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A remark on global sections of secant bundles of curves

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Abstract
We compute global sections of tensor products and symmetric products of several secant bundles of a nonsingular projective curve. We closely follow the approach of Gentiana Danila's work on the similar problem for nonsingular projective surfaces.

Keywords Secant bundles · Hilbert scheme · Symmetric products of vector bundles

Mathematics Subject Classification 14C05 · 14F05 · 14Q20

1 Introduction

Throughout this paper, we work over the field $\mathbb{C}$ of complex numbers. In the recent work [2] on secant varieties of curves, we use a result [2, Theorem 3.8] on symmetric products of secant bundles of curves without including a proof because one can easily adapt the proof given by Danila in [1] for the same result for surface case. However, it would be helpful to work out the details of the proof of [2, Theorem 3.8] in order to complete the literature even though the original idea is certainly due to Danila. We also slightly generalize this result to allow the mixed case involving several secant bundles.

Let X be a nonsingular projective curve, and fix an integer $n \geq 1$. Write $H := \mathrm{Hilb}^n X$ to be the Hilbert scheme parameterizing length n subschemes of X. Although H is the same as the n-th symmetric product X_n of the curve X, we use the notation $\mathrm{Hilb}^n X$ in this paper

Dedicated to Professor Fabrizio Catanese on the occasion of his 70th birthday.

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to be consistent with the notation used in [1]. Let Z be the universal family over H so that Z is the incidence subscheme in the product $H \times X$. One has the following diagram

$$\begin{array}{ccc} Z & \xrightarrow{\beta} & X \\ {\scriptstyle \alpha}\big\downarrow & & \\ H & & \end{array}$$

where the morphisms α and β are the restriction onto Z of the projections of $H \times X$ to the components, respectively. For a line bundle L on X, the *secant bundle* E_L associated to L is defined by

$$E_L := \alpha_*(\beta^* L)$$

which is a rank n locally free sheaf on the Hilbert scheme H. For any $\xi \in H$, the fiber E_L over ξ is $H^0(\xi, L|_\xi)$. Note also that $H^0(H, E_L) = H^0(X, L)$.

Let $\mathscr{L} = \{L_1, \ldots, L_k\}$ be a collection of (not necessarily distinct) k line bundles L_i on X with $k \geq 1$, and $P = \{S_1, \ldots, S_m\}$ be a partition of the set $\{1, \ldots, k\}$ of length m with $m \geq 1$. For each $1 \leq i \leq m$, we set

$$\mathscr{L}^{S_i} := \bigotimes_{\alpha \in S_i} L_\alpha$$

which is a line bundle on X. The following is the main theorem of this paper, generalizing [2, Theorem 3.8].

Theorem 1.1 *Let X be a nonsingular projective curve, and $H := \mathrm{Hilb}^n X$ be the Hilbert scheme of n points on X with $n \geq 1$. Let $\mathscr{L} = \{L_1, \ldots, L_k\}$ be a collection of (not necessarily distinct) k line bundles L_i on X with $k \geq 1$, and $P = \{S_1, \ldots, S_m\}$ be a partition of the set $\{1, \ldots, k\}$ of length m with $m \geq 1$. If $m \leq n$, then one has*

$$H^0(H, E_{\mathscr{L}^{S_1}} \otimes \cdots \otimes E_{\mathscr{L}^{S_m}}) = H^0(X, \mathscr{L}^{S_1}) \otimes \cdots \otimes H^0(X, \mathscr{L}^{S_m}).$$

In particular, for a line bundle L on X and for $1 \leq k \leq n$, one has

$$H^0(H, E_L^{\otimes k}) = H^0(X, L)^{\otimes k}.$$

Here we state a useful consequence of the theorem. A special case of the following corollary is used in [2] to show the projectively normality of secant varieties of nonsingular projective curves.

Corollary 1.2 *Let $L_1, \ldots, L_m$ be line bundles on a nonsingular projective curve X with $m \geq 1$, and $H := \mathrm{Hilb}^n X$ be the Hilbert scheme of n points on X with $n \geq 1$. For integers $k_1, \ldots, k_m \geq 1$ with $\sum_{i=1}^m k_i \leq n$, one has*

$$H^0(H, S^{k_1} E_{L_1} \otimes \cdots \otimes S^{k_m} E_{L_m}) = S^{k_1} H^0(X, L_1) \otimes \cdots \otimes S^{k_m} H^0(X, L_m).$$

In particular, for a line bundle L on X and for $1 \leq k \leq n$, one has

$$H^0(H, S^k E_L) = S^k H^0(X, L).$$

The proofs of the main theorem and its corollary are given in the next section.

