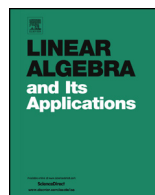




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Joins of circulant matrices



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ABSTRACT

We study the spectrum of the join of several circulant matrices. We apply our results to compute explicitly the spectrum of certain graphs obtained by joining several circulant graphs.

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1. Introduction

Circulant matrices provide a nontrivial, elegant, and simple set of objects in matrix theory. They appear quite naturally in many problems in spectral graph theory (see [1], [2], [7], [8], [13], [18]) and non-linear dynamics (see [11], [12], [20]). The Circulant Diagonalization Theorem describes the eigenspectrum and eigenspaces of a circulant matrix explicitly via the discrete Fourier transform. Consequently, many problems involving circulant matrices have closed-form or analytical solutions.

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For example, in many applications, a natural model of a network is a *ring graph*, in which nodes are regularly placed along a circle and, for a fixed number m , each node is connected to its m closest neighbours on each side. Networks such as this can be represented by adjacency matrices which are circulant, which opens the possibility for exact solutions for problems involving the structure or dynamics of these networks. More generally, a graph which has a circulant adjacency matrix with respect to a suitable ordering of the vertices is called a *circulant graph*.

Many real-world networks, however, display structure beyond that of circulant networks. For example, networks may be composed of several smaller modules, joined together in some way (see the final section for a particular example). From both a theoretical and an applied perspective, it is interesting and important to study the spectra of graphs obtained by joining together smaller subgraphs.

The combination of these previous observations naturally led us to investigate the spectrum of networks composed of several circulant graphs. While in general it is impossible to relate the spectrum of a graph with the spectra of its subgraphs, joins of circulant graphs provide an exception. Here we present a study of these spectra, and some applications. These results can provide analytical insight into the dynamics of composite networks (see e.g. [14]), which will be the subject of future work.

More precisely, we generalize the Circulant Diagonalization Theorem to the *joins of several circulant matrices*, by which we mean matrices of the shape

$$A = \left(\begin{array}{c|c|c|c} C_1 & a_{1,2}\mathbf{1} & \cdots & a_{1,d}\mathbf{1} \\ \hline a_{2,1}\mathbf{1} & C_2 & \cdots & a_{2,d}\mathbf{1} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{d,1}\mathbf{1} & a_{d,2}\mathbf{1} & \cdots & C_d \end{array} \right), \quad (*)$$

where, for each $1 \leq i, j \leq d$, C_i is a circulant matrix of size $k_i \times k_i$ (with complex entries), and $a_{i,j}\mathbf{1}$ is a $k_i \times k_j$ matrix with all entries equal to a constant $a_{i,j} \in \mathbb{C}$. We remark that, to simplify notation, $\mathbf{1}$ is used as the common symbol for all matrices with all entries equal to 1, independently of their sizes. However no confusion should occur as the submatrices $a_{i,j}\mathbf{1}$ are uniquely determined.

Our main theorem is

Theorem. *The spectrum of a matrix A as in $(*)$ is the union of the following multisets*

$$\text{Spec}(\overline{A}) \cup \bigcup_{i=1}^d \left\{ \lambda_j^{C_i} \mid 1 \leq j \leq k_i - 1 \right\},$$

where $\overline{A}$ is an explicit $d \times d$ matrix, whose entries are the row sums of the blocks of A , and the $\lambda_j^{C_i}$'s are the eigenvalues of each circulant block C_i , except for the eigenvalue given by the row sum. Furthermore, a generalized eigenbasis of A can be directly obtained

from eigenbases of the circulant blocks and a generalized eigenbasis of $\overline{A}$. In particular, A is diagonalizable if and only if $\overline{A}$ is.

This theorem completely solves our main problem of characterization of spectrum of the join of d circulant matrices. We note that the methods in this article can be generalized to a wider class of matrices, namely normal matrices with constant row sums. This extension will be discussed in a separate paper in preparation.

The structure of this article is as follows. In Section 2, we illustrate the join of two circulant matrices. This serves as a motivation for our study as well as to guide the readers to the more general case. In Section 3, we give the complete proof of the main theorem, which consists of several steps. First, we show how to extend eigenvectors of a circulant block to eigenvectors of the join. Secondly, we show that the generalized eigenspaces of $\overline{A}$ lift to the generalized eigenspaces of A . Finally, we prove that the collection of (generalized) eigenvectors for A , obtained from the previous two processes, form a generalized eigenbasis. In Section 4, we discuss some applications of our results to spectral graph theory. In the final section, we use the main theorem to study the dynamics of networks of coupled oscillators. Specifically, we construct a family of networks of Kuramoto oscillators with non-trivial equilibrium points.

2. Motivation: the join of two circulant matrices

A special instance of *joining circulant matrices* arises when we study the removal of one (directed) cycle from a complete graph. Recall that the *complete graph* of size n , denoted K_n , is the simple graph with an edge between any two distinct nodes. Its adjacency matrix A is given by

$$A_{ij} = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{otherwise.} \end{cases}$$

Moreover, a (directed) cycle of length k , or k -cycle, denoted C_k , is the simple graph on k nodes, in which the nodes can be ordered in such a way that each node is connected only with the subsequent one, and the last one only with the first one. Its adjacency matrix A is given by

$$A_{ij} = \begin{cases} 1 & \text{if } j = i + 1 \text{ or } (i, j) = (k, 1) \\ 0 & \text{otherwise.} \end{cases}$$

Finally, the complement of a graph G is the graph G^c with the same vertices as G and which has the edge between two distinct vertices if and only if that edge is not in G . In other words, the adjacency matrix A^c of G^c is related to the adjacency matrix A of G by

