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Singularities and syzygies of secant varieties of nonsingular projective curves

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Abstract In recent years, the equations defining secant varieties and their syzygies have attracted considerable attention. The purpose of the present paper is to conduct a thorough study on secant varieties of curves by settling several conjectures and revealing interaction between singularities and syzygies. The main results assert that if the degree of the embedding line bundle of a nonsingular curve of genus g is greater than $2g + 2k + p$ for nonnegative integers k and p, then the k-th secant variety of the curve has normal Du Bois singularities, is arithmetically Cohen–Macaulay, and satisfies the property $N_{k+2,p}$. In addition, the singularities of the secant varieties are further classified according to the genus of the curve, and the Castelnuovo–Mumford

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regularities are also obtained as well. As one of the main technical ingredients, we establish a vanishing theorem on the Cartesian products of the curve, which may have independent interests and may find applications elsewhere.

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1 Introduction

Throughout the paper, we work over an algebraically closed field $\Bbbk$ of characteristic zero. Let

$$C \subseteq \mathbb{P}(H^0(C, L)) = \mathbb{P}^r$$

be a nonsingular projective curve of genus $g \geq 0$ embedded by the complete linear system of a very ample line bundle L on C. For an integer $k \geq 0$, the *k-th secant variety*

$$\Sigma_k = \Sigma_k(C, L) \subseteq \mathbb{P}^r$$

to the curve C is defined to be the Zariski closure of the union of $(k+1)$-secant k-planes to C in $\mathbb{P}^r$. One has the natural inclusions

$$C = \Sigma_0 \subseteq \Sigma_1 \subseteq \cdots \subseteq \Sigma_{k-1} \subseteq \Sigma_k \subseteq \mathbb{P}^r.$$

If $\deg L \geq 2g + 2k + 1$, then

$$\dim \Sigma_k = 2k + 1 \quad \text{and} \quad \operatorname{Sing}(\Sigma_k) = \Sigma_{k-1}.$$

Note that Σ_{k-1} has codimension two in Σ_k. The geometric consequence of the condition $\deg L \geq 2g+2k+1$ is that any effective divisor on C of degree $k+1$ spans a k-plane in $\mathbb{P}^r$.

There has been a great deal of work on the secant varieties in the last three decades. The major part of the research focused on local properties, defining equations, and syzygies. Recently, classical questions on secant varieties find interesting applications to algebraic statistics and algebraic complexity theory. However, a lot of problems in this area are still widely open, and not much is known about general pictures. For the first secant variety of a curve, investigation has been conducted in a series of work by Vermeire [22–25] and the work with his collaborator Sidman [17,18]. Among other things, the issue whether secant varieties are normal attracted special attention, as normality is critical in establishing many other important properties. Only for the first secant variety, the normality problem was settled by Ullery [21] fairly recently for a nonsingular projective variety of any dimension under suitable conditions on the embedding line bundle. Soon afterwards Chou and Song [2] further showed that the first secant variety has Du Bois singularities under the setting of Ullery's study.

On the other hand, the classical questions on the projective normality and the defining equations of secant varieties are the initial case of a more general picture involving higher syzygies, under the frame of Green's pioneering work [10]. Keeping in mind that the curve can be viewed as its zeroth secant variety, the fundamental *Green's* $(2g+1+p)$*-theorem* (see [10,11]) asserts that if the embedding line bundle L has $\deg L \geq 2g+1+p$, then $C \subseteq \mathbb{P}^r$ is projectively normal and satisfies the property $N_{2,p}$, i.e., the curve is cut out by quadrics and the first p steps of its minimal graded free resolution are linear (see Sect. 2.2 for relevant definitions on syzygies). This result sheds the lights on understanding the full picture of syzygies of arbitrary order secant varieties.

In this paper, we give a thorough study on singularities and syzygies of the k-th secant variety Σ_k of the curve C for arbitrary integer $k \geq 0$. The general philosophy guiding our research can be summarized as that singularities and syzygies interact each other in the way that the singularities of Σ_k determine its syzygies while the syzygies of Σ_{k-1} determine the singularities of Σ_k, and so on and so forth. It turns out that all the sufficient conditions that guarantee each basic property of secant varieties are satisfied if the embedding line bundle is positive enough beyond an effective bound.

The first main result of the paper describes that the possible singularities of secant varieties are mild ones naturally appearing in birational geometry. We refer to Sect. 2.1 for the definitions of singularities.

Theorem 1.1 *Let C be a nonsingular projective curve of genus g, and L be a line bundle on C. For an integer $k \geq 0$, suppose that*

$$\deg L \geq 2g + 2k + 1.$$

Then $\Sigma_k = \Sigma_k(C, L)$ has normal Du Bois singularities. Furthermore, one has the following:

(1) *$g = 0$ if and only if Σ_k is a Fano variety with log terminal singularities.*
(2) *$g = 1$ if and only if Σ_k is a Calabi–Yau variety with log canonical singularities but not log terminal singularities.*
(3) *$g \geq 2$ if and only if there is no boundary divisor Γ on Σ_k such that (Σ_k, Γ) is a log canonical pair.*

The theorem therefore completely solves the normality problems mentioned above (see Ullery's conjecture [20, Conjecture E]), and answers Chou–Song's question [2, Question 1.6] for curves.

The second main result gives a description on syzygies of the k-th secant variety. It reveals one full picture hiding in the Green's $(2g + 1 + p)$-theorem aforementioned.

Theorem 1.2 *Let $C \subseteq \mathbb{P}(H^0(C, L)) = \mathbb{P}^r$ be a nonsingular projective curve of genus g embedded by the complete linear system of a very ample line bundle L on C. For integers $k, p \geq 0$, suppose that*

$$\deg L \geq 2g + 2k + 1 + p.$$

Then one has the following:

(1) *$\Sigma_k = \Sigma_k(C, L) \subseteq \mathbb{P}^r$ is arithmetically Cohen–Macaulay.*
(2) *$\Sigma_k \subseteq \mathbb{P}^r$ satisfies the property $N_{k+2,p}$.*
(3) *$\operatorname{reg}(\mathscr{O}_{\Sigma_k}) = 2k + 2$ unless $g = 0$, in which case $\operatorname{reg}(\mathscr{O}_{\Sigma_k}) = k + 1$.*
(4) *$h^0(\omega_{\Sigma_k}) = \dim K_{r-2k-1,2k+2}(\Sigma_k, \mathscr{O}_{\Sigma_k}(1)) = \binom{g+k}{k+1}$.*

The results in the theorem were conjectured by Sidman–Vermeire [18, Conjecture 1.3], [24, Conjectures 5 and 6]. The conjectures were quite wide open. For $g \leq 1$, the conjectures were settled by Graf von Bothmer–Hulek [26] and Fisher [8]. By work of Vermeire [23–25], Sidman–Vermeire [18], and Yang [27], the question about $N_{3,p}$ was finally settled for the first secant variety Σ_1.

Theorem 1.2 gives a complete picture for syzygies of arbitrary order secant varieties of curves. If $\deg L \geq 2g+2k+1$, then $\Sigma_k \subseteq \mathbb{P}^r$ is indeed projectively normal. If $\deg L \geq 2g + 2k + 2$, then Σ_k is ideal-theoretically cut out by the hypersurfaces of degree $k + 2$, as it cannot be contained in a smaller degree hypersurface. Furthermore, if $\deg L \geq 2g+2k+1+p$, then the first p steps of the minimal graded free resolution of Σ_k are linear.

We mention here several quick examples to show that the degree bounds on the line bundle L in the theorems are optimal. (i) Assume C has genus $g=4$ and take general points p, q, r, and s on C. The line bundle $L=\omega_C(p+q+r+s)$ embeds C in $\mathbb{P}^{g+2}$. Then the first secant variety Σ_1 is neither normal nor Cohen–Macaulay. See Example 5.14 for non-normal higher secant varieties Σ_k with $k \geq 2$. (ii) If C is an elliptic curve and $\deg L = 2k+3$, then the k-th secant variety Σ_k in $\mathbb{P}^{2k+2}$ is a hypersurface of degree $2k+3$. (iii) If C has genus 2 and degree 12 in $\mathbb{P}^{10}$, then Σ_1 satisfies $N_{3,5}$ but fails $N_{3,6}$, and Σ_2 satisfies $N_{4,3}$ but fails $N_{4,4}$. The last two examples are taken from [17,26], and one may find more examples there.

To prove the main results of the paper, we utilize Bertram's construction [1] to realize secant varieties as the images of projectivized vector bundles. To be more precise, we consider the k-th symmetric product C_{k+1} of C. We have a canonical morphism $\sigma_{k+1}\colon C_k \times C \to C_{k+1}$ defined by sending (ξ, x) to $\xi + x$ and the projection $p\colon C_k \times C \to C$. One defines the secant sheaf

$$E_{k+1,L} :=\sigma_{k+1,*}(p^*L),$$

which is a vector bundle on C_{k+1} of rank $k+1$, and the secant bundle

$$B^k(L) :=\mathbb{P}(E_{k+1,L}).$$

Notice that $E_{k+1,L}$ parameterizes $(k+1)$-secant k-planes, i.e., the fiber of $E_{k+1,L}$ over $\xi \in C_{k+1}$ can be identified with $H^0(\xi, L|_\xi)$. The complete linear system of the tautological line bundle of $B^k(L)$ determines a natural morphism to the projective space $\mathbb{P}^r$ such that the image is Σ_k. It gives rise to a resolution of singularities

$$\beta\colon B^k(L) \longrightarrow \Sigma_k.$$

We then consider the $(k-1)$-th relative secant variety Z_{k-1}, which is actually a divisor in the smooth variety $B^k(L)$. Our strategy is to pass computation for codimension two situation $\Sigma_{k-1} \subseteq \Sigma_k$ to the codimension one situation $Z_{k-1} \subseteq B^k(L)$. The picture for the first secant variety is rather simple, and Z_0 is just $C \times C$. Thus one can easily transfer cohomological computation from Σ_1 to C_2 through $B^1(L)$. However, for higher secant varieties, such method does not work directly in that Z_{k-1} is singular. Fortunately, after blowup consecutively along the stratification induced by the inclusions $C \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \cdots \subseteq \Sigma_{k-1}$, as exhibited in [1], we then arrive at a birational morphism

$$b_k : \operatorname{bl}_k(B^k(L)) \longrightarrow B^k(L),$$

which we prove to be a log resolution of the log pair $(B^k(L), Z_{k-1})$. Based on this setup, in Theorem 1.1, for instance, to prove the normality of Σ_k at a point x, we adapt the strategy of Ullery in [21] to consider the unique minimal m-secant plane containing x. It cuts the curve along a degree $m+1$ divisor ξ. By the formal function theorem, the normality of the k-th secant variety Σ_k at x follows from the normality and projective normality of the smaller order secant variety Σ_{k-m-1} in the space $\mathbb{P}(H^0(C, L(-2\xi)))$. This leads us to study a general question on the property $N_{k+2,p}$ or higher syzygies of Σ_k.

Turning to the proof of Theorem 1.2, we assume $\deg L \geq 2g+2k+1+p$, and consider the kernel bundle M_{Σ_k} in the exact sequence

$$0 \longrightarrow M_{\Sigma_k} \longrightarrow H^0(C, L) \otimes \mathscr{O}_{\Sigma_k} \longrightarrow \mathscr{O}_{\Sigma_k}(1) \longrightarrow 0,$$

induced by the evaluation map on the global sections of $\mathscr{O}_{\Sigma_k}(1)$. The critical observation we made here is that in order to establish the property $N_{k+2,p}$, one only needs cohomology vanishing involving the wedge product of M_{Σ_k} tensored with $I_{\Sigma_{k-1}|\Sigma_k}(k+1)$. More precisely, it is sufficient to show the following cohomology vanishing

$$H^i(\Sigma_k, \wedge^j M_{\Sigma_k} \otimes I_{\Sigma_{k-1}|\Sigma_k}(k+1)) = 0 \;\text{ for } i \geq j-p, i \geq 1, j \geq 0. \tag{1.1}$$

The next important technical step is to prove the following *Du Bois type conditions*:

$$R^i\beta_*\mathscr{O}_{\Sigma_k}(k+1)(-Z_{k-1}) = \begin{cases} I_{\Sigma_{k-1}|\Sigma_k} & \text{for } \; i=0, \\ 0 & \text{for } \; i>0. \end{cases} \tag{1.2}$$

Then the cohomology groups in (1.1) can be calculated on $B^k(L)$ by involving the sheaf $\beta^*\mathscr{O}_{\Sigma_k}(k+1)(-Z_{k-1})$. We observe that in fact this sheaf is the pullback of a line bundle $A_{k+1,L}$ on the symmetric product C_{k+1} of the curve C. Therefore, once we use the exact sequence

$$0 \longrightarrow M_{k+1,L} \longrightarrow H^0(C, L) \otimes \mathscr{O}_{C_{k+1}} \longrightarrow E_{k+1,L} \longrightarrow 0,$$

induced by the evaluation map on the global sections of $E_{k+1,L}$, we are able to further connect vanishing (1.1) with the following cohomological vanishing

$$H^i(C_{k+1}, \wedge^j M_{k+1,L} \otimes A_{k+1,L}) = 0 \;\text{ for } i \geq j-p,\; i \geq 1,\; j \geq 0, \tag{1.3}$$

on the symmetric product C_{k+1}. As the final ingredient of the proof, inspired by Rathmann's vanishing results in [16], we show the following vanishing

$$H^i\big(C^{k+1}, \wedge^j q^* M_{k+1,L} \otimes (\underbrace{L \boxtimes \cdots \boxtimes L}_{k+1 \text{ times}})(-\Delta)\big) = 0$$
$$\text{for} \quad i \geq j - p,\ i \geq 1,\ j \geq 0, \tag{1.4}$$

on the Cartesian product C^{k+1} of the curve C, where $q\colon C^{k+1} \to C_{k+1}$ is the natural quotient map and Δ is the sum of all pairwise diagonals. Now, (1.4) implies (1.3), and hence, we finally obtain (1.1). The vanishing result (1.4) may have independent interests, and we hope that it will find other applications somewhere in the future.

This paper is organized as follows. We begin in Sect. 2 with recalling basic definitions and properties of singularities and syzygies of algebraic varieties. In Sect. 3, we introduce several vector bundles on symmetric products of curves, review Bertram's blowup constructions for secant bundles, and show some useful results for the main results of the paper. In Sect. 4, one of the main technical ingredients, a vanishing theorem on the Cartesian products of curves, is established. Section 5 is then devoted to the proofs of the main results of the paper. Finally, we discuss some open problems on secant varieties in Sect. 6.

2 Preliminaries

We recall relevant definitions and properties of singularities and syzygies of algebraic varieties.

2.1 Singularities

The Deligne–Du Bois complex $\underline{\Omega}^{\bullet}_X$ for a singular variety X is a generalization of the de Rham complex for a nonsingular variety (see [13, Chapter 6] for detail). There is a natural map

$$\mathscr{O}_X \longrightarrow \underline{\Omega}^0_X = Gr^0_{\text{filt}} \underline{\Omega}^{\bullet}_X.$$

We say that X has *Du Bois* singularities if the above map is a quasi-isomorphism.

Let X be a normal projective variety, and Δ be a boundary divisor on X so that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Take a log resolution $f\colon Y \to X$ of the pair (X, Δ). We may write

$$K_Y = f^*(K_X + \Delta) + \sum_{E\colon \text{prime divisor on } Y} a(E; X, \Delta)E,$$

where $a(E; X, \Delta)$ is the discrepancy of the prime divisor E over X. It is easy to check that the discrepancy is independent of the choice of log resolutions. We say that (X, Δ) is a *klt* (resp. *log canonical*) pair if $a(E; X, \Delta) > -1$ (resp. $a(E; X, \Delta) \geq -1$) for every prime divisor E over X. We say that X has *log terminal* (resp. *log canonical*) singularities if $(X, 0)$ is a klt (resp. log canonical) pair. Note that log terminal singularities are rational singularities and (semi-)log canonical singularities are Du Bois singularities. We refer to [13] for more details of the various notions of singularities and log pairs.

2.2 Syzygies

Let $X \subseteq \mathbb{P}(H^0(X, L)) = \mathbb{P}^r$ be a projective variety embedded by the complete linear system of a very ample line bundle L on X. Let S be the homogeneous coordinate ring of $\mathbb{P}^r$, and

$$R = R(X, L) := \bigoplus_{m \geq 0} H^0(X, mL)$$

be the graded section ring associated to L, viewed as an S-module. Then R has a minimal graded free resolution $E_\bullet(X, L)$:

$$\begin{array}{ccccccccccccc} 0 & \longleftarrow & R & \longleftarrow & \bigoplus S(-a_{0,j}) & \longleftarrow & \bigoplus S(-a_{1,j}) & \longleftarrow & \cdots & \longleftarrow & \bigoplus S(-a_{r,j}) & \longleftarrow & 0. \\ & & & & \| & & \| & & & & \| & & \\ & & & & E_0 & & E_1 & & & & E_r & & \end{array}$$

We define the *Koszul cohomology group*

$$K_{p,q}(X, L) := \mathrm{Tor}_p^S(R, S/S_+)_{p+q},$$

where $S_+ \subseteq S$ denotes the irrelevant maximal ideal. Then we have

$$E_p = \bigoplus_q K_{p,q}(X, L) \otimes_{\Bbbk} S(-p-q).$$

Notice that $X \subseteq \mathbb{P}^r$ is projectively normal if and only if $K_{0,j}(X, L) = 0$ for all $j \geq 1$. The *Castelnuovo–Mumford regularity* of R, denoted by $\mathrm{reg}(R)$, is defined to be the minimal positive integer q such that $K_{p,j}(X, L) = 0$ for all $p \geq 0$ and $j \geq q + 1$. We say that R satisfies the *property* $N_{d,p}$ for some integer $d \geq 2$ if

$$K_{i,j}(X, L) = 0 \quad \text{for} \quad i \leq p \quad \text{and} \quad j \geq d.$$

Assume that $X \subseteq \mathbb{P}^r$ is projectively normal. Then R is the homogeneous coordinate ring of X so that R satisfies the property $N_{d,p}$ if and only if $X \subseteq \mathbb{P}^r$ satisfies the property $N_{d,p}$ in the sense of [7]. In this case, it satisfies the property $N_{d,1}$ if and only if the defining ideal of X in $\mathbb{P}^r$ is generated in degrees $\leq d$. In general, the property $N_{d,p}$ means that up to p stage, the i-th syzygy of the minimal graded free resolution $E_\bullet(X, L)$ is generated in degrees $\leq i - 1 + d$.

Consider now the evaluation map

$$\mathrm{ev}\colon H^0(X, L) \otimes \mathscr{O}_X \longrightarrow L,$$

which is surjective since L is base point free. Denote by M_L the kernel sheaf of the map ev, then one obtains a short exact sequence of vector bundles

$$0 \longrightarrow M_L \longrightarrow H^0(X, L) \otimes \mathscr{O}_X \xrightarrow{\mathrm{ev}} L \longrightarrow 0.$$

We use the following result to compute the Koszul cohomology group.