2 Proof of the main results

Let X be a nonsingular projective curve, $H := \mathrm{Hilb}^n X$ be the Hilbert scheme of n points on X with $n \geq 1$, and $\alpha\colon Z \to H$ be the universal family with the projection $\beta\colon Z \to X$. For an integer $k \geq 1$, let

$$X^k := \underbrace{X \times \cdots \times X}_{k \text{ times}}$$

be the k-th ordinary product of X with projection $p_i\colon X^k \to X$ to the i-th component of X^k for each $1 \leq i \leq k$. Define the k-th fiber product

$$(Z/H)^k := \underbrace{Z \times_H \cdots \times_H Z}_{k \text{ times}}.$$

with projection $q_i\colon (Z/H)^k \to Z$ to the i-th component of $(Z/H)^k$ for each $1 \leq i \leq k$. We may regard $(Z/H)^k$ as the incidence subscheme of $H \times X^k$, and we have set-theoretically

$$(Z/H)^k = \{(\xi, x_1, \ldots, x_k) \mid x_i \in \xi,\ \xi \in H\} \subseteq H \times X^k.$$

One obtains a diagram

$$\begin{array}{ccc} (Z/H)^k & \xrightarrow{\beta_k} & X^k \\ {\scriptstyle \alpha_k}\big\downarrow & & \\ H & & \end{array}$$

where α_k and β_k are induced by α and β respectively in natural way. Notice that α_k and β_k are surjective.

Proposition 2.1 *One has the following:*

(1) *The map α_k is finite, flat, and generically smooth.*
(2) *The scheme $(Z/H)^k$ is Cohen-Macaulay and reduced.*
(3) *For k line bundles $L_1, \ldots, L_k$ on X, one has*

$$\alpha_{k,*}\Big(\beta_k^*(p_1^* L_1 \otimes \cdots \otimes p_k^* L_k)\Big) = E_{L_1} \otimes \cdots \otimes E_{L_k}.$$

Proof (1) When $k = 1$, clearly $\alpha_1 = \alpha\colon Z \to H$ is flat and generically smooth. For any $k \geq 2$, we have a commutative diagram

$$\begin{array}{ccc} (Z/H)^k & \xrightarrow{\gamma_k} & (Z/H)^{k-1} \\ {\scriptstyle q_k}\big\downarrow & \searrow^{\alpha_k} & \big\downarrow{\scriptstyle \alpha_{k-1}} \\ Z & \xrightarrow[\alpha]{} & H \end{array}$$

where $\gamma_k\colon (Z/H)^k \to (Z/H)^{k-1}$ is the projection to the first $k-1$ components of $(Z/H)^k$. Note that $(Z/H)^k = (Z/H)^{k-1} \times_H Z$. By induction, we may assume that α_{k-1} is finite, flat, and generically smooth. By base change, q_k is finite, flat, and generically smooth, and so is $\alpha_k = \alpha \circ q_k$.
(2) The assertion (1) immediately implies that $(Z/H)^k$ is Cohen-Macaulay. In particular, $(Z/H)^k$ satisfies S_1 condition. The generic smoothness of α_k implies that $(Z/H)^k$ satisfies the condition R_0 so that it is reduced.

(3) This is a direct consequence of Künneth formula. Alternatively, one can prove it by induction as follows. When $k = 1$, it is trivial. We assume that $k \geq 2$ and

$$\alpha_{k-1,*}\beta_{k-1}^*(p_1^*L_1 \otimes \cdots \otimes p_{k-1}^*L_{k-1}) = E_{L_1} \otimes \cdots \otimes E_{L_{k-1}}.$$

We have

$$\beta_k^*(p_1^*L_1 \otimes \cdots \otimes p_{k-1}^*L_{k-1} \otimes p_k^*L_k) = \gamma_k^*\beta_{k-1}^*(p_1^*L_1 \otimes \cdots \otimes p_{k-1}^*L_{k-1}) \otimes q_k^*\beta^*L_k$$

