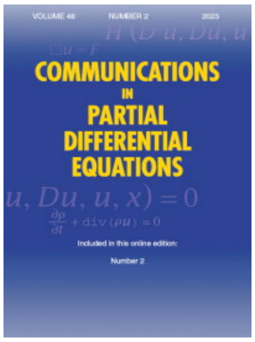

Communications in Partial Differential Equations

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/lpde20

The Yang-Mills heat flow with random distributional initial data

Sky Cao & Sourav Chatterjee

COMMUNICATIONS IN PARTIAL DIFFERENTIAL EQUATIONS
2023, VOL. 48, NO. 2, 209–251
https://doi.org/10.1080/03605302.2023.2169937

Taylor & Francis
Taylor & Francis Group

The Yang-Mills heat flow with random distributional initial data

Sky Cao[a] and Sourav Chatterjee[b]

[a]School of Mathematics, Institute for Advanced Study, Princeton, NJ, USA; [b]Department of Statistics, Stanford University, Stanford, CA, USA

ABSTRACT
We construct local solutions to the Yang–Mills heat flow (in the DeTurck gauge) for a certain class of random distributional initial data, which includes the 3D Gaussian free field. The main idea, which goes back to work of Bourgain as well as work of Da Prato–Debussche, is to decompose the solution into a rougher linear part and a smoother nonlinear part, and to control the latter by probabilistic arguments. In a companion work, we use the main results of this paper to propose a way toward the construction of 3D Yang–Mills measures.

ARTICLE HISTORY
Received 10 March 2022
Accepted 15 January 2023

KEY WORDS
Gaussian free field; Yang–Mills heat equation; Yang–Mills theory

2020 MATHEMATICS SUBJECT CLASSIFICATION
35A01; 60G60; 35R60; 81T13

1. Introduction

Take any dimension $d \geq 2$. Let G be a compact Lie group and let $\mathfrak{g}$ denote the Lie algebra of G. We assume that $G \subseteq \mathrm{U}(N)$ for some $N \geq 1$. A *connection* on the trivial principal G-bundle $\mathbb{T}^d \times G$ is a function $A : \mathbb{T}^d \to \mathfrak{g}^d$, that is, a d-tuple of functions $A = (A_1, ..., A_d)$, with $A_i : \mathbb{T}^d \to \mathfrak{g}$ for $1 \leq i \leq d$. Note that A can also be viewed as a $\mathfrak{g}$-valued 1-form on $\mathbb{T}^d$. Thus, in this paper, we will use "connection" and "1-form" interchangeably.

The main object of this paper is to prove local existence of solutions to the Yang–Mills heat flow with random distributional initial data. The Yang–Mills heat flow (also often called the Yang–Mills gradient flow, or Yang–Mills heat equation) is the following PDE on time-dependent connections $A(t)$ (in the following, we omit the time parameter t):

$$\begin{aligned}\partial_t A_i = \Delta A_i + \sum_{j=1}^{d}\big(-\partial_{ji}A_j + \big[A_j, 2\partial_j A_i - \partial_i A_j + [A_j, A_i]\big] \\ + \big[\partial_j A_j, A_i\big]\big), \quad 1 \leq i \leq d.\end{aligned} \tag{YM}$$

This equation can be obtained as the gradient flow of a certain action on the space of connections, analogous to how the heat equation can be obtained as the gradient flow

CONTACT Sky Cao skycao@ias.edu School of Mathematics, Institute for Advanced Study, Princeton NJ 08540, USA

of the Dirichlet energy. The Yang–Mills heat flow has played a central role in various areas of mathematics, starting with the paper by Atiyah and Bott [1]. See [2, Section 1] for a historical overview of this equation and its many applications in mathematics and physics, as well as for an encyclopedic account of existing results. See also [3–9] for some newer results.

Actually, in this paper, we will not directly work with (YM), but rather a certain well-known variant which is often treated as equivalent to (YM). This variant is the following PDE:

$$\partial_t A_i = \Delta A_i + \sum_{j=1}^{d} \big[A_j, 2\partial_j A_i - \partial_i A_j + [A_j, A_i]\big], \quad 1 \le i \le d. \tag{ZDDS}$$

We will refer to this as the ZDDS equation, named after the authors associated with this equation — Zwanziger [10], DeTurck [11], Donaldson [12], Sadun [13]. For a discussion as to why (ZDDS) is equivalent to (YM), see, e.g., [14, Section 1]. The advantage of (ZDDS) is that it is a parabolic equation, and thus local existence is often easier to establish for (ZDDS) than (YM). Indeed, one of the main methods for showing local existence of (YM) for various types of initial data is to first show it for (ZDDS), and then use a well-known procedure to obtain solutions to (YM) out of solutions to (ZDDS) (see, e.g., [3, Section 1.3]).

By now, local existence of solutions to (ZDDS) has been established for various classes of initial data — again, see the survey [2], as well as [3, 6]. However, as far as we can tell, there are no results for distributional initial data. In particular, there are no results which consider random distributional initial data that is too rough to be handled purely by deterministic methods. The present paper seeks to address this case. Our motivation is twofold. For one, we think that this case is of intrinsic interest — random initial data has been studied for a variety of PDEs, such as the nonlinear Schrödinger equation [15–19], the nonlinear wave equation [20, 21], and the Navier–Stokes equations [22]. (This list of references is woefully incomplete. See, e.g., [17, 19] for more.)

Second, we originally came upon this problem through the companion work [23]. In that paper, we give a proposal for constructing 3D Euclidean Yang–Mills theories (following a suggestion of Charalambous and Gross [3]), and in particular, we construct and study a state space that may potentially support 3D Yang–Mills measures. As evidence of this possibility, in [23] we apply the results of the present paper to give nontrivial elements of the state space, and additionally to give a road map for completing the program and actually constructing 3D Yang–Mills measures. See [23] for more background and discussion.

1.1. The main result

We begin to build toward the statement of the main result, Theorem 1.19. In this paper, we will often deal with functions $A(t, x)$ of both $t \in [0, \infty)$ and $x \in \mathbb{T}^d$. In an abuse of notation, for $t \in [0, \infty)$, we will write $A(t)$ to denote the function on $\mathbb{T}^d$ given by $x \mapsto A(t, x)$.

Definition 1.1. Let $0 < T \leq \infty$. We say that A is a solution to (ZDDS) on $[0, T)$ if $A \in C^\infty([0, T) \times \mathbb{T}^d, \mathfrak{g}^d)$ and A satisfies (ZDDS) on $(0, T) \times \mathbb{T}^d$. Similarly, we say that A is a solution to (ZDDS) on $(0, T)$ if $A \in C^\infty((0, T) \times \mathbb{T}^d, \mathfrak{g}^d)$ and A satisfies (ZDDS) on $(0, T) \times \mathbb{T}^d$.

We next state the following classical theorem and lemma, which give local existence and uniqueness for solutions to (ZDDS) with smooth initial data. Since these results concern smooth initial data, they are not new and quite classical. For instance, the local existence and regularity results stated below can be obtained by combining the various general results of [2, Sections 17.4, 17.5, and 20.1]. Thus, the proofs will be omitted.

Theorem 1.2. *Let A^0 be a smooth 1-form. There exists $T > 0$ and a solution $A \in C^\infty([0, T) \times \mathbb{T}^d, \mathfrak{g}^d)$ to (ZDDS) on $[0, T)$ with initial data $A(0) = A^0$.*

Lemma 1.3. *Let $T > 0$. Suppose that $A, \tilde{A} \in C^\infty([0, T) \times \mathbb{T}^d, \mathfrak{g}^d)$ are solutions to (ZDDS) on $[0, T)$ such that $A(0) = \tilde{A}(0)$. Then $A = \tilde{A}$.*

Because of Lemma 1.3, in circumstances where the smooth initial data has been specified, we will usually say “the solution to (ZDDS)” rather than “a solution to (ZDDS)”.

Before we proceed, we need some notation. For integers $n \geq 1$, let $[n] := \{1, ..., n\}$. For vectors $v \in \mathbb{R}^n$ for some n, we write $|v|$ for the Euclidean norm of v, and we write $|v|_\infty := \max_{1 \leq i \leq n} |v_i|$ for the ℓ^∞ norm of v. Next, since we assumed that $G \subseteq \mathrm{U}(N)$, the Lie algebra $\mathfrak{g}$ is a real finite-dimensional Hilbert space, with inner product given by $\langle S_1, S_2 \rangle = \mathrm{Tr}(S_1^* S_2) = -\mathrm{Tr}(S_1 S_2)$ (note that $S^* = -S$ for all $S \in \mathfrak{g}$, because $G \subseteq \mathrm{U}(N)$).

Definition 1.4. Let $d_\mathfrak{g}$ be the dimension of $\mathfrak{g}$. Throughout this paper, fix an orthonormal basis $(S^a, a \in [d_\mathfrak{g}])$ of $\mathfrak{g}$.