Proposition 2.1 (cf. [5, Proposition 3.2]) *Assume that $H^i(X, L^m) = 0$ for $i > 0$ and $m > 0$. Then one has*

$$K_{p,q}(X, L) = H^1(X, \wedge^{p+1} M_L \otimes L^{q-1}) \quad for \quad q \geq 2.$$

We conclude this section by reviewing Castelnuovo–Mumford regularity for a projective subscheme $X \subseteq \mathbb{P}^r$. We say that $\mathscr{O}_X$ (resp. $X \subseteq \mathbb{P}^r$) is *m-regular* if $H^i(X, \mathscr{O}_X(m - i)) = 0$ (resp. $H^i(\mathbb{P}^r, I_{X|\mathbb{P}^r}(m - i)) = 0$) for $i > 0$. We say that $X \subseteq \mathbb{P}^r$ is *m-normal* if the natural restriction map $H^0(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(m)) \to H^0(X, \mathscr{O}_X(m))$ is surjective. Note that $X \subseteq \mathbb{P}^r$ is $(m+1)$-regular if and only if $\mathscr{O}_X$ is m-regular and $X \subseteq \mathbb{P}^r$ is m-normal. By Mumford's regularity theorem, if $\mathscr{O}_X$ (resp. $X \subseteq \mathbb{P}^r$) is m-regular, then so is $(m + 1)$-regular. We denote by $\mathrm{reg}(\mathscr{O}_X)$ (resp. $\mathrm{reg}(X)$) the smallest integer m such that $\mathscr{O}_X$ (resp. $X \subseteq \mathbb{P}^r$) is m-regular. Notice that $\mathrm{reg}(\mathscr{O}_X) = \mathrm{reg}(R(X, \mathscr{O}_X(1)))$. We refer to [4,5,10] for more details on syzygies and Koszul cohomology of algebraic varieties.

3 Symmetric products, secant bundles, and secant varieties

In this section, we review relevant facts on symmetric products and basic constructions of secant bundles and secant varieties. We also show some useful results on secant bundles, which play important roles in proving the main results of the paper. The reader may also look Bertram's original paper [1, Sections 1 and 2] for more details.

Throughout the section, we fix a nonsingular projective curve C of genus $g \geq 0$ and a line bundle L on C. For an integer $k \geq 1$, we write the k-th symmetric product of the curve C as C_k and the k-th Cartesian (or ordinary) product of the curve C as C^k. We set $C^0 = C_0 = \emptyset$. Denote by

$$q_k : C^k \longrightarrow C_k$$

the quotient morphism from C^k to C_k. It is a finite flat surjective morphism of degree $k!$. We have the canonical morphism

$$\sigma_{k+1} : C_k \times C \longrightarrow C_{k+1}$$

defined by sending (ξ, x) to $\xi + x$. It is a finite flat surjective morphism of degree $k+1$.

3.1 Lemmas on symmetric products

We begin with defining the secant sheaf on C_{k+1} associated to a line bundle on C.

Definition 3.1 For an integer $k \geq 1$, let $p : C_k \times C \to C$ be the projection to C. For a line bundle L on C, we define the *secant sheaf* on C_{k+1} associated to L to be

$$E_{k+1,L} := \sigma_{k+1,*}(p^* L) = \sigma_{k+1,*}(\mathscr{O}_{C_k} \boxtimes L).$$

Notice that $E_{k+1,L}$ is a locally free sheaf on C_{k+1} of rank $k+1$ and the fiber of $E_{k+1,L}$ over $\xi \in C_{k+1}$ can be identified with $H^0(\xi, L|_\xi)$.

Next, we introduce several line bundles on the symmetric product C_{k+1}, which play a central role in this paper (see also [6,16] for the importance in the gonality conjecture).

Definition 3.2 Let $k \geq 1$ be an integer.

(1) Write $L^{\boxtimes k} := \underbrace{L \boxtimes \cdots \boxtimes L}_{k \text{ times}} = p_1^* L \otimes \cdots \otimes p_k^* L$ on C^k, where $p_i : C^k \to C$ is the projection to the i-th component. The symmetric group $\mathfrak{S}_k$ acts on $L^{\boxtimes k}$ in a natural way: $\mu \in \mathfrak{S}_k$ sends a local section $s_1 \otimes \cdots \otimes s_k$ to $s_{\mu(1)} \otimes \cdots \otimes s_{\mu(k)}$. Then $L^{\boxtimes k}$ is invariant under the action, so descends to a line bundle on C_k, denoted by $T_k(L)$.

(2) Define δ_{k+1} to be a divisor on C_{k+1} such that $\mathscr{O}_{C_{k+1}}(\delta_{k+1}) := \det \big(\sigma_{k+1,*}(\mathscr{O}_{C \times C_k})\big)^*$.

(3) Define $N_{k+1,L} := \det E_{k+1,L}$ on C_{k+1}.

(4) Define $A_{k+1,L} := T_{k+1}(L)(-2\delta_{k+1})$ on C_{k+1}.

When $k = 0$, we use the convention that $T_1(L) = E_{1,L} = L$ and $\delta_1 = 0$.

Remark 3.3 Due to the lack of reference, we list several basic properties of the line bundles defined above. Those are well known to experts, and are not hard to prove. Let $k \geq 1$ be an integer.

(1) $N_{k+1,L} = T_{k+1}(L)(-\delta_{k+1})$.
(2) $H^0(C_{k+1}, T_{k+1}(L)) = S^{k+1}H^0(C, L)$ and $H^0(C_{k+1}, N_{k+1}(L)) = \wedge^{k+1}H^0(C, L)$.
(3) $q_{k+1}^*\mathscr{O}_{C_{k+1}}(\delta_{k+1}) = \mathscr{O}_{C^{k+1}}(\Delta_{k+1})$, where $\Delta_{u,v} := \{(x_1, \ldots, x_k) \in C^{k+1} \mid x_u = x_v\}$ is the pairwise diagonal on C^{k+1} and $\Delta_{k+1} := \sum_{1 \leq u < v \leq k+1} \Delta_{u,v}$. When $k = 1$, we let $\Delta_1 = 0$.
(4) $\sigma_{k+1}^*\mathscr{O}_{C_{k+1}}(\delta_{k+1}) = (\mathscr{O}_{C_k}(\delta_k) \boxtimes \mathscr{O}_C)(D_k)$, where D_k is the divisor on $C_k \times C$ defined to be the image of the morphism $C_{k-1} \times C \to C_k \times C$ sending (ξ, p) to $(\xi + p, p)$.
(5) $q_k^*T_k(L) = p_1^*L \otimes \cdots \otimes p_k^*L = L^{\boxtimes k}$. Since $q_{k,*}\mathscr{O}_{C^k}$ contains $\mathscr{O}_{C_k}$ as a direct summand, $T_k(L)$ is a direct summand of $q_{k,*}L^{\boxtimes k}$.
(6) For any two line bundles L_1 and L_2 on C, one has $T_k(L_1) \otimes T_k(L_2) = T_k(L_1 \otimes L_2)$.
(7) Given a point $p \in C$, the divisor X_p on C_{k+1} is defined to be the image of the morphism $C_k \to C_{k+1}$ sending ξ to $\xi + p$. It is ample, and $\mathscr{O}_{C_{k+1}}(X_p) = T_{k+1}(\mathscr{O}_C(p))$. For any line bundle L on C, we have $T_{k+1}(L)|_{X_p} = T_k(L)$. (See the proof of Lemma 3.4.)
(8) The canonical bundle of C_{k+1} is given by $\omega_{C_{k+1}} = T_{k+1}(\omega_C)(-\delta_{k+1}) = N_{k+1,\omega_C}$.

We now prove some useful lemmas.

Lemma 3.4 *Let $k \geq 1, m \geq 0$ be integers. Fix a degree $m+1$ divisor ξ_{m+1} on C, and consider C_{k-m} as a subscheme of C_{k+1} embedded by sending a divisor ξ to $\xi + \xi_{m+1}$. Then one has*

$$A_{k+1,L}|_{C_{k-m}} = A_{k-m,L(-2\xi_{m+1})}.$$

Proof Fix a point $p \in \xi_{m+1}$ so that we can write $\xi_{m+1} = \xi_m + p$ for some degree m divisor ξ_m on C. Consider the embeddings $C_{k-m} \subseteq C_k \subseteq C_{k+1}$, where $C_k \subseteq C_{k+1}$ is embedded by sending a divisor ξ to $\xi + p$ and $C_{k-m} \subseteq C_k$ is embedded by sending a divisor ξ to $\xi + \xi_m$. Thus, inductively, we only need to show that

$$A_{k+1,L}|_{C_k} = A_{k,L(-2p)}. \tag{3.1}$$

Regard $X_p = C_k$ as a divisor in C_{k+1}. Recall by definition that $A_{k+1,L} = T_{k+1}(L)(-2\delta_{k+1})$. Thus it suffices to prove the following: (1) $T_{k+1}(L)|_{X_p} = T_k(L)$ and (2) $\delta_{k+1}|_{X_p} = \delta_k + T_k(p)$. To see (1), we use the commutative diagram

$$\begin{tikzcd} C^k \arrow[r] \arrow[d, "q_k"'] & C^k \times C \arrow[d, "\sigma_{k+1}"] \\ X_p \arrow[r, hook] & C_{k+1}, \end{tikzcd}$$

where the upper horizontal map is given by sending $(x_1, \ldots, x_k)$ to $(x_1, \ldots, x_k, p)$. We can check that $q_k^*(T_{k+1,L}|_{X_p}) = L^{\boxtimes k}$, which proves (1) as q_k^* is an injection on Picard groups. To see (2), we use the adjunction formula $K_{X_p} = (K_{C_{k+1}} + X_p)|_{X_p}$. Since $K_{C_{k+1}} = T_{k+1}(K_C) - \delta_{k+1}$ and $K_{X_p} = T_k(K_C) - \delta_k$, we deduce that $\delta_{k+1}|_{X_p} = \delta_k + X_p|_{X_p}$. Note that $X_p|_{X_p} = T_{k+1}(p)|_{X_p} = T_k(p)$. Thus (2) is proved. $\square$

Lemma 3.5 *For any integer $k \geq 1$, the line bundle $\mathscr{O}_{C_{k+1}}(-\delta_{k+1})$ is a direct summand of the locally free sheaf $q_{k+1,*}\mathscr{O}_{C^{k+1}}$.*

Proof We prove the lemma by the induction on k. For $k = 1$, it is well known that $q_{2,*}\mathscr{O}_{C^2}$ splits as $\mathscr{O}_{C_2} \oplus \mathscr{O}_{C_2}(-\delta_2)$. Since the quotient map $q_{k+1}\colon C^{k+1} \to C_{k+1}$ factors through $C_k \times C$, one only needs to show that $\mathscr{O}_{C_{k+1}}(-\delta_{k+1})$ is a direct summand of $\sigma_{k+1,*}(\mathscr{O}_{C_k}(-\delta_k) \boxtimes \mathscr{O}_C)$. Observe that $\mathscr{O}_{C_{k+1}}(-\delta_{k+1})$ is a direct summand of $(\sigma_{k+1,*}\mathscr{O}_{C_k \times C})^*(-\delta_{k+1})$. By the relative duality with the relative canonical line bundle $\omega_{C_k \times C/C_{k+1}} = \mathscr{O}_{C_k \times C}(D_k)$, one obtains $(\sigma_{k+1,*}\mathscr{O}_{C_k \times C})^* = \sigma_{k+1,*}\mathscr{O}_{C_k \times C}(D_k)$, so

$$(\sigma_{k+1,*}\mathscr{O}_{C_k \times C})^*(-\delta_{k+1}) = \sigma_{k+1,*}\mathscr{O}_{C_k \times C}(D_k) \otimes \mathscr{O}_{C_{k+1}}(-\delta_{k+1}).$$

Recall that $\sigma_{k+1}^*\mathscr{O}_{C_{k+1}}(-\delta_{k+1}) = (\mathscr{O}_{C_k}(-\delta_k) \boxtimes \mathscr{O}_C)(-D_k)$. By the projection formula, we have

$$\sigma_{k+1,*}\mathscr{O}_{C_k \times C}(D_k) \otimes \mathscr{O}_{C_{k+1}}(-\delta_{k+1}) = \sigma_{k+1,*}(\mathscr{O}_{C_k}(-\delta_k) \boxtimes \mathscr{O}_C),$$

and thus, the lemma is proved. $\square$

Remark 3.6 We give an alternative proof of Lemma 3.5 by group actions, which may be of independent interest. Write the divisor $\delta = \delta_{k+1}$ and the structure sheaf $\mathscr{O} = \mathscr{O}_{C_{k+1}}$. Let $\mathfrak{A}_{k+1}$ be the alternating subgroup of the symmetric group $\mathfrak{S}_{k+1}$, and $f\colon C^{k+1} \to Y$ be the quotient morphism under the natural induced action of $\mathfrak{A}_{k+1}$ on C^{k+1}. There is a natural degree two morphism $g\colon Y \to C_{k+1}$ through which the quotient map $q = q_{k+1}\colon C^{k+1} \to C_{k+1}$ factors, i.e., $q = g \circ f$. Note that Y has quotient singularities, which are rational singularities. Thus Y is Cohen–Macaulay, so the map g is flat and $g_*\mathscr{O}_Y$ splits as $\mathscr{O} \oplus \mathscr{O}(-\delta')$ for some divisor δ' on C_{k+1}. We claim that δ'

is actually linearly equivalent to δ. To see this, notice that f is unramified at codimension one points. Then $q^*\mathscr{O}(-2\delta) \cong q^*\mathscr{O}(-2\delta')$, which means that $\delta - \delta'$ is a 2-torsion divisor. So if the genus of C is zero, then C_{k+1} has no nontrivial torsion line bundle and therefore $\mathscr{O}(\delta - \delta') = \mathscr{O}$. If the genus of C is positive, then since $H^0(\mathscr{O}(\delta)) = 0$ and $g_*(g^*\mathscr{O}(\delta)) = \mathscr{O}(\delta) \oplus \mathscr{O}(\delta - \delta')$, we see that $\mathscr{O}(\delta - \delta') = \mathscr{O}$ if and only if $H^0(g^*\mathscr{O}(\delta)) \neq 0$. But this follows from the fact that the section defining $q^*\delta = \Delta$ is invariant under the group $\mathfrak{A}_{k+1}$, and therefore, it gives a nonzero global section of $g^*\mathscr{O}(\delta)$. Thus the claim is proved. Finally, note that $\mathscr{O}_Y$ is a direct summand of $f_*\mathscr{O}_{C^{k+1}}$. The lemma then follows.

The following seems to be well known to experts, but we include the proof.

Lemma 3.7 *For any integers $k \geq 1$ and $i \geq 0$, one has*

$$H^i(C_{k+1}, T_{k+1}(L)) \cong S^{k+1-i} H^0(C, L) \otimes \wedge^i H^1(C, L).$$

In particular, the following hold:

$$\begin{aligned} &H^0(C_{k+1}, T_{k+1}(\omega_C)) \cong S^{k+1} H^0(C, \omega_C), \\ &H^1(C_{k+1}, T_{k+1}(\omega_C)) \cong S^k H^0(C, \omega_C), \\ &H^i(C_{k+1}, T_{k+1}(\omega_C)) = 0 \quad \textit{for} \quad i \geq 2. \end{aligned}$$

Proof By [14, Proposition 1.1], we have

$$H^i(C_{k+1}, T_{k+1}(L)) = H^i(C^{k+1}, L^{\boxtimes k+1})^{\mathfrak{S}_{k+1}} \quad \text{for any} \quad i \geq 0,$$

where the right-hand-side is the invariant subspace under the action of $\mathfrak{S}_{k+1}$. By Künneth formula, the vector space $V := H^i(C^{k+1}, L^{\boxtimes k+1})$ is a direct sum of the subspace $W := T^{k+1-i} H^0(C, L) \otimes T^i H^1(C, L)$ with some other isomorphic summands, where the notation T^a means the a-times tensor products. Write $\mathfrak{G} = \mathfrak{S}_{k+1-i} \times \mathfrak{S}_i$ as the subgroup of $\mathfrak{S}_{k+1}$ fixing the subspace W. Then one has the following commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\beta} & W^{\mathfrak{G}} \\ \big\downarrow & & \big\downarrow \alpha \\ V & \xrightarrow{\alpha} & V^{\mathfrak{S}_{k+1}}, \end{array}$$

where $\alpha(x) = \frac{1}{(k+1)!} \sum_{g \in \mathfrak{S}_{k+1}} g(x)$ and $\beta(x) = \frac{1}{(k+1-i)! i!} \sum_{g \in \mathfrak{G}} g(x)$. Since every invariant cohomological class must be of the form

$$s + g_1(s) + g_2(s) + \cdots$$

where $s \in W$ and g_i are suitable elements in $\mathfrak{S}_{k+1}$, it follows that the right-hand-side vertical map $\alpha\colon W^{\mathfrak{G}} \to V^{\mathfrak{S}_{k+1}}$ in the above diagram is surjective. Hence $W^{\mathfrak{G}} = V^{\mathfrak{S}_{k+1}}$. But note that the action of the subgroup $\mathfrak{G}$ is symmetric on $T^{k+1-i}H^0(C, L)$ part but alternating on $T^i H^1(C, L)$ part of the space W. Therefore, the invariant subspace $H^i(C^{k+1}, L^{\boxtimes k+1})^{\mathfrak{S}_{k+1}}$ is isomorphic to $S^{k+1-i}H^0(C, L) \otimes \wedge^i H^1(C, L)$. □

The following theorem will be applied to checking the projective normality of higher secant varieties of curves. Danila [3] considers the Hilbert schemes of points on surfaces, but the proof smoothly works for the symmetric products of curves.

Theorem 3.8 (Danila [3]) *For integers $k \geq 1$ and $1 \leq \ell \leq k+1$, one has*

$$H^0\big(C_{k+1}, E_{k+1,L}^{\otimes \ell}\big) \cong H^0(C, L)^{\otimes \ell},$$

where the isomorphism is $\mathfrak{S}_{k+1}$-equivariant. In particular,

$$H^0\big(C_{k+1}, S^\ell E_{k+1,L}\big) \cong S^\ell H^0(C, L).$$

3.2 Secant varieties via secant bundles

We first recall the following definition.

Definition 3.9 We say that a line bundle L on C *separates k points* (or equivalently, *L is $(k-1)$-very ample*) for an integer $k \geq 1$ if the restriction map

$$H^0(C, L) \longrightarrow H^0(\xi, L|_\xi)$$

is surjective for all $\xi \in C_k$.

For instance, L separates 1 point if and only if L is globally generated, and L separates 2 points if and only if L is very ample. By Riemann–Roch theorem, it is elementary to see that if $\deg L \geq 2g + k$, then L separates $k+1$ points. It can be also shown that if B is an effective line bundle and $x_1, \ldots, x_{g+2k+1}$ are general points on C, then $B\big(\sum_{i=1}^{g+2k+1} x_i\big)$ separates $k+1$ points.

Directly from the definition of secant sheaves, one has $H^0(C_{k+1}, E_{k+1,L}) = H^0(C, L)$. Recall that the fiber of $E_{k+1,L}$ over $\xi \in C_{k+1}$ is $H^0(\xi, L|_\xi)$. We then see that if L separates $k+1$ points, then $E_{k+1,L}$ is globally generated. Thus one obtains a short exact sequence of vector bundles

$$0 \longrightarrow M_{k+1,L} \longrightarrow H^0(C, L) \otimes \mathscr{O}_{C_{k+1}} \xrightarrow{\text{ev}} E_{k+1,L} \longrightarrow 0,$$

where $M_{k+1,L}$ is the kernel bundle of the evaluation map $\text{ev}\colon H^0(C, L) \otimes \mathscr{O}_{C_{k+1}} \to E_{k+1,L}$ on the global sections of $E_{k+1,L}$.

Definition 3.10 For an integer $k \geq 0$, define the *secant bundle of k-planes* over C_{k+1} to be

$$B^k(L) := \mathbb{P}(E_{k+1,L})$$

equipped with the natural projection $\pi_k \colon B^k(L) \to C_{k+1}$.