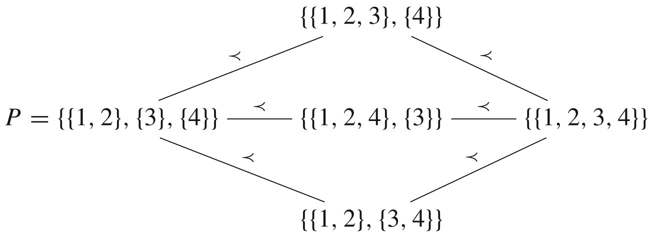
Note that $q_{k,*}\gamma_k^*\mathcal{F} = \alpha^*\alpha_{k-1,*}\mathcal{F}$ for any coherent sheaf $\mathcal{F}$ on $(Z/H)^{k-1}$ by base change. By the projection formula, we have

$$\begin{aligned}
&\alpha_{k,*}\Big(\beta_k^*(p_1^*L_1 \otimes \cdots \otimes p_{k-1}^*L_{k-1} \otimes p_k^*L_k)\Big)\\
&= \alpha_*q_{k,*}\Big(\gamma_k^*\beta_{k-1}^*(p_1^*L_1 \otimes \cdots \otimes p_{k-1}^*L_{k-1}) \otimes q_k^*\beta^*L_k\Big)\\
&= \alpha_*\Big(q_{k,*}\gamma_k^*\beta_{k-1}^*(p_1^*L_1 \otimes \cdots \otimes p_{k-1}^*L_{k-1}) \otimes \beta^*L_k\Big)\\
&= \alpha_*\Big(\alpha^*\alpha_{k-1,*}\beta_{k-1}^*(p_1^*L_1 \otimes \cdots \otimes p_{k-1}^*L_{k-1}) \otimes \beta^*L_k\Big)\\
&= \alpha_{k-1,*}\beta_{k-1}^*(p_1^*L_1 \otimes \cdots \otimes p_{k-1}^*L_{k-1}) \otimes \alpha_*\beta^*L_k\\
&= E_{L_1} \otimes \cdots \otimes E_{L_{k-1}} \otimes E_{L_k}.
\end{aligned}$$

as desired. □

Definition 2.2 Let P and Q be two partitions of a set. We say that P is a *refinement* of Q and we write $P \prec Q$ if P can be obtained by splitting sets in Q by finitely many steps. Equivalently, $P \prec Q$ if every set in P is a subset of a set in Q. In this case, we also say that Q is a *coarsening* of P.

Example 2.3 Consider a partition $P = \{\{1, 2\}, \{3\}, \{4\}\}$ of the set $\{1, 2, 3, 4\}$. We list all coarsening of P as follows:



In the rest of this section, we use the following notations. For an integer $k \geq 1$, let

$$\mathscr{L} = \{L_1, \ldots, L_k\}$$

be a collection of (not necessarily distinct) line bundles L_i on X, and

$$P = \{S_1, \ldots, S_m\}$$

be a partition of $\mathbb{N}_k := \{1, 2, \ldots, k\}$ of length $m \leq n$ so that S_i are pairwise disjoint subsets of $\mathbb{N}_k$ and $S_1 \cup \cdots \cup S_m = \mathbb{N}_k$. For a set S, we denote by $|S|$ the number of elements in S. For each $1 \leq i \leq m$, we set $\mathscr{L}^{S_i} := \bigotimes_{\alpha \in S_i} L_\alpha$, which is a line bundle on X.

The partition $P = \{S_1, \ldots, S_m\}$ of $\mathbb{N}_k$ can be used to define a subscheme of the variety X^k. All such subschemes arose from partitions induce a stratification of X^k, which further induces a stratification of the reduced scheme $(Z/H)^k$. To see this, for each $S_i \in P$, we define the following closed subscheme of X^k:

$$\Delta_{S_i} := \begin{cases} X^k & \text{if } |S_i| = 1, \\ \{(x_1, \ldots, x_k) \mid x_a = x_b \text{ if } a, b \in S_i\} & \text{if } |S_i| \geq 2. \end{cases}$$

Then we define a closed subscheme Δ_P of X^k associated to the partition P as

$$\Delta_P := \Delta_{S_1} \cap \cdots \cap \Delta_{S_m}.$$

We also define an open subscheme Δ_P° of Δ_P as

$$\Delta_P^\circ := \{(x_1, \cdots, x_k) \in \Delta_P \mid x_a \neq x_b \text{ if } a \in S_i, b \in S_j \text{ and } i \neq j\}.$$

It is clear that $\Delta_P \cong X^m$ and Δ_P° is isomorphic to $(X^m)^\circ$ that has closed points with distinct coordinates. Then the family

$$\{\Delta_P \mid P \text{ is a partition of } \mathbb{N}_k\}$$

gives a stratification of X^k. It is easy to see that for a partition Q, if $P \prec Q$, then $\Delta_Q \subseteq \Delta_P$.