We may thus equivalently view a ($\mathfrak{g}$-valued) 1-form $A : \mathbb{T}^d \to \mathfrak{g}^d$ as a collection $(A_j^a, a \in [d_\mathfrak{g}], j \in [d])$ of functions $A_j^a : \mathbb{T}^d \to \mathbb{R}$, satisfying the relation

$$A_j = \sum_{a \in [d_\mathfrak{g}]} A_j^a S^a, \quad j \in [d]. \tag{1.1}$$

Next, we recall the notation for Fourier coefficients. Let $\{e_n\}_{n \in \mathbb{Z}^d}$ be the Fourier basis on $\mathbb{T}^d$. Explicitly, if we identify functions on $\mathbb{T}^d$ with 1-periodic functions on $\mathbb{R}^d$, then $e_n(x) = e^{\mathrm{i}2\pi n \cdot x}$. Given $f \in L^1(\mathbb{T}^d, \mathbb{R})$, define the Fourier coefficient

$$\widehat{f}(n) := \int_{\mathbb{T}^d} f(x) \overline{e_n(x)} dx \in \mathbb{C}, \quad n \in \mathbb{Z}^d.$$

Note (since f is $\mathbb{R}$-valued) that $\widehat{f}(-n) = \overline{\widehat{f}(n)}$ for all $n \in \mathbb{Z}^d$. We note that this all generalizes to the case where f takes values in some finite-dimensional normed linear space $(V, |\cdot|)$, in which case $\widehat{f}(n) \in V^\mathbb{C}$, where $V^\mathbb{C} := \{v_1 + \mathrm{i}v_2 : v_1, v_2 \in V\}$ is the “complexified” version of V, with norm $|v_1 + \mathrm{i}v_2| := (|v_1|^2 + |v_2|^2)^{1/2}$. Moreover, defining $\overline{v_1 + \mathrm{i}v_2} := v_1 - \mathrm{i}v_2$, we have that $\widehat{f}(-n) = \overline{\widehat{f}(n)}$ for all $n \in \mathbb{Z}^d$.

Throughout this paper, given a normed linear space $(V, |\cdot|_V)$, we will abuse notation and write $|v|$ instead of $|v|_V$ for the norm of $v \in V$. Similarly, when $(V, \langle \cdot, \cdot \rangle_V)$ is an inner product space, we will write $\langle v_1, v_2 \rangle$ instead of $\langle v_1, v_2 \rangle_V$. The main examples of this are when V is one of $\mathfrak{g}, \mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^d, (\mathfrak{g}^d)^{\mathbb{C}}$ (note that the inner product that we defined on $\mathfrak{g}$ induces inner products on the latter three spaces).

Definition 1.5. For $N \geq 0$, define the Fourier truncation operator F_N on distributions as follows. Given a distribution ϕ on $\mathbb{T}^d$, define $F_N\phi \in C^\infty(\mathbb{T}^d)$ by

$$F_N\phi := \sum_{\substack{n \in \mathbb{Z}^d \\ |n|_\infty \leq N}} \widehat{\phi}(n) e_n,$$

where $|n|_\infty$ is the ℓ^∞ norm of n. (Here $\widehat{\phi}(n) := (\phi, e_{-n})$ for $n \in \mathbb{Z}^d$.)

Definition 1.6 (Quadratic forms). Let A^0 be a smooth 1-form. Let $\mathbb{I} := [d_{\mathfrak{g}}] \times [d] \times \mathbb{T}^d$. Let λ be the measure on $\mathbb{I}$ defined by taking the product of counting measure on $[d_{\mathfrak{g}}]$, counting measure on $[d]$, and Lebesgue measure on $\mathbb{T}^d$. We say that $K : \mathbb{I}^2 \to \mathbb{R}$ is a smooth function if for any $a_1, a_2 \in [d_{\mathfrak{g}}]$, $j_1, j_2 \in [d]$, the function on $(\mathbb{T}^d)^2$ defined by $(x, y) \mapsto K((a_1, j_1, x), (a_2, j_2, y))$ is smooth. In this case, we write $K \in C^\infty(\mathbb{I}^2, \mathbb{R})$. Given a smooth function $K \in C^\infty(\mathbb{I}^2, \mathbb{R})$, define

$$(A^0, KA^0) := \int_{\mathbb{I}} \int_{\mathbb{I}} A^{0,a_1}_{j_1}(x) K(i_1, i_2) A^{0,a_2}_{j_2}(y) d\lambda(i_1) d\lambda(i_2) \in \mathbb{R}, \tag{1.2}$$

where $i_1 = (a_1, j_1, x)$, $i_2 = (a_2, j_2, y)$.

We next begin to state the assumptions that are involved in the statement of our main result, Theorem 1.19. First, we make some definitions.

Definition 1.7. By a random $\mathfrak{g}^d$-valued distribution $\mathbf{A}^0$, we mean a stochastic process $((\mathbf{A}^0, \phi), \phi \in \mathbb{C}^\infty(\mathbb{T}^d, \mathbb{R}))$ of $\mathfrak{g}^d$-valued random variables such that for all $\phi_1, \phi_2 \in C^\infty(\mathbb{T}^d, \mathbb{R})$, $c_1, c_2 \in \mathbb{R}$, we have that

$$(\mathbf{A}^0, c_1\phi_1 + c_2\phi_2) \overset{a.s.}{=} c_1(\mathbf{A}^0, \phi_1) + c_2(\mathbf{A}^0, \phi_2). \tag{1.3}$$

Remark 1.8. There are several different ways one can view $\mathbf{A}^0$. By linearity, we may also view $\mathbf{A}^0$ as a $(\mathfrak{g}^d)^{\mathbb{C}}$-valued stochastic process $((\mathbf{A}^0, \phi), \phi \in C^\infty(\mathbb{T}^d, \mathbb{C}))$ indexed by $\mathbb{C}$-valued test functions, which satisfy (1.3) for $\phi_1, \phi_2 \in C^\infty(\mathbb{T}^d, \mathbb{C})$ and $c_1, c_2 \in \mathbb{C}$, and which also satisfy $\overline{(\mathbf{A}^0, \phi)} = (\mathbf{A}^0, \bar{\phi})$ for all $\phi \in C^\infty(\mathbb{T}^d, \mathbb{C})$.

Also, as is the case with 1-forms, we may equivalently view $\mathbf{A}^0$ as a $\mathbb{R}$-valued process $((\mathbf{A}^{0,a}_j, \phi), \phi \in C^\infty(\mathbb{T}^d, \mathbb{R}), a \in [d_{\mathfrak{g}}], j \in [d])$. Then again by linearity, we may view $\mathbf{A}^0$ as a $\mathbb{C}$-valued process $((\mathbf{A}^{0,a}_j, \phi), \phi \in C^\infty(\mathbb{T}^d, \mathbb{C}), a \in [d_{\mathfrak{g}}], j \in [d])$.

We will use these different viewpoints (i.e., $\mathfrak{g}^d$-valued, $(\mathfrak{g}^d)^{\mathbb{C}}$-valued, $\mathbb{R}$-valued, $\mathbb{C}$-valued) interchangeably. As much as possible, we will try to take the vector-valued (i.e. $\mathfrak{g}^d$

-valued or $(\mathfrak{g}^d)^{\mathbb{C}}$-valued) viewpoint, but we will find it convenient later on (in particular in Section 4.2) to take the scalar-valued viewpoint for certain arguments.

In what follows, let $\mathbf{A}^0$ be a random $\mathfrak{g}^d$-valued distribution.

Definition 1.9 (Fourier truncations). Let $N \geq 0$ be a finite integer. Define the Fourier truncation $\mathbf{A}^0_N := F_N \mathbf{A}^0$, which is a $\mathfrak{g}^d$-valued stochastic process with smooth sample paths.

Remark 1.10. Since $\mathbf{A}^0_N$ is a random smooth 1-form, we may (recalling (1.1)) equivalently view $\mathbf{A}^0_N$ as a $\mathbb{R}$-valued stochastic process with smooth sample paths $\mathbf{A}^0_N = (\mathbf{A}^{0,a}_{N,j}(x), a \in [d_{\mathfrak{g}}], j \in [d], x \in \mathbb{T}^d)$.

Definition 1.11 (Fourier coefficients). Define the Fourier coefficients of $\mathbf{A}^0$ by

$$\widehat{\mathbf{A}}^0(n) := (\mathbf{A}^0, e_{-n}), \quad a \in [d_{\mathfrak{g}}], j \in [d], n \in \mathbb{Z}^d.$$

Note that $\overline{\widehat{\mathbf{A}}^0(n)} = \widehat{\mathbf{A}}^0(-n)$ for all $n \in \mathbb{Z}^d$.

Definition 1.12. For $\phi \in C^\infty(\mathbb{T}^d, \mathbb{R})$, define $\sigma_\phi := (\mathbb{E}[|(\mathbf{A}^0, \phi)|^2])^{1/2}$. For $K \in C^\infty(\mathbb{I}^2, \mathbb{R})$, let $\sigma_{N,K} := (\mathbb{E}[(\mathbf{A}^0_N, K\mathbf{A}^0_N)^2])^{1/2}$ (recall Definition 1.6).