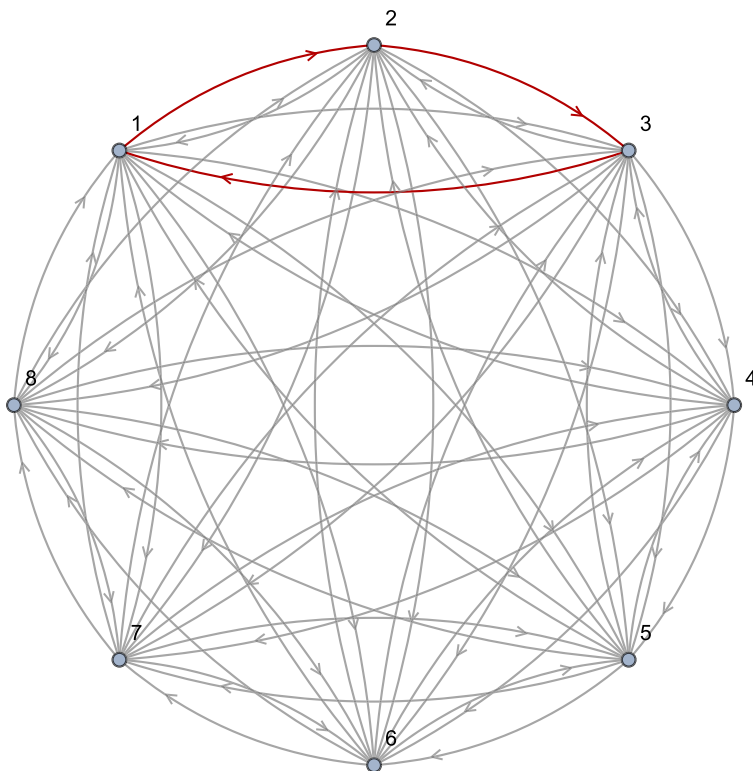


Fig. 1. The complete graph K_8 with a cycle of length 3 removed. (For interpretation of the colours in the figures, the reader is referred to the web version of this article.)

$A^c = \mathbf{1} - I - A$, where $\mathbf{1}$ is a square matrix of ones and I is an identity matrix, of suitable size. In particular, the adjacency matrices of complete graphs, cycles, and complements of cycles are all circulant.

We illustrate the general phenomenon of cycle removal on a small concrete example. Let us remove a 3-cycle C_3 from the complete graph K_8 with 8 nodes, and call the resulting graph $\mathcal{K} = K_8 - C_3$. We choose to remove the cycle

$$(1, 2), (2, 3), (3, 1)$$

which is highlighted in red in Fig. 1.

We note that removing any other cycle of length 3 would produce an isomorphic graph. Another representation of this graph is depicted in the figure below. We have two circulant graphs G and H (in green and grey respectively) and all nodes from each ring graph are adjacent to all nodes of the other ring graph. This is an instance of the *join of two circulant graphs* (Fig. 2), which we will define in Section 4.

In matrix terms, the adjacency matrix of $\mathcal{K}$ is a block matrix, with circulant diagonal blocks and 1 everywhere else.

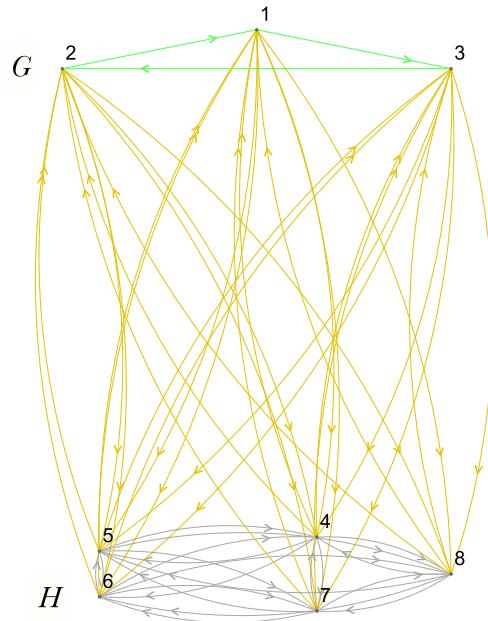


Fig. 2. The join of two circulant graphs G and H .

$$\left(\begin{array}{ccc|ccccc} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{array} \right)$$

The position of the eigenvalues of the adjacency matrix of $\mathcal{K}$ in the complex plane highlights a nontrivial interplay between the eigenvalues of the two circulant blocks, which motivates our investigations of the eigenspectra of joins of circulant matrices (Fig. 3).

It is worth noticing that the eigenvalues for the graph $\mathcal{K}$ have been obtained through the software **Mathematica**, but in the course of the paper we will derive analytical expressions for them.

To begin our investigation, we recall the Circulant Diagonalization Theorem (see [6] for a more thorough discussion about circulant matrices). In the following, ω_k denotes a fixed primitive k -th root of unity.

Theorem 1 (*Circulant Diagonalization Theorem, [6]*). *Let*

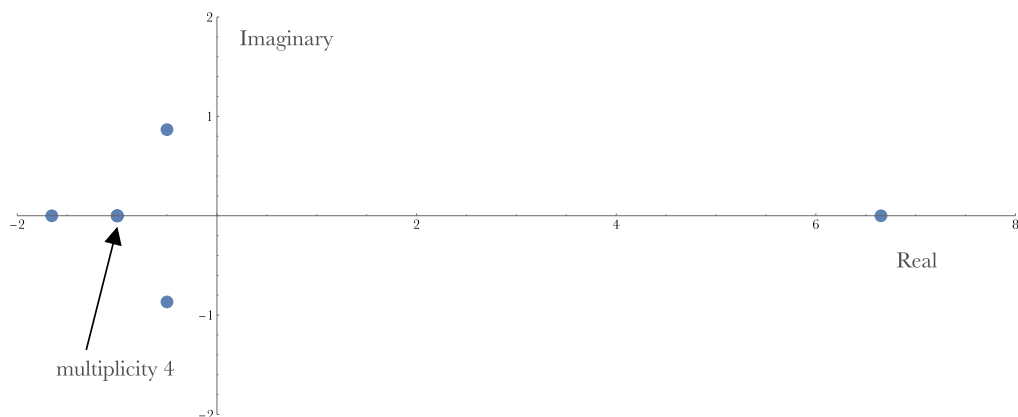


Fig. 3. The eigenvalues of the resulting graph.

