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Extension of multivalued holomorphic functions on a Stein space

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Abstract

A version of the classical Kerner's theorem for a singular Stein space Ω with a compact strongly pseudoconvex boundary has been recently established by Huang–Xiao (J Reine Angew Math 770:183–203, 2021) when $\dim_{\mathbb{C}}\Omega \geq 3$. A partial result for the case of complex dimension two was also obtained in Huang and Xiao (J Reine Angew Math 770:183–203, 2021). In this paper, we answer the two dimensional case left open in Huang and Xiao (J Reine Angew Math 770:183–203, 2021) in its full generality.

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1 Introduction

Let Ω be a Stein space of complex dimension $n \geq 2$ with possibly isolated singularities in its interior and with a connected compact strongly pseudoconvex smooth boundary $M = \partial\Omega$. We denote by $\mathrm{Sing}(\Omega)$ the set of singularities of Ω. Write $\overline{\Omega} = \Omega \cup M$, $\mathrm{Reg}(\Omega) = \Omega \setminus \mathrm{Sing}(\Omega)$, $\mathrm{Reg}(\overline{\Omega}) = \overline{\Omega} \setminus \mathrm{Sing}(\Omega)$. If Ω does not have singularity, the classical Kerner's theorem can be used to prove that if a germ of CR function can be extended along any curve in $\partial\Omega$ then the CR germ can be holomorphically continued along any curve in $\overline{\Omega}$. Kerner's argument is of global nature and relies

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on the smoothness of the manifold. The Kerner-type extension theorem has important applications in several complex variables such as the uniformization of strongly psedo-convex domains, construction of hyperbolic metrics with constant sectional curvature and proof of the Cheng's conjecture (see [3, 6, 8, 13] and the references therein). Recently, Huang–Xiao [6] generalized the Kerner-type extension theorem to Stein spaces with singularities. They proved that if a germ of CR function can be CR continued along any curve in M, then it can be holomorphically extended along any curve in $\mathrm{Reg}(\overline{\Omega})$ when $\dim_{\mathbb{C}}\Omega \geq 3$. However, for $\dim_{\mathbb{C}}(\Omega) = 2$, they need the assumption that the CR germ on M admits a uniformly bounded continuation along any curve in M. Making use of the Kerner-type extension theorem on Stein spaces with possible singularities, Huang–Xiao [6] constructed a hyperbolic metric on the regular part of the Stein space with a compact spherical boundary. In this paper, we will show that when $\dim_{\mathbb{C}}(\Omega) = 2$, the assumption that the CR germ admits uniformly bounded continuation inside M can be removed. The tool which we use to replace the Phragmé-Lindelöff principle crucially used in [6] is the Kerner theorem [7] established in the smooth category.

Before stating our main result, we first introduce some standard terminology in the study of extensions of CR functions and holomorphic functions. (f, D) is said to be a continuous CR map element, or simply a CR element, over M into $\mathbb{C}^N$ if D is a connected open piece of M and $f : D \to \mathbb{C}^N$ is a continuous CR map for a certain N. Similarly, we say (g, U) is a holomorphic map element, or simply a holomorphic element, over $\mathrm{Reg}(\overline{\Omega})$ into $\mathbb{C}^N$ if U is connected open subset of $\mathrm{Reg}(\overline{\Omega})$ and $g : U \to \mathbb{C}^N$ is a continuous map and holomorphic in $U \cap \Omega$.

Fix a CR map element (f, D) on M and a curve $\sigma : [0, 1] \to \mathrm{Reg}(\overline{\Omega})$ such that $\sigma(0) \in D$ (respectively, a curve $\sigma : [0, 1] \to M$ such that $\sigma(0) \in D$). We make the following definition.

Definition 1.1 We say (f, D) admits a holomorphic continuation along a curve $\sigma : [0, 1] \to \mathrm{Reg}(\overline{\Omega})$ with $\sigma(0) \in D$ (respectively, admits a continuous CR continuation along σ) if there exists a collection of holomorphic map elements $\{(f_j, U_j)\}_{j=0}^{k}$ on $\mathrm{Reg}(\overline{\Omega})$(respectively, a collection of CR elements $\{(f_j, U_j)\}_{j=0}^{k}$ on M) such that

1. $f_0 = f$ in a neighborhood of $\sigma(0)$ in $U_0 \cap D$.
2. There is a partition $0 = t_0 < t_1 < \cdots < t_{k+1} = 1$ such that $\sigma([t_j, t_{j+1}]) \subset U_j$ for all $0 \leq j \leq k$.
3. $f_j = f_{j+1}$ on $U_j \cap U_{j+1}$ for $0 \leq j \leq k - 1$.