Definition 1.13. Let $\alpha \in (0, d)$, and define the bivariate distribution

$$G^\alpha(x, y) := \sum_{\substack{n \in \mathbb{Z}^d \\ n \neq 0}} \frac{1}{|n|^\alpha} e_n(x - y).$$

Define $G^\alpha_0(x) := G^\alpha(0, x)$.

Remark 1.14. Note that G^α is the Green's function for the fractional negative Laplacian $(-\Delta)^{\alpha/2}$ on $\mathbb{T}^d$. In particular, G^2 is the Green's function for $-\Delta$ (which is the covariance function of the GFF, to be introduced a bit later).

We quote the following lemma giving properties of G^α. See [24, Theorem 2.17] for a proof.

Lemma 1.15. *Let $\alpha \in (0, d)$. The distribution G^α_0 is smooth on $\mathbb{T}^d - \{0\}$, with the following properties.*

1. *G^α_0 is bounded from below.*
2. *As $x \to 0$, we have that $G^\alpha_0(x) \sim d_{\mathbb{T}^d}(0, x)^{-(d-\alpha)}$. Here "$\sim$" means that the ratio tends to a positive constant.*

We now make the following assumptions on $\mathbf{A}^0$. One should think of these assumptions as saying that $\mathbf{A}^0$ qualitatively behaves like a Gaussian field. Indeed, the assumptions were all abstracted from properties of the Gaussian free field, which will be introduced later.

(A) (L^2 regularity). For all $\phi \in C^\infty(\mathbb{T}^d, \mathbb{R})$, we have $\mathbb{E}[|(\mathbf{A}^0, \phi)|^2] < \infty$. Moreover, we have that as $N \to \infty$, $\mathbb{E}[|(\mathbf{A}^0, \phi) - (\mathbf{A}^0_N, \phi)|^2] \to 0$.

(B) (Tail bounds). There exist constants $C_{\mathrm{B}} > 0$ and $\beta_{\mathrm{B}} > 0$ such that the following hold. For any $\phi \in C^\infty(\mathbb{T}^d, \mathbb{R})$, we have that

$$\mathbb{P}(|(\mathbf{A}^0, \phi)| > u) \le C_{\mathrm{B}} \exp\left(-(u/\sigma_\phi)^{\beta_{\mathrm{B}}}/C_{\mathrm{B}}\right), \quad u \ge 0.$$

Additionally, for any $N \ge 0$, and for any smooth function $K \in C^\infty(\mathbb{I}^2, \mathbb{R})$ such that $K((a, j_1, x), (a, j_2, y)) = 0$ for all $a \in [d_\mathfrak{g}]$, $j_1, j_2 \in [d]$, $x, y \in \mathbb{T}^d$, we have that

$$\mathbb{P}(|(\mathbf{A}^0_N, K\mathbf{A}^0_N)| > u) \le C_{\mathrm{B}} \exp\left(-(u/\sigma_{N,K})^{\beta_{\mathrm{B}}}/C_{\mathrm{B}}\right), \quad u \ge 0.$$

(C) (Translation invariance of covariance function) There is an integrable function $\rho : (\mathbb{T}^d)^2 \to L(\mathfrak{g}^d, \mathfrak{g}^d)$ (here $L(\mathfrak{g}^d, \mathfrak{g}^d)$ is the space of linear maps $\mathfrak{g}^d \to \mathfrak{g}^d$) such that for any test functions $\phi_1, \phi_2 \in C^\infty(\mathbb{T}^d, \mathbb{R})$, and any linear map $K : \mathfrak{g}^d \to \mathfrak{g}^d$,

$$\mathbb{E}\left[\langle (\mathbf{A}^0, \phi_1), K(\mathbf{A}^0, \phi_2)\rangle\right] = \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \phi_1(x)\phi_2(y)\mathrm{Tr}(K\rho(x,y)^t)dxdy.$$

Moreover, we assume that ρ is translation invariant, i.e. $\rho(x, y) = \rho(x - y, 0) = \rho(0, y - x)$ for $x, y \in \mathbb{T}^d$. Here, $\mathrm{Tr}(K\rho(x,y)^t)$ is (for instance) computed by representing $M, \rho(x, y)$ as matrices with respect to the basis of $\mathfrak{g}^d$ induced by the orthonormal basis $(S^a, a \in [d_\mathfrak{g}])$ of $\mathfrak{g}$. Similarly, "integrable" in this context can be taken to mean that all matrix entry functions of ρ are integrable.

(D) (Covariance is only as singular as G^α). For some $\alpha \in (0, d)$, there is some constant C_{D} such that for any $x, y \in \mathbb{T}^d$, $x \neq y$,

$$|\mathrm{Tr}(\rho(x, y))| \le C_{\mathrm{D}}(G^\alpha(x, y) + C_{\mathrm{D}}).$$

We assume without loss of generality that $G^\alpha + C_{\mathrm{D}} \ge 1$ (this is possible since G^α_0 is bounded from below, by Lemma 1.15).

(E) (Four product assumption). There is some constant $C_{\mathrm{E}} \ge 0$ such that the following holds. Let $a_1, a_2 \in [d_\mathfrak{g}]$, $j_1, j_2 \in [d]$, $\phi_1, \phi_2, \phi_3, \phi_4 \in C^\infty(\mathbb{T}^d, \mathbb{C})$. Assume that $a_1 \neq a_2$. Let $Z_1 = (\mathbf{A}^{0,a_1}_{j_1}, \phi_1)$, $Z_2 = (\mathbf{A}^{0,a_2}_{j_2}, \phi_2)$, $Z_3 = (\mathbf{A}^{0,a_1}_{j_1}, \phi_3)$, $Z_4 = (\mathbf{A}^{0,a_2}_{j_2}, \phi_4)$. Then

$$|\mathbb{E}\left[Z_1 Z_2 \overline{Z_3 Z_4}\right]| \le C_{\mathrm{E}}(|\mathbb{E}\left[Z_1\overline{Z_3}\right]\mathbb{E}\left[Z_2\overline{Z_4}\right]| + |\mathbb{E}\left[Z_1\overline{Z_4}\right]\mathbb{E}\left[Z_2\overline{Z_3}\right]|).$$

Remark 1.16. Assumption (C) is motivated by the following fact. Let X, Y be random vectors in $\mathbb{R}^n$, and let $\Sigma := \mathbb{E}[XY^t]$ (so Σ is an $n \times n$ matrix). Then for any $n \times n$ matrix M, we have that $\mathbb{E}[X^t M Y] = \mathrm{Tr}(M\Sigma^t)$.

Also, to be more concrete, instead of working with the function ρ in Assumption (C), one may work with scalar functions $\rho^{a_1 a_2}_{j_1 j_2} : (\mathbb{T}^d)^2 \to \mathbb{R}$ for $a_1, a_2 \in [d_\mathfrak{g}]$, $j_1, j_2 \in [d]$, which are defined by requiring that

$$\mathbb{E}\left[(\mathbf{A}^{0,a_1}_{j_1}, \phi_1)(\mathbf{A}^{0,a_2}_{j_2}, \phi_2)\right] = \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \phi_1(x)\rho^{a_1 a_2}_{j_1 j_2}(x, y)\phi_2(y)dxdy.$$

The function $\rho^{a_1 a_2}_{j_1 j_2}$ is then interpreted as the $((a_1, j_1), (a_2, j_2))$ matrix entry of ρ.

Remark 1.17. In fact, we don't really need to assume that $\alpha < d$; G_0^α can be defined for $\alpha \geq d$ as well. However, we will just assume that $\alpha \in (0, d)$, because this will simplify our proofs later on. In any case, the regime $\alpha \in (0, d)$ is the more nontrivial setting, since G_0^α becomes less singular as α increases (see, e.g., [24, Section 6.1]).

Remark 1.18. In assumption (E), it is important that we don't have the term $\mathbb{E}[Z_1 Z_2]\mathbb{E}[\bar{Z}_3 \bar{Z}_4]$, because this term will lead to divergences in Section 4.2.

We now state the main result of this paper.

Theorem 1.19. *Let $\mathbf{A}^0$ be a random $\mathfrak{g}^d$-valued distribution that satisfies Assumptions (A)–(E). Moreover, suppose that Assumption (D) is satisfied with $\alpha \in (\max\{d - 4/3, d/2\}, d)$. Then there exists a $\mathfrak{g}^d$-valued stochastic process $\mathbf{A} = (\mathbf{A}(t, x), t \in (0, 1), x \in \mathbb{T}^d)$, and a random variable $T \in (0, 1]$, such that the following hold. The function $(t, x) \mapsto \mathbf{A}(t, x)$ is in $C^\infty((0, T) \times \mathbb{T}^d, \mathfrak{g}^d)$, and moreover, it is a solution to (ZDDS) on $(0, T)$. Also, $\mathbb{E}[T^{-p}] < \infty$ for all $p \geq 1$.*

The process $\mathbf{A}$ relates to $\mathbf{A}^0$ in the following way. There exists a sequence $\{T_N\}_{N \geq 0}$ of $(0, 1]$-valued random variables such that the following hold. First, for any $p \geq 1$, we have that $\sup_{N \geq 0} \mathbb{E}[T_N^{-p}] < \infty$, and that $\mathbb{E}[|T_N^{-1} - T^{-1}|^p] \to 0$. Also, let $\{\mathbf{A}_N^0\}_{N \geq 0}$ be the sequence of Fourier truncations of $\mathbf{A}^0$ (defined in Definition 1.11). Then there is a sequence $\{\mathbf{A}_N\}_{N \geq 0}$ of $\mathfrak{g}^d$-valued stochastic processes $\mathbf{A}_N = (\mathbf{A}_N(t, x), t \in [0, 1), x \in \mathbb{T}^d)$ such that for each $N \geq 0$, the function $(t, x) \mapsto \mathbf{A}_N(t, x)$ is in $C^\infty([0, T_n) \times \mathbb{T}^d, \mathfrak{g}^d)$ and is the solution to (ZDDS) on $[0, T_n)$ with initial data $\mathbf{A}_N(0) = \mathbf{A}_N^0$.