$$C = \begin{pmatrix} c_0 & c_{k-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{k-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{k-2} & & \ddots & \ddots & c_{k-1} \\ c_{k-1} & c_{k-2} & \cdots & c_1 & c_0 \end{pmatrix} = \text{Circ}(c_0, c_1, \dots, c_{k-1})$$

be the circulant matrix formed by the vector $(c_0, c_1, \dots, c_{k-1})^T \in \mathbb{C}^k$. Let

$$v_{k,j} = \left(1, \omega_k^j, \omega_k^{2j}, \dots, \omega_k^{(k-1)j} \right)^T, \quad j = 0, 1, \dots, k-1.$$

Then $v_{k,j}$ is an eigenvector of C associated with the eigenvalue

$$\lambda_j = c_0 + c_{k-1}\omega_k^j + c_{k-2}\omega_k^{2j} + \cdots + c_1\omega_k^{(k-1)j}$$

Remark 2. For any choice of $k \in \mathbb{N} \setminus \{0\}$, the vectors $v_{k,0}, \dots, v_{k,k-1}$ are linearly independent. This can be seen by noticing that the matrix formed by the vectors is a Vandermonde matrix.

In the following, the operator $*$ denotes vector concatenation:

$$(x_1, \dots, x_m)^T * (y_1, \dots, y_n)^T = (x_1, \dots, x_m, y_1, \dots, y_n)^T$$

Proposition 3. Let C be a $k \times k$ circulant matrix, D be any $(n-k) \times (n-k)$ matrix, let $\mathbf{1}_{k_1, k_2}$ denote the $k_1 \times k_2$ matrix entirely made of ones, and let A be the $n \times n$ matrix

$$A = \begin{pmatrix} C & \mathbf{1}_{k, n-k} \\ \mathbf{1}_{n-k, k} & D \end{pmatrix}.$$

For $1 \leq j \leq k-1$ let

$$w_j = (1, \omega_k^j, \omega_k^{2j}, \dots, \omega_k^{(k-1)j}, \underbrace{0, \dots, 0}_{n-k \text{ zeros}})^T = v_{k,j} * \vec{0}_{n-k}.$$

Then w_j is an eigenvector of A associated with the eigenvalue

$$\lambda_j^C = c_0 + c_{k-1}\omega_k^j + c_{k-2}\omega_k^{2j} + \dots + c_1\omega_k^{(k-1)j}$$

Proof. When we directly calculate Aw_j we see that the first k elements of this vector are $Cv_{k,j}$ and the remaining $n - k$ elements are equal to the sum

$$t_j = \sum_{i=0}^{k-1} \omega_k^{ij}.$$

In other words, we have

$$Aw_j = Cv_{k,j} * \underbrace{(t_j, t_j, \dots, t_j)^T}_{n-k \text{ terms}} = \lambda_j^C v_j * \underbrace{(t_j, t_j, \dots, t_j)^T}_{n-k \text{ terms}}$$

Since, for $1 \leq j \leq k-1$,

$$t_j = \sum_{i=0}^{k-1} \omega_k^{ij} = \frac{(\omega_k^j)^k - 1}{\omega_k^j - 1} = 0,$$

it follows that $Aw_j = \lambda_j \omega_j$. We conclude that w_j , $1 \leq j \leq k-1$, are eigenvectors of A with associated eigenvalue λ_j^C as asserted. $\square$

If D is also circulant, $D = \text{Circ}(d_0, d_1, \dots, d_{k_2-1})$ with $k_2 = n - k$, an analogous argument applies. In summary, recalling Remark 2 for the claim on linear independence, we have proved the following statement.

Proposition 4. Let A be a $(k_1 + k_2) \times (k_1 + k_2)$ matrix of the form

$$A = \begin{pmatrix} C & \mathbf{1}_{k_1, k_2} \\ \mathbf{1}_{k_2, k_1} & D \end{pmatrix},$$

with $C = \text{Circ}(c_0, \dots, c_{k_1-1})$ and $D = \text{Circ}(d_0, \dots, d_{k_2-1})$ circulant matrices of dimension $k_1 \times k_1$ and $k_2 \times k_2$ respectively. For $1 \leq j \leq k_1 - 1$ let

$$w_j = (1, \omega_{k_1}^j, \omega_{k_1}^{2j}, \dots, \omega_{k_1}^{(k_1-1)j}, \underbrace{0, \dots, 0}_{k_2 \text{ zeros}})^T = v_{k_1,j} * \vec{0}_{k_2}.$$

Then w_j is an eigenvector of A associated with the eigenvalue

$$\lambda_j^C = c_0 + c_{k_1-1}\omega_{k_1}^j + c_{k_1-2}\omega_{k_1}^{2j} + \dots + c_1\omega_{k_1}^{(k_1-1)j}.$$

For $1 \leq j \leq k_2 - 1$, let

$$z_j = (\underbrace{0, \dots, 0}_{k_1 \text{ zeros}}, 1, \omega_{k_2}^j, \omega_{k_2}^{2j}, \dots, \omega_{k_2}^{(k_2-1)j})^T = \vec{0}_{k_1} * v_{k_2, j}.$$

Then z_j is an eigenvector associated with the eigenvalue

$$\lambda_j^D = d_0 + d_{k_2-1}\omega_{k_2}^j + d_{k_2-2}\omega_{k_2}^{2j} + \dots + d_1\omega_{k_2}^{(k_2-1)j}$$

Furthermore, the system of $k_1 + k_2 - 2$ eigenvectors $\{w_j\}_{j=1}^{k_1-1} \cup \{z_j\}_{j=1}^{k_2-1}$ is linearly independent.

In order to find the two remaining eigenvalues and corresponding eigenvectors of the matrix A , we introduce an auxiliary matrix.

Proposition 5. Keeping the notation of the previous proposition, let $C_s = \sum_{i=0}^{k_i-1} c_i$ be the sum of each row in C , and similarly let $D_s = \sum_{i=0}^{k_2-1} d_i$. Let us consider the 2×2 matrix

$$\overline{A} = \begin{pmatrix} C_s & k_2 \\ k_1 & D_s \end{pmatrix}.$$

Let $(x, y) \in \mathbb{C}^2$ be an eigenvector for $\overline{A}$ with respect to an eigenvalue λ . Then

$$v = (\underbrace{x, x, \dots, x}_{k_1 \text{ terms}}, \underbrace{y, y, \dots, y}_{k_2 \text{ terms}})^T$$

is an eigenvector of A with respect to the eigenvalue λ .