Suppose that L separates $k+1$ points. Then the tautological bundle $\mathscr{O}_{\mathbb{P}(E_{k+1,L})}(1)$ of $B^k(L)$ is also globally generated, and therefore, it induces a morphism

$$\beta_k \colon B^k(L) \longrightarrow \mathbb{P}(H^0(C, L)).$$

Definition 3.11 For $k \geq 0$, assume that a line bundle L on the curve C separates $k+1$ points. The *k-th secant variety* $\Sigma_k = \Sigma_k(C, L)$ of C in $\mathbb{P}(H^0(C, L))$ is the image of the morphism $\beta_k \colon B^k(L) \to \mathbb{P}(H^0(C, L))$. We have a morphism

$$\beta_k \colon B^k(L) \longrightarrow \Sigma_k.$$

We use the convention that $B^{-1}(L) = \Sigma_{-1} = \emptyset$.

Geometrically, if the curve C is embedded by the complete linear system $|L|$ in the projective space $\mathbb{P}(H^0(C, L))$, then the k-th secant variety Σ_k is nothing but the variety swept out by the $(k+1)$-secant k-planes of C. If L separates $k+1$ points, then a $(k+1)$-secant k-plane of C is spanned by a divisor ξ on C of degree $k+1$.

Definition 3.12 Assume that a line bundle L on the curve C separates $2k+2$ points for an integer $k \geq 0$. Let m be an integer with $0 \leq m \leq k$, and $x \in \Sigma_m \setminus \Sigma_{m-1}$ be a point. Since L also separates $2m+2$ points, the morphism $\beta_m \colon B^m(L) \to \Sigma_m$ is an isomorphism over $U^m(L)$. Hence x can be viewed as a point in $B^m(L)$. Then projecting x by $\pi_m \colon B^m(L) \to C_{m+1}$, one gets a divisor $\xi_{m+1,x}$ on C of degree $m+1$. It is uniquely determined by x. We call $\xi_{m+1,x}$ the *degree* $m+1$ *divisor on* C *determined by* x.

The above definition can be interpreted geometrically. The m-plane in $\mathbb{P}(H^0(C, L))$ spanned by $\xi_{m+1,x}$ is the unique $(m+1)$-secant m-plane of C containing x.

Let $x \in \Sigma_k$ be a general point so that $\xi_{k+1,x}$ contains distinct $k+1$ general points of C. The classical Terracini's lemma asserts that the projective tangent space of Σ_k at x in $\mathbb{P}^r$ is spanned by the projective tangent lines of C at the points of $\xi_{k+1,x}$. Hence the conormal space of Σ_k in $\mathbb{P}^r$ at x is isomorphic to $H^0(C, L(-2\xi_{k+1,x}))$. We will prove a more general version of this statement in Proposition 3.13 below.

For $0 \leq m \leq k$, there is a natural morphism

$$\alpha_{k,m}\colon B^m(L) \times C_{k-m} \longrightarrow B^k(L)$$

defined in [1, p. 432, line-5], which we recall here. For any $\xi_{m+1} \in C_{m+1}$ and $\xi_{k-m} \in C_{k-m}$, let $\xi := \xi_{m+1} + \xi_{k-m} \in C_{k+1}$. Note that the $(m+1)$-secant m-plane $\mathbb{P}(H^0(L|_{\xi_{m+1}}))$ spanned by ξ_{m+1} is naturally embedded in the $(k+1)$-secant k-plane $\mathbb{P}(H^0(L|_{\xi}))$ spanned by ξ. Fiberwisely, $\alpha_{k,m}$ maps $\mathbb{P}(H^0(L|_{\xi_{m+1}})) \times \xi_{k-m}$ into $\mathbb{P}(H^0(L|_{\xi}))$. Next, we define the *relative secant variety* Z_m^k *of* m*-planes* in $B^k(L)$ to be the image of the morphism $\alpha_{k,m}\colon B^m(L) \times C_{k-m} \to B^k(L)$. If the number k is clear from the context, then we simply write Z_m instead of Z_m^k. Define

$$U^k(L) := B^k(L) \backslash Z_{k-1}^k,$$

which is the complement of the largest relative secant variety (see [1, p. 434])

The morphism $\alpha_{k,m}$ is compatible with the morphisms β_k and β_m, i.e., one has a commutative diagram

$$\begin{tikzcd}
B^m(L) \times C_{k-m} \arrow[r, "\alpha_{m,k}"] \arrow[d, "\pi_{B^m(L)}"'] & B^k(L) \arrow[d, "\beta_k"] \\
B^m(L) \arrow[r, "\beta_m"'] & \mathbb{P}(H^0(L)),
\end{tikzcd}$$

where $\pi_{B^m(L)}$ is the projection.

It has been showed in [1, Lemma 1.4(a) and Corollary followed] that if L separates $2k+2$ points, the morphism $\beta_k\colon B^k(L) \to \Sigma_k$ is birational. In particular, the restricted morphism

$$\beta_k|_{U^k(L)}\colon U^k(L) \longrightarrow \mathbb{P}(H^0(C, L))$$

is an immersion. Especially, $\Sigma_m \backslash \Sigma_{m-1}$ is isomorphic to $U^m(L)$ for $0 \leq m \leq k$. It is clear that $\beta_k(Z_m) = \Sigma_m$, so one has a commutative diagram

$$\begin{tikzcd}
Z_0 \arrow[r, hook] \arrow[d] & Z_1 \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] & Z_{k-1} \arrow[r, hook] \arrow[d] & B^k(L) \arrow[d, "\beta_k"] & \\
C \arrow[r, hook] & \Sigma_1 \arrow[r, hook] & \cdots \arrow[r, hook] & \Sigma_{k-1} \arrow[r, hook] & \Sigma_k \arrow[r, hook] & \mathbb{P}(H^0(L)).
\end{tikzcd}$$

It is easy to check that set-theoretically $\beta_k^{-1}(\Sigma_m) = Z_m$. The set of secant varieties $\{\Sigma_i\}_{i=0}^{k-1}$ gives a stratification of Σ_k, which in turn induces a stratification by relative secant varieties $\{Z_i\}_{i=0}^{k-1}$ for $B^k(L)$. Therefore, for a point $x \in \Sigma_k$, there exists a unique integer m with $0 \leq m \leq k$ such that $x \in \Sigma_m \backslash \Sigma_{m-1}$.

The following is the main result of this subsection. It plays an important role in proving the normality of higher secant varieties of curves. The crucial point is the computation of the conormal sheaf $N^*_{F_x/B^k(L)}$. The obstruction lies on the fact that Z_m is quite singular. To overcome this difficulty, we work on suitable nonsingular open subset of Z_m.

Proposition 3.13 *Fix an integer $k \geq 1$, and suppose that a line bundle L on the curve C separates $2k+2$ points. Let m be an integer with $0 \leq m \leq k$. Then the following hold true:*

(1) *The commutative diagram*

$$\begin{tikzcd} U^m(L) \times C_{k-m} \arrow[r, "\alpha_{m,k}"] \arrow[d, "\pi_{U^m(L)}"'] & B^k(L) \arrow[d, "\beta_k"] \\ U^m(L) \arrow[r, "\beta_m"'] & \mathbb{P}(H^0(C, L)) = \mathbb{P}^r \end{tikzcd}$$

is a fiber product diagram.

(2) *Let $x \in \Sigma_m \backslash \Sigma_{m-1}$ be a point, $\xi_{m+1,x}$ be the unique degree $m+1$ divisor determined by x, and $F_x := \beta_k^{-1}(x)$ be the fiber over x. Then one has the following:*
   (a) *$F_x \cong C_{k-m}$.*
   (b) *$N^*_{\Sigma_m/\mathbb{P}^r} \otimes \Bbbk(x) \cong H^0(C, L(-2\xi_{m+1,x}))$.*
   (c) *$N^*_{Z_m/B^k(L)}\Big|_{F_x} = E_{k-m, L(-2\xi_{m+1,x})}$.*
   (d) *$N^*_{F_x/B^k(L)} \cong \mathscr{O}_{F_x}^{\oplus 2m+1} \oplus E_{k-m, L(-2\xi_{m+1,x})}$.*
   (e) *The natural morphism*

$$T_x^*\mathbb{P}^r \longrightarrow H^0(F_x, N^*_{F_x/B^k(L)})$$

   is surjective, and is an isomorphism if $m \neq k$.

Proof (1) Let $U := \mathbb{P}(H^0(C, L)) \backslash \Sigma_{m-1}$ which is an open subset of $\mathbb{P}(H^0(C, L))$, and $V := \beta_k^{-1}(U)$. Then we obtain a commutative diagram

$$\begin{tikzcd} U^m(L) \times C_{k-m} \arrow[r, "\alpha_{m,k}"] \arrow[d, "\pi_{U^m(L)}"'] & V \arrow[d, "\beta_k"] \\ U^m(L) \arrow[r, "\beta_m"'] & U \end{tikzcd}$$

in which $\alpha_{m,k}$ and β_m are closed immersions by [1, Lemma 1.2]. Write $Z := \beta_k^{-1}(U^m(L))$. Then we see that $U^m(L) \times C_{k-m} \subseteq Z$. First, we claim that set-theoretically, $U^m(L) \times C_{k-m} = Z$. To see this, let $x \in \Sigma_m \subseteq \Sigma_k$ be a point. Then every $(m+1)$-secant m-plane containing x is spanned by a unique degree $m+1$ divisor ξ_{m+1} on C. By letting ξ_{k-m} run through all points in C_{k-m}, one creates all possible $(k+1)$-secant k-plane containing x spanned by $\xi_{m+1} + \xi_{k-m}$. But such $(m+1)$-secant m-planes are parameterized by $\beta_m^{-1}(x)$. Hence $\beta_k^{-1}(x)$ is the image of $\beta_m^{-1}(x) \times C_{k-m}$ under $\alpha_{m,k}$ as sets. This proves the claim. Next, we shall show that scheme-theoretically, $U^m(L) \times C_{k-m} = Z$. To this end, it is enough to show the natural morphism

$$\beta_k^*(N^*_{U^m(L)/U}) \longrightarrow N^*_{U^m(L)\times C_{k-m}/V}$$

of conormal sheaves is surjective. Take $x \in U^m(L)$. By base change, it is enough to show that

$$\pi^*_{B^m(L)}(N^*_{U^m(L)/U} \otimes \Bbbk(x)) \longrightarrow N^*_{U^m(L)\times C_{k-m}/V}|_{\{x\}\times C_{k-m}} \tag{3.2}$$

is surjective. Following notation in [1, Lemmas 1.3 and 1.4], we have

$$\begin{aligned} N^*_{U^m(L)\times C_{k-m}/V}|_{\{x\}\times C_{k-m}} &= N^*_{\alpha_{k,m}}(\{x\} \times C_{k-m}) \quad \text{and} \\ N^*_{U^m(L)/U} \otimes \Bbbk(x) &= N^*_{\beta_m}(x). \end{aligned}$$

The morphism in (3.2) is the same as

$$\mu_{m,k} \colon \pi^*_{B^m(L)} N^*_{\beta_m}(x) \longrightarrow N^*_{\alpha_{m,k}}(\{x\} \times C_{k-m}) \tag{3.3}$$

Hence by [1, Lemma 1.4(c)], $\mu_{m,k}$ is surjective, which completes the proof.
(2) (a) This follows directly from (1).
(b) We identify $U^m(L) = \Sigma_m \setminus \Sigma_{m-1}$. Recall that if x is a general point of $U^m(L)$ and $\xi_{m+1,x}$ contains distinct $m+1$ general points of C, then the classical Terracini's lemma implies that $N^*_{\Sigma_m/\mathbb{P}^r} \otimes \Bbbk(x) \cong H^0(C, L(-2\xi_{m+1,x}))$.

Next write π_C and $\pi_{C_{m+1}}$ to be the projections from $C_{m+1} \times C$ to the indicated factors. Let $D_{m+1} \subseteq C_{m+1} \times C$ be the universal divisor over C_{m+1}. Consider the sheaf $\mathscr{M} = \pi_{C_{m+1},*}(\pi_C^*(L)(-2D_{m+1}))$ on C_{m+1}. We have

$$\begin{tikzcd} & & \pi_m^*\mathscr{M}|_{U^m(L)} \arrow[d, hook] \arrow[dr, "\eta"] & & \\ 0 \arrow[r] & N^*_{\Sigma_m/\mathbb{P}^r}(1)|_{U^m(L)} \arrow[r] & H^0(C, L) \otimes \mathscr{O}_{U^m(L)} \arrow[r] & P^1(\mathscr{O}_{\Sigma_m}(1))|_{U^m(L)} \arrow[r] & 0, \end{tikzcd}$$

where $P^1(\mathscr{O}_{\Sigma_m}(1))$ is the first principal part bundle. As the map η is generically zero, it is zero. This implies that $\pi_m^*\mathscr{M} \cong N^*_{\Sigma_m/\mathbb{P}^r}(1)|_{U^m(L)}$, and the result follows.
(c) This is included in the proof of [1, Lemma 1.3] implicitly. For reader's convenience, we outline the proof here. For a positive integer i, write

$$D_{i+1} = C \times C_i \subseteq C \times C_{i+1}$$

to be the universal family of divisors of degree $i+1$, embedded via $(x,\xi) \mapsto (x, x+\xi)$. In the space $C \times C_{m+1} \times C_{k-m}$, we define two divisors $\mathscr{D}_{m+1}$ and $\mathscr{D}_{k-m}$ as follows

$$\mathscr{D}_{m+1} :=D_{m+1} \times C_{k-m}, \quad \text{and} \quad \mathscr{D}_{k-m} :=C_{m+1} \times D_{k-m}.$$

They are nonsingular and meet transversally. Let π_C, $\pi_{C_{m+1}}$, $\pi_{C_{k-m}}$ be the projections of $C \times C_{m+1} \times C_{k-m}$ to the indicated factors, and π^C, $\pi^{C_{m-1}}$, $\pi^{C_{k-m}}$ be the projections to the complement of the indicated factors. Then $B^m(L) \times C_{k-m}$ can be realized as a projectivized vector bundle over $C_{m+1} \times C_{k-m}$ with a projection π, i.e.,

$$\pi : B^m(L) \times C_{k-m} = \mathbb{P}\Big(\pi^C_*(\pi_C^* L \otimes \mathscr{O}_{\mathscr{D}_{m+1}})\Big) \longrightarrow C_{m+1} \times C_{k-m}.$$

Let $\mathscr{O}_{B^m(L)\times C_{k-m}}(1)$ be the tautological line bundle on $\mathbb{P}\Big(\pi^C_*(\pi_C^* L \otimes \mathscr{O}_{\mathscr{D}_{m+1}})\Big)$. Consider the vector bundle

$$\mathscr{H} = \pi^C_*(\pi_C^* L \otimes \mathscr{O}_{\mathscr{D}_{k-m}}(-2\mathscr{D}_{m+1})).$$

The key point proved in [1, p. 439] is that

$$N^*_{Z_m/B^k(L)}|_{U^m(L)\times C_{k-m}} \cong \pi^*\mathscr{H} \otimes \mathscr{O}_{B^m(L)\times C_{k-m}}(-1)|_{U^m(L)\times C_{k-m}}.$$

Thus we obtain

$$N^*_{Z_m/B^k(L)}\Big|_{F_x} = \pi^*\mathscr{H} \otimes \mathscr{O}_{B^m(L)\times C_{k-m}}(-1)|_{F_x}$$

as $F_x \subseteq U^m(L) \times C_{k-m}$. Since $\mathscr{O}_{B^m(L)\times C_{k-m}}(-1)|_{F_x} = \mathscr{O}_{F_x}$ and $\pi^*\mathscr{H}|_{F_x} = E_{k-m, L(-2\xi_{m+1,x})}$ by base change, the result follows immediately.
(d) By (1), we see the morphism

$$\beta_k : U^m(L) \times C_{k-m} = Z_m \backslash Z_{m-1} \longrightarrow U^m(L) = \Sigma_m \backslash \Sigma_{m-1}$$

is a smooth morphism with fibers C_{k-m}. Thus we have

$$N^*_{F_x/Z_m} = T^*_x \Sigma_m \otimes \mathscr{O}_{F_x} = \mathscr{O}_{F_x}^{\oplus 2m+1}$$

since Σ_m is nonsingular at x and has dimension $2m+1$. In particular, $H^0(N^*_{F_x/Z_m}) = T^*_x \Sigma_m$. Consider the short exact sequence

$$0 \longrightarrow N^*_{Z_m/B^k(L)}|_{F_x} \longrightarrow N^*_{F_x/B^k(L)} \longrightarrow N^*_{F_x/Z_m} \longrightarrow 0. \tag{3.4}$$

We claim that the above short exact sequence splits. To this end, consider the diagram

$$\begin{tikzcd}
T^*_x\mathbb{P}(H^0(C,L)) \arrow[r, two heads] \arrow[d] & T^*_x\Sigma_m \arrow[d, "="] \\
H^0(F_x, N^*_{F_x/B^k(L)}) \arrow[r] & H^0(F_x, N^*_{F_x/Z_m}).
\end{tikzcd}$$

We see that the morphism $H^0(F_x, N^*_{F_x/B^k(L)}) \to H^0(F_x, N^*_{F_x/Z_m})$ is surjective. Thus the short exact sequence (3.4) splits because $N^*_{F_x/Z_m}$ is a direct sum of $\mathscr{O}_{F_x}$. Hence, we obtain

$$N^*_{F_x/B^k(L)} = N^*_{Z_m/B^k(L)}|_{F_x} \oplus N^*_{F_x/Z_m} = E_{k-m,L(-2\xi_{m+1,x})} \oplus \mathscr{O}_{F_x}^{\oplus 2m+1},$$

as desired.

(e) Now we use (b), (d) and the sequence (3.4) to form the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(C, L(-2\xi_{m+1,x})) \arrow[r] \arrow[d, "="] & T^*_x\mathbb{P}^r \arrow[r] \arrow[d] & T^*_x\Sigma_m \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & H^0(C_{k-m}, E_{k-m,L(-2\xi_{m+1,x})}) \arrow[r] & H^0(F_x, N^*_{F_x/B^k(L)}) \arrow[r] & T^*_x\Sigma_m \arrow[r] & 0.
\end{tikzcd}$$

The result then follows immediately. □

Remark 3.14 In the proposition above, it is worth noting that $Z_m \backslash Z_{m-1} = U^m(L) \times C_{k-m}$ and $U^m(L) = \Sigma_m \backslash \Sigma_{m-1}$. Therefore, we actually obtain a fiber product diagram

$$\begin{tikzcd}
Z_m\backslash Z_{m-1} \arrow[r, hook] \arrow[d] & B^k(L) \arrow[d, "\beta_k"] \\
\Sigma_m\backslash\Sigma_{m-1} \arrow[r, hook] & \mathbb{P}(H^0(C,L))
\end{tikzcd}$$

which means that $Z_m \backslash Z_{m-1}$ is the scheme-theoretical preimage of $\Sigma_m \backslash \Sigma_{m-1}$.