Finally, for any $k \in \{0, 1\}$, $p \geq 1$, $\delta \in (0, 1), \varepsilon > 0$, we have that

$$\lim_{N \to \infty} \mathbb{E}\left[\sup_{t \in (0, (1-\delta)T)} t^{p((k/2)+(1/4)(d-\alpha)+\varepsilon)} ||\mathbf{A}_N(t) - \mathbf{A}(t)||_{C^k}^p \right] = 0.$$

Remark 1.20. A closely related result was obtained by Chandra et al. as part of their recent work [25] – see Section 1.2 and Remarks 2.8 and 3.15 in their paper for some similarities and differences. While there is some variation in the ways the results and the proofs are phrased, ultimately we are both (as far as we can tell) exploiting the same phenomenon, which is probabilistic smoothing, which we describe next.

Remark 1.21. The assumption that $\alpha > \max\{d - 4/3, d/2\}$ ensures that $\mathbf{A}^0$ is not too singular, so that there is no need for renormalization when defining the solution to (ZDDS) with initial data $\mathbf{A}^0$.

We now give a quick overview of the proof of the local existence part of Theorem 1.19. The proof of the other part of the theorem has a similar main idea. As usual with local existence for parabolic PDEs, we would like to try to realize the solution $\mathbf{A}$ as the fixed point of some contraction map W. Then, we could for instance obtain $\mathbf{A}$ by taking the limit of $W^{(n)}(\mathbf{A}^1)$, where $W^{(n)}$ is the n-fold composition of W, and $\mathbf{A}^1$ is the linear part of $\mathbf{A}$, i.e., $\mathbf{A}^1$ is the solution to the heat equation with initial data $\mathbf{A}^0$. However, the problem is that the initial data $\mathbf{A}^0$ is too rough, so that deterministic arguments break down already in the first step of the Picard iteration — that is, when trying to obtain estimates on $W(\mathbf{A}^1)$ by deterministic (worst-case) methods, we get divergent integrals.

The saving grace is that $W(\mathbf{A}^1)$ behaves better than the worst-case. So instead of bounding $W(\mathbf{A}^1)$ deterministically, we bound it probabilistically, which allows us to take advantage of probabilistic cancelations which occur. To give an analogy with an elementary example, note that if $\{X_n\}_{n\geq 1}$ is a sequence of i.i.d. random variables with mean 0 variance 1, then the series $\sum_{n=1}^{\infty} X_n/n$ converges a.s. (by Kolmogorov's two series theorem). However, if we were to try to bound this deterministically, the sum $\sum_{n=1}^{\infty} n^{-1} = \infty$ would inevitably appear. Analogously, we show that $W(\mathbf{A}^1)$ can be defined in a probabilistic sense, and in fact the difference $W(\mathbf{A}^1) - \mathbf{A}^1$ is more regular than the linear part $\mathbf{A}^1$. Once this regularity gain is established, we can then obtain the local existence of $\mathbf{A}$ by a deterministic fixed point argument (i.e., Picard iteration).

This general idea to exploit the effects of probabilistic smoothing was (as far as we can tell) first used by Bourgain [15, 16] to analyze the nonlinear Schrödinger equation with GFF initial data. A similar idea was later used by Da Prato and Debussche [26, 27] in the stochastic PDE setting. There is by now a wide body of work building on this idea in many different settings – see [17, Section 1.2.2] for a much more complete list of references.

We next introduce the Gaussian free field (GFF), which will be the main example of random distributional initial data in this paper. Standard references are [28–30]. A d-dimensional mean zero GFF on $\mathbb{T}^d$ is a mean zero Gaussian process $h = ((h, \phi), \phi \in C^\infty(\mathbb{T}^d, \mathbb{R}))$ such that for all test functions $\phi_1, \phi_2 \in C^\infty(\mathbb{T}^d, \mathbb{R})$, the covariance is given by

$$\mathbb{E}[(h, \phi_1)(h, \phi_2)] = \sum_{\substack{n\in\mathbb{Z}^d \\ n\neq 0}} \frac{1}{|n|^2} \widehat{\phi}_1(n)\overline{\widehat{\phi}_2(n)}. \tag{1.4}$$

For $N \geq 0$, let the Fourier truncation $h_N = (h_N(x), x \in \mathbb{T}^d)$ be the mean zero Gaussian process with smooth sample paths defined as $h_N := F_N h$. By standard properties of F_N and the GFF, we have that for any $\phi \in C^\infty(\mathbb{T}^d, \mathbb{R})$, $(h_N, \phi) \to (h, \phi)$ both a.s. and in L^2 (actually, the a.s. convergence holds simultaneously for all ϕ). Therefore h_N converges to h in a natural sense. (Another viewpoint is that if we view h as a random element of a negative Sobolev space, then h_N a.s. converges to h in that space.)

Since (ZDDS) is a PDE on 1-forms, the initial data we take must also be a 1-form. Recalling that we may view a 1-form A as a collection of functions $(A_j^a, a \in [d_{\mathfrak{g}}], j \in [d])$ satisfying (1.1), this motivates the following definition of the d-dimensional $\mathfrak{g}^d$-valued GFF. We say that $\mathbf{A}^0$ is a d-dimensional $\mathfrak{g}^d$-valued GFF if it is a collection of stochastic processes

$$\mathbf{A}^0 = (\mathbf{A}_j^{0,a}, a \in [d_{\mathfrak{g}}], j \in [d]),$$

where $\mathbf{A}_j^{0,a}, a \in [d_{\mathfrak{g}}], j \in [d]$ are independent d-dimensional GFFs. This is $\mathfrak{g}^d$-valued, because given $\phi \in C^\infty(\mathbb{T}^d, \mathbb{R})$, we may define a $\mathfrak{g}^d$-valued random variable $(\mathbf{A}^0, \phi) = ((\mathbf{A}^0, \phi)_j, j \in [d])$ through the relation (1.1).

As previously mentioned, the assumptions of Theorem 1.19 were abstracted from properties of the GFF. Thus, naturally, we will be able to obtain the following corollary of Theorem 1.19.

Corollary 1.22. *Let $d = 3$, and let $\mathbf{A}^0$ be a $\mathfrak{g}^3$-valued GFF on $\mathbb{T}^3$. Then the statement of Theorem 1.19 applies to $\mathbf{A}^0$.*

Remark 1.23. The above corollary will also hold for $d = 2$ by a simpler deterministic argument. However, once $d \geq 4$, the same result does not necessarily apply, because Assumption (D) will only be satisfied for α small (e.g., $\alpha = 2$ when $d = 4$), which is to say that the GFF becomes too singular once $d \geq 4$.

1.2. Additional notation

We introduce some additional notation. Throughout this paper, C will denote a generic constant that may depend only on G. It may change from line to line, and even within a line. To express dependence on some additional parameter, say α, we will write C_α. In these situations, we always understand C_α to also depend on G. Similarly, if we say that C_α depends only on α, we really mean that C_α depends only on α and G.

The metric on $\mathbb{T}^d$ (that is, the metric induced by the standard Euclidean metric of $\mathbb{R}^d$) will be denoted by $d_{\mathbb{T}^d}$. Explicitly, if $\Pi : \mathbb{R}^d \to \mathbb{T}^d$ is the canonical projection map, then $d_{\mathbb{T}^d}(x, y) := \inf\{|x_0 - y_0| : \Pi(x_0) = x, \Pi(y_0) = y\}$. Here $|x_0 - y_0|$ is the Euclidean distance between $x_0, y_0 \in \mathbb{R}^d$.

Fix a real finite-dimensional normed linear space $(V, |\cdot|)$ (for instance, we may take $V = \mathfrak{g}$ or $\mathfrak{g}^d$). For $r \geq 0$, we write $C^r(\mathbb{T}^d, V)$ for the usual Hölder space and for $p \geq 0$, we write $L^p(\mathbb{T}^d, V)$ for the usual L^p space. We will write the respective norms as $||f||_{C^r}$ and $||f||_p$ for brevity.