Example 2.4 Consider X^3 with $k = 3$. We have four partitions of $\mathbb{N}_3$: $P_1 = \{\{1\}, \{2\}, \{3\}\}$, $P_2 = \{\{1, 2\}, \{3\}\}$, $P_3 = \{\{1, 3\}, \{2\}\}$, $P_4 = \{\{2, 3\}, \{1\}\}$, $P_5 = \{\{1, 2, 3\}\}$. Then we have $\Delta_{P_1} = X^3$, $\Delta_{P_2} = \Delta_{1,2}$, $\Delta_{P_3} = \Delta_{1,3}$, $\Delta_{P_4} = \Delta_{2,3}$, $\Delta_{P_5} = \Delta_{1,2,3}$.

Now, we show how the stratification of X^k induces a stratification of $(Z/H)^k$, on which our cohomological computation will be done. For any partition P of $\mathbb{N}_k$, we define subschemes of $(Z/H)^k$:

$$W_P^\circ := \beta_k^{-1}(\Delta_P^\circ) \text{ and } W_P := \overline{W_P^\circ}.$$

Then the family

$$\{W_P \mid P \text{ is a partition of } \mathbb{N}_k\}$$

gives a stratification of $(Z/H)^k$.

Proposition 2.5 *One has the following:*

(1) *The restriction morphism*

$$\beta_k|_{W_P^\circ} : W_P^\circ \longrightarrow \Delta_P^\circ$$

has fibers set-theoretically equal to $\mathrm{Hilb}^{n-m} X$.

(2) *W_P is irreducible of dimension n, and $\beta_k(W_P) = \Delta_P$.*

(3) *$\beta_{P,*}\mathscr{O}_{W_P} = \mathscr{O}_{\Delta_P}$ where $\beta_P := \beta_k|_{W_P}$.*

Proof Recall $P = \{S_1, \ldots, S_m\}$. Let $x = (x_1, \ldots, x_k) \in \Delta_P^\circ$ be a closed point so that each x_a is a closed point of X and $x_a \neq x_b$ if $a \in S_i$, $b \in S_j$ and $i \neq j$. Take m distinct points on X in the coordinates of x to form a degree m divisor $\xi_x \in \mathrm{Hilb}^m X$. Now, $\alpha_k|_{(\beta_k|_{W_P^0})^{-1}(x)} : (\beta_k|_{W_P^0})^{-1}(x) \to H$ is an injective map, and

$$\alpha_k\big((\beta_k|_{W_P^0})^{-1}(x)\big) = \{\delta + \xi_x \mid \delta \in \mathrm{Hilb}^{n-m} X\}.$$

This proves (1). Thus W_P° is irreducible of dimension n, and then, (2) follows immediately.

To prove (3), we consider the Stein factorization $W_P \xrightarrow{f} \Delta' \xrightarrow{h} \Delta_P$ of the morphism β_P. Then $f_*\mathscr{O}_{W_p} = \mathscr{O}_{\Delta'}$, and h is a finite morphism. As we showed above that a generic fiber of β_P is connected, the morphism h is birational and finite. But Δ_P is nonsingular, and therefore, h must be an isomorphism. This proves (3). □

Remark 2.6 One should note that the fibers of the morphism $\beta_P \colon W_P \to \Delta_P$ are not all equal to $\mathrm{Hilb}^{n-m} X$. Fibers over the points in $\Delta_P - \Delta_P^\circ$ could have different dimensions.

Proposition 2.5 says that the partition $P = \{S_1, \dots, S_m\}$ of $\mathbb{N}_k$ gives an irreducible component W_P of $(Z/H)^k$. Let

$$W_P^+ := \bigcup_{P \prec Q} W_Q \quad \text{and} \quad \beta_P^+ := \beta_k|_{W_P^+}.$$

It is easy to see that both morphisms

$$\beta_P^+ \colon W_P^+ \longrightarrow \Delta_P \quad \text{and} \quad \beta_P \colon W_P \longrightarrow \Delta_P$$

are surjective. We define a function l_P for the partition P by

$$l_P \colon \{1, \dots, k\} \longrightarrow \{1, \dots, m\}$$

such that $l(i) = j$ if $i \in S_j$. Certainly the function l_p depends on the ordering of the sets S_j in P. But it should be clear in the rest of the paper that this is not essential. Then we obtain a closed embedding