3.3 Blowup construction of secant bundles

We keep assuming that $k \geq 1$ and $\deg L \geq 2g+2k+1$. We use the blowup construction of secant bundles established in [1, Propostitions 2.2, 2.3 and Corollary 2.4]. For each $0 \leq m \leq k$, we will consecutively blowup $B^m(L)$ along smooth centers m-times to obtain smooth varieties

$$\mathrm{bl}_1(B^m(L)),\ \mathrm{bl}_2(B^m(L)),\ \ldots,\ \mathrm{bl}_m(B^m(L)).$$

If $m = 0$, then there is nothing to blowup. We simply set $\mathrm{bl}_0(B^0(L)) := B^0(L) = C$. Thus we now start with constructing $\mathrm{bl}_1(B^m(L))$ for $m \geq 1$. Notice that the natural morphism $\alpha_{m,0}\colon B^0(L) \times C_m \to B^m(L)$ is a closed embedding for $m \geq 1$. We then define

$$\mathrm{bl}_1(B^m(L)) := \text{blowup of } B^m(L) \text{ along } B^0(C) \times C_m.$$

If $m = 1$, then we are done. Otherwise, if $m \geq 2$, then suppose that $\mathrm{bl}_i(B^m(L))$ has been defined for any $1 \leq i \leq m-1$. By [1, Proposition 2.2] and its proof (for instance, the claim in the last two lines on page 444 of [1]), we see that the natural morphism $\mathrm{bl}_i(B^i(L)) \times C_{m-i} \to \mathrm{bl}_i(B^m(L))$ is a closed embedding. We then define

$$\mathrm{bl}_{i+1}(B^m(L)) := \text{blowup of } \mathrm{bl}_i(B^m(L)) \text{ along } \mathrm{bl}_i(B^i(C)) \times C_{m-i}.$$

This construction works for any integer m with $0 \leq m \leq k$. We write

$$b_m\colon \mathrm{bl}_m(B^m(L)) \longrightarrow B^m(L)$$

the composition map of blowups. Denote by E_i for $0 \leq i \leq m-1$ the exceptional divisor on $\mathrm{bl}_m(B^m(l))$ which is from the $(i+1)$-th blowup. Note that $\beta_m(b_m(E_i)) = \Sigma_i$. It has been showed in [1] that in each stage of blowups, the exceptional divisors always meet transversally with the center of the next blowup. Therefore, the divisor $E_0 + \cdots + E_{m-1}$ on $\mathrm{bl}_m(B^m(L))$ has a simple normal crossing support. As proved in [1], we have

$$E_i \cap E_{i+1} \cap \cdots \cap E_{m-1} = \mathrm{bl}_i(B^i(L)) \times C^{m-i} \quad \text{for} \quad 0 \leq i \leq m-1.$$

For example, $E_{m-1} = \mathrm{bl}_{m-1}(B^{m-1}(L)) \times C$ and $E_0 \cap \cdots \cap E_{m-1} = \mathrm{bl}_0(B^0(L)) \times C^m = C^{m+1}$. In particular, for $m = k$ we get the following diagram describing blowups of $B^k(L)$:

$$
\begin{tikzcd}
 & & & & & \mathrm{bl}_k(B^k(L)) \arrow[d, "\cong"] \\
 & & & & \mathrm{bl}_{k-1}(B^{k-1}(L)) \times C \arrow[r, hook] \arrow[d] & \mathrm{bl}_{k-1}(B^k(L)) \arrow[d] \\
 & & & & \vdots \arrow[d] & \vdots \arrow[d] \\
 & & \mathrm{bl}_2(B^2(L)) \times C_{k-2} \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] & \mathrm{bl}_2(Z_{k-1}) \arrow[r, hook] \arrow[d] & \mathrm{bl}_2(B^k(L)) \arrow[d] \\
 & \mathrm{bl}_1(B^1(L)) \times C_{k-1} \arrow[r, hook] \arrow[d] & \mathrm{bl}_1(Z_2) \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] & \mathrm{bl}_1(Z_{k-1}) \arrow[r, hook] \arrow[d] & \mathrm{bl}_1(B^k(L)) \arrow[d] \\
B^0(L) \times C_k \arrow[r, hook] \arrow[d] & Z_1 \arrow[r, hook] \arrow[d] & Z_2 \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] & Z_{k-1} \arrow[r, hook] \arrow[d] & B^k(L) \arrow[d, "\beta_k"] \\
C & \Sigma_1 & \Sigma_2 & \cdots & \Sigma_{k-1} & \Sigma_k .
\end{tikzcd}
$$

where $\mathrm{bl}_i(Z_l)$ is the strict transform of the variety Z_l in $\mathrm{bl}_i(B^k(L))$. The variety on the left end of each row in the diagram is the center of the blowup for the next step. If we focus on the final step of blowups of $B^k(L)$, we obtain the following digram

$$
\begin{tikzcd}
E_0 \cap \cdots \cap E_{k-1} \arrow[d, equal] & E_1 \cap \cdots \cap E_{k-1} \arrow[d, equal] & E_2 \cap \cdots \cap E_{k-1} \arrow[d, equal] & \cdots & E_{k-1} \arrow[d, equal] & \\
\mathrm{bl}_0(B^0(L)) \times C^k \arrow[r, hook] \arrow[d] & \mathrm{bl}_1(B^1(L)) \times C^{k-1} \arrow[r, hook] \arrow[d] & \mathrm{bl}_2(B^2(L)) \times C^{k-2} \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] & \mathrm{bl}_{k-1}(B^{k-1}(L)) \times C \arrow[r, hook] \arrow[d] & \mathrm{bl}_k(B^k(L)) \arrow[d, "b_k"] \\
B^0(L) \times C_k \arrow[r, hook] \arrow[d] & Z_1 \arrow[r, hook] \arrow[d] & Z_2 \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] & Z_{k-1} \arrow[r, hook] \arrow[d] & B^k(L) \arrow[d, "\beta_k"] \\
C & \Sigma_1 & \Sigma_2 & \cdots & \Sigma_{k-1} & \Sigma_k .
\end{tikzcd}
$$

The following is the main result of this subsection. It plays a crucial role in the proofs of the main theorems of the paper.

Proposition 3.15 *Fix an integer $k \geq 1$, and let L be a line bundle on the curve C with* $\deg L \geq 2g + 2k + 1$. *Recall that $\pi_k \colon B^k(L) \to C_{k+1}$ is the canonical projection. Then the following hold true:*

(1) *Z_{k-1} is flat over C_{k+1}.*
(2) *Let H be the tautological divisor on $B^k(L) = \mathbb{P}(E_{k+1,L})$ so that $\mathscr{O}_{B^k(L)}(H) := \beta_k^* \mathscr{O}_{\Sigma_k}(1)$. Then one has*

$$
\begin{aligned}
&\mathscr{O}_{B^k(L)}((k+1)H - Z_{k-1}) = \pi_k^* A_{k+1,L}, \\
&R^i \pi_{k,*} \mathscr{O}_{B^k(L)}(\ell H - Z_{k-1}) = \begin{cases} 0 & \text{for } \; i \geq 0, \; 0 < \ell \leq k \\ 0 & \text{for } \; i > 0, \; \ell \geq k+1. \end{cases}
\end{aligned}
$$

(3) $b_k\colon \mathrm{bl}_k(B^k(L)) \to B^k(L)$ *is a log resolution of the pair* $(B^k(L), Z_{k-1})$ *such that*

$$
\begin{aligned}
&K_{\mathrm{bl}_k(B^k(L))} = b_k^*(K_{B^k(L)} + Z_{k-1}) - E_0 - E_1 - \cdots - E_{k-1},\\
&b_k^* Z_{k-1} = kE_0 + (k-1)E_1 + \cdots + E_{k-1}.
\end{aligned}
$$

Proof We keep using the blowup construction of secant varieties.

(1) Recall that Z_{k-1} is the image of the map $\alpha_{k-1,k}\colon B^{k-1}(L) \times C \to B^k(L)$ and $\alpha_{k-1,k}$ is birational to Z_{k-1} since L separates $2k+2$ points (see [1, Lemma 1.2]). Hence Z_{k-1} is an irreducible divisor in $B^k(L)$, and therefore, is Cohen–Macaulay. Now for any point $\xi \in C_{k+1}$, the fiber of the map $Z_{k-1} \to C_{k+1}$ over ξ, at least set-theoretically, is the union of the linear spaces spanned by the length k subschemes of ξ. Hence the fiber over ξ has dimension $k-1$. By [15, 23.1], we see that Z_{k-1} is flat over C_{k+1}.

(2) Take a general point $\xi \in C_{k+1}$. Without loss of generality, we may assume that $\xi = x_1 + \cdots + x_{k+1}$ is a sum of distinct $k+1$ points on C. Write $F_\xi := \pi_k^{-1}(\xi)$ the fiber over ξ. Note that $F_\xi = \mathbb{P}^k$, which can be regarded as a linear subspace of $\mathbb{P}(H^0(C, L))$ spanned by $x_1, \ldots, x_{k+1}$. In other words, F_ξ is the k-plane secant to C along $x_1, \ldots, x_{k+1}$. Write $\widetilde{F_\xi}$ the strict transform of F_ξ under the birational morphism b_k. Write $\Lambda_i = F_\xi \cap Z_i$ for $0 \le i \le k-1$.

We note that

$$
\begin{aligned}
\Lambda_0 &= F_\xi \cap Z_0 = F_\xi \cap B^0(L) \times C_k = \{x_1, x_2, \ldots, x_{k+1}\},\\
\Lambda_1 &= F_\xi \cap Z_1 = \bigcup_{i \neq j} \overline{x_i x_j},\\
&\vdots\\
\Lambda_{k-1} &= F_\xi \cap Z_{k-1} = \bigcup_{i_1 \neq i_2 \neq \cdots \neq i_k} \overline{x_{i_1} x_{i_2} \cdots x_{i_k}}.
\end{aligned}
$$

To obtain $\widetilde{F_\xi}$, we blowup F_ξ along Λ_0 and then blowup along the strict transform of Λ_1, and so on. Now, the number of irreducible components of Λ_{k-1} containing $\overline{x_{i_1} \cdots x_{i_m}}$ is $\binom{k+1-m}{k-m}$ for all $1 \le m \le k$. This allows us to calculate the total transform of Λ_{k-1} in $\widetilde{F_\xi}$, which in turn implies that

$$
\begin{aligned}
b_k^* Z_{k-1} &= \binom{k}{k-1} E_0 + \binom{k-1}{k-2} E_1 + \cdots + \binom{1}{0} E_{k-1}\\
&= kE_0 + (k-1)E_1 + \cdots + E_{k-1} \qquad (3.5)
\end{aligned}
$$

because $\widetilde{F}_\xi$ meets all the divisors $E_0, \dots, E_{k-1}$ transversally and $\widetilde{F}_\xi \cap E_{m-1}$ is the union of strict transforms of the exceptional divisors over Λ_{m-1} for all $1 \le m \le k$.

For a coherent sheaf $\mathscr{F}$ (resp. a subscheme Z) on $B^k(L)$ and for a point $\xi' \in C_{k+1}$, we denote by $\mathscr{F}_{\xi'}$ (resp. $Z_{\xi'}$) the fiber over ξ'. In this notation, $Z_{k-1,\xi} = \Lambda_{k-1}$ is a union of $k+1$ distinct linear spaces $\mathbb{P}^{k-1}$ in $B^k(L)_\xi = \mathbb{P}^k$. Therefore $Z_{k-1,\xi}$ is a degree $k+1$ divisor in $B^k(L)_\xi$. By the result (1), Z_{k-1} is flat over C_{k+1}, so the degree of $Z_{k-1,\xi'}$ in $B^k(L)_{\xi'}$ is $k+1$ for all $\xi' \in C_{k+1}$. This implies that

$$\mathscr{O}_{B^k(L)}(\ell H - Z_{k-1})_{\xi'} \cong \mathscr{O}_{\mathbb{P}^k}(\ell - (k+1)) \quad \text{for all} \quad \ell \in \mathbb{Z}.$$

Hence the function $h^0(\mathscr{O}_{B^k(L)}((k+1)H - Z_{k-1})_{\xi'}) = 1$ for all $\xi' \in C_{k+1}$. Thus

$$A := \pi_{k,*}\mathscr{O}_{B^k(L)}((k+1)H - Z_{k-1})$$

is a line bundle on C_{k+1}. Since $\pi_k \colon \mathbb{P}(E_{k+1,L}) \to C_{k+1}$ is the natural projection, we have

$$\pi_k^* A \cong \mathscr{O}_{B^k(L)}((k+1)H - Z_{k-1}).$$

Similarly, if $0 < \ell \le k$, then $h^i(\mathscr{O}_{B^k(L)}(\ell H - Z_{k-1})_{\xi'}) = 0$ for all $i \ge 0$, and if $\ell \ge k+1$, then $h^i(\mathscr{O}_{B^k(L)}(\ell H - Z_{k-1})_{\xi'}) = 0$ for all $i > 0$. Thus we obtain the second result in (2).

Next, we show that $A = A_{k+1,L}$. We focus on the following commutative diagram

$$\begin{tikzcd}
C^{k+1} \arrow[r, hook] \arrow[rrrdd, "q:=q_{k+1}"'] & \mathrm{bl}_1(B^1(L)) \times C^2 \arrow[r, hook] & \mathrm{bl}_2(B^2(L)) \times C \arrow[r, hook] & \mathrm{bl}_k(B^k(L)) \arrow[d, "b_k"] \\
& & & B^k(L) \arrow[d, "\pi_k"] \\
& & & C_{k+1}.
\end{tikzcd}$$

We have

$$\begin{aligned}
& b_k^*(\pi_k^* A)|_{C^{k+1}} = q^* A, \\
& b_k^*((k+1)H - Z_{k-1})|_{C^{k+1}} \\
& \quad = (k+1)H - (kE_0 + (k-1)E_1 + \dots + E_{k-1})|_{C^{k+1}},
\end{aligned}$$

where by abuse of notation we write $H = b_k^* H|_{C^{k+1}}$. Hence, on C^{k+1}, we have

$$(k+1)H - (kE_0 + (k-1)E_1 + \cdots + E_{k-1})|_{C^{k+1}} \sim_{\text{lin}} q^* A.$$

Recall that C^{k+1} is a complete intersection in $\text{bl}_k(B^k(L))$ cut out by the divisors $E_0, E_1, \ldots, E_{k-1}$. Thus we have

$$\det N^*_{C^{k+1}/\,\text{bl}_k(B^k(L))} = \mathscr{O}_{C^{k+1}}(-E_0 - E_1 - \cdots - E_{k-1}).$$

Using the formula $\det N^*_{C^{k+1}/\,\text{bl}_k(B^k(L))} = \omega_{\text{bl}_k(B^k(L))}|_{C^{k+1}} \otimes \omega^{-1}_{C^{k+1}}$, we get

$$-(E_0 + E_1 + \cdots + E_{k-1})|_{C^{k+1}} = K_{\text{bl}_k(B^k(L))}|_{C^{k+1}} - K_{C^{k+1}}. \tag{3.6}$$

Recall that $\text{bl}_k(B^k(L))$ is obtained by consecutively blowing up the smooth centers $\text{bl}_i(B^i(L)) \times C_{k-i}$ which has codimension $k-i$. Thus we find

$$-((k-1)\cdot E_0 + \cdots + 1\cdot E_{k-2} + 0\cdot E_{k-1}) = -K_{\text{bl}_k(B^k(L))} + b_k^* K_{B^k(L)}. \tag{3.7}$$

Combining (3.6) and (3.7), we obtain

$$-(kE_0 + (k-1)E_1 + \cdots + E_{k-1})|_{C^{k+1}} = -K_{C^{k+1}} + b_k^* K_{B^k(L)}|_{C^{k+1}}.$$

Recall that $B^k(L) = \mathbb{P}(E_{k+1,L})$ is a projectivized vector bundle over C_{k+1}. Thus we have

$$\begin{aligned} K_{B^k(L)} &= -(k+1)H + \pi_k^* \det E_{k+1,L} + \pi_k^* K_{C_{k+1}} \\ &= -(k+1)H + \pi_k^* T_{k+1}(L)(-\delta_{k+1}) + \pi_k^* T_{k+1}(K_C)(-\delta_{k+1}). \end{aligned}$$

Finally, we compute

$$\begin{aligned} &(k+1)H - (kE_0 + (k-1)E_1 + \cdots + E_{k-1})|_{C^{k+1}} \\ &= (k+1)H - K_{C^{k+1}} + \pi_k^* K_{B^k(L)}|_{C^{k+1}} \\ &= (k+1)H - K_{C^{k+1}} + [-(k+1)H + q^* T_{k+1}(L)(-\delta_{k+1}) \\ &\qquad + q^* T_{k+1}(K_C)(-\delta_{k+1})] \\ &= q^*(T_{k+1}(L)(-2\delta_{k+1})). \end{aligned}$$

Thus $q^* A \cong q^*(T_{k+1}(L)(-2\delta_{k+1}))$. Since $q^* \colon \text{Pic}\, C_{k+1} \to \text{Pic}\, C^{k+1}$ is injective, one gets $A \cong T_{k+1}(L)(-2\delta_{k+1}) = A_{k+1,L}$. This proves the first result of (2).

(3) Recall that $E_0 + \cdots + E_k$ has a simple normal crossing support. Thus the birational morphism $b_k\colon \operatorname{bl}_k(B^k(L)) \to B^k(L)$ is a log resolution of the pair $(B^k(L), Z_{k-1})$. The remaining assertions follow from (3.5) and (3.7). $\square$

4 A vanishing theorem on Cartesian products of curves

The aim of this section is to establish a vanishing theorem on the product of a curve. It is inspired by Rathmann's vanishing results in [16, Section 3]. A similar result on C^2 has been proved by Yang [27].

Let us keep the notations introduced in previous sections. Let $k \geq 0$ be an integer. Recall that given a line bundle L on the curve C separating $k+1$ points, there is a short exact sequence

$$0 \longrightarrow M_{k+1,L} \longrightarrow H^0(C, L) \otimes \mathscr{O}_{C_{k+1}} \longrightarrow E_{k+1,L} \longrightarrow 0$$

on C_{k+1} (see Sect. 3.2). Recall also the quotient morphism $q_{k+1}\colon C^{k+1} \to C_{k+1}$, the pairwise diagonal $\Delta_{u,v} :=\{(x_1, \ldots, x_k) \in C^{k+1} \mid x_u = x_v\}$ on C^{k+1}, and $\Delta_{k+1} := \sum_{1\leq u<v\leq k+1} \Delta_{u,v}$. We define the locally free sheaf

$$Q_{k+1,L} :=q_{k+1}^* M_{k+1,L}.$$

on the Cartesian product C^{k+1} of the curve C. Note that

$$Q_{k+1,L} = p_* \left((\mathscr{O}_{C^{k+1}} \boxtimes L) \left(-\sum_{u=1}^{k+1} \Delta_{u,k+2} \right) \right),$$

where $p\colon C^{k+2} \to C^{k+1}$ is the projection to the first $k+1$ components.

