & x = 0 \end{cases}$$

Clearly ψ_1 is a bijection, and both ψ_1 and ψ_1^{-1} are continuous, so that $T(\mathbb{T}^2)$ is homeomorphic to $\tilde{\mathcal{R}}$.

To see that $\tilde{\mathcal{R}}$ is homeomorphic to \mathcal{R} , it is enough to show that the graph $G_{[0,1)}$ of $\tilde{\mathfrak{T}}$ on $[0, 1)$ is homeomorphic to the graph $G_{(0,1)}$ of \mathfrak{T} on $(0, 1)$. Consider the bijection $\psi_2 : G_{[0,1)} \rightarrow G_{(0,1)}$ which maps $(0, 1) \mapsto (\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{n}, \frac{1}{n}) \mapsto (\frac{1}{n+1}, \frac{1}{n+1})$ for $n \in \mathbb{N}_{\geq 2}$, and is the identity otherwise. It is clear that ψ_2 is a bijection, both ψ_2 and ψ_2^{-1} are continuous, and hence ψ_2 is a homeomorphism between $G_{[0,1)}$ and $G_{(0,1)}$. Therefore, the revolutions of these sets, $\tilde{\mathcal{R}}$ and \mathcal{R} , are also homeomorphic. ■

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