Proof. We have

$$Av = (\underbrace{C_s x + k_2 y, \dots, C_s x + k_2 y}_{k_1 \text{ terms}}, \underbrace{k_1 x + D_s y, \dots, k_1 x + D_s y}_{k_2 \text{ terms}})^T.$$

By assumption, $C_s x + k_2 y = \lambda x$, and $D_s x + k_1 y = \lambda y$.

Therefore, we see that

$$Av = \lambda (\underbrace{x, x, \dots, x}_{k_1 \text{ terms}}, \underbrace{y, y, \dots, y}_{k_2 \text{ terms}})^T = \lambda v. \quad \square$$

Proposition 6. Keeping the notation of the previous proposition, suppose further that $\overline{A}$ is diagonalizable with eigenvectors (x_1, y_1) and (x_2, y_2) . Let

$$v_1 = (\underbrace{x_1, x_1, \dots, x_1}_{k_1 \text{ terms}}, \underbrace{y_1, y_1, \dots, y_1}_{k_2 \text{ terms}})^T, \quad v_2 = (\underbrace{x_2, x_2, \dots, x_2}_{k_1 \text{ terms}}, \underbrace{y_2, y_2, \dots, y_2}_{k_2 \text{ terms}})^T.$$

Then the system $\{w_j\}_{j=1}^{k_1-1} \cup \{z_j\}_{j=1}^{k_2-1} \cup \{v_1, v_2\}$ of eigenvectors of A is linearly independent. In other words, A is diagonalizable by these eigenvectors.

Proof. For each k let

$$E_k = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_k & \omega_k^2 & \dots & \omega_k^{k-1} \\ 1 & \omega_k^2 & \omega_k^4 & \dots & \omega_k^{2(k-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_k^{k-1} & \omega_k^{2(k-1)} & \dots & \omega_k^{(k-1)(k-1)} \end{pmatrix}$$

be the matrix that is used to diagonalize a $k \times k$ circulant matrix, and $\widehat{E}_k$ be the submatrix of E_k with the first column removed. Let $e_k = \det(E_k) \neq 0$. The system $\{w_j\}_{j=1}^{k_1-1} \cup \{z_j\}_{j=1}^{k_2-1} \cup \{v_1, v_2\}$ can be arranged to create the following matrix

$$E = \left(\begin{array}{c|c|c|c} x_1 & \widehat{E}_{k_1} & x_2 & 0 \\ \vdots & & \vdots & \\ x_1 & & x_2 & \\ \hline y_1 & 0 & y_2 & \widehat{E}_{k_2} \\ \vdots & & \vdots & \\ y_1 & & y_2 & \end{array} \right)$$

Using the Laplace expansion of the determinant (see [15, Theorem 2.4.1]), we obtain the term $x_1 y_2 e_{k_1} e_{k_2}$ as the product of the determinant of the left top corner block matrix of size $k_1 \times k_1$ with the determinant of the right down corner matrix of the size $k_2 \times k_2$. The only other non-zero summand in the Laplace expansion is the product

$$\det \left(\begin{bmatrix} \widehat{E}_{k_1} & \begin{bmatrix} x_2 \\ \vdots \\ x_2 \end{bmatrix} \end{bmatrix} \right) \cdot \det \left(\begin{bmatrix} y_1 & \widehat{E}_{k_2} \end{bmatrix} \right) = -x_2 y_1 e_{k_1} e_{k_2}.$$

Consequently,

$$\det(E) = e_{k_1} e_{k_2} (x_1 y_2 - x_2 y_1) = e_{k_1} e_{k_2} \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \neq 0. \quad \square$$

In addition, there is a relationship between the eigenvalues of A and $\overline{A}$, to prove which we need a preliminary lemma.

Lemma 7. Let $M = \text{Circ}(m_0, \dots, m_{k-1})$ be a circulant matrix. Let $M_s = \sum_{i=0}^{k-1} m_i$. Let $\{\lambda_j^M\}_{j=0}^{k-1}$ be the set of eigenvalues of M described in the Circulant Diagonalization Theorem 1. Then

- (1) $\sum_{j=1}^{k_1-1} \lambda_j^M = \text{Tr}(M) - M_s.$
 (2) $\sum_{j=1}^{k_1-1} (\lambda_j^M)^2 = \text{Tr}(M^2) - M_s^2.$

Proof. Both equalities are direct consequences of the facts that, when $j = 0$, $\lambda_j^M = M_s$, and that for all $k \geq 0$

$$\text{Tr}(M^k) = \sum_{j=0}^{k-1} (\lambda_j^M)^k. \quad \square$$

Proposition 8. *Keeping the notation of the previous proposition, let λ_1, λ_2 be the two remaining eigenvalues of A , that is, the eigenvalues not coming from the circulant blocks C and D . Then λ_1 and λ_2 are eigenvalues of $\overline{A}$.*

Proof. It is enough to show that

$$\lambda_1 + \lambda_2 = C_s + D_s, \quad \text{and} \quad \lambda_1 \lambda_2 = C_s D_s - k_1 k_2.$$

First, by Proposition 4 we have

$$\lambda_1 + \lambda_2 + \sum_{j=1}^{k_1-1} \lambda_j^C + \sum_{j=1}^{k_2-1} \lambda_j^D = \text{Tr}(A) = \text{Tr}(C) + \text{Tr}(D).$$

By Lemma 7, we have

$$\sum_{j=1}^{k_1-1} \lambda_j^C = \text{Tr}(C) - C_s, \quad \text{and} \quad \sum_{j=1}^{k_2-1} \lambda_j^D = \text{Tr}(D) - D_s.$$