Theorem 4.1 *Let C be a nonsingular projective curve of genus g, and L be a line bundle on C. For an integer $k \geq 0$, let $B = B'\big(\sum_{i=1}^{g+2k+1} x_i\big)$ be a line bundle on C, where B' is an effective line bundle and $x_1, \ldots, x_{g+2k+1}$ are general points on C. For integers $i > 0$ and $j \geq 0$, suppose that*

$$\deg L \geq 2g + 2k + 1 - i + j.$$

Then one has

$$H^i\big(C^{k+1}, \wedge^j Q_{k+1,B} \otimes L^{\boxtimes k+1}\big(-\Delta_{k+1}\big)\big) = 0. \tag{4.1}$$

Proof Suppose that $B' \neq \mathscr{O}_C$ so that $b :=\deg B' > 0$. We can write $B' = \mathscr{O}_C\big(\sum_{i=1}^{b} x_i'\big)$, where $x_1', \ldots, x_b'$ are (possibly non-distinct) points on C. We set $B_0 :=\mathscr{O}_C\big(\sum_{i=1}^{g+2k+1} x_i\big)$ and $B_\ell :=B_0\big(\sum_{i=1}^{\ell} x_i'\big)$ for $1 \leq \ell \leq b$. Then B_ℓ

separates $k+1$ points for each $0 \leq \ell \leq b$, and $B_b = B$. For $0 \leq \ell \leq b-1$, we have an exact sequence

$$0 \longrightarrow Q_{k+1,B_\ell} \longrightarrow Q_{k+1,B_{\ell+1}} \longrightarrow \mathscr{O}_C(-x'_{\ell+1})^{\boxtimes k+1} \longrightarrow 0,$$

which induces an exact sequence

$$\begin{aligned} 0 \longrightarrow \wedge^j Q_{k+1,B_\ell} &\longrightarrow \wedge^j Q_{k+1,B_{\ell+1}} \\ &\longrightarrow \wedge^{j-1} Q_{k+1,B_\ell} \otimes \mathscr{O}_C(-x'_{\ell+1})^{\boxtimes k+1} \longrightarrow 0. \end{aligned}$$

Then we see that the cohomology vanishing

$$H^i\big(C^{k+1}, \wedge^j Q_{k+1,B_{\ell+1}} \otimes L^{\boxtimes k+1}(-\Delta_{k+1})\big) = 0$$

follows from the cohomology vanishing

$$\begin{aligned} &H^i\big(C^{k+1}, \wedge^j Q_{k+1,B_\ell} \otimes L^{\boxtimes k+1}(-\Delta_{k+1})\big) = 0, \\ &H^i\big(C^{k+1}, \wedge^{j-1} Q_{k+1,B_\ell} \otimes L(-x'_{\ell+1})^{\boxtimes k+1}(-\Delta_{k+1})\big) = 0. \end{aligned}$$

Note that $\deg L \geq 2g+2k+1-i+j$ and $\deg L(-x'_{\ell+1}) \geq 2g+2k+1-i+(j-1)$. For each k, by the induction on ℓ, we can conclude that the cohomology vanishing (4.1) for $B = B_0$ (or equivalently, $B' = \mathscr{O}_C$) implies the cohomology vanishing (4.1) for arbitrary B.

We now proceed by the induction on k. First, we consider the case that $k=0$ and $B' = \mathscr{O}_C$. Since $B = \mathscr{O}_C\big(\sum_{i=1}^{g+1} x_i\big)$ is base point free, we have an exact sequence

$$0 \longrightarrow Q_{1,B} \longrightarrow H^0(C,B) \otimes \mathscr{O}_C \longrightarrow B \longrightarrow 0.$$

By Riemann–Roch theorem, we find $h^0(C,B) = 2$, so $Q_{1,B} = B^{-1}$ is a line bundle. In this case, the required cohomology vanishing (4.1) for $B = B_0$ is nothing but

$$\begin{aligned} &H^1(C,L) = 0 \quad \text{when} \quad i=1,\ j=0,\ \deg L \geq 2g, \\ &H^1(C, L \otimes B^{-1}) = 0 \quad \text{when} \quad i=1,\ j=1,\ \deg L \geq 2g+1. \end{aligned}$$

The first vanishing is trivial, and the second vanishing follows from that $\deg L \otimes B^{-1} \geq g$. Thus the cohomology vanishing (4.1) holds for $B = B_0$, and so does for arbitrary B when $k=0$.

Suppose now that $k > 0$. By the induction on k, for smaller k, we assume that the cohomology vanishing (4.1) holds for arbitrary B. We consider the case that $B = B_0 = \mathscr{O}_C\big(\sum_{i=1}^{g+2k+1} x_i\big)$.

Assume that $j = \operatorname{rank}(Q_{k+1,B}) = k+1$. Note that $\det Q_{k+1,B} = (B^{-1})^{\boxtimes k+1}(\Delta_{k+1})$. Then the desired cohomology vanishing (4.1) is nothing but

$$H^i(C^{k+1}, (L \otimes B^{-1})^{\boxtimes k+1}) = 0 \quad \text{for} \quad i > 0.$$

Since $\deg L \geq 2g + 2k + 1 - i + (k+1)$, we have

$$\deg L \otimes B^{-1} \geq 2g + 3k + 2 - i - (g + 2k + 1) = g + k + 1 - i \geq g.$$

Thus $H^1(C, L \otimes B^{-1}) = 0$. By Künneth formula, the above vanishing holds.

Assume that $j < \operatorname{rank}(Q_{k+1,B})$. From the definition of $Q_{k,L}$ one can deduce a short exact sequence

$$0 \longrightarrow Q_{k+1,B} \longrightarrow Q_{k,B} \boxtimes \mathscr{O}_C \longrightarrow (\mathscr{O}_{C^k} \boxtimes B)\left(-\sum_{u=1}^{k} \Delta_{u,k+1}\right) \longrightarrow 0.$$

The Koszul complex then gives rise to a resolution of $\wedge^j Q_{k+1,B}$:

$$\begin{aligned} \cdots \to (\wedge^{j+2} Q_{k,B} \boxtimes B^{-2})\left(2\sum_{u=1}^{k} \Delta_{u,k+1}\right) \\ \to (\wedge^{j+1} Q_{k,B} \boxtimes B^{-1})\left(\sum_{u=1}^{k} \Delta_{u,k+1}\right) \to \wedge^j Q_{k+1,B} \to 0 \end{aligned}$$

(see also [16, Proposition 3.1]). Thus to show the required cohomology vanishing (4.1), it suffices to check that

$$\begin{aligned} H^{i+\ell}\Big(C^{k+1}, \big((\wedge^{j+\ell+1} Q_{k,B} \otimes L^{\boxtimes k}) \boxtimes (L \otimes B^{-\ell-1})\big) \\ \left((\ell+1)\sum_{u=1}^{k} \Delta_{u,k+1} - \Delta_{k+1}\right)\Big) = 0 \end{aligned} \tag{4.2}$$

for $\ell \geq 0$. In the sequel, we establish (4.2) under the assumption $\deg L \geq 2g + 2k + 1 - i + j$ and $B = B_0 = \mathscr{O}_C\big(\sum_{i=1}^{g+2k+1} x_i\big)$.

Consider the case that $i + \ell \leq 1$, i.e., $i = 1$, $\ell = 0$. In this case, we have

$$\deg L \otimes B^{-1} \geq 2g + 2k + 1 - 1 + j - (g + 2k + 1) = g - 1 + j \geq g - 1$$

so that $H^1(C, L \otimes B^{-1}) = 0$. Note that

$$\sum_{u=1}^{k} \Delta_{u,k+1} - \Delta_{k+1} = - \sum_{1 \le u < v \le k} \Delta_{u,v} = -\Delta_k.$$

Since we have

$$\deg L \ge 2g + 2k + j \ge 2g + 2k - 1 + j = 2g + 2(k-1) + 1 - 1 + (j+1),$$

it follows from the induction on k that

$$H^1\big(C^k, \wedge^{j+1} Q_{k,B} \otimes L^{\boxtimes k}(-\Delta_k)\big) = 0.$$

By Künneth formula, we obtain the desired vanishing (4.2)

$$H^1\big(C^{k+1}, \big(\wedge^{j+1}\, Q_{k,B} \otimes L^{\boxtimes k}(-\Delta_k)\big) \boxtimes (L \otimes B^{-1})\big) = 0.$$

Consider the case that $i + \ell \ge 2$. Let $\mathrm{pr}_{k+1} \colon C^{k+1} \to C$ be the projection to the $(k+1)$-th component. The fiber of

$$R^{i'} \mathrm{pr}_{k+1,*} \Big(\big((\wedge^{j+\ell+1} Q_{k,B} \otimes L^{\boxtimes k}) \boxtimes (L \otimes B^{-\ell-1}) \big) \left((\ell+1) \sum_{u=1}^{k} \Delta_{u,k+1} - \Delta_{k+1} \right) \Big)$$

over $x \in C$ is

$$H^{i'}\big(C^k, \wedge^{j+\ell+1} Q_{k,B} \otimes L(\ell x)^{\boxtimes k}(-\Delta_k)\big). \tag{4.3}$$

By considering the Leray spectral sequence for $\mathrm{pr}_{k+1,*}$, to show the desired vanishing (4.2)

$$H^{i+\ell} \Big(C^{k+1}, \big((\wedge^{j+\ell+1} Q_{k,B} \otimes L^{\boxtimes k}) \boxtimes (L \otimes B^{-\ell-1}) \big) \left((\ell+1) \sum_{u=1}^{k} \Delta_{u,k+1} - \Delta_{k+1} \right) \Big) = 0,$$

it is enough to prove that the cohomology (4.3) vanishes for $i' = i+\ell-1, i+\ell$. For this i', we have $i' \ge i - 1$, so we find

$$\deg L(\ell x) \ge 2g + 2k + 1 - i + j + \ell \ge 2g + 2(k-1) + 1 - i' + (j + \ell + 1).$$

By the induction on k, we see that the cohomology (4.3) vanishes for $i' = i+\ell-1, i+\ell$. Thus we obtain the desired vanishing (4.2). Therefore, the cohomology vanishing (4.1) for $B = B_0$ follows, and so does for arbitrary B. We complete the proof. □

5 Properties of secant varieties of curves

This section is devoted to the study of various properties of secant varieties of curves. In particular, we prove the main results of the paper; Theorem 1.1 follows from Theorem 5.2 and Proposition 5.4, and Theorem 1.2 follows from Theorems 5.2, 5.8, and Corollary 5.10.

We keep using notations introduced before. Recall that C is a nonsingular projective curve of genus g embedded by a very ample line bundle L in the space $\mathbb{P}(H^0(C, L)) = \mathbb{P}^r$. Consider the k-th secant variety $\Sigma_k = \Sigma_k(C, L)$ in $\mathbb{P}^r$. As $\mathscr{O}_{\Sigma_k}(1)$ is globally generated by the linear forms of $\mathbb{P}^r$, the evaluation map on the global sections of $\mathscr{O}_{\Sigma_k}(1)$ induces an short exact sequence

$$0 \longrightarrow M_{\Sigma_k} \longrightarrow H^0(C, L) \otimes \mathscr{O}_{\Sigma_k} \longrightarrow \mathscr{O}_{\Sigma_k}(1) \longrightarrow 0, \tag{5.1}$$

where M_{Σ_k} is the kernel bundle. Moreover, we also need to consider the $(k-1)$-th secant variety $\Sigma_{k-1} = \Sigma_{k-1}(C, L)$, and use the following exact sequence

$$0 \longrightarrow I_{\Sigma_{k-1}|\Sigma_k} \longrightarrow \mathscr{O}_{\Sigma_k} \longrightarrow \mathscr{O}_{\Sigma_{k-1}} \longrightarrow 0, \tag{5.2}$$

where $I_{\Sigma_{k-1}|\Sigma_k}$ is the defining ideal sheaf of Σ_{k-1} in Σ_k. Recall the birational morphism $\beta_k \colon B^k(L) \to \Sigma_k$ and the relative secant variety Z_{k-1} on $B^k(L)$. Suppose that Σ_k is normal. By Zariski's main theorem, $\beta_{k,*}\mathscr{O}_{B^k(L)} = \mathscr{O}_{\Sigma_k}$, and hence,

$$\beta_{k,*}\mathscr{O}_{B^k(L)}(-Z_{k-1}) = I_{\Sigma_{k-1}|\Sigma_k}.$$

The following lemma is a consequence of the vanishing theorem established in Sect. 4.

Lemma 5.1 *Let $k \geq 0$ and $p \geq 0$ be integers, and L be a line bundle on C. Assume that*

$$\deg L \geq 2g + 2k + 1 + p.$$

Consider the k-th secant variety $\Sigma_k = \Sigma_k(C, L)$ in the space $\mathbb{P}(H^0(C, L)) = \mathbb{P}^r$. If Σ_k is normal and $R^i\beta_{k,*}\mathscr{O}_{B^k(L)}(-Z_{k-1}) = 0$ for all $i > 0$, then one has

$$H^i(\Sigma_k, \wedge^j M_{\Sigma_k} \otimes I_{\Sigma_{k-1}|\Sigma_k}(k+1)) = 0 \;\; \textit{for} \;\; i \geq j - p, \; i \geq 1, \; j \geq 0.$$

Proof Recall that $B^k(L) = \mathbb{P}(E_{k+1,L})$ with the natural projection $\pi_k\colon B^k(L) \to C_{k+1}$. Let H be the tautological divisor on $B^k(L)$ so that $\mathscr{O}_{B^k(L)}(H) = \mathscr{O}_{B^k(L)}(1) = \beta_k^*\mathscr{O}_{\Sigma_k}(1)$. One can identify $H^0(B^k(L), \mathscr{O}_{B^k(L)}(H)) = H^0(C_{k+1}, E_{k+1,L}) = H^0(C, L)$. Write $M_H := \beta_k^* M_{\Sigma_k}$. By the snake lemma, one can form the following commutative diagram

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & K \arrow[d] & \\
0 \arrow[r] & \pi_k^* M_{k+1,L} \arrow[r] \arrow[d] & H^0(C, L) \otimes \mathscr{O}_{B^k(L)} \arrow[r] \arrow[d, equal] & \pi_k^* E_{k+1,L} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_H \arrow[r] \arrow[d] & H^0(C, L) \otimes \mathscr{O}_{B^k(L)} \arrow[r] & \mathscr{O}_{B^k(L)}(H) \arrow[r] \arrow[d] & 0 \\
 & K \arrow[d] & & 0 & \\
 & 0, & & &
\end{tikzcd}
\tag{5.3}
$$

in which the right-hand-side vertical exact sequence is the relative Euler sequence. By Bott's formula on projective spaces, we obtain

$$R^i \pi_{k,*} \wedge^j K = 0 \quad \text{for all } i \geq 0 \text{ and } j > 0. \tag{5.4}$$

Since Σ_k is normal and $R^i \beta_{k,*}\mathscr{O}_{B^k(L)}(-Z_{k-1}) = 0$ for all $i > 0$, we have

$$
\begin{aligned}
&H^i(\Sigma_k, \wedge^j M_{\Sigma_k} \otimes I_{\Sigma_{k-1}|\Sigma_k}(k+1)) \\
&\quad = H^i(B^k(L), \wedge^j M_H \otimes \mathscr{O}_{B^k(L)}((k+1)H - Z_{k-1}))
\end{aligned}
\tag{5.5}
$$

for $i \geq 0$ and $j \geq 0$. Now, the left-hand-side vertical exact sequence of (5.3) induces a filtration

$$\wedge^j M_H = F^0 \supseteq F_1 \supseteq \cdots \supseteq F^j \supseteq F^{j+1} = 0$$

such that $F^{\ell}/F^{\ell+1} = \pi_k^* \wedge^{\ell} M_{k+1,L} \otimes \wedge^{j-\ell} K$ for $0 \leq \ell \leq j$. By (5.4) and the projection formula, we find

$$\begin{aligned} &H^i(B^k(L), \pi_k^* \wedge^{\ell} M_{k+1,L} \otimes \wedge^{j-\ell} K) \\ &\quad = H^i(C_{k+1}, \wedge^{\ell} M_{k+1,L} \otimes \pi_{k,*} \wedge^{j-\ell} K) = 0 \end{aligned}$$

for $i \geq 0$, $j > 0$ and $0 \leq \ell \leq j-1$. We have $\mathscr{O}_{B^k(L)}((k+1)H - Z_{k-1}) = \pi_k^* A_{k+1,L}$ by Proposition 3.15 (2). Thus we see that

$$H^i(C_{k+1}, \wedge^j M_{k+1,L} \otimes A_{k+1,L}) = 0 \ \text{ for } i \geq j-p,\ i \geq 1,\ j \geq 0, \tag{5.6}$$

implies the cohomology vanishing

$$\begin{aligned} &H^i(B^k(L), \wedge^j M_H \otimes \mathscr{O}_{B^k(L)}((k+1)H - Z_{k-1})) = 0 \\ &\quad \text{for } i \geq j-p,\ i \geq 1,\ j \geq 0. \end{aligned}$$

Hence by (5.5), to prove the lemma, it suffices to show the cohomology vanishing (5.6).

To this end, we consider the natural quotient map $q_{k+1} \colon C^{k+1} \to C_{k+1}$. Note that

$$q_{k+1}^*(\wedge^j M_{k+1,L} \otimes N_{k+1,L}) = \wedge^j Q_{k+1,L} \otimes L^{\boxtimes k+1}\big(-\Delta_{k+1}\big).$$

By projection formula, we have

$$\begin{aligned} &\wedge^j M_{k+1,L} \otimes N_{k+1,L} \otimes q_{k+1,*}\mathscr{O}_{C^{k+1}} \\ &\quad = q_{k+1,*}\big(\wedge^j Q_{k+1,L} \otimes L^{\boxtimes k+1}\big(-\Delta_{k+1}\big)\big). \end{aligned}$$

Recall that $A_{k+1,L} = N_{k+1,L}(-\delta_{k+1})$. Lemma 3.5 implies that $\wedge^i M_{k+1,L} \otimes A_{k+1,L}$ is a direct summand of $\wedge^j M_{k+1,L} \otimes N_{k+1,L} \otimes q_{k+1,*}\mathscr{O}_{C^{k+1}}$. Thus the desired cohomology vanishing (5.6) follows from

$$H^i\big(C^{k+1}, \wedge^j Q_{k+1,L} \otimes L^{\boxtimes k+1}(-\Delta_{k+1})\big) = 0 \ \text{ for } i \geq j-p,\ i \geq 1,\ j \geq 0.$$

which is nothing but Theorem 4.1 because $L\big(-\sum_{i=1}^{g+2k+1} x_i\big)$ is effective for general points $x_1, \ldots, x_{g+2k+1}$ on C. We finish the proof. $\square$

5.1 Normality, projective normality, and property $N_{k+2,p}$

The following is the main result of the paper. It is worth noting that all of the claimed properties in the theorem are proved at the same time to make the induction work.

Theorem 5.2 *Let $k \geq 0$ and $p \geq 0$ be integers, and L be a line bundle on C. Assume that*

$$\deg L \geq 2g + 2k + 1 + p.$$

Consider the k-th secant variety $\Sigma_k = \Sigma_k(C, L)$ in the space $\mathbb{P}(H^0(C, L)) = \mathbb{P}^r$. Then one has the following:

(1) *Σ_k is normal.*
(2) *$R^i \beta_{k,*}\mathscr{O}_{B^k(L)}(-Z_{k-1}) = 0$ for all $i > 0$.*
(3) *$H^i(\Sigma_k, I_{\Sigma_{k-1}|\Sigma_k}(\ell)) = H^i(\Sigma_k, \mathscr{O}_{\Sigma_k}(\ell)) = 0$ for all $i > 0, \ell > 0$.*
(4) *$\Sigma_k \subseteq \mathbb{P}^r$ is projectively normal, and satisfies the property $N_{k+2,p}$.*

Proof We proceed by the induction on the number k. The statements (1), (2), (3) in the theorem are trivial for the case $k = 0$ while the statement (4) is Green's theorem. Thus, in the sequel, we assume that $k \geq 1$ and the theorem holds for smaller k. For a number m with $0 \leq m \leq k$, we let $\Sigma_m := \Sigma_m(C, L)$.