Here, (f_k, U_k) is called a branch of (f, D) obtained by holomorphic (respectively, CR) continuation of (f, D) along σ. Fix a plurisubharmonic function $\Psi : \mathbb{C}^N \to \mathbb{R}$ such that $\Psi(f) \leq 0$. We further say (f, D) admits a holomorphic (respectively, CR) continuation along σ with Ψ-estimate if for any branch (g, U) described above, $\Psi(g) \leq 0$ on U.

Definition 1.2 Let (f, D) and Ψ be as above and let $\widetilde{\Omega}$ be an open connected subset of $\mathrm{Reg}(\overline{\Omega})$ containing D. We say that (f, D) admits unrestricted holomorphic continuation in $\widetilde{\Omega}$ (respectively, admits unrestricted CR continuation in M) with Ψ-estimate if (f, D) admits holomorphic continuation with Ψ-estimate along every curve σ in

$\widetilde{\Omega}$ with $\sigma(0) \in D$ (respectively, if (f, D) admits a CR continuation with Ψ-estimate along any curve γ in M with $\gamma(0) \in D$).

We similarly define the notion of a holomorphic branch of a CR or holomorphic map element in $\widetilde{\Omega}$. We now state the main result of this paper.

Theorem 1.3 *Let Ω be a Stein space of complex dimension two with possibly isolated singularities and with a connected compact strongly pseudoconvex smooth boundary $M = \partial\Omega$. Let (f, D) be a continuous CR map element on M into $\mathbb{C}^N$ and let $\Psi : \mathbb{C}^N \to \mathbb{R}$ be a plurisubharmonic function such that $\Psi(f) \leq 0$ on D. Suppose that (f, D) admits unrestricted CR continuation in M with Ψ-estimate. Then (f, D) admits unrestricted holomorphic continuation in* $\mathrm{Reg}(\overline{\Omega})$ *with Ψ-estimate. Moreover, assume that there is a holomorphic branch (h, U) of (f, D) in* $\mathrm{Reg}(\overline{\Omega})$ *such that $\Psi(h(p)) = 0$ at some $p \in U \setminus M$. Then $\Psi(g) = 0$ for any holomorphic brach (g, V) of (f, D). In particular, $\Psi(f) \equiv 0$ on D.*

The above statement was proved by Huang–Xiao in [6, Theorem 1.2] in the case of complex dimension at least 3. The two dimensional case was proved when any CR extension inside M is assumed to be uniformly bounded. When Ω is smooth, the multiple valued extension from an open subset of Ω to its envelope of holomorphy was proved by Kerner in [7] (see Theorem 2.2 stated later).

Extensions of a germ of a CR map from a boundary piece of a complex space along curves into its interior have attracted much attentions in the past many years. Among many applications, results of this type can be used to construct invariant metrics [6]. Related to our paper here include recent works done in [1, 4, 5, 9, 13] and [11], etc, to name a few. In particular, Theorem 1.4 of Mir–Zaitsev ([11]) states that any germ of a smooth CR map from an open piece of a real-analytic generic minimal submanifold M into a sphere can be CR-continued inside M. Thus Theorem 1.3 of [6] (also Theorem 1.3) can be applied to prove that if M bounds a Stein space with isolated singularities then such a CR germ can also be holomorphically continued inside the smooth part of the Stein space.

With Theorem 1.3 at our disposal, it is easy to see that the extension result in Corollary 4.1 (A) of [6] also holds in the case of complex dimension two.

To prove Theorem 1.3, we first fix a Morse plurisubharmonic defining function ρ of $\overline{\Omega}$. More precisely, we choose a plurisubharmonic function $\rho : \overline{\Omega} \to [-\infty, 0]$ such that $\rho \equiv 0$ on M, $\rho < 0$ in Ω and $\rho(z) = -\infty$ if and only if z is a singular point of Ω. In addition, $d\rho|_M \neq 0$ and ρ is a smooth strongly plurisubharmonic function on $\mathrm{Reg}(\overline{\Omega})$. Moreover, ρ has only finitely many critical points in $\mathrm{Reg}(\Omega)$ and they are all non-degenerate. Such a defining function ρ can be roughly constructed as follows [6]: The local existence of such a function near a singular point can be found in Milnor [10]. Away from singular points, the construction of such a function can be found, for example, in the book of Forstneric [2]. Then one glues these functions and applies the Morse approximation to get the desired ρ.

2 Envelope of holomorphy and Kerner's theorem

We first recall the notation of envelope of holomorphy [12, Chap. 6]. Let X be a Stein manifold. A (non-branched) Riemann domain D over X is a triple (D, p_D, X) consisting of a connected complex manifold D and a local biholomorphic map $p_D : D \to X$. If $X = \mathbb{C}^n$, then $(D, p_D, \mathbb{C}^n)$ is the classical non-branched Riemann domain spread over $\mathbb{C}^n$. In this paper, all Riemann domains are assumed to be non-branched.