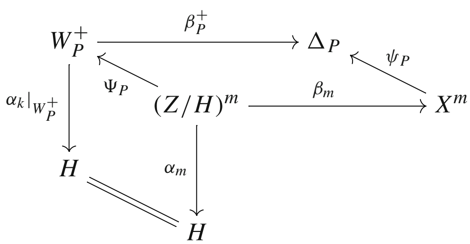
$$\psi_P \colon X^m \longrightarrow X^k, \quad (x_1, \dots, x_m) \longmapsto (x_{l(1)}, \dots, x_{l(k)}).$$

The map $\psi_P \colon X^m \to \psi_P(X_m) = \Delta_P$ is an isomorphism. Then ψ_P induces a closed embedding

$$\Psi_P \colon H \times X^m \longrightarrow H \times X^k, \quad (\xi, x_1, \dots, x_m) \longmapsto (\xi, x_{l(1)}, \dots, x_{l(k)}).$$

Proposition 2.7 *One has the following:*

(1) $\Psi_P|_{(Z/H)^m} \colon (Z/H)^m \to \Psi_P((Z/H)^m) = W_P^+$ *is an isomorphism, and it fits into the following commutative diagram:*



(2) *There is a one-to-one correspondence between the coarsenings of P and the irreducible components of W_P^+.*
(3) *One has*

$$\alpha_k|_{W_P^+,*}((\beta_k^*(p_1^*L_1 \otimes \cdots \otimes p_k^*L_k))|_{W_P^+}) = E_{\mathscr{L}^{S_1}} \otimes \cdots \otimes E_{\mathscr{L}^{S_m}}.$$

(4) *One has*

$$\begin{aligned}\beta_{P,*}\big(\beta_k^*(p_1^*L_1\otimes\cdots\otimes p_k^*L_k)\big)|_{W_P} &= (p_1^*L_1\otimes\cdots\otimes p_k^*L_k)|_{\Delta_P}\\ &= p_1^*\mathscr{L}^{S_1}\otimes\cdots\otimes p_m^*\mathscr{L}^{S_m}\end{aligned}$$

where we identify Δ_P with X_m via the morphism ψ_P in the second identification.

Proof (1) and (2): Observe that any coarsening Q of P can be obtained by combining finitely many sets in P. In this way, there is a one-to-one correspondence between the coarsenings of P and the partitions of the set $\mathbb{N}_m=\{1,\ldots,m\}$. To prove the results (1) and (2), it is sufficient to establish that there is a one-to-one correspondence between the partitions of $\mathbb{N}_m$ and the irreducible components of $(Z/H)^m$.

Let P' be a partition of $\mathbb{N}_m$. Applying Proposition 2.5 (2) with $k=m$, we see that the subvariety $W_{P'}$ of $(Z/H)^m$ is an irreducible component of $(Z/H)^m$. It is clear that different partitions give rise to different irreducible components. Conversely, let W be an irreducible component of $(Z/H)^m$. Since $(Z/H)^m$ has only dimension n irreducible components, there exists a dense open subset U of W such that $\alpha_m(U)$ is dense in H and U does not touch any other irreducible component of $(Z/H)^m$. Let $\xi\in H$ be a general point so that it consists of n distinct points of X. Take any $(\xi,x_1,\ldots,x_m)\in\alpha_m^{-1}(\xi)\cap U$. We then obtain a partition P' by the rule that i and j belong to the same subset of $\mathbb{N}_m$ if and only if $x_i=x_j$. Observe that $\beta_m(\xi,x_1,\ldots,x_m)=(x_1,\ldots,x_m)\in\Delta_{P'}^{\circ}$. This shows that the point $(\xi,x_1,\ldots,x_m)\in W_{P'}$. Since U is an open subset of an irreducible component W of $(Z/H)^m$, it follows that $W=W_{P'}$.
(3) By (1) and Proposition 2.1 (3), we have

$$\begin{aligned}\alpha_k|_{W_P^+,*}((\beta_k^*(p_1^*L_1\otimes\cdots\otimes p_k^*L_k))|_{W_P^+}) &= \alpha_k|_{W_P^+,*}\beta_P^{+,*}((p_1^*L_1\otimes\cdots\otimes p_k^*L_k)|_{\Delta_P})\\ &= \alpha_{m,*}\beta_m^*\psi_P^*\big((p_1^*L_1\otimes\cdots\otimes p_k^*L_k)|_{\Delta_P}\big)\\ &= \alpha_{m,*}\beta_m^*(p_1^*\mathscr{L}^{S_1}\otimes\cdots\otimes p_m^*\mathscr{L}^{S_m})\\ &= E_{\mathscr{L}^{S_1}}\otimes\cdots\otimes E_{\mathscr{L}^{S_m}}.\end{aligned}$$