(1) The proof here follows the proofs of Lemma 2.1 and Theorems D of [21]. The question is local. For a closed point $x \in \Sigma_k$, it is enough to show that Σ_k is normal at x. As $\Sigma_k \backslash \Sigma_{k-1}$ is nonsingular, we assume that $x \in \Sigma_m \backslash \Sigma_{m-1}$ for some $0 \leq m \leq k-1$. Let $\xi := \xi_{m+1,x} \in C_{m+1}$ be the degree $m+1$ divisor on C determined by x. The morphism $\beta = \beta_k \colon B^k(L) \to \Sigma_k$ induces the morphisms for sheaves

$$\begin{tikzcd} \mathscr{O}_{\mathbb{P}^r} \arrow[r, two heads] \arrow[rr, bend left] & \mathscr{O}_{\Sigma_k} \arrow[r, hook] & \beta_*\mathscr{O}_{B^k(L)}. \end{tikzcd}$$

Thus it suffices to prove that the natural morphism $\mathscr{O}_{\mathbb{P}^r} \to \beta_*\mathscr{O}_{B^k(L)}$ is surjective at $x \in \Sigma_m \backslash \Sigma_{m-1}$. Let $F := \beta^{-1}(x)$ be the fiber over x. Then $F \cong C_{k-m}$ (Proposition 3.13 (2.a)). By the formal function theorem, it is sufficient to show that the induced morphism

$$\Psi_x \colon \varprojlim (\mathscr{O}_{\mathbb{P}^r}/\mathfrak{m}^\ell) \longrightarrow \varprojlim H^0(\mathscr{O}_{B^k(L)}/I_F^\ell)$$

is surjective, where $\mathfrak{m} = \mathfrak{m}_x$ is the ideal sheaf of $x \in \mathbb{P}^r$ and I_F is the ideal sheaf of F in $B^k(L)$. Using the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{m}^\ell/\mathfrak{m}^{\ell+1} \arrow[r] \arrow[d, "\alpha_\ell"] & \mathscr{O}_{\mathbb{P}^r}/\mathfrak{m}^{\ell+1} \arrow[r] \arrow[d] & \mathscr{O}_{\mathbb{P}^r}/\mathfrak{m}^\ell \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^0(I_F^\ell/I_F^{\ell+1}) \arrow[r] & H^0(\mathscr{O}_{B^k(L)}/I_F^{\ell+1}) \arrow[r] & H^0(\mathscr{O}_{B^k(L)}/I_F^\ell) \arrow[r] & \cdots \end{tikzcd}$$

and the induction on ℓ, we further reduce to show that the map

$$\alpha_\ell \colon \mathfrak{m}^\ell/\mathfrak{m}^{\ell+1} \longrightarrow H^0(I_F^\ell/I_F^{\ell+1})$$

is surjective for all $\ell \geq 0$. Note that

$$\mathfrak{m}^\ell/\mathfrak{m}^{\ell+1} = S^\ell(T_x^*\mathbb{P}^r) \quad \text{and} \quad I_F^\ell/I_F^{\ell+1} \cong S^\ell N^*_{F/B^k(L)}.$$

The map α_ℓ factors as follows

$$\begin{tikzcd} S^\ell(T_x^*\mathbb{P}^r) \arrow[r, "S^\ell\alpha_1"] \arrow[dr, "\alpha_\ell"'] & S^\ell H^0\big(N^*_{F/B^k(L)}\big) \arrow[d, "\theta_l"] \\ & H^0\big(S^\ell N^*_{F/B^k(L)}\big). \end{tikzcd}$$

But Proposition 3.13 (2.e) says that the map $\alpha_1 \colon T_x^*\mathbb{P}^r \to H^*(N^*_{F/B^k(L)})$ is an isomorphism. Thus in order to show that α_ℓ is surjective, it suffices to show that the morphism θ_ℓ is surjective. To this end, we use Proposition 3.13 (2.d), which says that

$$N^*_{F/B^k(L)} \cong \mathscr{O}_F^{\oplus 2m+1} \oplus E_{n-m,L(-2\xi)}.$$

Thus the surjectivity of θ_ℓ would follow from the surjectivity of the morphism

$$S^i H^0(E_{k-m,L(-2\xi)}) \longrightarrow H^0(S^i E_{k-m,L(-2\xi)}) \text{ for } 0 \leq i \leq \ell.$$

But this follows from the inductive hypothesis because $\deg L(-2\xi) \geq 2g + 2(k-m-1)+1+p$ and therefore the secant variety $\Sigma_{k-m-1}(C, L(-2\xi))$ in the space $\mathbb{P}(H^0(C, L(-2\xi)))$ is normal and projective normality.

(2) The question is local. For a closed point $x \in \Sigma_k$, we shall show that $R^i\beta_*\mathscr{O}_{B^k(L)}(-Z_{k-1})_x = 0$ for all $i > 0$. Since $\beta \colon B^k(L) \to \Sigma_k$ is isomorphic over $x \in \Sigma_k \setminus \Sigma_{k-1}$, we may assume $x \in \Sigma_m \setminus \Sigma_{m-1}$ for some $0 \leq m \leq k-1$. Let $\xi := \xi_{m+1,x} \in C_{m+1}$ be the degree $m+1$ divisor on C determined by x. Let $F := \beta^{-1}(x)$ be the fiber of β over x, and I_F be the ideal sheaf of F in $B^k(L)$. Recall that $F \cong C_{k-m}$ (Proposition 3.13 (2.a)). By the formal function theorem, it suffices to show that

$$\varprojlim H^i(F, \mathscr{O}_{B^k(L)}(-Z_{k-1}) \otimes \mathscr{O}_{B^k(L)}/I_F^\ell) = 0 \text{ for } i > 0.$$

To this end, we need to prove that

$$H^i(F, \mathscr{O}_{B^k(L)}(-Z_{k-1}) \otimes \mathscr{O}_{B^k(L)}/I_F^\ell) = 0 \text{ for } i > 0 \text{ and } \ell \geq 1.$$

which can be deduced from the vanishing

$$H^i(F, \mathscr{O}_{B^k(L)}(-Z_{k-1}) \otimes I_F^\ell/I_F^{\ell+1}) = 0 \text{ for } i > 0 \text{ and } \ell \geq 0. \tag{5.7}$$

One can calculate that $\mathscr{O}_{B^k(L)}(-Z_{k-1})|_F = A_{k+1,L}|_F = A_{k-m,L(-2\xi)}$ by Lemma 3.4 and that $I_F^\ell/I_F^{\ell+1} = S^\ell N^*_{F/B^k(L)}$ for $\ell \geq 0$, where $N^*_{F/B^k(L)} \cong \mathscr{O}_F^{\oplus 2m+1} \oplus E_{k-m,L(-2\xi)}$ by Proposition 3.13 (2.d). Thus vanishing (5.7) can be reduced further to show

$$H^i(C_{k-m}, A_{k-m,L(-2\xi)} \otimes S^\ell E_{k-m,L(-2\xi)}) = 0 \text{ for } i > 0 \text{ and } \ell \geq 0. \tag{5.8}$$

Now, as $\deg L(-2\xi) \geq 2g + 2(k-m-1) + 1 + p$, the line bundle $L(-2\xi)$ is very ample. Accordingly, we consider the secant varieties $\Sigma'_{k-m-1} := \Sigma_{k-m-1}(C, L(-2\xi))$ and $\Sigma'_{k-m-2} := \Sigma'_{k-m-2}(C, L(-2\xi))$ in the space $H^0(C, L(-2\xi))$. By inductive hypothesis, the proposition holds for Σ'_{k-m-1}. Recall that $B^{k-m-1}(L(-2\xi)) = \mathbb{P}(E_{k-m,L(-2\xi)})$ with the projection π_{k-m-1} to C_{k-m} and there is a birational morphism $\beta_{k-m-1}\colon B^{k-m-1}(L(-2\xi)) \to \Sigma'_{k-m-1}$. Write H to be the tautological divisor on $B^{k-m-1}(L(-2\xi))$. Notice that

$$\begin{aligned}
&\pi_{k-m-1,*}\mathscr{O}_{B^{k-m-1}(L(-2\xi))}((k-m)H - Z_{k-m-2}) = A_{k-m,L(-2\xi)},\\
&\beta_{k-m-1,*}\mathscr{O}_{B^{k-m-1}(L(-2\xi))}(-Z_{k-m-2}) = I_{\Sigma'_{k-m-2}|\Sigma'_{k-m-1}}.
\end{aligned}$$

By applying the inductive hypothesis for Σ'_{k-m-1}, we have

$$\begin{aligned}
&H^i(C_{k-m}, S^{\ell-k+m}E_{k-m,L(-2\xi)} \otimes A_{k-m,L(-2\xi)})\\
&= H^i(B^{k-m-1}(L(-2\xi)), \mathscr{O}_{B^{k-m-1}(L(-2\xi))}(\ell H - Z_{k-m-2}))\\
&= H^i(\Sigma'_{k-m-1}, I_{\Sigma'_{k-m-2}|\Sigma'_{k-m-1}}(\ell))
\end{aligned}$$

for all $i \geq 0$ and $\ell \in \mathbb{Z}$. Hence, vanishing (5.8) follows from the vanishing for $I_{\Sigma'_{k-m-2}|\Sigma'_{k-m-1}}$, which holds by the inductive hypothesis. This completes the proof of (2).

(3) By the inductive hypothesis, we have $H^i(\Sigma_{k-1}, \mathscr{O}_{\Sigma_{k-1}}(\ell)) = 0$ for $i > 0$ and $\ell > 0$. Grant for the time being the following claim:

$$H^i(\Sigma_k, I_{\Sigma_{k-1}|\Sigma_k}(\ell)) = 0 \text{ for all } i > 0 \text{ and } 1 \leq \ell \leq 2k+2-i. \tag{5.9}$$

Chasing through the associated long exact sequence to the short exact sequence (5.2), we obtain

$$H^i(\Sigma_k, \mathscr{O}_{\Sigma_k}(\ell)) = 0 \text{ for all } i > 0 \text{ and } 1 \leq \ell \leq 2k+2-i.$$

In particular, $\mathscr{O}_{\Sigma_k}$ is $(2k+2)$-regular, so the assertion (3) follows.

We next turn to the proof of the claim (5.9). Let H be the tautological divisor on $B^k(L) = \mathbb{P}(E_{k+1,L})$. By (1), Σ_k is normal. Thus we have

$$\beta_{k,*}\mathscr{O}_{B^k(L)}(-Z_{k-1}) = I_{\Sigma_{k-1}|\Sigma_k} \quad \text{and} \quad \pi_{k,*}\mathscr{O}_{B^k(L)}((k+1)H - Z_{k-1}) = A_{k+1,L}.$$

By (2), $R^i\beta_{k,*}\mathscr{O}_{B^k(L)}(-Z_{k-1}) = 0$ for $i > 0$, so we obtain

$$\begin{aligned} H^i(\Sigma_k, I_{\Sigma_{k-1}|\Sigma_k}(\ell)) &= H^i(B^k(L), \mathscr{O}_{B^k(L)}(\ell H - Z_{k-1})) \\ &= H^i(C_{k+1}, S^{\ell-k-1}E_{k+1,L} \otimes A_{k+1,L}). \end{aligned}$$

Thus (5.9) holds automatically when $i \geq k+2$ or $1 \leq \ell \leq k$. It only remains to consider the case that $1 \leq i \leq k+1$ and $k+1 \leq \ell \leq 2k+2-i$.

Now, the short exact sequence (5.1) induces a short exact sequence

$$0 \longrightarrow \wedge^{j+1}M_{\Sigma_k} \longrightarrow \wedge^{j+1}H^0(C, L) \otimes \mathscr{O}_{\Sigma_k} \longrightarrow \wedge^j M_{\Sigma_k} \otimes \mathscr{O}_{\Sigma_k}(1) \longrightarrow 0.$$

Tensoring with $I_{\Sigma_{k-1}|\Sigma_k}$, we obtain a short exact sequence

$$\begin{aligned} 0 &\longrightarrow \wedge^{j+1}M_{\Sigma_k} \otimes I_{\Sigma_{k-1}|\Sigma_k} \longrightarrow \wedge^{j+1}H^0(C, L) \otimes I_{\Sigma_{k-1}|\Sigma_k} \\ &\longrightarrow \wedge^j M_{\Sigma_k} \otimes I_{\Sigma_{k-1}|\Sigma_k}(1) \longrightarrow 0. \end{aligned}$$

This gives a long exact sequence of cohomology groups

$$\begin{aligned} \cdots &\longrightarrow \wedge^{j+1}H^0(C, L) \otimes H^i(\Sigma_k, I_{\Sigma_{k-1}|\Sigma_k}(\ell)) \\ &\longrightarrow H^i(\Sigma_k, \wedge^j M_{\Sigma_k} \otimes I_{\Sigma_{k-1}|\Sigma_k}(\ell+1)) \\ &\longrightarrow H^{i+1}(\Sigma_k, \wedge^{j+1}M_{\Sigma_k} \otimes I_{\Sigma_{k-1}|\Sigma_k}(\ell)) \longrightarrow \cdots. \end{aligned}$$

It follows that the statement

$$H^i(\Sigma_k, \wedge^j M_{\Sigma_k} \otimes I_{\Sigma_{k-1}|\Sigma_k}(\ell)) = 0 \;\text{ for }\; i \geq 1,\; j \geq 0 \text{ and } i \geq j-p \qquad (*)_\ell$$

implies the corresponding statement $(*)_{\ell+1}$. Since Lemma 5.1 says that $(*)_{k+1}$ is true, we conclude that $(*)_\ell$ holds for $\ell \geq k+1$, i.e.,

$$H^i(\Sigma_k, \wedge^j M_{\Sigma_k} \otimes I_{\Sigma_{k-1}|\Sigma_k}(\ell)) = 0 \text{ for } i \geq 1,\; j \geq 0,\; i \geq j-p \text{ and } \ell \geq k+1. \tag{5.10}$$

When $j = 0$, this implies (5.9) for $i \geq 1$ and $\ell \geq k+1$. This finishes the proof of (3).

(4) We first show that $\Sigma_k \subseteq \mathbb{P}^r$ is projectively normal. By Danila's theorem (Theorem 3.8),

$$\begin{aligned} H^0(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(\ell)) = S^\ell H^0(C, L) &= H^0(B^k(L), \mathscr{O}_{B^k(L)}(\ell)) \\ &= H^0(\Sigma_k, \mathscr{O}_{\Sigma_k}(\ell)) \ \text{ for } 0 \leq \ell \leq k+1. \end{aligned}$$

For $0 \leq \ell \leq k+1$, this implies that $H^0(\mathbb{P}^r, I_{\Sigma_k}(\ell)) = H^1(\mathbb{P}^r, I_{\Sigma_k}(\ell)) = 0$, where $I_{\Sigma_m} = I_{\Sigma_m|\mathbb{P}^r}$ is the defining ideal sheaf of Σ_m in $\mathbb{P}^r$ for $0 \leq m \leq k$. We have a short exact sequence

$$0 \longrightarrow I_{\Sigma_k} \longrightarrow I_{\Sigma_{k-1}} \longrightarrow I_{\Sigma_{k-1}|\Sigma_k} \longrightarrow 0. \tag{5.11}$$

We then obtain $H^0(\mathbb{P}^r, I_{\Sigma_{k-1}}(\ell)) = H^0(\Sigma_k, I_{\Sigma_{k-1}|\Sigma_k}(\ell))$ for $0 \leq \ell \leq k+1$. For $\ell \geq k+1$, consider the following commutative diagram

$$\begin{tikzcd} S^{\ell-k-1}H^0(C, L) \otimes H^0(\Sigma_k, I_{\Sigma_{k-1}}(k+1)) \arrow[r] \arrow[d, equal] & H^0(\Sigma_k, I_{\Sigma_{k-1}}(\ell)) \arrow[d] \\ S^{\ell-k-1}H^0(C, L) \otimes H^0(\Sigma_k, I_{\Sigma_{k-1}|\Sigma_k}(k+1)) \arrow[r] & H^0(\Sigma_k, I_{\Sigma_{k-1}|\Sigma_k}(\ell)). \end{tikzcd} \tag{5.12}$$

By (5.10), $H^1(\Sigma_k, M_{\Sigma_k} \otimes I_{\Sigma_{k-1}|\Sigma_k}(\ell)) = 0$ for $\ell \geq k+1$. Then the multiplication map in the bottom of (5.12) is surjective, and hence, the right vertical map of (5.12) is surjective. We then conclude that the map $H^0(\mathbb{P}^r, I_{\Sigma_{k-1}}(\ell)) \to H^0(\Sigma_k, I_{\Sigma_{k-1}|\Sigma_k}(\ell))$ is surjective for $\ell \geq 0$. By induction, $\Sigma_{k-1} \subseteq \mathbb{P}^r$ is projectively normal, so $H^1(\mathbb{P}^r, I_{\Sigma_{k-1}}(\ell)) = 0$ for $\ell \geq 0$. Therefore, by considering (5.11), we obtain $H^1(\mathbb{P}^r, I_{\Sigma_k}(\ell)) = 0$ for $\ell \geq 0$, which means that $\Sigma_k \subseteq \mathbb{P}^r$ is projectively normal.

Next we show that $\Sigma_k \subseteq \mathbb{P}^r$ satisfies $N_{k+2,p}$. Recall from (3) that $H^i(\Sigma_k, \mathscr{O}_{\Sigma_k}(\ell)) = 0$ for $i \geq 1$ and $\ell \geq 1$. By Proposition 2.1, we only need to show that $H^1(\Sigma_k, \wedge^j M_{\Sigma_k} \otimes \mathscr{O}_{\Sigma_k}(\ell)) = 0$ for $\ell \geq k+1$ and $1 \leq j \leq p+1$. Consider the short exact sequence

$$0 \longrightarrow \wedge^j M_{\Sigma_k} \otimes I_{\Sigma_{k-1}|\Sigma_k} \longrightarrow \wedge^j M_{\Sigma_k} \longrightarrow \wedge^j M_{\Sigma_{k-1}} \longrightarrow 0.$$

Since $\deg L \geq 2g + 1 + 2(k-1) + 1 + p + 2$, we may assume by induction that $\Sigma_{k-1} \subseteq \mathbb{P}^r$ satisfies $N_{k+1,p+2}$. So by Proposition 2.1, we have $H^1(\Sigma_{k-1}, \wedge^j M_{\Sigma_{k-1}}(\ell)) = 0$ for $\ell \geq k$ and $1 \leq j \leq p+3$. Combine this with (5.10), we get $H^1(\Sigma_k, \wedge^j M_{\Sigma_k}(\ell)) = 0$ for $1 \leq j \leq p+1$ and $\ell \geq k+1$ as desired. □

Remark 5.3 We have seen in the above proof that Danila's theorem (Theorem 3.8) shows $H^0(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(\ell)) = H^0(\Sigma_k, \mathscr{O}_{\Sigma_k}(\ell))$ for all $1 \leq \ell \leq k+1$. This in particular implies that the defining ideal of the k-th secant variety Σ_k in $\mathbb{P}^r$ has no forms of degree $\leq k+1$.

5.2 Singularities

Proposition 5.4 *Let $k \geq 0$ be an integer, and L be a line bundle on C. Assume that*

$$\deg L \geq 2g + 2k + 1.$$

Consider the k-th secant variety $\Sigma_k = \Sigma_k(C, L)$ in the space $\mathbb{P}(H^0(C, L))$. Then one has the following:

(1) *Σ_k has normal Du Bois singularities.*
(2) *$g = 0$ if and only if there exists a boundary divisor Γ on Σ_k such that (Σ_k, Γ) is a klt pair. In this case, Σ_k is a Fano variety with log terminal singularities and of Picard rank one.*
(3) *$g = 1$ if and only if there exists a boundary divisor Γ on Σ_k such that (Σ_k, Γ) is a log canonical pair but it cannot be a klt pair. In this case, Σ_k is a Calabi–Yau variety with log canonical singularities.*

In particular, $g \geq 2$ if and only if there is no boundary divisor Γ on Σ_k such that (Σ_k, Γ) is a log canonical pair.