A Riemann domain (D, p_D, X) is said to be contained in another Riemann domain (G, p_G, X) if there is a local biholomorphism $j : D \to G$ such that

$$p_G \circ j = p_D.$$

Definition 2.1 The envelope of holomorphy of a Riemann domain (D, p_D, X) is a Riemann domain $(H(D), p_{H(D)}, X)$ with the following properties:

1. There is a local biholomorphic map $\alpha_D : D \to H(D)$ such that

$$p_{H(D)} \circ \alpha_D = p_D.$$

2. For any $f \in \mathcal{O}(D)$, there is an $F \in \mathcal{O}(H(D))$ such that

$$F \circ \alpha_D = f,$$

   where $\mathcal{O}(D)$ and $\mathcal{O}(H(D))$ are the spaces of holomorphic functions on D and $H(D)$, respectively.
3. If there is a Riemann domain (G, p_G, X) and a map $\beta : D \to G$ such that the above Property 1 and Property 2 hold when $H(D)$ is replaced by G, then there is a locally biholomorphic map $\gamma : G \to H(D)$ such that

$$\alpha_D = \gamma \circ \beta, \;\; p_G = p_{H(D)} \circ \gamma.$$

The envelope of holomorphy exists and is unique up to an isomorphism by Thullen's theorem (see [12, Theorem 1, Chap. 6]). For our later application, we describe briefly how $(H(D), p_{H(D)}, \mathbb{C}^n)$ is constructed:

For an open subset U of $\mathbb{C}^n$, we define the set $\mathcal{O}^{\mathbb{R}}(U)$ of $\mathbb{R}$-tuples of holomorphic functions over U as the set of maps from the set of real numbers $\mathbb{R}$ into the ring of holomorphic functions $\mathcal{O}(U)$ over U. For an $\mathbb{R}$-tuple $(f_t)_{t\in\mathbb{R}} \in \mathcal{O}^{\mathbb{R}}(U)$, we call f_t the t-component of the tuple. For $z \in U$, we can define in a standard way the germ of (f_t) at z, denoted by $[(f_t)]_z \in \mathcal{O}_z^{\mathbb{R}}$. Define $\mathcal{O}^{\mathbb{R}} := \cup_{z\in\mathbb{C}^n} \mathcal{O}_z^{\mathbb{R}}$ and write

$$X_{((f_t),U)} = \{[(f_t)]_z : z \in U\} \subset \mathcal{O}^{\mathbb{R}}.$$

We use the set of all such $X_{((f_t),U)}$ to form a basis of open subsets in $\mathcal{O}^{\mathbb{R}}$. Then the natural projection map $\pi : \mathcal{O}^{\mathbb{R}} \to \mathbb{C}^n$ is a local homemorphism. A connected open subset of $\mathcal{O}^{\mathbb{R}}$ is always Hausdorff and second countable by the Poincaré–Volterra theorem. It also has a natural complex structure to make π a local biholomorphism.

Now, let $(D, p_D, \mathbb{C}^n)$ be a Riemann domain. Since a holomorphic function is uniquely determined by the jets at a given point, hence $\mathcal{O}(D)$ is equivalent to a subset of $\mathbb{R}$. On the other hand, it contains all constant functions. Hence, by the Cantor–Bernstein theorem, there is a bijection map S from $\mathbb{R}$ to $\mathcal{O}(D)$. Namely, for each $s \in \mathbb{R}$, $S(s) = f_s \in \mathcal{O}(D)$ and S is one-to-one and onto. Now, for $p \in D$, let $p_0 = p_D(p)$. Assume π is a homoemorphism from $p \in U$ to $p_0 \in U_0$. The envelope of holomorphy $H(D)$ of D (with respect to S) is the largest connected open subset of $\mathcal{O}^{\mathbb{R}}$ containing

$$X_{((f_s \circ (p_D|_U)^{-1})_{s\in\mathbb{R}}, U_0)}.$$

Apparently, $H(D)$ is independent of the choice of p and U. The envelope of holomorphy defined in this way is not unique (depending on S). But they all are naturally biholomorphic to each other. We write $(H(D), p_{H(D)}, \mathbb{C}^n)$ with $p_{H(D)} = \pi|_{H(D)}$ for the envelope of holomorphy of $(D, p_D, \mathbb{C}^n)$. Notice that α_D is defined by sending each $p \in D$ to $[(f_s \circ (p_D|_U)^{-1})_{s\in\mathbb{R}}]_p$. For any $f_s \in \mathcal{O}(D)$, the corresponding F_s takes value at $q \in \mathcal{O}^{\mathbb{R}}$ to be that of its s-component of the germ at $\pi(q)$.