Combining these equalities, we conclude that

$$\lambda_1 + \lambda_2 = C_s + D_s.$$

To prove the equality $\lambda_1 \lambda_2 = C_s D_s - k_1 k_2$, we first compute $\lambda_1^2 + \lambda_2^2$, using A^2 . We have

$$A^2 = \begin{pmatrix} C^2 + k_2 \mathbf{1}_{k_1} & * \\ * & D^2 + k_1 \mathbf{1}_{k_2} \end{pmatrix},$$

where $\mathbf{1}_k$ denotes a $k \times k$ matrix with all entries equal to 1. This implies that

$$\text{Tr}(A^2) = \text{Tr}(C^2) + \text{Tr}(D^2) + 2k_1 k_2.$$

Additionally, we have

$$\begin{aligned}\mathrm{Tr}(A^2) &= \lambda_1^2 + \lambda_2^2 + \sum_{j=1}^{k_1-1} (\lambda_j^C)^2 + \sum_{j=1}^{k_2-1} (\lambda_j^D)^2, \\ \mathrm{Tr}(C^2) &= \sum_{j=1}^{k_1-1} (\lambda_j^C)^2 + C_s^2, \\ \mathrm{Tr}(D^2) &= \sum_{j=1}^{k_2-1} (\lambda_j^D)^2 + D_s^2.\end{aligned}$$

Combining these equalities, we get

$$\lambda_1^2 + \lambda_2^2 = C_s^2 + D_s^2 + 2k_1k_2.$$

Therefore, by Newton's formula we have

$$\begin{aligned}\lambda_1\lambda_2 &= \frac{1}{2} [(\lambda_1 + \lambda_2)^2 - \lambda_1^2 - \lambda_2^2] \\ &= \frac{1}{2} [(C_s + D_s)^2 - (C_s^2 + D_s^2 + 2k_1k_2)] \\ &= C_sD_s - k_1k_2.\end{aligned}$$

This completes the proof. $\square$

We discuss a significant case in which $\overline{A}$ is diagonalizable.

Proposition 9. *Keeping the notation of the previous proposition, suppose that C_s and D_s are real numbers (or complex numbers with the same real part, or with the same imaginary part). Then $\overline{A}$ is diagonalizable. Consequently, A is diagonalizable by the system of eigenvectors discussed in Proposition 6.*

Proof. The characteristic polynomial of $\overline{A}$ is

$$X^2 - (C_s + D_s)X + (C_sD_s - k_1k_2).$$

The discriminant of this polynomial is

$$\Delta(\overline{A}) = (C_s + D_s)^2 - 4(C_sD_s - k_1k_2) = (C_s - D_s)^2 + 4k_1k_2.$$

Since $C_s - D_s$ is either real or purely imaginary, $(C_s - D_s)^2 \in \mathbb{R}$, hence $\Delta(\overline{A}) > 0$. Therefore, $\overline{A}$ has two distinct eigenvalues and hence is diagonalizable. For the sake of completion, the two eigenvalues are

$$\lambda_i = \frac{C_s + D_s \pm \sqrt{(C_s - D_s)^2 + 4k_1k_2}}{2}. \quad \square$$

3. The general case

In the previous section we considered joins of 2 circulant matrices of a special important shape. In this section, we extend our results to general finite joins of circulant matrices. In our main theorem, we completely characterize the spectrum of these matrices. First, let us introduce some notations and conventions.

Let $d, k_1, k_2, \dots, k_d \in \mathbb{N} \setminus \{0\}$. Set also $n = k_1 + k_2 + \dots + k_d$. Thus n is a partition of n into d non-zero summands. We shall consider $n \times n$ matrices of the following form

$$A = \left(\begin{array}{c|c|c|c} C_1 & a_{1,2}\mathbf{1} & \cdots & a_{1,d}\mathbf{1} \\ \hline a_{2,1}\mathbf{1} & C_2 & \cdots & a_{2,d}\mathbf{1} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{d,1}\mathbf{1} & a_{d,2}\mathbf{1} & \cdots & C_d \end{array} \right),$$

where for each $1 \leq i, j \leq d$ $C_i = \text{Circ}(c_{i,0}, \dots, c_{i,k_i-1})$ is a circulant matrix of size $k_i \times k_i$, and $a_{i,j}\mathbf{1}$ is a $k_i \times k_j$ matrix with all entries equal to a constant $a_{i,j}$.

We have a direct generalization of Proposition 4:

Proposition 10. For each $1 \leq i \leq d$ and $1 \leq j \leq k_i - 1$ let

$$\begin{aligned} w_{i,j} &= \vec{0}_{k_1} * \dots * \vec{0}_{k_{i-1}} * v_{k_i,j} * \vec{0}_{k_{i+1}} * \dots * \vec{0}_{k_d} \\ &= \vec{0}_{k_1} * \dots * \vec{0}_{k_{i-1}} * \underbrace{(1, \omega_{k_i}^j, \omega_{k_i}^{2j}, \dots, \omega_{k_i}^{(k_i-1)j})^T}_{i\text{-th block}} * \vec{0}_{k_{i+1}} * \dots * \vec{0}_{k_d}. \end{aligned}$$

Then $w_{i,j}$ is an eigenvector of A associated with the eigenvalue

$$\lambda_j^{C_i} = c_{i,0} + c_{i,k_i-1}\omega_{k_i}^j + c_{i,k_i-2}\omega_{k_i}^{2j} + \dots + c_{i,1}\omega_{k_i}^{(k_i-1)j}$$

Furthermore, the system of $\sum_{i=1}^d k_i - d$ eigenvectors $\{w_{i,j}\}$ is linearly independent.

We introduce the following terminology.

Definition 11. Keeping the previous notation, we will refer to the $w_{i,j}$'s and to the associated eigenvalues as the *circulant eigenvectors and eigenvalues* of A .