Let $(e^{t\Delta})_{t>0}$ be the semigroup generated by the Laplacian Δ. Explicitly, given $f \in L^1(\mathbb{T}^d, V)$ and $t > 0$, we have that

$$e^{t\Delta} f = \sum_{n \in \mathbb{Z}^d} e^{-4\pi^2 |n|^2 t} \widehat{f}(n) e_n. \tag{1.5}$$

Additionally, $e^{t\Delta} f$ has an explicit representation in terms of convolution with the heat kernel Φ, i.e., for all $t > 0$, $x \in \mathbb{T}^d$, we have that

$$(e^{t\Delta} f)(x) = \int_{\mathbb{T}^d} f(y) \Phi(t, x - y) dy, \tag{1.6}$$

where

$$\Phi(t, x) := \sum_{n \in \mathbb{Z}^d} e^{-4\pi^2 |n|^2 t} e_n(x), \quad t > 0, x \in \mathbb{T}^d. \tag{1.7}$$

We also know that $\Phi(t, \cdot)$ is a probability density, and so it is non-negative, and thus we have the following monotonicity property for integrable $\mathbb{R}$-valued functions f, g:

$$|f| \leq g \Rightarrow |e^{t\Delta}f| \leq e^{t\Delta}g. \tag{1.8}$$

(In the above, $|f| \leq g$ means $|f(x)| \leq g(x)$ for all $x \in \mathbb{T}^d$ — actually, a.e. $x \in \mathbb{T}^d$ suffices — and similarly for $|e^{t\Delta}f| \leq e^{t\Delta}g$.)

Recall the orthonormal basis $(S^a,\ a \in [d_\mathfrak{g}])$ of $\mathfrak{g}$ from Definition 1.4. Let $(f^{abc},\ a, b, c \in [d_\mathfrak{g}])$ be the corresponding structure constants, i.e.,

$$[S^a, S^b] = \sum_{c \in [d_\mathfrak{g}]} f^{abc} S^c. \tag{1.9}$$

By starting from the definition above (and using that the inner product is given by $\langle S_1, S_2 \rangle = \mathrm{Tr}(S_1^* S_2) = -\mathrm{Tr}(S_1 S_2)$), we obtain for $a, b, c \in [d_\mathfrak{g}]$, $\langle [S^a, S^b], S^c \rangle = \langle [S^b, S^c], S^a \rangle$. This shows that $f^{abc} = f^{bca}$. Proceeding similarly, we may obtain

$$f^{abc} = f^{cab} = -f^{acb} = -f^{bac} = -f^{cba} = f^{bca}. \tag{1.10}$$

Remark 1.24. Even though we are introducing structure constants here, the results of this paper do not really rely on the specific bracket structure of (ZDDS). Indeed, we expect that the arguments could be adapted to the case where $\mathfrak{g}$ is replaced by a finite-dimensional normed algebra, and (ZDDS) is replaced by an equation of the form

$$\partial_t A = \Delta A + A(\nabla A) + A^3.$$

1.3. Organization of the paper

We now give a summary of the rest of the paper. In Section 2, we state Theorem 2.9, which is a deterministic result that gives local existence of solutions to (ZDDS) with distributional intial data, assuming certain conditions are met. We also state various other useful deterministic lemmas in Section 2.1. Given Theorem 2.9, the remainder of the paper is then concerned with showing that the conditions of the theorem are indeed met, for random distributional initial data with certain properties, as listed just before Theorem 1.19. Sections 3.1 and 3.2 collect the main intermediate steps toward the proof of Theorem 1.19. Given these intermediate steps, Theorem 1.19 is proven in Section 3.3. Corollary 1.22 is obtained as an application of Theorem 1.19 in the same section. In Section 3.4, we state and prove Proposition 3.18, which is a variant of Theorem 1.19 that will be used in [23]. Sections 4.1 and 4.2 contain the technical arguments needed to prove the intermediate results of Sections 3.1 and 3.2.

We will reiterate this at several later points, but we also mention here that the proofs of many intermediate results in this paper are omitted. For the full proofs, please see the complete version of this paper on arXiv.

2. Deterministic results

In this section, we collect the deterministic results that are needed later on in the paper. We emphasize here that the results of this section may be read independently of the rest of the paper (although of course the main reason for these results is to use them to deduce Theorem 1.19). The main result of this section (Theorem 2.9) shows local existence of solutions to (ZDDS) with distributional initial data. The proofs of most results in this section are small variations of proofs of classical results in the theory of local existence for nonlinear parabolic PDEs, and thus they will be omitted. For full proofs, please see the complete version of this paper on arXiv. We first define the notation that will be needed in Theorem 2.9 and in other parts of this paper.

Definition 2.1. For $r \geq 0$, $T > 0$, define the path space $\mathcal{P}_T^r$ to be the space of continuous functions $A : [0, T] \to C^r(\mathbb{T}^d, \mathfrak{g}^d)$. Define the norm $||\cdot||_{\mathcal{P}_T^r}$ on $\mathcal{P}_T^r$ by

$$||A||_{\mathcal{P}_T^r} := \sup_{0 \leq t \leq T} ||A||_{C^r}, \quad A \in \mathcal{P}_T^r.$$

Note that $(\mathcal{P}_T^r, ||\cdot||_{\mathcal{P}_T^r})$ is a Banach space.

Definition 2.2. Let $\gamma \geq 0$, $T > 0$. Define the path space $\mathcal{Q}_T^\gamma$ to be the space of continuous functions $A : (0, T] \to C^1(\mathbb{T}^d, \mathfrak{g}^d)$ such that

$$||A||_{\mathcal{Q}_T^\gamma} := \sup_{t \in (0,T]} t^\gamma ||A(t)||_{C^0} + \sup_{t \in (0,T]} t^{(1/2)+\gamma} ||A(t)||_{C^1} < \infty.$$

For $R \geq 0$, define $\mathcal{Q}_{T,R}^\gamma := \left\{ A \in \mathcal{Q}_T^\gamma : ||A||_{\mathcal{Q}_T^\gamma} \leq R \right\}$.

Remark 2.3. We thank one of the referees for pointing out here that the regularity parameter γ is flipped, in that larger γ allows for more irregularity.

By standard arguments, one can show that $(\mathcal{Q}_T^\gamma, ||\cdot||_{\mathcal{Q}_T^\gamma})$ is a Banach space.

Definition 2.4. Given a 1-form $A \in C^1(\mathbb{T}^d, \mathfrak{g}^d)$, define $X(A) \in C^0(\mathbb{T}^d, \mathfrak{g}^d)$ by $X(A) = (X_i(A), i \in [d])$, where

$$X_i(A) := \sum_{j \in [d]} \left[A_j, 2\partial_j A_i - \partial_i A_j + [A_j, A_i] \right], \quad i \in [d].$$

Define $X^{(2)}(A), X^{(3)}(A) \in C^0(\mathbb{T}^d, \mathfrak{g}^d)$ as follows. For $i \in [d]$, let

$$X_i^{(2)}(A) := \sum_{j \in [d]} [A_j, 2\partial_j A_i - \partial_i A_j], \quad X_i^{(3)}(A) := \sum_{j \in [d]} [A_j, [A_j, A_i]].$$

Note by construction that $X(A) = X^{(2)}(A) + X^{(3)}(A)$.

Definition 2.5. Let $T > 0$. Let $j \in \{2, 3\}$. Let $A : [0, T] \to C^1(\mathbb{T}^d, \mathfrak{g}^d)$ be a continuous function. Suppose that

$$\int_0^T ||e^{(t-s)\Delta} X^{(j)}(A(s))||_{C^1} ds < \infty. \tag{2.1}$$

Define $\rho^{(j)}(A) : [0, T] \to C^1(\mathbb{T}^d, \mathfrak{g}^d)$ by

$$\rho^{(j)}(A)(t) := \int_0^t e^{(t-s)\Delta} X^{(j)}(A(s)) ds, \quad t \in [0, T].$$

We say that $\rho^{(j)}(A)$ is well-defined for A if (2.1) holds. Now if $\rho^{(j)}(A)$ is well-defined for A for $j=2, 3$, define $\rho(A) : [0,T] \to C^1(\mathbb{T}^d, \mathfrak{g}^d)$ by $\rho(A) := \rho^{(2)}(A) + \rho^{(3)}(A)$. In this case, we say that $\rho(A)$ is well-defined for A. Note since $X = X^{(2)} + X^{(3)}$, we have that

$$\rho(A)(t) = \int_0^t e^{(t-s)\Delta} X(A(s))ds, \quad t \in [0,T].$$

We also will use these definitions in the case where $[0,T]$ is replaced by $(0,T]$ everywhere.

The next lemma shows that $\rho(A)$ is well-defined for A if $A \in \mathcal{P}^1_T$, and moreover that $\rho(A) \in \mathcal{P}^1_T$.