Let $(D, p_D, \mathbb{C}^n)$ be a Riemann domain and let D' be a connected open subset of D. Assume that any holomorphic function over D' extends to a holomorphic function in D. Let $S : \mathbb{R} \to \mathcal{O}(D')$ be a bijective map that also naturally gives rise to a bijective map from $\mathbb{R}$ to $\mathcal{O}(D)$. For $p \in D' \subset D$ with $p_0 = p_D(p)$, then the envelopes of holomorphy of D and D', respectively, are both defined as the largest open subset of $\mathcal{O}^{\mathbb{R}}$ containing $X_{((f_s \circ (p_D|_U)^{-1})_{s\in\mathbb{R}}, U_0)}$, where $U \subset D'$ with p_D being a homeomorphism when restricted over U. Hence, they give the same complex manifold sitting inside $\mathcal{O}^{\mathbb{R}}$.

We mention that α_D may be not globally injective. It is easy to see that holomorphic functions over $H(D)$ separates points. It is a classical and remarkable theorem of Oka that the envelope of holomorphy of a Riemann domain $(D, p_D, \mathbb{C}^n)$ is holomorphically convex and thus is a Stein manifold. Let $(D, p_D, \mathbb{C}^n)$ and $(D', p_{D'}, \mathbb{C}^n)$ be two Riemann domains and let $(H(D), p_{H(D)}, \mathbb{C}^n)$, $(H(D'), p_{H(D')}, \mathbb{C}^n)$ be their envelopes of holomorphy, respectively. Let $u : D \to D'$ be a holomorphic map which is a local biholomorphism. Then there exists a unique local biholomorphism $\widetilde{u} : H(D) \to H(D')$ such that

$$\alpha_{D'} \circ u = \widetilde{u} \circ \alpha_D. \tag{2.1}$$

Let $(G, p_G, \mathbb{C}^n)$ be a Riemann domain. Let $\pi : \widehat{G} \to G$ be a universal cover of G. Then $(\widehat{G}, p_{\widehat{G}} = p_G \circ \pi, \mathbb{C}^n)$ is automatically a Riemann domain. Still use, as before, the notation α_G and $\alpha_{\widehat{G}}$ for the natural local isomorphisms: $\alpha_G : G \to H(G)$ and $\alpha_{\widehat{G}} : \widehat{G} \to H(\widehat{G})$. The following is the main theorem of Kerner proved in [7]:

Theorem 2.2 ([7, Theorem 1 b.]) *Let $(G, p_G, \mathbb{C}^n)$ be a Riemann domain. Let $\pi : \widehat{G} \to G$ be a universal cover of G. Let $H(\widehat{G})$, $H(G)$ be the envelopes of holomorphy of $\widehat{G}$, G respectively. Then $H(\widehat{G})$ is a universal cover of $H(G)$. Moreover, there is a covering map $\widehat{\pi} : H(\widehat{G}) \to H(G)$ such that $\widehat{\pi} \circ \alpha_{\widehat{G}} = \alpha_G \circ \pi$.*

By the monodromy theorem and the lifting lemma for a universal cover, we notice that a holomorphic function on the universal cover $\widehat{G}$ corresponds to a germ of a holomorphic function on G which can be holomorphically continued along any path in G. Hence, Theorem 2.2 shows that a germ of holomorphic function can be holomorphically continued along each path in $H(G)$ if it extends holomorphically along paths inside G. The following example shows that this is no longer the case when G is sitting in a singular Stein space even with a normal quotient singularity:

Example 2.3 ([6]) Let $\overline{\Omega}$ be the Stein space with a spherical boundary defined by

$$\overline{\Omega} = \{W = (w_1, w_2, w_3) \in \mathbb{C}^3 : \sum_{j=1}^{3} |w_j|^2 \leq 1, w_2^2 = 2w_1w_3\}.$$

Let $\pi : \overline{\mathbb{B}^2} \to \overline{\Omega}$ be given by $\pi(z_1, z_2) = (z_1^2, \sqrt{2}z_1z_2, z_2^2)$. Note that π is a 2 to 1 covering map from $\mathbb{B}^2\backslash\{0\}$ to $\Omega\backslash\{0\}$ and $\pi(0) = 0$. Fix $p_0 = (\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}) \in M := \partial\Omega$. Let D be a small simply connected open piece of M containing p_0 and let (f, D) be a CR mapping element given by $f(W) = (\sqrt{w_1}, \sqrt{w_3})$. Here $\sqrt{w} = \sqrt{|w|}e^{i\frac{\theta}{2}}$ for $w = |w|e^{i\theta}$ with $-\pi < \theta < \pi$. Notice that f maps D into $\partial\mathbb{B}^2$. Notice that M is spherical. By a result in [6], (f, D) admits uniformly bounded holomorphic map continuation along curves inside $\mathrm{Reg}(\overline{\Omega})$. However, it does not admit a holomorphic map continuation along a certain curve γ in $\overline{\Omega}$ with $\gamma(0) = p_0$ and $\gamma(1) = 0$. Notice that the singularity of Ω is a normal quotient singularity at 0. Hence, for a single-valued continuous CR function defined over M, it always extends holomorphically over the whole space Ω.