Let $\lambda_1, \lambda_2, \dots, \lambda_d$ be the (not necessarily distinct) remaining eigenvalues of A . The *reduced characteristic polynomial* of A is

$$\bar{p}_A(X) = \prod_{i=1}^d (X - \lambda_i) = \frac{p_A(X)}{\prod_{\substack{1 \leq i \leq d, \\ 1 \leq j \leq k_i-1}} (X - \lambda_j^{C_i})}.$$

Motivated by the findings of Section 2, we look for the missing eigenvectors of A in a special form, namely

$$v = (x_1)_{k_1} * \dots * (x_i)_{k_i} * \dots * (x_d)_{k_d}, \quad (3.1)$$

where

$$(x_i)_{k_i} = \underbrace{(x_i, \dots, x_i)^T}_{k_i \text{ terms}}.$$

For $1 \leq i \leq d$, we denote the row sum of the matrix C_i by

$$C_{is} = \sum_{j=0}^{k_i-1} c_{i,j}.$$

A direct calculation shows that

$$Av = (C_{1s}x_1 + a_{12}k_2x_2 + \dots + a_{1d}k_dx_d)_{k_1} * \dots * (a_{d1}k_1x_1 + a_{d2}k_2x_2 + \dots + C_{ds}x_d)_{k_d}.$$

Therefore, the equation $Av = \lambda v$ can be equivalently written as

$$\overline{A}(x_1, x_2, \dots, x_d)^T = \lambda(x_1, \dots, x_d)^T,$$

where $\overline{A}$ is the $d \times d$ matrix

$$\overline{A} = \begin{pmatrix} C_{1s} & a_{12}k_2 & \cdots & a_{1n}k_d \\ a_{21}k_1 & C_{2s} & \cdots & a_{2n}k_d \\ \vdots & \vdots & \ddots & \vdots \\ a_{d1}k_1 & a_{d2}k_2 & \cdots & C_{ds} \end{pmatrix}.$$

In other words, an eigenvector of A of the form (3.1) can be “condensed” to an eigenvector $(x_1, \dots, x_d)^T$ of $\overline{A}$ with respect to the same eigenvalue. A strong converse statement also holds: to prove it, we need a preliminary lemma.

Lemma 12. Let $X = (x_{ij})$ be a $d \times d$ matrix. Let M be the $(k_1 + \dots + k_d) \times (k_1 + \dots + k_d)$ matrix formed by the following column vectors (in this order)

$$\begin{aligned} &(x_{11})_{k_1} * \dots * (x_{d1})_{k_d}, \quad w_{j,1} \quad (1 \leq j \leq k_1 - 1), \\ &(x_{12})_{k_1} * \dots * (x_{d2})_{k_d}, \quad w_{j,2} \quad (1 \leq j \leq k_2 - 1), \\ &(x_{1n})_{k_1} * \dots * (x_{dn})_{k_d}, \quad w_{j,d} \quad (1 \leq j \leq k_d - 1). \end{aligned}$$

Then

$$\det(M) = \det(E_{k_1}) \dots \det(E_{k_d}) \det(X),$$

where E_k is the nonsingular matrix

$$E_k = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_k & \omega_k^2 & \dots & \omega_k^{k-1} \\ 1 & \omega_k^2 & \omega_k^4 & \dots & \omega_k^{2(k-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_k^{k-1} & \omega_k^{2(k-1)} & \dots & \omega_k^{(k-1)(k-1)} \end{pmatrix}$$

In particular, M is non-singular iff X is non-singular.

Proof. By induction and the Laplace expansion formula, analogously to the proof of Proposition 6. $\square$

Definition 13. For $v = (x_1, \dots, x_d)^T \in \mathbb{C}^d$, $k_1, \dots, k_d \in \mathbb{N} \setminus \{0\}$ and $n = k_1 + \dots + k_d$, we refer to the vector

$$v^{\otimes} = (\underbrace{x_1, \dots, x_1}_{k_1 \text{ terms}}, \dots, \underbrace{x_d, \dots, x_d}_{k_d \text{ terms}})^T \in \mathbb{C}^n$$

as the *tensor expansion* of v .

Proposition 14. The tensor expansions of the generalized eigenspaces of $\overline{A}$ are generalized eigenspaces of A . More precisely, if $(\overline{A} - \lambda I)^m v = 0$ for some $v \in \mathbb{C}^d$, $\lambda \in \text{Spec}(\overline{A})$ and $m \in \mathbb{N}$, then $(A - \lambda I)^m v^{\otimes} = 0$.

Proof. Note preliminarily that, by the construction of the matrix $\overline{A}$, for any $v \in \mathbb{C}^d$ and any $\lambda \in \mathbb{C}$

$$[(\overline{A} - \lambda I)v]^{\otimes} = (A - \lambda I)v^{\otimes}. \quad (3.2)$$

We proceed by induction on m . The case $m = 1$, that is, of ordinary eigenvectors, is a direct consequence of Equation (3.2). Now suppose by inductive hypothesis that for $w \in \mathbb{C}^d$ and $\lambda \in \text{Spec}(\overline{A})$

$$(\overline{A} - \lambda I)^{m-1} w = 0 \Rightarrow (A - \lambda I)^{m-1} w^{\otimes} = 0, \quad (3.3)$$

and let $v \in \mathbb{C}^d$ satisfy $(\overline{A} - \lambda I)^m v = 0$. Then $w = (\overline{A} - \lambda I)v$ satisfies the premise of (3.3). Consequently,

$$(A - \lambda I)^m v^{\otimes} = (A - \lambda I)^{m-1} ((A - \lambda I)v)^{\otimes} \stackrel{(3.2)}{=} (A - \lambda I)^{m-1} w^{\otimes} \stackrel{(3.3)}{=} 0. \quad \square$$

Proposition 15. Let $\{u_1, \dots, u_d\}$ be a basis of generalized eigenvectors of $\overline{A}$. Then the set made of the circulant eigenvectors $w_{i,j}$ of A introduced in Proposition 10, together with $u_1^{\otimes}, \dots, u_d^{\otimes}$, is linearly independent.