Lemma 2.6. *Let $T \in (0,1]$. Let $A \in \mathcal{P}^1_T$. Then $\rho^{(j)}(A)$ is well-defined for A for $j \in \{2,3\}$, and moreover $\rho^{(2)}(A), \rho^{(3)}(A) \in \mathcal{P}^1_T$. Thus also $\rho(A) \in \mathcal{P}^1_T$.*

Definition 2.7. Let $A^1 : (0,1] \to C^1(\mathbb{T}^d, \mathfrak{g}^d)$ be such that $A^1(t) = e^{(t-s)\Delta}A^1(s)$ for all $s,t \in (0,1]$, $s<t$. Let $B^1 : (0,1] \to C^1(\mathbb{T}^d, \mathfrak{g}^d)$ be a continuous function. We say that B^1 is a first nonlinear part for A^1 if the following holds. For $t_0 \in (0,1)$, let $\tilde{A}^1(t) := A^1(t_0+t)$, $\tilde{B}^1(t) := B^1(t_0+t)$, $t \in [0, 1-t_0]$. Then for all $t_0 \in (0,1)$ and all $t \in [0,1-t_0]$, we have that $\tilde{B}^1(t) = e^{t\Delta}\tilde{B}^1(0) + \rho(\tilde{A}^1)(t)$.

Remark 2.8. To see where Definition 2.7 comes from, suppose A^0 is a smooth 1-form, and let $A^1(t) = e^{t\Delta}A^0$, $t \in (0,1]$. Then one can verify that $B^1 = \rho(A^1)$ is a first nonlinear part of A^1. Definition 2.7 abstracts this relation to the setting where $\rho(A^1)$ is not necessarily well-defined, which will be the case for us, because we are considering (random) distributional initial data.

We can now state the main result of this section. This theorem is the deterministic part of the argument outlined just after the statement of Theorem 1.19. In essence, this theorem says the following. In usual local existence arguments via contraction mapping, given A^1 as in Definition 2.7, we would want to bound $\rho(A^1)$, and moreover, show that $\rho(A^1)$ is more regular than A^1. However, for us, $\rho(A^1)$ will not even be well-defined, because A^1 will be too rough. On the other hand, if we are able to obtain a proxy B^1 for $\rho(A^1)$, such that B^1 is more regular than A^1, then we can still run a fixed point argument to obtain a solution to (ZDDS). If it helps, one can think of this strategy as running a fixed point argument on an "enhanced space" consisting of pairs (A^1, B^1), instead of just A^1.

Theorem 2.9. *Let $\gamma_1 \in [0,1/2)$, $\gamma_2 \in [0,1/4)$ be such that $\gamma_1 + \gamma_2 < 1/2$. Then, there is a continuous non-increasing function $\tau_{\gamma_1\gamma_2} : [0,\infty) \to (0,1]$ (which only depends on γ_1, γ_2, d) such that the following holds. Let $A^1 : (0,1] \to C^1(\mathbb{T}^d, \mathfrak{g}^d)$ be such that $A^1(t) = e^{(t-s)\Delta}A^1(s)$ for all $s,t \in (0,1]$, $s<t$. Suppose $A^1 \in \mathcal{Q}^{\gamma_1}_1$. Suppose that there exists $B^1 \in \mathcal{Q}^{\gamma_2}_1$ which is a first nonlinear part for A^1. Let $R := \max\left\{||A^1||_{\mathcal{Q}^{\gamma_1}_1}, ||B^1||_{\mathcal{Q}^{\gamma_2}_1}\right\}$, and let $T := \tau_{\gamma_1\gamma_2}(R)$. Then there exists $B \in \mathcal{Q}^{\gamma_2}_{T,3R}$ such that $A := A^1 + B$ is in $C^\infty((0,T) \times \mathbb{T}^d, \mathfrak{g}^d)$, and moreover A is a solution to (ZDDS) on $(0,T)$.*

Additionally, we have continuity in the data, in the following sense. Suppose that we have a sequence $\{A_n^1\}_{n\geq 1} \subseteq \mathcal{Q}_1^{\gamma_1}$ *such that for each* $n \geq 1$, $A_n^1(t) = e^{(t-s)\Delta}A_n^1(s)$ *for all* $s, t \in (0, 1]$, $s < t$. *Suppose we have a sequence* $\{B_n^1\}_{n\geq 1} \subseteq \mathcal{Q}_1^{\gamma_2}$, *such that for each* $n \geq 1$, B_n^1 *is a first nonlinear part for* A_n^1. *Suppose that* $||A_n^1 - A^1||_{\mathcal{Q}_1^{\gamma_1}} \to 0$ *and* $||B_n^1 - B^1||_{\mathcal{Q}_1^{\gamma_2}} \to 0$. *Let* $R_n := \max\left\{||A_n^1||_{\mathcal{Q}_1^{\gamma_1}}, ||B_n^1||_{\mathcal{Q}_1^{\gamma_2}}\right\}$, *and* $T_n := \tau_{\gamma_1\gamma_2}(R_n)$. *For each* $n \geq 1$, *let* $B_n \in \mathcal{Q}_{T_n,3R_n}^{\gamma_2}$ *be as constructed by the first part of the theorem, so that* $A_n := A_n^1 + B_n$ *is a solution to (ZDDS) on* $(0, T_n)$. *Then for all* $T_0 \in (0, T)$, *we have that* $||B_n - B||_{\mathcal{Q}_{T_0}^{\gamma_2}} \to 0$, *which implies that* $||A_n - A||_{\mathcal{Q}_{T_0}^{\max\{\gamma_1,\gamma_2\}}} \to 0$.

We next state several auxiliary results that arise from the proof of Theorem 2.9.

Lemma 2.10. *Let* $\gamma_1, \gamma_2, \tau_{\gamma_1\gamma_2}$ *be as in Theorem 2.9. Then*

$$\tau_{\gamma_1\gamma_2}(R)^{-1} \leq C_{\gamma_1,\gamma_2,d}(1 + R^{4/(1-2\max\{\gamma_1,\gamma_2\})}), \quad R \geq 0.$$

The following lemma shows that for smooth initial data A^0, the solution to (ZDDS) given by Theorem 2.9 coincides with the solution to (ZDDS) with initial data A^0 (which is given by Theorem 1.2).

Lemma 2.11. *Let* $\gamma_1, \gamma_2, \tau_{\gamma_1\gamma_2}$ *be as in Theorem 2.9. Let* A^0 *be a smooth 1-form. Let* $A^1(t) = e^{t\Delta}A^0$, $t \geq 0$. *Let* $B^1 = \rho(A^1)$. *(Recall that by Remark 2.8,* B^1 *is a first nonlinear part for* A^1*.) Let* $R := \max\left\{||A^1||_{\mathcal{Q}_1^{\gamma_1}}, ||B^1||_{\mathcal{Q}_1^{\gamma_2}}\right\}$, $T := \tau_{\gamma_1\gamma_2}(R) > 0$. *Then, there exists* $A \in C^\infty([0, T) \times \mathbb{T}^d, \mathfrak{g}^d)$ *such that* A *is the solution to (ZDDS) on* $[0, T)$ *with initial data* $A(0) = A^0$. *Moreover, on* $(0, T) \times \mathbb{T}^d$, A *is equal to the solution to (ZDDS) given by Theorem 2.9.*

2.1. Useful lemmas

In this section, we introduce some deterministic lemmas which will be needed later. For the first lemma, recall the definition of $\mathcal{P}_T^r$ from Definition 2.1, as well as the definitions of $\rho^{(2)}$ and $\rho^{(3)}$ from Definition 2.5.

Lemma 2.12. *Let* $\{A_n^0\}_{n\leq\infty} \subseteq C^1(\mathbb{T}^d, \mathfrak{g}^d)$ *be a sequence of 1-forms. For* $n \leq \infty$, *let* $A_n^1(t) = e^{t\Delta}A_n^0$, $t \geq 0$. *Let* $T \in (0, 1]$, *and suppose that* $||A_n^1 - A_\infty^1||_{\mathcal{P}_T^1} \to 0$. *Then for* $j \in \{2, 3\}$, *we have that* $||\rho^{(j)}(A_n^1) - \rho^{(j)}(A_\infty^1)||_{\mathcal{P}_T^1} \to 0$, *and consequently, we also have that* $||\rho(A_n^1) - \rho(A_\infty^1)||_{\mathcal{P}_T^1} \to 0$.

Lemma 2.13. *Let* $\gamma \in [0, 1/3)$, $T \in (0, 1]$, $R \geq 0$. *Let* $A \in \mathcal{Q}_{T,R}^\gamma$. *Then* $\rho^{(3)}(A)$ *is well-defined for* A, *and moreover* $\rho^{(3)}(A) \in \mathcal{Q}_T^0$, *and*

$$||\rho^{(3)}(A)||_{\mathcal{Q}_T^0} \leq CT^{1-3\gamma}R^3.$$

Additionally, for $A_1, A_2 \in \mathcal{Q}^{\gamma}_{T,R}$, we have that

$$||\rho^{(3)}(A_1) - \rho^{(3)}(A_2)||_{\mathcal{Q}^0_T} \le CT^{1-3\gamma}R^2||A_1 - A_2||_{\mathcal{Q}^{\gamma}_T}.$$

In the remainder of Section 2.1, we will give an explicit formula for $\rho(A^1)$, where A^1 is defined as $A^1(t) = e^{t\Delta}A^0$, and A^0 is a smooth 1-form. This formula will be in terms of the Fourier coefficients of A^0. It will be used in Section 4.