We sketch Kerner's proof of Theorem 2.2 in the following in order to prepare for notations which will be used later. (The reader may find detailed proof of Theorem 2.2 in [7]). First, we show that the envelope of holomorphy of a simply connected Riemann domain is also simply connected.

Lemma 2.4 *Let $(G, p_G, \mathbb{C}^n)$ be a non-branched Riemann domain. If G is simply connected, then so is $H(G)$.*

Proof In fact, if we denote by $\pi : E \to H(G)$ a universal covering map, then there is a lift $\tau : G \to E$ such that

$$\pi \circ \tau = \alpha_G.$$

Since both α_G and π are local homemorphism, thus τ is a local homemorphism. Since $H(G)$ is Stein, thus E is also a Stein manifold by a theorem of Stein [14]. We only need show that π is injective. If not, there exist x, y such that $\pi(x) = \pi(y)$. Since E is Stein, there exists a holomorphic function f on E such that $f(x) \neq f(y)$. Since $f \circ \tau$ is a holomorphic function on G, then it can be holomorphically extended to $H(G)$ denoted by F which satisfies $F \circ \alpha_G = f \circ \tau$. Thus,

$$F \circ \pi \circ \tau = f \circ \tau.$$

By uniqueness of holomorphic functions, $F \circ \pi = f$. Then

$$f(x) = F \circ \pi(x) = F \circ \pi(y) = f(y)$$

and thus we achieve a contradiction. □

Now, we let $\pi : \widehat{G} \to G$ be a universal covering map. Let Δ be the deck transformation group with respect to this universal cover. Then for any $\sigma \in \Delta$, $\sigma : \widehat{G} \to \widehat{G}$ is a biholomorphic map. By (2.1), there exists a map $\widetilde{\sigma} : H(\widehat{G}) \to H(\widehat{G})$ which is biholomorphic such that

$$\alpha_{\widehat{G}} \circ \sigma = \widetilde{\sigma} \circ \alpha_{\widehat{G}}. \tag{2.2}$$

We denote by $\widetilde{\Delta} = \{\widetilde{\sigma} : \sigma \in \Delta\}$. Then $\widetilde{\Delta} \subset \mathrm{Aut}(H(\widehat{G}))$ is a subgroup of the automorphism group of $H(\widehat{G})$. Let $Z = H(\widehat{G})/\widetilde{\Delta}$ be the quotient space with the quotient topology induced from $H(\widehat{G})$ and let $\eta : H(\widehat{G}) \to Z$ be the natural quotient map which is an open map. Then $(H(\widehat{G}), \eta, Z)$ is a regular covering space and Z is a Stein manifold.

By (2.1), there is a holomorphic map $\widehat{\pi} : H(\widehat{G}) \to H(G)$ such that

$$\widehat{\pi} \circ \alpha_{\widehat{G}} = \alpha_G \circ \pi. \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$\widehat{\pi} \circ \widetilde{\sigma} \circ \alpha_{\widehat{G}} = \widehat{\pi} \circ \alpha_{\widehat{G}} \circ \sigma = \alpha_G \circ \pi \circ \sigma = \alpha_G \circ \pi = \widehat{\pi} \circ \alpha_{\widehat{G}}.$$

Since $\alpha_{\widehat{G}}$ is locally biholomorphic, by the uniqueness of holomorphic functions, we have

$$\widehat{\pi} \circ \widetilde{\sigma} = \widehat{\pi}, \forall \widetilde{\sigma} \in \widetilde{\Delta}.$$

It follows that there is map $\rho : Z \to H(G)$ such that $\rho \circ \eta = \widehat{\pi}$ and ρ is a local homemorphism and surjective. Moreover, ρ is a biholomorphic map. Thus, $\widehat{\pi} : H(\widehat{G}) \to H(G)$ is a regular cover which is also universal as $H(\widehat{G})$ is simply connected by Lemma 2.4.

3 Proof of Theorem 1.3

Let Ω and (f, D) be as in Theorem 1.3. Assume the hypotheses in Theorem 1.3. Write $\overline{\Omega}_{r_2,r_1} = \{p \in \overline{\Omega} : r_2 < \rho \le r_1\}$, where ρ is the plurisubharmonic Morse defining function of $\overline{\Omega}$ mentioned in Sect. 1. Then by [6, Lemma 3.3], the CR map element (f, D) admits unrestricted holomorphic continuation with Ψ-estimate in $\overline{\Omega}_{-\varepsilon_2,0}$ for some small $\varepsilon_2 > 0$. Let $\widetilde{\Omega} \subset \mathrm{Reg}(\overline{\Omega})$ be a connected open subset of $\mathrm{Reg}(\overline{\Omega})$ containing $M = \partial\Omega$. We say $\widetilde{\Omega}$ has the extendable property if (f, D) admits unrestricted holomorphic continuation in $\widetilde{\Omega}$ with Ψ-estimate. To prove Theorem 1.3, we need to prove $\mathrm{Reg}(\overline{\Omega})$ has the extendable property. Now set

$$A = \{a < 0 : \overline{\Omega}_{a,0} \text{ has the extendable property}\}.$$

Then A is not empty, as $[-\varepsilon_2, 0) \subset A$. Set $b = \inf(A)$.