Proof. We claim that $\text{span}\{w_{i,j} \mid i = 1, \dots, d, j = 1, \dots, k_i\} \cap \text{span}\{u_1^\otimes, \dots, u_d^\otimes\} = \{0\}$. In fact, the latter span is included in the subspace $U = \{v = (y_1, \dots, y_n) \in \mathbb{C}^n \mid y_1 = \dots = y_{k_1}, y_{k_1+1} = \dots = y_{k_1+k_2}, \dots, y_{k_1+\dots+k_{d-1}+1} = \dots = y_{k_1+\dots+k_d}\}$. If by contradiction we assume a nontrivial linear combination $\sum_{i=1}^d \sum_{j=1}^{k_i} \alpha_{i,j} w_{i,j}$ to lie in U , then by direct inspection each partial linear combination $\sum_{j=1}^{k_i} \alpha_{i,j} w_{i,j}$ (with fixed i) has to lie in U . Suppose, without loss of generality, that the partial linear combination $\sum_{j=1}^{k_1} \alpha_{1,j} w_{1,j}$ is nontrivial. Then, for some $c \in \mathbb{C}$,

$$\sum_{j=1}^{k_1} \alpha_{1,j} w_{1,j} = (\underbrace{c, c, \dots, c}_{k_1}, 0, \dots, 0)^T,$$

which implies a nontrivial linear relation between $v_{k_1,0}, v_{k_1,1}, v_{k_1,k_1-1}$, in contradiction with Remark 2. $\square$

Now a counting argument on dimensions shows that there is no room for any (generalized) eigenvector of A other than the circulant eigenvectors and the tensor expansions of the (generalized) eigenvectors of $\overline{A}$. We collect several direct consequences of this fact.

Corollary 16. *A is diagonalizable if and only if $\overline{A}$ is. In particular, if $\overline{A}$ is diagonalizable with eigenvalue-eigenvector pairs $(\lambda_1, v_1), \dots, (\lambda_d, v_d)$, then A is diagonalizable with the following system of eigenvalue-eigenvector pairs:*

$$\begin{aligned} &(\lambda_j, v_j^\otimes) \text{ for } 1 \leq j \leq d, \\ &(\lambda_j^{C_1}, w_{1,j}) \text{ for } 1 \leq j \leq k_1 - 1, \\ &(\lambda_j^{C_2}, w_{2,j}) \text{ for } 1 \leq j \leq k_2 - 1, \\ &\quad \dots \\ &(\lambda_j^{C_d}, w_{d,j}) \text{ for } 1 \leq j \leq k_d - 1. \end{aligned}$$

Corollary 17. *The reduced characteristic polynomial of A coincides with the characteristic polynomial of $\overline{A}$, namely*

$$\overline{p}_A(X) = p_{\overline{A}}(X).$$

4. Some applications to network theory

In this section, we apply the main results to study the spectrum of several (directed) graphs by the join and edge-removal procedures. In particular, we provide a conceptual explanation for the spectrum of the graph described in the second section.

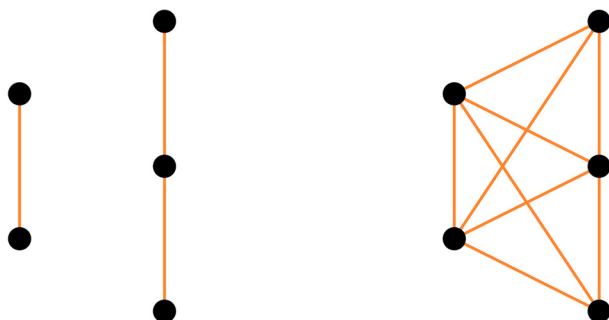
First, we recall a graph construction, namely the join construction (see [10, Chapter 2] and [23]).

Definition 18. Let $G = (V(G), E(G))$, $H = (V(H), E(H))$ be two graphs. The join of G and H , denoted by $G + H$, is the graph with vertex set $V = V(G) \cup V(H)$, and in which

two vertices u and v are adjacent if and only if

- $u, v \in V(G)$ and $uv \in E(G)$.
- $u, v \in V(H)$ and $uv \in E(H)$.
- $u \in V(G)$ and $v \in V(H)$.
- $u \in V(H)$ and $v \in V(G)$.

Here is a pictorial illustration:



Let $A(G)$ and $A(H)$ be the adjacency matrices of G and H respectively. Then the adjacency matrix of $G + H$ is given by

$$A(G + H) = \begin{pmatrix} A(G) & \mathbf{1}_{k_1, k_2} \\ \mathbf{1}_{k_2, k_1} & A(H) \end{pmatrix},$$

with $k_1 = |V(G)|$ and $k_2 = |V(H)|$. Therefore, in any case in which $A(G)$ and $A(H)$ are circulant, the spectrum of $A(G + H)$ is completely determined by Propositions 4, 8, 9. Here are some interesting instances.

Example 19 (Ring graphs). For two positive integers k, m , the *ring graph* $\text{RG}(k, m)$ is the undirected graph whose k vertices can be arranged in a circle in such a way that each vertex is connected to its m closest neighbours on each side (with the understanding that, for $k \leq 2m + 1$, $\text{RG}(k, m)$ is the complete graph K_k). In particular, ring graphs are regular. We choose a total order of the vertices which goes along the aforementioned circle. This produces a circulant adjacency matrix, whose eigenvalue corresponding to the eigenvector $(1, 1, \dots, 1)^T$ is the graph valency $2m$.

Consequently, if $k_1 > 2m_1 + 1$ or $k_2 > 2m_2 + 1$, the spectrum of $\text{RG}(k_1, m_1) + \text{RG}(k_2, m_2)$ is the union of three multisets

$$(\text{Spec}(\text{RG}(k_1, m_1)) \setminus \{2m_1\}) \cup (\text{Spec}(\text{RG}(k_2, m_2)) \setminus \{2m_2\}) \cup \{\lambda_1, \lambda_2\},$$

with

$$\begin{aligned}\lambda_1, \lambda_2 &= \frac{(2m_1 + 2m_2) \pm \sqrt{(2m_1 - 2m_2)^2 + 4k_1k_2}}{2} \\ &= m_1 + m_2 \pm \sqrt{(m_1 - m_2)^2 + k_1k_2}.\end{aligned}$$

For the sake of completion, if $k_1 \geq 2m_1 + 1$ and $k_2 \geq 2m_2 + 1$, then clearly $\text{RG}(k_1, m_1) + \text{RG}(k_2, m_2)$ is the complete graph on $k_1 + k_2$ vertices, so its spectrum is well known.