Proof (1) By Theorem 5.2 (1), we know that Σ_k is normal. By proceeding by the induction on k, we show that Σ_k has Du Bois singularities. If $k = 0$, then $\Sigma_0 = C$ so that the assertion is trivial. In the sequel, we assume that $k \geq 1$ and the assertion (1) holds for $k-1$. By [13, Corollary 6.28], it suffices to check the following:

(a) Σ_{k-1} has Du Bois singularities.
(b) Z_{k-1} has Du Bois singularities.
(c) $\beta_{k,*}\mathscr{O}_{B^k(L)}(-Z_{k-1}) = I_{\Sigma_{k-1}|\Sigma_k}$ and $R^i\beta_{k,*}\mathscr{O}_{B^k(L)}(-Z_{k-1}) = 0$ for $i > 0$.

By inductive hypothesis, (a) holds. For (b), consider the composition map $b_k \colon \mathrm{bl}_k(B^k(L)) \to B^k(L)$ of blowups (see Sect. 3.3). Recall from Proposition 3.15 (3) that

$$K_{\mathrm{bl}_k(B^k(L))} = b_k^*(K_{B^k(L)} + Z_{k-1}) - (E_0 + \cdots + E_{k-1}).$$

Thus the log pair $(B^k(L), Z_{k-1})$ is log canonical, and hence, Z_{k-1} has semi-log canonical singularities. Then, by [13, Corollary 6.32], Z_{k-1} has Du Bois singularities, i.e., (b) holds. Finally, (c) holds by Theorem 5.2.

(2), (3) Recall that $\beta_k : B^k(L) \to \Sigma_k$ is a resolution of singularities and Σ_k is normal. For a general point $x \in \Sigma_{k-1} \setminus \Sigma_{k-2}$, we denote by $F_x := \beta_k^{-1}(x)$ the fiber of β_k over x. Note that $F_x \cong C$. Let H be the tautological divisor on $B^k(L) = \mathbb{P}(E_{k+1,L})$, i.e., $\mathscr{O}_{B^k(L)}(H) = \mathscr{O}_{B^k(L)}(1)$. Recall from Proposition 3.15 (2) that $Z_{k-1} \sim_{\text{lin}} (k+1)H - \pi_k^*(T_{k+1}(L) - 2\delta_{k+1})$. We can easily check that

$$K_{B^k(L)} + Z_{k-1} \sim_{\text{lin}} \pi_k^*(K_{C_{k+1}} + \delta_{k+1}) = \pi_k^* T_{k+1}(K_C). \tag{5.13}$$

We first prove (2). Suppose that $C = \mathbb{P}^1$. It is well known that $C_{n+1} \cong \mathbb{P}^{n+1}$. For a sufficiently small rational number $\epsilon > 0$, by (5.13), we have

$$\begin{aligned} -(K_{B^k(L)} + (1-\epsilon)Z_{k-1}) &\sim_{\mathbb{Q}\text{-lin}} \epsilon(k+1)H \\ &+\pi_k^*(T_{k+1}(-K_C - \epsilon L) + 2\epsilon\delta_{k+1}). \end{aligned}$$

We may assume that $T_{k+1}(-K_C - \epsilon L) + 2\epsilon\delta_{k+1}$ is ample on C_{k+1}. Now, $B^k(L)$ has Picard rank two, and the nef cone of $B^k(L)$ is generated by H and $\pi_k^*(T_{k+1}(-K_C - \epsilon L) + 2\epsilon\delta_{k+1})$. Thus $-(K_{B^k(L)} + (1-\epsilon)Z_{k-1})$ is ample. By considering the log resolution of $(B^k(L), (1-\epsilon)Z_{k-1})$ in Proposition 3.15 (3), we see that $(B^k(L), (1-\epsilon)Z_{k-1})$ is a klt pair. Hence $B^k(L)$ is of Fano type. By [9, Theorem 5.1], Σ_k is also of Fano type. Now, Σ_k has Picard rank one. Therefore, it is a Fano variety with log terminal singularities. For the converse, suppose that there exists a boundary divisor Γ such that (Σ_k, Γ) is a klt pair. By [12, Corollary 1.5], $F_x \cong C$ is rationally chain connected, so C is a rational curve.

We finally prove (3). Suppose that C is an elliptic curve. By (5.13), we have

$$K_{B^k(L)} + Z_{k-1} \sim_{\text{lin}} \pi_k^* T_{k+1}(K_C) = 0.$$

Then the 'only if' direction immediately follows from [9, Lemma 1.1]. In this case, we actually have $K_{\Sigma_k} = \beta_{k,*}(K_{B^k(L)} + Z_{k-1}) = 0$. Thus Σ_k is a Calabi–Yau variety with log canonical singularities. For the converse, suppose that there exists a boundary divisor Γ such that (Σ_k, Γ) is a log canonical pair. We have

$$K_{B^k(L)} + Z_{k-1} + \beta_k^{-1}\Gamma = \beta_k^*(K_{\Sigma_k} + \Gamma) + (1+a)Z_{k-1},$$

where $a = a(Z_{k-1}; \Sigma_k, \Gamma) \geq -1$ is the discrepancy of the β_k-exceptional divisor Z_{k-1}. By restricting the above divisor to $F_x \cong C$, we obtain

$$K_C + (\beta_k^{-1}\Gamma)|_C = -(1+a)(L - 2\xi),$$

where $\xi := \xi_{k,x}$ is the degree k divisor on C determined by x. Then

$$-K_C = (1+a)(L-2\xi) + (\beta_k^{-1}\Gamma)|_C$$

is effective so that C is either a rational curve or an elliptic curve. This proves the converse direction, and hence, we complete the proof. □

Remark 5.5 It is easy to check that $g = 0$ if and only if Σ_k has rational singularities (cf. [24, Proposition 9]).

Remark 5.6 When $g = 1$, we see that Σ_k is Gorenstein with $\omega_{\Sigma_k} \cong \mathscr{O}_{\Sigma_k}$ (this is also proved in [26, 8.14]). In the next subsection, we show that $\Sigma_k \subseteq \mathbb{P}(H^0(C, L))$ is arithmetically Cohen–Macaulay, and therefore, its cone is Gorenstein. For instance, one can deduce that the k-th secant variety Σ_k of an elliptic curve embedded by a degree $2k+4$ line bundle is a complete intersection in $\mathbb{P}^{2k+3}$.

Remark 5.7 In contrast to the smaller genus case, if $g \geq 2$, then Σ_k is not $\mathbb{Q}$-Gorenstein, i.e., K_{Σ_k} is not $\mathbb{Q}$-Cartier. To show this, suppose that K_{Σ_k} is $\mathbb{Q}$-Cartier. For a sufficiently divisible integer $m > 0$, we have $mK_{B^k(L)} - maZ_{k-1} \sim_{\text{lin}} \beta_k^*(mK_{\Sigma_k})$, where $a = a(Z_{k-1}; \Sigma_k, 0) < -1$ is the discrepancy of Z_{k-1}. By restricting to $\beta_k^{-1}(x) \cong C_k$ for any point $x \in C \subseteq \Sigma_k$, we see that

$$m\big(T_k(K_C + (1-a)L - 2(1-a)x\big) - 2(1-a)\delta_k \sim_{\text{lin}} 0.$$

Thus we obtain $2m(1-a)x \sim_{\text{lin}} 2m(1-a)y$ for any points $x, y \in C$, but it is impossible.

5.3 Arithmetic Cohen–Macaulayness and Castelnuovo–Mumford regularity

Theorem 5.8 *Let $k \geq 0$ be an integer, and L be a line bundle on C. Assume that*

$$\deg L \geq 2g + 2k + 1.$$

Consider the k-th secant variety $\Sigma_k = \Sigma_k(C, L)$ in the space $\mathbb{P}(H^0(C, L)) = \mathbb{P}^r$. Then one has the following:

(1) $H^i(\Sigma_k, \mathscr{O}_{\Sigma_k}(-\ell)) = 0$ *for* $1 \leq i \leq 2k$ *and* $\ell \geq 0$.
(2) $H^{2k+1}(\Sigma_k, \mathscr{O}_{\Sigma_k}) = S^{k+1}H^0(C, \omega_C)^*$.

In particular, $\Sigma_k \subseteq \mathbb{P}^r$ is arithmetically Cohen–Macaulay.

Proof We first recall from Proposition 5.4 (1) that Σ_k has Du Bois singularities. By [13, Theorem 10.42], we have

$$h^i(\Sigma_k, \mathscr{O}_{\Sigma_k}(-\ell)) = h^i(\Sigma_k, \mathscr{O}_{\Sigma_k}(-1)) \quad \text{for} \quad 1 \leq i \leq 2k \;\; \text{and} \;\; \ell \geq 1.$$

Therefore, the result (1) is equivalent to the cohomology vanishing

$$H^i(\Sigma_k, \mathscr{O}_{\Sigma_k}(-\ell)) = 0 \quad \text{for} \quad 1 \leq i \leq 2k \;\; \text{and} \;\; \ell = 0, 1.$$

We now proceed by the induction on k. Note that the case with $k = 0$ is trivial. For $k \geq 1$, we assume that $\Sigma_{k-1} \subseteq \mathbb{P}^r$ has results (1) and (2). Concerning the cohomological long exact sequence associated to the short exact sequence (5.2), we make the following:

Claim 5.9 (a) $H^i(\Sigma_k, I_{\Sigma_{k-1}|\Sigma_k}(-\ell)) = 0$ *for* $1 \leq i \leq 2k-1$ *and* $\ell = 0, 1$.
(b) *The connection map* τ_ℓ *of the cohomological groups*

$$\cdots \longrightarrow H^{2k-1}(\mathscr{O}_{\Sigma_{k-1}}(-\ell)) \xrightarrow{\tau_\ell} H^{2k}(I_{\Sigma_{k-1}|\Sigma_k}(-\ell)) \longrightarrow \cdots$$

is an isomorphism for $\ell = 0, 1$.

Granted the claim for the moment, using inductive hypothesis on Σ_{k-1} and chasing through the long exact sequence associated to (5.2), we immediately obtain from (a) that

$$H^i(\Sigma_k, \mathscr{O}_{\Sigma_k}(-\ell)) = 0 \quad \text{for} \quad 1 \leq i \leq 2k-2 \quad \text{and} \quad \ell = 0, 1.$$

Furthermore, we arrive at an exact sequence involving the connection map τ_ℓ as follows

$$\begin{aligned} 0 \longrightarrow H^{2k-1}(\mathscr{O}_{\Sigma_k}(-\ell)) \longrightarrow H^{2k-1}(\mathscr{O}_{\Sigma_{k-1}}(-\ell)) \xrightarrow{\tau_\ell} H^{2k}(I_{\Sigma_{k-1}|\Sigma_k}(-\ell)) \\ \longrightarrow H^{2k}(\mathscr{O}_{\Sigma_k}(-\ell)) \longrightarrow 0. \end{aligned}$$

The statement (b) then implies that

$$H^i(\Sigma_k, \mathscr{O}_{\Sigma_k}(-\ell)) = 0 \quad \text{for} \quad 2k-1 \leq i \leq 2k \;\; \text{and} \;\; \ell = 0, 1,$$

which proves (1).

For the result (2), chasing through the long exact sequence would yield

$$H^{2k+1}(\Sigma_k, \mathscr{O}_{\Sigma_k}) = H^{2k+1}(\Sigma_k, I_{\Sigma_{k-1}|\Sigma_k}).$$

By Theorem 5.2 (2) and Serre duality, for any i and ℓ, we have

$$\begin{aligned} H^i(I_{\Sigma_{k-1}|\Sigma_k}(-\ell)) &= H^i(\mathscr{O}_{B^k(L)}(-\ell H - Z_{k-1})) \\ &= H^{2k+1-i}(\mathscr{O}_{B^k(L)}(K_{B^k(L)} + Z_{k-1} + \ell H))^*, \end{aligned}$$

where H is the tautological divisor on $B^k(L) = \mathbb{P}(E_{k+1,L})$. Recall from (5.13) that

$$K_{B^k(L)} + Z_{k-1} \sim_{\text{lin}} \pi_k^*(K_{C_{k+1}} + \delta_{k+1}) = \pi_k^* T_{k+1}(K_C).$$

Thus we obtain

$$H^i(\Sigma_k, I_{\Sigma_{k-1}|\Sigma_k}(-\ell)) = H^{2k+1-i}(C_{k+1}, S^\ell E_{k+1,L} \otimes T_{k+1}(\omega_C))^*. \tag{5.14}$$

In particular, when $i = 2k+1$, we find

$$H^{2k+1}(\Sigma_k, \mathscr{O}_{\Sigma_k}) = H^{2k+1}(\Sigma_k, I_{\Sigma_{k-1}|\Sigma_k}) = H^0(C_{k+1}, T_{k+1}(\omega_C))^*.$$

By Lemma 3.7, we get the result (2).

We now prove Claim 5.9 (a). Assume that $\ell = 0$. As calculated in (5.14), we have

$$H^i(\Sigma_k, I_{\Sigma_{k-1}|\Sigma_k}) = H^{2k+1-i}(C_{k+1}, T_{k+1}(\omega_C))^*.$$

Then Lemma 3.7 implies Claim 5.9 (a) for $\ell = 0$. Assume that $\ell = 1$. By (5.14), we have

$$H^i(\Sigma_k, I_{\Sigma_{k-1}|\Sigma_k}(-1)) = H^{2k+1-i}(C_{k+1}, E_{k+1,L} \otimes T_{k+1}(\omega_C))^*.$$

Recall that we have a canonical morphism $\sigma_{k+1}\colon C_k \times C \to C_{k+1}$. We observe that

$$\sigma_{k+1,*}(T_k(\omega_C) \boxtimes (\omega \otimes L)) = E_{k+1,L} \otimes T_{k+1}(\omega_C).$$

Then we find

$$H^{2k+1-i}(C_{k+1}, E_{k+1,L} \otimes T_{k+1}(\omega_C)) = H^{2k+1-i}(C_k \times C, T_k(\omega_C) \boxtimes (\omega_C \otimes L)). \tag{5.15}$$

For $1 \leq i \leq 2k-1$, we have $2k+1-i \geq 2$. By Lemma 3.7 and Künneth formula, we get

$$H^{2k+1-i}(C_k \times C, T_k(\omega_C) \boxtimes (\omega_C \otimes L)) = 0.$$

This implies Claim 5.9 (a) for $\ell = 1$.

We next turn to the proof of Claim 5.9 (b). By Theorem 5.2 (2) for both Σ_k and Σ_{k-1} and calculation in (5.14), we recall that

$$\begin{aligned}
H^{2k}(I_{\Sigma_{k-1}|\Sigma_k}(-\ell))^* &= H^1(\omega_{B^k(L)}(Z_{k-1}+\ell H)) = H^1(S^\ell E_{k+1,L}\otimes T_{k+1}(\omega_C)),\\
H^{2k-1}(\mathscr{O}_{\Sigma_{k-1}}(-\ell))^* &= H^0(\omega_{B^{k-1}(L)}(Z_{k-2}+\ell H)) = H^0(S^\ell E_{k,L}\otimes T_k(\omega_C)).
\end{aligned}$$

For $\ell = 0$, by Lemma 3.7, we have $h^{2k}(\Sigma_k, I_{\Sigma_{k-1}|\Sigma_k}) = h^{2k-1}(\Sigma_{k-1}, \mathscr{O}_{\Sigma_{k-1}})$. For $\ell = 1$, by (5.15) and Künneth formula, we see that

$$\begin{aligned}
&H^1(E_{k+1,L}\otimes T_{k+1}(\omega_C)) = H^1(T_k(\omega_C)\boxtimes(\omega_C\otimes L))\\
&\quad = H^1(T_k(\omega_C))\otimes H^0(\omega_C\otimes L),\\
&H^0(E_{k,L}\otimes T_k(\omega_C)) = H^0(T_{k-1}(\omega_C)\boxtimes(\omega_C\otimes L))\\
&\quad = H^0(T_{k-1}(\omega_C))\otimes H^0(\omega_C\otimes L).
\end{aligned} \tag{5.16}$$

Lemma 3.7 then implies that $h^{2k}(\Sigma_k, I_{\Sigma_{k-1}|\Sigma_k}(-\ell)) = h^{2k-1}(\Sigma_{k-1}, \mathscr{O}_{\Sigma_{k-1}}(-\ell))$. Thus, to show Claim 5.9 (b), it is sufficient to show that τ_ℓ is injective for $\ell = 0, 1$.

To this end, recall that we have the following commutative diagram

$$\begin{tikzcd}
C_k\times C \arrow[rr, "\sigma_{k+1}"] & & C_{k+1}\\
B^{k-1}(L)\times C \arrow[u, "\pi_k\times \mathrm{id}_C"] \arrow[r, "\alpha_{k,k-1}"] & Z_{k-1} \arrow[r, hook] \arrow[d, "\beta_k|_{Z_{k-1}}"] & B^k(L) \arrow[u, "\pi_k"'] \arrow[d, "\beta_k"]\\
& \Sigma_{k-1} \arrow[r, hook] & \Sigma_k.
\end{tikzcd}$$

Note that $\alpha_{k,k-1}^*\omega_{Z_{k-1}} = \omega_{B^{k-1}(L)}(Z_{k-2})\boxtimes\omega_C$ and there is a natural injection

$$\begin{aligned}
&H^0(B^{k-1}(L), \omega_{B^{k-1}(L)}(Z_{k-2}+\ell H)) \hookrightarrow H^1(B^{k-1}(L)\\
&\quad \times C, \omega_{B^{k-1}(L)}(Z_{k-2}+\ell H))\boxtimes\omega_C).
\end{aligned}$$

Then we obtain the following commutative diagram

$$\begin{tikzcd}
H^1(S^\ell E_{k_1}\otimes T_{k+1}(\omega_C)) \arrow[rr] \arrow[d, equal] & & H^1(S^\ell E_{k,L}\otimes T_k(\omega_C)\boxtimes\omega_C) \arrow[d, equal]\\
H^1(\omega_{B^k(L)}(Z_{k-1}+\ell H)) \arrow[r] \arrow[d, equal] & H^1(\omega_{Z_{k-1}}(\ell H)) \arrow[r] \arrow[d] & H^1(\omega_{B^{k-1}(L)}(Z_{k-2}+\ell H))\boxtimes\omega_C)\\
H^{2k}(I_{\Sigma_{k-1}|\Sigma_k}(-\ell))^* \arrow[r, "\tau_\ell^*"] & H^{2k-1}(\mathscr{O}_{\Sigma_{k-1}}(-\ell))^* \arrow[r, equal] & H^0(\omega_{B^{k-1}(L)}(Z_{k-2}+\ell H)). \arrow[u, hook]
\end{tikzcd}$$

It is enough to check that the map on the top is injective. This is clear for $\ell = 0$. For $\ell = 1$, by (5.16) and Lemma 3.7, we have the following injection

$$H^1(E_{k+1} \otimes T_{k+1}(\omega_C)) \cong H^0(E_{k,L} \otimes T_k(\omega_C)) \hookrightarrow H^1(E_{k,L} \otimes T_k(\omega_C) \boxtimes \omega_C).$$

Thus the map on the top for $\ell = 1$ is injective as required.