If $b = -\infty$, Theorem 1.3 holds trivially. We will therefore assume $b > -\infty$ in what follows. Notice that it follows from the definition of b that $\overline{\Omega}_{b,0}$ has the extendable property. Write $\inf(\rho) := \inf\{\rho(z) : z \in \overline{\Omega} \setminus \mathrm{sing}(\Omega)\}$. ($\inf(\rho)$ is $-\infty$ when $\mathrm{Sing}(\Omega) \neq \emptyset$.)

Case I: $b = \inf \rho$;

Case II: $b > \inf(\rho)$ and $M_b := \{q \in \mathrm{Reg}(\overline{\Omega}) : \rho(q) = b\}$ is smooth;

When $b > \inf(\rho)$ and M_b contains critical points of ρ, let $p \in M_b$ be a critical point. Then we choose a small neighborhood U_p of p such that p is the only critical point of p in U_p. We choose holomorphic coordinates $z = (z_1, z_2)$ on U_p such that $z(p) = (0, 0)$ and the defining function ρ of $\overline{\Omega}$ takes the normal form in U_p:

$$\rho = |z|^2 + 2\mathrm{Re}(\lambda_1 z_1^2 + \lambda_2 z_2^2) + O(|z|^3) + b$$

where $|z|^2 = |z_1|^2 + |z_2|^2$. Here, we have $0 \leq \lambda_1, \lambda_2 < \infty$ and $\lambda_1,\ \lambda_2 \neq \frac{1}{2}$ by the non-degeneracy of the Morse defining function ρ at its critical points. Recall that the z_j-direction is called elliptic if $0 \leq \lambda_j < \frac{1}{2}$ and hyperbolic if $\lambda_j > \frac{1}{2}$.

Case III: $b > \inf(\rho)$ and M_b contains critical points of ρ with at least one elliptic direction.

Cases I, II and III can not hold as already proved in [6]. The only case we need to consider in this paper is the following:

Case IV: $b > \inf(\rho)$ and M_b contains critical points of ρ with only hyperbolic directions.

Let $p \in M_b$ be a hyperbolic critical point. Then choose a neighborhood U_p of p with p being the only critical point of ρ in U_p as above such that $z(p) = 0$ and ρ takes the following normal form near p:

$$\rho = |z|^2 + 2\mathrm{Re}(\lambda_1 z_1^2 + \lambda_2 z_2^2) + O(|z|^3) + b.$$

Here we have $\lambda_j > \frac{1}{2}$, $j = 1, 2$. Also, by the Morse lemma, in a certain smooth coordinates $x = (x_1, x_2, x_3, x_4)$ over U_p with $x(p) = 0$, we have

$$\rho(x) = x_1^2 - x_2^2 + x_3^2 - x_4^2 + b.$$

For a sufficiently small $\varepsilon > 0$, set $V_\varepsilon := \{q \in U_p : |x(q)|^2 < \varepsilon^2\} \Subset U_p$, where $|x(q)|^2 = \sum_{j=1}^4 |x_j(q)|^2$. Then one checks easily that $V_\varepsilon \cap \overline{\Omega}_{b,0}$ is connected. Notice that $V_\varepsilon \cap \overline{\Omega}_{b,0}$ has a fundamental group isomorphic to $\mathbb{Z}$. In the following, we fix such a small ε.

We now prove the following key proposition:

Proposition 3.1 *For a sufficiently small $\varepsilon_1 > 0$ with $\varepsilon_1 \ll \varepsilon$ and for any $q \in V_{\varepsilon_1} \cap \overline{\Omega}_{b,0}$, if $[g]_q$ is the germ of a holomorphic branch of (f, D) obtained by holomorphic continuation along a curve in $\overline{\Omega}_{b,0}$, then $[g]_q$ extends to a single-valued holomorphic function in V_{ε_1} with the Ψ-estimate.*

Proposition 3.1 was proved in [6] with an extra assumption that all branches of (f, D) have a uniform bound by making use of the Phragmé–Lindelöff principle. We will circumvent this difficulty caused by the non-uniform boundedness by employing Theorem 2.2.