Recall from Section 1.1 that we may view 1-forms $A^0 : \mathbb{T}^d \to \mathfrak{g}^d$ equivalently as collections of $\mathbb{R}$-valued functions $(A^{0,a}_j, a \in [d_{\mathfrak{g}}], j \in [d])$ which satisfy the relation (1.1). In the following, recall also the structure constants $(f^{abc},\ \ a, b, c \in [d_{\mathfrak{g}}])$ defined at the end of Section 1.2.

Definition 2.14. For $m = (n^1, n^2) \in (\mathbb{Z}^d)^2$, $t \ge 0$, define

$$I(m,t) := \int_0^t e^{-4\pi^2|n^1+n^2|^2(t-s)}e^{-4\pi^2(|n^1|^2+|n^2|^2)s}ds.$$

Additionally, for $a = (a_0, a_1, a_2) \in [d_{\mathfrak{g}}]^3$, $j = (j_0, j_1, j_2) \in [d]^3$, define

$$d(m,a,j) := \mathrm{i}2\pi f^{a_0a_1a_2}(\delta_{j_0j_2}n^2_{j_1} - \delta_{j_0j_1}n^1_{j_2} + (1/2)\delta_{j_1j_2}(n^1_{j_0} - n^2_{j_0})).$$

Here, $\delta_{jk} = \mathbb{1}(j = k)$ for $j, k \in [d]$.

Remark 2.15. Note that $0 \le I(m,t) \le t$. Note that by (1.10), if a_0, a_1, a_2 are not distinct, then $f^{a_0a_1a_2} = 0$, and thus also $d(m,a,j) = 0$. Also, note that if we let $-m := (-n^1, -n^2)$, then (here we use that $f^{a_0a_1a_2}$ is real, which follows by definition – recall (1.9))

$$d(-m,a,j) = -d(m,a,j) = \overline{d(m,a,j)}. \tag{2.2}$$

Finally, note that $|d(m,a,j)| \le C(|n^1| + |n^2|)$.

The proof of the following lemma is a long but straightforward calculation, and thus it is omitted.

Lemma 2.16. *Let $A^0 \in C^\infty(\mathbb{T}^d, \mathfrak{g}^d)$ be a smooth 1-form. Let $A^1(t) = e^{t\Delta}A^0$, $t \ge 0$. Let $A^2 = \rho^{(2)}(A^1)$. For any $a_0 \in [d_{\mathfrak{g}}]$, $j_0 \in [d]$, $t \ge 0$, we have that*

$$A^{2,a_0}_{j_0}(t) = \sum_{\substack{a_1,a_2 \in [d_{\mathfrak{g}}] \\ j_1,j_2 \in [d]}} \sum_{n^1,n^2 \in \mathbb{Z}^d} I(m,t)d(m,a,j)\widehat{A}^{0,a_1}_{j_1}(n^1)\widehat{A}^{0,a_2}_{j_2}(n^2)e_{n^1+n^2},$$

where, for brevity, we have taken $m = (n^1, n^2)$, $a = (a_0, a_1, a_2)$, $j = (j_0, j_1, j_2)$. Additionally, we have that

$$\partial_t A^{2,a_0}_{j_0}(t) = \Delta A^{2,a_0}_{j_0}(t) + \sum_{\substack{a_1,a_2 \in [d_{\mathfrak{g}}] \\ j_1,j_2 \in [d]}} \sum_{n^1,n^2 \in \mathbb{Z}^d} d(m,a,j)e^{-4\pi^2|n^1|^2t}\widehat{A}^{0,a_1}_{j_1}(n^1)e^{-4\pi^2|n^2|^2t}\widehat{A}^{0,a_2}_{j_2}(n^2)e_{n^1+n^2}.$$

From the previous lemma, one can show the following. Again, the proof is omitted, as it is a calculation. Recall the notation from Definition 1.6.

Lemma 2.17. *Let A^0, A^2 be as in Lemma 2.16. Suppose that for some $N \geq 0$, we have that $\widehat{A}^0(n) = 0$ for all $|n|_\infty > N$. Then for any $a_0 \in [d_{\mathfrak{g}}]$, $j_0 \in [d]$, $t_0 \in (0,1]$, $x_0 \in \mathbb{T}^d$, there exists a smooth function $K \in C^\infty(\mathbb{I}^2, \mathbb{R})$ such that*

$$A^{2,a_0}_{j_0}(t_0, x_0) = (A^0, KA^0).$$

Additionally, for $l \in [d]$, there exists a smooth function $L \in C^\infty(\mathbb{I}^2, \mathbb{R})$ such that

$$\partial_l A^{2,a_0}_{j_0}(t_0, x_0) = (A^0, LA^0).$$

Finally, for any $a_1 \in [d_{\mathfrak{g}}]$, $j_1, j_2 \in [d]$, $x, y \in \mathbb{T}^d$, we have that

$$K((a_1, j_1, x), (a_1, j_2, y)) = L((a_1, j_1, x), (a_1, j_2, y)) = 0.$$

3. Outline of intermediate results and proof of Theorem 1.19

In this section, we outline the intermediate results that are needed in the proof of Theorem 1.19. Then, in Section 3.3, we show how to use these intermediate results to deduce Theorem 1.19. The proofs of these intermediate results are deferred until Section 4.

3.1. Linear part

Throughout this section, let $\mathbf{A}^0$ be a random $\mathfrak{g}^d$-valued distribution satisfying Assumptions (A), (B), (C), and (D). For this section, we just assume that Assumption (D) holds for some $\alpha \in (0, d)$, i.e., we do not need the restriction $\alpha > \max\{d - 4/3, d/2\}$ that appears in Theorem 1.19. These assumptions hold, even if they are not explicitly stated in the various lemmas or propositions. The proofs of all results stated in this section are in Section 4.1.

Recall the definition of the heat kernel Φ (Eq. (1.7)). We proceed to define $\mathbf{A}^1$, which may be interpreted as $\mathbf{A}^1(t) = e^{t\Delta}\mathbf{A}^0$.

Definition 3.1. Define the $\mathfrak{g}^d$-valued stochastic process $\mathbf{A}^1 = (\mathbf{A}^1(t,x), t \in (0,1], x \in \mathbb{T}^d)$ by $\mathbf{A}^1(t,x) := (e^{t\Delta}\mathbf{A}^0)(x) = (\mathbf{A}^0, \Phi(t, x - \cdot))$.

We will first show the following result about regularity of $\mathbf{A}^1$.

Lemma 3.2. *There exists a modification of $\mathbf{A}^1$ which has smooth sample paths, and which is a solution to the heat equation on $(0,1] \times \mathbb{T}^d$.*

Thus from here on out, we will assume that $\mathbf{A}^1$ has been modified to have smooth sample paths which are solutions to the heat equation (so that $\mathbf{A}^1(t) = e^{(t-s)\Delta}\mathbf{A}^1(s)$ for all $s, t \in (0,1]$, $s < t$). Next, we define the natural notion of Fourier truncations of $\mathbf{A}^1$.

Definition 3.3. Let $N \geq 0$. Define the $\mathfrak{g}^d$-valued stochastic process $\mathbf{A}^1_N = (\mathbf{A}^1_N(t,x), t \in (0,1], x \in \mathbb{T}^d)$ by $\mathbf{A}^1_N := F_N \mathbf{A}^1$.

Remark 3.4. Since F_N is linear and $\mathbf{A}^1 = e^{t\Delta}\mathbf{A}^0$, we have that $\mathbf{A}^1_N = e^{t\Delta} F_N \mathbf{A}^0 = e^{t\Delta}\mathbf{A}^0_N$ (recall we defined $\mathbf{A}^0_N = F_N \mathbf{A}^0$ in Definition 1.9). Also, we have that $\mathbf{A}^1_N(t,x) =$

$(F_N\mathbf{A}^0, \Phi(t,x-\cdot)) = (\mathbf{A}^0, F_N\Phi(t,x-\cdot))$, and so by Assumption (A), we have that for any $t \in (0,1]$, $x \in \mathbb{T}^d$, $\mathbf{A}^1_N(t,x) \xrightarrow{L^2} \mathbf{A}^1(t,x)$.

We now state the main result of Section 3.1.

Proposition 3.5. *For $\varepsilon > 0$, let $\gamma_\varepsilon := (1/4)(d-\alpha)+\varepsilon$. For any $\varepsilon > 0$, $p \geq 1$, we have that*

$$\sup_{N\geq 0} \mathbb{E}\left[||\mathbf{A}^1_N||^p_{\mathcal{Q}^{\gamma_\varepsilon}_1}\right], \mathbb{E}\left[||\mathbf{A}^1||^p_{\mathcal{Q}^{\gamma_\varepsilon}_1}\right] \leq C_{\varepsilon,p} < \infty.$$

Here, $C_{\varepsilon,p}$ depends only on ε, p, d, and the various constants in Assumptions (A)–(D); i.e., α, β_{B}, C_{B}, etc. Additionally, we have that

$$\lim_{N\to\infty} \mathbb{E}\left[||\mathbf{A}^1_N - \mathbf{A}^1||^p_{\mathcal{Q}^{\gamma_\varepsilon}_1}\right] = 0.$$

Remark 3.6. By Proposition 3.5, upon replacing $\mathbf{A}^1$ by a suitable modification, we may assume that $||\mathbf{A}^1||_{\mathcal{Q}_1^{(1/4)(d-\alpha)+\varepsilon}} < \infty$ for all $\varepsilon > 0$. Hereafter, we assume that this holds for $\mathbf{A}^1$.