Finally, recall the well known fact that a projective variety $X \subseteq \mathbb{P}^r$ is arithmetically Cohen–Macaulay if and only if the following hold:

(i) $X \subseteq \mathbb{P}^r$ is projectively normal.
(ii) $H^i(X, \mathscr{O}_X(\ell)) = 0$ for $0 < i < \dim X$ and $\ell \in \mathbb{Z}$.

By Theorem 5.2 (3), (4) and the vanishing property (1) imply that $\Sigma_k \subseteq \mathbb{P}^r$ is arithmetically Cohen–Macaulay. We complete the proof. □

Corollary 5.10 *Let $k \geq 0$ be an integer, and L be a line bundle on C. Assume that*

$$\deg L \geq 2g + 2k + 1.$$

Consider the secant variety $\Sigma_k = \Sigma_k(C, L)$ in the space $\mathbb{P}(H^0(C, L)) = \mathbb{P}^r$. Then one has the following:

(1) $h^0(\omega_{\Sigma_k}) = \dim K_{r-2k-1,2k+2}(\Sigma_k, \mathscr{O}_{\Sigma_k}(1)) = \binom{g+k}{k+1}$.
(2) *If* $g = 0$, *then* $\operatorname{reg}(\mathscr{O}_{\Sigma_k}) = k + 1$ *and* $\operatorname{reg}(\Sigma_k) = k + 2$.
(3) *If* $g \geq 1$, *then* $\operatorname{reg}(\mathscr{O}_{\Sigma_k}) = 2k + 2$ *and* $\operatorname{reg}(\Sigma_k) = 2k + 3$.

Proof (1) As $\Sigma_k \subseteq \mathbb{P}(H^0(C, L)) = \mathbb{P}^r$ is arithmetically Cohen–Macaulay by Theorem 5.8, dualizing the minimal graded free resolution of $R(\Sigma_k, \mathscr{O}_{\Sigma_k}(1))$ and shifting by $-r-1$ gives the minimal graded free resolution of the canonical module. This implies that

$$\dim K_{r-2k-1,2k+2}(\Sigma_k, \mathscr{O}_{\Sigma_k}(1)) = h^0(\Sigma_k, \omega_{\Sigma_k}).$$

By the Serre duality and Theorem 5.8, we obtain

$$h^0(\Sigma_k, \omega_{\Sigma_k}) = h^{2k+1}(\Sigma_k, \mathscr{O}_{\Sigma_k}) = \dim S^{k+1} H^0(C, \omega_C) = \binom{g+k}{k+1}.$$

(2), (3) By Theorem 5.2 (3), (4), we see that

$$\operatorname{reg}(\Sigma_k) = \operatorname{reg}(\mathscr{O}_{\Sigma_k}) + 1 \leq 2k + 3.$$

By Theorem 5.2 (3) and Theorem 5.8 (1), we know that $H^i(\Sigma_k, \mathscr{O}_{\Sigma_k}(\ell)) = 0$ for $1 \leq i \leq 2k$ and $\ell \in \mathbb{Z}$. Thus we only have to consider the (non)vanishing of $H^{2k+1}(\Sigma_k, \mathscr{O}_{\Sigma_k}(\ell))$.

For (2), suppose that $g = 0$. It is enough to show that $H^{2k+1}(\Sigma_k, \mathscr{O}_{\Sigma_k}(-k)) = 0$ and $H^{2k+1}(\Sigma_k, \mathscr{O}_{\Sigma_k}(-k-1)) \neq 0$. By Proposition 5.4 (2), Σ_k has log terminal singularities, and hence, it has rational singularities, i.e., $R^i\beta_{k,*}\mathscr{O}_{B^k(L)} = 0$ for $i > 0$. Then we obtain

$$H^{2k+1}(\Sigma_k, \mathscr{O}_{\Sigma_k}(\ell)) = H^{2k+1}(B^k(L), \mathscr{O}_{B^k(L)}(\ell)) = H^0(B^k(L), \omega_{B^k(L)}(-\ell))^*.$$

It is elementary to see that $H^0(B^k(L), \omega_{B^k(L)}(k)) = 0$ but $H^0(B^k(L), \omega_{B^k(L)}(k+1)) \neq 0$.

For (3), suppose that $g \geq 1$. It is enough to prove that $H^{2k+1}(\Sigma_k, \mathscr{O}_{\Sigma_k}) \neq 0$. By Theorem 5.8 (2), we find $H^{2k+1}(\Sigma_k, \mathscr{O}_{\Sigma_k}) = S^{k+1}H^0(C, \omega_C) \neq 0$. We finish the proof. □

5.4 Further properties of secant varieties

We have shown the main theorems of the paper. In this subsection, we discuss further properties of secant varieties of curves.

Proposition 5.11 *Let $k \geq 0$ be an integer, and L be a line bundle on C. Assume that*

$$\deg L \geq 2g + 2k + 1.$$

Consider the k-th secant variety $\Sigma_k = \Sigma_k(C, L)$ in the space $\mathbb{P}(H^0(C, L)) = \mathbb{P}^r$. Then one has the following:

(1) *The degree of $\Sigma_k \subseteq \mathbb{P}^r$ is given by*

$$\deg \Sigma_k = \sum_{i=0}^{\min(k+1,g)} \binom{\deg L - g - k - i}{k+1-i}\binom{g}{i}.$$

(2) *The multiplicity of Σ_k at a point $x \in \Sigma_m \setminus \Sigma_{m-1}$ with $0 \leq m \leq k$ is given by*

$$\begin{aligned}\operatorname{mult}_x \Sigma_k &= \deg \Sigma_{k-m-1}(C, L(-2\xi_{m+1,x}))\\ &= \sum_{i=0}^{\min(k-m,g)} \binom{\deg L - g - m - 1 - k - i}{k-m-i}\binom{g}{i}.\end{aligned}$$

Proof (1) follows from [19, Proposition 1]. In fact, $\deg \Sigma_k$ is the Segre class $s_{k+1}(E^*_{k+1,L})$. For (2), notice that $\operatorname{mult}_x \Sigma_k$ is the Segre class $s_0(\{x\}, \Sigma_k)$, which is invariant under a birational morphism. Recall that $F := \beta_k^{-1}(x) \cong$

C_{k-m} and $N_{F/B^k(L)} \cong \mathscr{O}_F^{\oplus 2m+1} \oplus E^*_{k-m,L(-2\xi_{m+1,x})}$ (Proposition 3.13 (2.a, 2.d)). Thus we have

$$\operatorname{mult}_x \Sigma_k = s_{k-m}(F, B^k(L)) = s_{k-m}(N_{F/B^k(L)}) = s_{k-m}(E^*_{k-m,L(-2\xi_{m+1,x})}).$$

Consider the secant variety $\Sigma_{k-m-1}(C, L(-2\xi_{m+1,x}))$ in the space $\mathbb{P}(H^0(C, L(-2\xi_{m+1,x})))$. Then we obtain

$$s_{k-m}(E^*_{k-m,L(-2\xi_{m+1,x})}) = \deg \Sigma_{k-m-1}(C, L(-2\xi_{m+1,x})),$$

which completes the proof by (1) since $\deg L(-2\xi_{m+1,x}) \geq 2g + 2(k-m-1) + 1$. $\square$

Next, we show that $B^k(L)$ is the normalization of the blowup of Σ_k along Σ_{k-1}. For this purpose, we prove the following lemma.

Lemma 5.12 *For any integer $k \geq 0$, one has the following:*

(1) $A_{k+1,L}$ *is globally generated if* $\deg L \geq 2g + 2k$.
(2) $A_{k+1,L}$ *is globally generated and ample if* $\deg L \geq 2g + 2k + 1$.

Proof For a point $p \in C$, consider the short exact sequence

$$0 \longrightarrow A_{k+1,L}(-X_p) \longrightarrow A_{k+1,L} \longrightarrow A_{k+1,L}|_{X_p} \longrightarrow 0.$$

Note that $A_{k+1,L}|_{X_p} = A_{k,L(-2p)}$ and $A_{k+1,L}(-X_p) = A_{k+1,L(-p)}$. By induction on k, we only need to show $H^1(C_{k+1}, A_{k+1,L(-p)}) = 0$. Pulling back the involved line bundle to C^{k+1} and applying Lemma 3.5, we can reduce the problem to prove the following cohomology vanishing

$$H^1(C^{k+1}, L^{\boxtimes k+1}(-\Delta_{k+1})) = 0 \;\text{ if }\; \deg L \geq 2g + 2k - 1. \tag{5.17}$$

If $k = 0$, then (5.17) is clear. Assume $k \geq 1$. Then L separates k points. Let $p\colon C^{k+1} \to C^k$ be the projection to the first k components. Then

$$p_* L^{\boxtimes k+1}(-\Delta_{k+1}) = Q_{k,L} \otimes L^{\boxtimes k}(-\Delta_k)$$

so that $H^1(C^{k+1}, L^{\boxtimes k+1}(-\Delta_{k+1})) = H^1(C^k, Q_{k,L} \otimes L^{\boxtimes k}(-\Delta_k))$. As $\deg L \geq 2g + 2k - 1 = 2g + 2(k-1) + 1$, the desired cohomology vanishing (5.17) follows from Theorem 4.1, proving (1). For (2), notice that $A_{k+1,L} = A_{k+1,L(-p)} \otimes T_{k+1}(\mathscr{O}_C(p))$. By (1), $A_{k+1,L(-p)}$ is globally generated, and we know that $T_{k+1}(\mathscr{O}_C(p))$ is ample. Hence (2) follows. $\square$

Proposition 5.13 *Let $k \geq 0$ be an integer, and L be a line bundle on C. Assume that*

$$\deg L \geq 2g + 2k + 1.$$

Consider the k-th secant variety $\Sigma_k = \Sigma_k(C, L)$ in the space $\mathbb{P}(H^0(C, L)) = \mathbb{P}^r$. Then one has the following:

(1) *$\beta_k \colon B^k(L) \to \Sigma_k$ factors through the blowup $\mathrm{Bl}_{\Sigma_{k-1}} \Sigma_k$ of Σ_k along Σ_{k-1}.*
(2) *$B^k(L)$ is the normalization of $\mathrm{Bl}_{\Sigma_{k-1}} \Sigma_k$.*
(3) *$\beta_{k,*}\mathscr{O}_{B^k(L)}(-mZ_{k-1}) = \overline{I^m_{\Sigma_{k-1}|\Sigma_k}}$ for $m \geq 0$, where $\overline{\mathfrak{a}}$ denotes the integral closure of an ideal sheaf $\mathfrak{a}$.*

Proof Recall the projection $\pi_k \colon B^k(L) \to C_{k+1}$. We write $\mathscr{O}_{B^k(L)}(H)$ to be the tautological bundle of $B^k(L)$, which also equals to $\beta_k^*\mathscr{O}_{\mathbb{P}^r}(1)$. For simplicity, we set $I := I_{\Sigma_k|\Sigma_{k-1}}$ and $Y := \mathrm{Bl}_{\Sigma_{k-1}} \Sigma_k$.

(1) It is enough to show that the natural morphism $\beta_k^* I \to \mathscr{O}_{B^k(L)}(-Z_{k-1})$ is surjective. Thus we only have to show $I \cdot \mathscr{O}_{B^k(L)} = \mathscr{O}_{B^k(L)}(-Z_{k-1})$. As we have seen in Proposition 3.15 (2) that $\mathscr{O}_{B^k(L)}((k+1)H - Z_{k-1}) = \pi_k^* A_{k+1,L}$, we can form the following commutative diagram

$$\begin{tikzcd} H^0(I(k+1)) \arrow[r, equal] \arrow[d] & H^0(\mathscr{O}_{B^k(L)}((k+1)H - Z_{k-1})) \arrow[d] \\ I \cdot \mathscr{O}_{B^k(L)}((k+1)H) \arrow[r] & \pi_k^* A_{k+1,L} . \end{tikzcd}$$

But $A_{k+1,L}$ is globally generated by Lemma 5.12. Therefore $I \cdot \mathscr{O}_{B^k(L)}((k+1)H) = \pi^* A_{k+1,L}$, which implies $I \cdot \mathscr{O}_{B^k(L)} = \mathscr{O}_{B^k(L)}(-Z_{k-1})$ as desired.
(2) We have the following factorization

$$\begin{tikzcd} & Y = \mathrm{Bl}_{\Sigma_{k-1}} \Sigma_k \arrow[d, "\varphi"] \\ B^k(L) \arrow[ur, "\alpha_k"] \arrow[r, "\beta_k"'] & \Sigma_k . \end{tikzcd}$$

Let E be the exceptional divisor on Y. As $I(k+1)$ is globally generated, $\varphi^*\mathscr{O}_{\Sigma_k}(k+1)(-E)$ is globally generated, and $\varphi^*\mathscr{O}_{\Sigma_k}(k+2)(-E)$ is very ample. For any point $x \in \Sigma_m \backslash \Sigma_{m-1}$, the fiber $\beta_k^{-1}(x) \cong C_{k-m}$

(Proposition 3.13 (2.a)). Let $\alpha_{k,x}\colon \beta_k^{-1}(x) \to \varphi^{-1}(x)$ be the induced morphism on fibers. We see that

$$\alpha_{k,x}^*(\varphi^*\mathscr{O}_{\Sigma_k}(k+2)(-E)) \cong A_{k+1,L}|_{C_{k-m}} \cong A_{k-m-1},\ L(-2\xi_{m+1,x}),$$

where $\xi_{m+1,x}$ is the unique degree $m+1$ divisor on C determined by x. But the last line bundle is ample by Lemma 5.12. So $\alpha_{k,x}$ is finite, and therefore, α_k is finite. Hence $B^k(L)$ is the normalization of Y.

(3) This is a direct consequence of (2).

Finally, we construct secant varieties of curves which are neither normal nor Cohen–Macaulay when $\deg L = 2g+2k < 2g+2k+1$. This shows that the degree bounds on embedding line bundle in Theorems 1.1 and 1.2 are optimal.

Example 5.14 Let $k \geq 1$ be an integer, and C be a nonsingular projective curve of genus $g \geq 2k+2$. Take an effective divisor D consisting of $2k+2$ general points of C such that $h^0(C, \mathscr{O}_C(D)) = 1$. Consider a very ample line bundle

$$L = \omega_C(D) \quad \text{with} \quad \deg L = 2g+2k.$$

Observe that L separates $2k+1$ points, and L separates $2k+2$ points except of D. We show that the k-th secant variety

$$\Sigma_k = \Sigma_k(C, L) \subseteq \mathbb{P}(H^0(C, L)) = \mathbb{P}^{g+2k}$$

is neither normal nor Cohen–Macaulay.

For any effective divisor ξ on C, we denote by Λ_ξ the linear space spanned by ξ in the space $\mathbb{P}^{g+2k}$. Let D_1 and D_2 be two effective divisors of degree $k+1$ such that $D_1 + D_2 = D$. By Riemann–Roch, $h^0(C, L(-D_1 - D_2)) = g$. Thus $D_1 + D_2$ span a linear space $\Lambda_{D_1+D_2}$ of dimension $2k$. This means that Λ_{D_1} and Λ_{D_2} span $\Lambda_{D_1+D_2}$ and intersect at a single point $q \in \Sigma_k \backslash C$. Let Z be an effective divisor of degree $k+1$, and suppose $D_1 + Z \neq D$. Then L separates $D_1 + Z$, and therefore, the space Λ_{D_1+Z} has dimension $2k+1$. Hence $\Lambda_{D_1} \cap \Lambda_Z = \emptyset$. This implies that $q \in \Sigma_k \backslash \Sigma_{k-1}$ and except of Λ_{D_1} and Λ_{D_2}, there is no any other $(k+1)$-secant k-plane of C passing through q. For any two degree $k+1$ effective divisors D_1' and D_2' such that $D_1' + D_2' = D$, the k-secant planes $\Lambda_{D_1'}$ and $\Lambda_{D_2'}$ intersect at a single point in $\Sigma_k \backslash \Sigma_{k-1}$. Let Q be the set of all such intersection points. Then Q contains only finitely many points.

Consider the morphism $\beta_k\colon B^k(L) \to \Sigma_k$. Let $x \in \Sigma_k \backslash \Sigma_{k-1}$. If $x \in Q$, then the fiber $\beta_k^{-1}(x)$ contains two points. If $x \notin Q$, then the fiber $\beta_k^{-1}(x)$ contains only one point y. In this case, we can show that the induced morphism

$\beta_k^{\#}\colon T_x^*\mathbb{P}^r \longrightarrow \mathfrak{m}_{B^k(L),y}/\mathfrak{m}^2_{B^k(L),y}$ on cotangent spaces is surjective. Therefore β_k is unramified at y, so it is isomorphic over x. In conclusion, β_k is an isomorphism over $\Sigma_k \backslash (\Sigma_{k-1} \cup Q)$. Then we have the short exact sequence

$$0 \longrightarrow \mathscr{O}_{\Sigma_k} \longrightarrow \beta_{k,*}\mathscr{O}_{B^k(L)} \longrightarrow \mathscr{Q} \longrightarrow 0,$$

where the support of the quotient sheaf $\mathscr{Q}$ has zero-dimensional components supported on Q. This means that Σ_k is not normal at any point in Q. Moreover, $H^1(\Sigma_k, \mathscr{O}_{\Sigma_k}(-\ell)) \neq 0$ for all $\ell \geq 0$, so Σ_k is not Cohen–Macaulay.

6 Open problems

To conclude this paper, we present a number of open problems. We keep using notations introduced before; thus C is a nonsingular projective curve of genus g embedded by a very ample line bundle L in the space $\mathbb{P}(H^0(C, L)) = \mathbb{P}^r$.

One of critical steps in the proof of the main results is to establish the Du Bois type condition (1.2). We have shown that $B^k(L)$ is the normalization of the blowup of Σ_k along Σ_{k-1}. For better understanding of the geometry of $B^k(L)$, one observes that if $k = 1$, then the variety $B^1(L)$ is indeed the blowup of Σ_1 along the curve C. This leads us to ask the following:

Problem 6.1 Can the secant bundle $B^k(L)$ be realized as the blowup of Σ_k along Σ_{k-1}?

The Danila's theorem (Theorem 3.8) handles the initial steps of projectively normality of secant varieties. It gives precise values of global sections of the symmetric products of the secant bundle $E_{k+1,L}$. On the other hand, the techniques used in Sect. 4 may offer an alternative approach to compute cohomology groups of the symmetric products of $E_{k+1,L}$. As an independent question, we wonder if one can deal with the following:

Problem 6.2 Compute cohomology groups of the symmetric products of the secant bundle $E_{k+1,L}$ on C_{k+1}.

If we view the classic theorem of Ein–Lazarsfeld [4] as a higher dimensional generalization of Green's result in [10], then we may ask a similar generalization of the results of the present paper to higher dimensional varieties. For a nonsingular projective variety X, consider the adjoint line bundle $L = K_X + dA$ where A is an ample line bundle and d is a natural number. For d sufficiently large, L embeds X into a projective space. We expect that in this case the secant varieties of X would have nice geometric and algebraic properties.

Problem 6.3 Extend the results of present paper to secant varieties of a non-singular projective variety X embedded in a projective space by a sufficiently positive line bundle.

This problem has two major essential difficulties. First of all, there is no a good construction involving secant bundles as the one in Bertram's work [1]. Secondly, the projectively normality of X embedded by the adjoint line bundle is still unsolved. One may further impose the condition that A is very ample so [4] can be applied or may follow the idea in [5] to study the asymptotic behavior of secant varieties. However, the surface case seems a reasonable starting point toward the arbitrary dimensional case.

Problem 6.4 Study secant varieties of a surface X embedded by the adjoint line bundle $K_X + dA$ where A is ample and d is a large integer.

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