(4) By considering the commutative diagram

$$\begin{array}{ccc}(Z/H)^k & \xrightarrow{\beta_k} & X^k\\ \uparrow & & \uparrow\\ W_P & \xrightarrow{\beta_P} & \Delta_P\end{array}$$

we have

$$\big(\beta_k^*(p_1^*L_1\otimes\cdots\otimes p_k^*L_k)\big)|_{W_P}=\beta_P^*\big((p_1^*L_1\otimes\cdots\otimes p_k^*L_k)|_{\Delta_P}\big).$$

Since $\beta_{P,*}\mathscr{O}_{W_P}=\mathscr{O}_{\Delta_P}$ by Proposition 2.5 (3), the first identification follows. Now, the second identification is clear. ◻

Proposition 2.8 *Let $s\in H^0(W_P^+,(\beta_k^*(p_1^*L_1\otimes\cdots\otimes p_k^*L_k))|_{W_P^+})$ be a section. If $s|_{W_P}=0$, then for any coarsening Q of P, one has $s|_{W_Q}=0$. In particular, the restriction map*

$$H^0(W_P^+,(\beta_k^*(p_1^*L_1\otimes\cdots\otimes p_k^*L_k))|_{W_P^+})\longrightarrow H^0(W_P,(\beta_k^*(p_1^*L_1\otimes\cdots\otimes p_k^*L_k))|_{W_P})$$

is injective.

Proof We have $q := |Q| \leq m = |P|$. We proceed by the reverse induction on q. It is clear that if $q = m$, then $Q = P$ and the result holds. We assume that $q < |P|$. Possibly by reordering indices, we may assume that $Q = \{S'_1, \dots, S'_q\}$ such that at least two subsets S_1, S_2 in P are contained in S'_1. By splitting S'_1 into two subsets S''_1, S''_2 such that $S_1 \subseteq S''_1, S_2 \subseteq S''_2$, we obtain a refinement $Q' = \{S''_1, S''_2, \dots, S''_{q+1}\}$ of Q with $S''_{i+1} = S'_i$ for all $2 \leq i \leq q$. Note that Q' is a coarsening of P. By the induction hypothesis, $s|_{W_{Q'}} = 0$.

Claim 2.8.1 There is a subvariety $C \subseteq W_Q \cap W_{Q'}$ such that $\beta_k(C) = \Delta_Q$.

We consider a morphism

$$f_{Q'}: X^n \longrightarrow H \times X^k, \ \ (x_1, \dots, x_n) \longmapsto \Psi_{Q'}(x_1 + \cdots + x_n, x_1, \dots, x_{q+1}).$$

Note that $f_{Q'}(X^n) = W_{Q'}$. Now consider $\Delta_{1,2} \subseteq X^n$. Let $C := f_{Q'}(\Delta_{1,2}) \subseteq W_{Q'}$. Then one can check that $\beta_k(C) = \Delta_Q$ and $C \subseteq W_Q$. We have shown the claim.

We have the following commutative diagram induced by restriction maps on global sections

$$\begin{array}{ccc} H^0(W_P^+, (\beta_k^*(p_1^*L_1 \otimes \cdots \otimes p_k^*L_k))|_{W_P^+}) & \longrightarrow & H^0(W_{Q'}, (\beta_k^*(p_1^*L_1 \otimes \cdots \otimes p_k^*L_k))|_{W_{Q'}}) \\ \downarrow & & \downarrow \\ H^0(W_Q, (\beta_k^*(p_1^*L_1 \otimes \cdots \otimes p_k^*L_k))|_{W_Q}) & \hookrightarrow & H^0(C, (\beta_k^*(p_1^*L_1 \otimes \cdots \otimes p_k^*L_k))|_C) \end{array}$$

where the bottom map is injective by Proposition 2.7 (4) and Claim 2.8.1. Since $s|_{W_{Q'}} = 0$, it follows that $s|_{W_Q} = 0$, completing the proof. □

We are ready to prove the main theorem and its corollary.