Example 20 (*Cycle removal 1*). Let us consider the graph obtained by removing an undirected cycle of length k from the complete graph K_n with $n > k$. Up to a reordering of the vertices, the resulting graph is the join of a circulant graph G , with k vertices and $k - 3$ edges and with adjacency matrix $A(G) = \text{Circ}(0, 0, 1, 1, \dots, 1, 0)$, and the complete graph $H = K_{n-k}$. Since

$$\text{Spec}(G) = \left\{ \left[\sum_{r=2}^{k-2} \omega_k^{rj} \right]_1 \mid j = 0, \dots, k \right\}$$

and

$$\text{Spec}(H) = \{[-1]_{n-k-2}, [n-k-1]_1\},$$

with lower indices after square brackets denoting algebraic multiplicity, the spectrum of $G + H$ is the multiset

$$\left\{ \left[\sum_{r=2}^{k-2} \omega_k^{rj} \right]_1 \mid j = 1, \dots, k \right\} \cup \{[-1]_{n-k-2}\} \cup \{\lambda_1, \lambda_2\}$$

with

$$\lambda_1, \lambda_2 = \frac{(n-4) + \sqrt{(n+2)^2 - 8k}}{2}.$$

Example 21 (*Cycle removal 2*). Similarly, the graph obtained by removing a directed cycle of length k from the complete graph K_n with $n > k$ is the join of a circulant graph G , with k vertices and $k - 2$ edges and with adjacency matrix $A(G) = \text{Circ}(0, 1, 1, \dots, 1, 0)$, and the complete graph $H = K_{n-k}$. Its spectrum is the multiset

$$\left\{ \left[\sum_{r=2}^{k-1} \omega_k^{rj} \right]_1 \mid j = 1, \dots, k \right\} \cup \{[-1]_{n-k-2}\} \cup \{\lambda_1, \lambda_2\}$$

with

$$\lambda_1, \lambda_2 = \frac{(n-3) + \sqrt{(n+1)^2 - 4k}}{2}.$$

5. Applications to non-linear dynamics on oscillator networks

To illustrate potential applications of this approach, we can now consider a dynamical system on the join of several circulant graphs. Specifically, we consider oscillators coupled on a graph on a matrix A , defined by joining d identical circulant graphs. We consider the Kuramoto model:

$$\frac{d\theta_i}{dt} = \omega_i + \epsilon \sum_{j=1}^N A_{ij} \sin(\theta_j - \theta_i), \quad (5.1)$$

which is a central tool in the description of synchronization in nature, from the behaviour of insects (see [4], [9]), patterns of social behaviour (see [16], [17]), neural systems (see [3], [5]), and physical systems (see [21], [22]). Here, θ_i is the state of oscillator $i \in [1, N]$ at time t , ω_i is the intrinsic angular frequency, κ scales the coupling strength, and element a_{ij} represents the weighted connection between oscillators i and j . We focus on the case where all oscillators have the same natural frequency, that is, $\omega_i = \omega$ for all $i \in [1, N]$. Under this condition, we can assume further that $\omega = 0$.

An important question in this area is the study of equilibrium points on a network of Kuramoto oscillators (see [19], [20]). Furthermore, it is known that the stability of these equilibrium points depends strongly on the specific pattern of connections, highlighting the importance of the network's structure on the Kuramoto dynamics (see [20]). In [14], we utilize an algebraic approach to study equilibrium points of this dynamical system. By studying a related complex-valued model introduced in [12], we prove the following theorem.

Theorem 22. (See [14, Proposition 2].) Suppose $\mathbf{x}_0 = e^{i\theta_0}$ is an eigenvector of $\mathbf{A} = (a_{ij})$ associated with a real eigenvalue λ . Then $\theta_0 = (\theta_1, \theta_2, \dots, \theta_N)$ is an equilibrium point of the following Kuramoto model.

$$\frac{d\theta_i}{dt} = \epsilon \sum_{j=1}^N a_{ij} \sin(\theta_j - \theta_i).$$

We will now use this result and the main theorem of this article to construct networks with interesting equilibrium points. More precisely, let C be a real symmetric circulant matrix of size $k \times k$. Let A be a join of d -identical copies of C , namely A is a network with the following weighted adjacency matrix

$$A = \left(\begin{array}{c|c|c|c} C & a_{1,2}\mathbf{1} & \cdots & a_{1,d}\mathbf{1} \\ \hline a_{2,1}\mathbf{1} & C & \cdots & a_{2,d}\mathbf{1} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{d,1}\mathbf{1} & a_{d,2}\mathbf{1} & \cdots & C \end{array} \right).$$

Let $(\varphi_1, \varphi_2, \dots, \varphi_d) \in [-\pi, \pi]^d$. For each $1 \leq j \leq k-1$ and $1 \leq i \leq d$ let us define

$$\omega_{i,j}^{(\varphi_i)} = e^{i\varphi_i} \omega_{i,j},$$

where $\omega_{i,j}$ is an eigenvector of A associated with the eigenvalue λ_j^C as described in Proposition 10. Because these eigenvectors for a fixed j are associated with a single eigenvalue, their sum $\sum_{i=1}^d \omega_{i,j}^{(\varphi_i)}$ is also an eigenvector associated with the eigenvalue λ_j^C . Note further that by the definition of $\omega_{i,j}$, we have

$$\sum_{i=1}^d \omega_{i,j}^{(\varphi_i)} = e^{i\theta_{0,j}^{(\varphi_1, \dots, \varphi_d)}},$$

where

$$\begin{aligned} & \theta_{0,j}^{(\varphi_1, \dots, \varphi_d)} \\ &= \left(\varphi_1, \frac{2\pi j}{k} + \varphi_1, \dots, \frac{2\pi(k-1)j}{k} + \varphi_1, \dots, \varphi_d, \frac{2\pi j}{k} + \varphi_d, \dots, \frac{2\pi(k-1)j}{k} + \varphi_d \right)^T. \end{aligned}$$

By Theorem 22, we conclude that

Proposition 23. *For all $1 \leq j \leq k-1$ and $(\varphi_1, \varphi_2, \dots, \varphi_d) \in [-\pi, \pi]^d$, $\theta_{0,j}^{(\varphi_1, \dots, \varphi_d)}$ is an equilibrium point of the KM associated with the adjacency matrix A .*

Declaration of competing interest

The authors declare no competing interests.

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