Proof of Proposition 3.1 Write

$$G = \overline{\Omega}_{b,0} \cap V_\varepsilon$$

for a small ε. We can treat G as an open subset of $\mathbb{C}^2$. Thus, $(G, j, \mathbb{C}^2)$ is a Riemann domain where $j : G \to \mathbb{C}^2$ is the natural inclusion map. Recall $V_\varepsilon = \{q \in U_p : |x(q)|^2 < \varepsilon^2\}$ where $x = (x_1, x_2, x_3, x_4)$ are smooth coordinates on U_p centered at p such that

$$\rho(x) = x_1^2 - x_2^2 + x_3^2 - x_4^2 + b.$$

Notice that $p \in \partial G$. Also, as we mentioned before, G is connected but not simply connected. We need the following lemma:

Lemma 3.2 *There exists a sufficiently small ε_1 with $0 < \varepsilon_1 \ll \varepsilon$ such that for any $f \in \mathcal{O}(G)$, f can be holomorphically continued to $G \cup V_{\varepsilon_1}$.*

Proof The key ingredient in the proof of this lemma is to apply the method of attaching a family of annuli in [6]. Fix $0 < \tilde{\varepsilon} < \eta \ll 1$. Consider a continuous family (parametrized by t) of Riemann surfaces with boundaries $\overline{E_t}$, $0 < t < \tilde{\varepsilon}$ in U_p defined by

$$\overline{E}_t := \{q \in U_p : \lambda_1 z_1^2 + \lambda_2 z_2^2 = t,\ 2\lambda_1|z_1|^2 + 2\lambda_2|z_2|^2 \le \eta^2\}$$

where $\lambda_1, \lambda_2 > \frac{1}{2}$. It is shown in [6] that for each t, $\overline{E_t}$ is biholomorphic to an annulus with two circles as its smooth boundaries. We assume that $\cup_t \overline{E}_t \Subset V_\varepsilon$ by choosing sufficiently small η. For any $(z_1, z_2) \in \overline{E}_t$, one has for a certain $\delta < 1$

$$\rho(z) \ge (1-\delta)|z|^2 + 2t + b > b.$$

Hence, $\overline{E_t} \Subset \overline{\Omega}_{b,0} \cap V_\varepsilon$. On the other hand, if $z = (z_1, z_2) \in \partial E_t$ one has

$$\rho(z) \ge (1-\delta)|z|^2 + 2t + b > (1-\delta)\frac{\eta^2}{2\lambda_1 + 2\lambda_2} + b.$$

This implies that there exists an $A_1 > 0$ independent of t such that

$$d(\partial E_t, \partial(\overline{\Omega}_{b,0} \cap V_\varepsilon)) \ge A_1 \text{ for any small } t. \tag{3.1}$$

Also,

$$d(\overline{E}_t, \partial V_\varepsilon) \ge A_2, \forall t$$

where A_2 is independent of t. Set $A = \min\{A_1, A_2\}$. Let $\varepsilon_1 > 0$, $\varepsilon_1 \ll 1$ be such that $V_{\varepsilon_1} \Subset U_p$ and $d(z, 0) < A$, $\forall z \in V_{\varepsilon_1}$. Write

$$z_t = \left(\sqrt{\frac{t}{\lambda_1}}, 0\right), \quad 0 < t < \tilde{\varepsilon}.$$

Then $z_t \in \overline{E}_t$ and $z_t \to 0$ as $t \to 0$. Let f be a holomorphic function on G. By (3.1) and the maximum principle, one has

$$|f|_{\max, \overline{E}_t} \leq |f|_{\max, \partial E_t} \leq C_f \tag{3.2}$$

for some constant C_f which does not depend on t. Since $\overline{E}_t \Subset \overline{\Omega}_{b,0} \cap V_\varepsilon$, by (3.1), one can choose an open set V such that $\cup_t \partial \overline{E}_t \Subset V$ with its closure $\overline{V} \Subset \overline{\Omega}_{b,0} \cap V_\varepsilon$. We denote by $C_1 := |f|_{\max, \overline{V}}$ and $\mathbb{P}(z, r) := \{(w_1, w_2) \in \mathbb{C}^2 : |w_1 - z_1| < r, |w_2 - z_2| < r\}$ the polydisc centered at $z = (z_1, z_2)$. Then there exists an r sufficiently small such that $\mathbb{P}(z, r) \Subset V$ for all $z \in \partial \overline{E}_t$ with $0 < t \ll 1$. We mention that such an r does not depend on t. By the Cauchy estimates, one has

$$|D^\alpha f(z)| \leq \frac{C_1 \alpha!}{r^{|\alpha|}}, \forall z \in \partial E_t, \forall t \tag{3.3}$$

where $\alpha = (\alpha_1, \alpha_2) \in (\mathbb{Z}^+)^2$, $|\alpha| = \alpha_1 + \alpha_2$ and $D^\alpha = \partial_{z_1}^{\alpha_1} \partial_{z_2}^{\alpha_2}$. Over E_t, by the maximum principle, one has

$$|D^\alpha f(z_t)| \leq \frac{C_1 \alpha!}{r^{|\alpha|}}, \forall t. \tag{3.4}$$