Proof of Theorem 1.1 For simplicity, we write $L_1 \boxtimes \cdots \boxtimes L_k = p_1^*L_1 \otimes \cdots \otimes p_k^*L_k$. First of all, consider the subvariety W_P and the subscheme W_P^+ of $(Z/H)^k$ with the induced surjective morphisms $\beta_P^+ : W_P^+ \to \Delta_P$ and $\beta_P : W_P \to \Delta_P$. Since there is a natural injective map $\mathscr{O}_{\Delta_P} \to \beta_{P,*}^+ \mathscr{O}_{W_P^+}$, we obtain an injective map

$$(L_1 \boxtimes \cdots \boxtimes L_k)|_{\Delta_P} \longrightarrow \beta_{P,*}^+ \beta_P^{+,*}((L_1 \boxtimes \cdots \boxtimes L_k)|_{\Delta_P}) = \beta_{P,*}^+((\beta_k^*(L_1 \boxtimes \cdots \boxtimes L_k))|_{W_P^+}). \tag{1}$$

Then we have the following commutative diagram on the global sections

$$\begin{array}{ccc} H^0(\Delta_P, (L_1 \boxtimes \cdots \boxtimes L_k)|_{\Delta_P}) & \xleftarrow{\ u\ } & \\ \Big\downarrow w & & \\ H^0(W_P^+, (\beta_k^*(L_1 \boxtimes \cdots \boxtimes L_k))|_{W_P^+}) & \xhookrightarrow{\ v\ } & H^0(W_P, (\beta_k^*(L_1 \boxtimes \cdots \boxtimes L_k))|_{W_P}) \end{array}$$

where w is an injection since (1) is an injection, u is an isomorphism by Proposition 2.7 (4), and v is an injection by Proposition 2.8. Hence the following three groups are all the same

$$\begin{aligned} H^0(W_P^+, (\beta_k^*(L_1 \boxtimes \cdots \boxtimes L_k))|_{W_P^+}) &= H^0(W_P, (\beta_k^*(L_1 \boxtimes \cdots \boxtimes L_k))|_{W_P}) \\ &= H^0(\Delta_P, (L_1 \boxtimes \cdots \boxtimes L_k)|_{\Delta_P}). \end{aligned}$$

On the other hand, by Proposition 2.7 (3), we have

$$H^0(W_P^+, (\beta_k^*(L_1 \boxtimes \cdots \boxtimes L_k))|_{W_P^+}) = H^0(H, E_{\mathscr{L}^{S_1}} \otimes \cdots \otimes E_{\mathscr{L}^{S_m}}),$$

and we have

$$H^0(\Delta_P, (L_1 \boxtimes \cdots \boxtimes L_k)|_{\Delta_P}) = H^0(X, \mathscr{L}^{S_1}) \otimes \cdots \otimes H^0(X, \mathscr{L}^{S_m}).$$

The theorem then follows immediately. □

Proof of Corollary 1.2 Let $k := \sum_{i=1}^m k_i$, and take the partition $P = \{\{1\}, \{2\}, \cdots, \{k\}\}$. Let

$$\mathscr{L} := \{\underbrace{L_1, \ldots, L_1}_{k_1 \text{ times}}, \ldots, \underbrace{L_m, \ldots, L_m}_{k_m \text{ times}}\}.$$

By Theorem 1.1, we have

$$H^0(H, E_{L_1}^{\otimes k_1} \otimes \cdots \otimes E_{L_m}^{\otimes k_m}) = H^0(X, L_1)^{\otimes k_1} \otimes \cdots \otimes H^0(X, L_m)^{\otimes k_m}. \tag{2}$$

Now, consider the action of $G := \mathbb{S}_{k_1} \times \cdots \times \mathbb{S}_{k_m}$ on the tensor product $E_{L_1}^{\otimes k_1} \otimes \cdots \otimes E_{L_m}^{\otimes k_m}$ in the natural way. Notice that the identification in (2) is G-equivariant. Thus we obtain

$$H^0(H, S^{k_1} E_{L_1} \otimes \cdots \otimes S^{k_m} E_{L_m}) = S^{k_1} H^0(X, L_1) \otimes \cdots \otimes S^{k_m} H^0(X, L_m)$$

as desired. □

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