Hence, by (3.4), the Taylor expansion of f at z_t converges in $\mathbb{P}(z_t, r)$ for all t. In particular, there exists a positive $\varepsilon_1 \ll 1$ such that f can be holomorphically continued to V_{ε_1}, as $V_{\varepsilon_1} \subset \mathbb{P}(z_t, r)$ when $t \to 0$. Since $V_{\varepsilon_1} \cap G$ is connected, by uniqueness of holomorphic functions, there exists an $F \in \mathcal{O}(G \cup V_{\varepsilon_1})$ such that $F|_G = f$. Hence we reached the conclusion of the Lemma 3.2. □

It thus follows from Lemma 3.2 and the construction of envelopes of holomorphy discussed in Sect. 2 that

$$H(G) = H(G \cup V_{\varepsilon_1}). \tag{3.5}$$

We next claim that $G \cup V_{\varepsilon_1}$ is simply connected.

Let $\gamma(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$, $t \in [0, 1]$, be a closed curve in $G \cup V_{\varepsilon_1}$. Set

$$\gamma_\tau(t) := (\tau x_1(t), \tau x_2(t), \tau x_3(t), \tau x_4(t)), \forall \tau \in [0, 1], t \in [0, 1].$$

Then $\{\gamma_\tau(t)\}_{0 \leq \tau \leq 1}$ is a continuous family of closed curves in $G \cup V_{\varepsilon_1}$ with $\gamma_0 = p$ and $\gamma_1 = \gamma$. Thus, $G \cup V_{\varepsilon_1}$ is simply connected.

By Lemma 2.4, the envelope of holomorphy $H(G \cup V_{\varepsilon_1})$ of $G \cup V_{\varepsilon_1}$ is simply connected. Hence $H(G)$ is simply connected by (3.5).

Let $\pi : \widehat{G} \to G$ be a universal cover and let Δ be the deck transformation group with respect to this cover. By Theorem 2.2, $H(\widehat{G})$ is a universal cover of $H(G)$ and thus

$$H(\widehat{G}) \cong H(G)$$

for $H(G)$ is simply connected. More precisely, the quotient map $\eta : H(\widehat{G}) \to Z := H(\widehat{G})/\widetilde{\Delta}$ is an isomorphism. Thus, $\widetilde{\Delta}$ is trivial. Recall that

$$\alpha_{\widehat{G}} \circ \sigma = \widetilde{\sigma} \circ \alpha_{\widehat{G}}, \forall \sigma \in \Delta.$$

Hence, one has

$$\alpha_{\widehat{G}} \circ \sigma = \alpha_{\widehat{G}}, \forall \sigma \in \Delta. \tag{3.6}$$

We mention that $\alpha_{\widehat{G}}$ is not globally injective but is a local biholomorphsim. Notice that Δ is not trivial as G is not simply connected. For $q \in V_{\varepsilon_1} \cap \overline{\Omega}_{b,0}$, let $[g]_q$ be the germ of a branch of (f, D). Assume that g is defined on an open neighborhood U of q. By the assumption on G, (g, U) can be holomorphically continued along any curve in G. Assume U is sufficiently small such that each connected component of $\pi^{-1}(U)$ is homeomorphic to U. Choose one connected component $\widehat{U}$ of $\pi^{-1}(U)$. There exists $\hat{q} \in \widehat{U}$ such that $\pi(\hat{q}) = q$ and $g \circ \pi$ is a holomorphic function on $\widehat{U}$. Since g can be holomorphically continued along any curve in G, we have that $(g \circ \pi, \widehat{U})$ can be holomorphically continued along curves in $\widehat{G}$. Since $\widehat{G}$ is simply connected, by the monodromy theorem there exists a holomorphic function F on $\widehat{G}$ such that $F|_{\widehat{U}} = g \circ \pi$. Moreover, F can be holomorphically continued to $H(\widehat{G})$, i.e., there exists a holomorphic function $\widetilde{F}$ on $H(\widehat{G})$ such that

$$\widetilde{F} \circ \alpha_{\widehat{G}} = F.$$

By (3.6), one has

$$F \circ \sigma = F, \forall \sigma \in \Delta.$$

Thus, there exists a holomorphic function $\tilde{g}$ on G such that $\tilde{g} \circ \pi = F$. By uniqueness of holomorphic functions, (g, U) can be holomorphically continued to a single valued holomorphic function $\tilde{g}$ in G. By Lemma 3.2 , $\tilde{g}$ can be holomorphically continued to a single valued holomorphic function in V_{ε_1}. The Ψ-estimate of $\tilde{g}$ can be deduced by the same argument in the proof of Lemma 3.5 in [6]. We thus have completed the proof of Proposition 3.1. □

After having established Proposition 3.1, we follow the same arguments in [6, pp. 199–200] without any change to prove that Case IV can not hold neither. Thus we have completed the proof of Theorem 1.3.

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