3.2. Nonlinear part

As in Section 3.1, throughout this section, we assume that $\mathbf{A}^0$ is a random $\mathfrak{g}^d$-valued distribution satisfying Assumptions (A), (B), (C), and (D). For this section, we assume (as in Theorem 1.19) that Assumption (D) holds for some $\alpha \in (\max\{d-4/3, d/2\}, d)$. Additionally, we assume that Assumption (E) is satisfied. These assumptions hold, even if they are not explicitly stated in the various lemmas, corollaries, or propositions.

Recall the process $\mathbf{A}^1$ constructed in Section 3.1. This process is such that $\mathbf{A}^1(t) = e^{(t-s)\Delta}\mathbf{A}^1(s)$ for all $0 < s < t \leq 1$. In this section, we construct a first nonlinear part $\mathbf{B}^1$ for $\mathbf{A}^1$, in the sense of Definition 2.7. We will do this by constructing $\mathbf{A}^2 = \rho^{(2)}(\mathbf{A}^1)$ and $\mathbf{A}^3 = \rho^{(3)}(\mathbf{A}^1)$ (recall Definition 2.5 for the definitions of $\rho^{(2)}, \rho^{(3)}$), and then letting $\mathbf{B}^1 = \mathbf{A}^2 + \mathbf{A}^3$. The construction of $\mathbf{A}^3$ is easier, so we handle it first.

Definition 3.7. Recall Remark 3.6 that $\mathbf{A}^1 \in \mathcal{Q}_1^{(1/4)(d-\alpha)+\varepsilon}$ for all $\varepsilon > 0$. Thus by Lemma 2.13, and the assumption that $\alpha > d - 4/3$, we may define a $\mathfrak{g}^d$-valued stochastic process $(\mathbf{A}^3(t,x), t \in (0,1], x \in \mathbb{T}^d)$ by $\mathbf{A}^3 := \rho^{(3)}(\mathbf{A}^1)$, and moreover this process is such that $\mathbf{A}^3 \in \mathcal{Q}^0_1$. Also, by the definition of $\rho^{(3)}$, the following holds. Take any $T_0 \in (0,1)$, and let $\widetilde{\mathbf{A}}^3 : [0, 1-T_0] \to C^1(\mathbb{T}^d, \mathfrak{g}^d)$ be defined by $\widetilde{\mathbf{A}}^3(t) := \mathbf{A}(T_0+t)$, $t \in [0, 1-T_0]$. Then

$$\widetilde{\mathbf{A}}^3(t) = e^{t\Delta}\widetilde{\mathbf{A}}^3(0) + \int_0^t e^{(t-s)\Delta} X^{(3)}(\mathbf{A}^1(T_0+s))ds, \quad t \in [0, 1-T_0]. \tag{3.1}$$

For $N \geq 0$, define also the stochastic process $\mathbf{A}^3_N = (\mathbf{A}^3_N(t,x), t \in (0,1], x \in \mathbb{T}^d)$ by $\mathbf{A}^3_N := \rho^{(3)}(\mathbf{A}^1_N)$.

The next result shows that $\mathbf{A}^3_N$ converges to $\mathbf{A}^3$ as $N \to \infty$, as expected.

Lemma 3.8. *For any $p \geq 1$, we have that*

$$\sup_{N \geq 0} \mathbb{E}\left[||\mathbf{A}_N^3||_{\mathcal{Q}_1^0}^p\right], \mathbb{E}\left[||\mathbf{A}^3||_{\mathcal{Q}_1^0}^p\right] \leq C_p < \infty.$$

Here C_p depends only on p, d, and the various constants in Assumptions (A)–(D); i.e., α, β_{B}, C_{B}, etc. Additionally,

$$\lim_{N \to \infty} \mathbb{E}\left[||\mathbf{A}_N^3 - \mathbf{A}^3||_{\mathcal{Q}_1^0}^p\right] = 0.$$

Proof. Both claims follow by combining Lemma 2.13, Hölder's inequality, and Proposition 3.5 with large enough p. □

We next proceed to construct $\mathbf{A}^2 = \rho^{(2)}(\mathbf{A}^1)$. We cannot just construct this deterministically as we did for $\mathbf{A}^3 = \rho^{(3)}(\mathbf{A}^1)$, because $\mathbf{A}^1$ is too rough, so that $\rho^{(2)}(\mathbf{A}^1)$ will not be well-defined. Instead, $\mathbf{A}^2$ will be constructed probabilistically.

Definition 3.9. For $N \geq 0$, define the process $\mathbf{A}_N^2 = (\mathbf{A}_N^2(t, x), t \in (0, 1], x \in \mathbb{T}^d)$ by $\mathbf{A}_N^2 := \rho^{(2)}(\mathbf{A}_N^1)$.

We proceed to construct $\mathbf{A}^2$ as an appropriate limit of $\mathbf{A}_N^2$. First, we show the following result. The proof is in Section 4.2.

Lemma 3.10. *For any $t \in (0, 1]$, $x \in \mathbb{T}^d$, we have that $\{\mathbf{A}_N^2(t, x)\}_{N \geq 0}$ is a Cauchy sequence in L^2.*

This leads directly to the following definition.

Definition 3.11. Define the $\mathfrak{g}^d$-valued stochastic process $\mathbf{A}^2 = (\mathbf{A}^2(t, x), t \in (0, 1], x \in \mathbb{T}^d)$ as follows. By Lemma 3.10, we have that $\{\mathbf{A}_N^2(t, x)\}_{N \geq 0}$ is Cauchy in L^2, and thus the sequence converges in L^2. Define $\mathbf{A}^2(t, x)$ to be the limit.

Having defined $\mathbf{A}^2$, the next step is the following. The proof is in Section 4.2.

Lemma 3.12. *The process $\mathbf{A}^2$ has a modification such that the function $t \mapsto \mathbf{A}^2(t)$ is a continuous function from $(0, 1]$ into $C^1(\mathbb{T}^d, \mathfrak{g}^d)$.*

Thus hereafter, we assume that (after a suitable modification) $\mathbf{A}^2$ is such that the function $t \mapsto \mathbf{A}^2(t)$ is a continuous function from $(0, 1]$ into $C^1(\mathbb{T}^d, \mathfrak{g}^d)$.

Definition 3.13. Define the $\mathfrak{g}^d$-valued stochastic process $\mathbf{B}^1 = (\mathbf{B}^1(t, x), t \in (0, 1], x \in \mathbb{T}^d)$ by $\mathbf{B}^1(t, x) := \mathbf{A}^2(t, x) + \mathbf{A}^3(t, x)$. For $N \geq 0$, let $\mathbf{B}_N^1 := \rho(\mathbf{A}_N^1) = \rho^{(2)}(\mathbf{A}_N^1) + \rho^{(3)}(\mathbf{A}_N^1) = \mathbf{A}_N^2 + \mathbf{A}_N^3$.

The next result shows that $\mathbf{B}^1$ is indeed a first nonlinear part for $\mathbf{A}^1$, in the sense of Definition 2.7. The proof is in Section 4.2.

Lemma 3.14. *On an event of probability 1, we have that for all $t_0, t_1 \in (0, 1]$, $t_0 < t_1$, $x \in \mathbb{T}^d$,*

$$\mathbf{B}^1(t_1,x) = (e^{(t_1-t_0)\Delta}\mathbf{B}^1(t_0))(x) + \int_0^{t_1-t_0} (e^{(t_1-t_0-s)\Delta}X(\mathbf{A}^1(t_0+s)))(x)ds.$$

In light of Lemma 3.14, hereafter, we assume that $\mathbf{A}^1, \mathbf{A}^2, \mathbf{A}^3$ have been modified so that $\mathbf{B}^1$ is a first nonlinear part of $\mathbf{A}^1$ (in the sense of Definition 2.7). We can now finally state the main result of Section 3.2. The proof is in Section 4.2.

Proposition 3.15. *For $\varepsilon > 0$, let $\gamma_\varepsilon := (1/2)(d-1-\alpha)+\varepsilon$. For any $\varepsilon > 0, p \geq 1$, we have that*

$$\sup_{N\geq 0}\mathbb{E}\Big[||\mathbf{A}^2_N||^p_{\mathcal{Q}_1^{\gamma_\varepsilon}}\Big], \mathbb{E}\Big[||\mathbf{A}^2||^p_{\mathcal{Q}_1^{\gamma_\varepsilon}}\Big] \leq C_{\varepsilon,p} < \infty, \quad \lim_{N\to\infty}\mathbb{E}\Big[||\mathbf{A}^2_N - \mathbf{A}^2||^p_{\mathcal{Q}_1^{\gamma_\varepsilon}}\Big] = 0.$$

Consequently, we have that

$$\sup_{N\geq 0}\mathbb{E}\Big[||\mathbf{B}^1_N||^p_{\mathcal{Q}_1^{\gamma_\varepsilon}}\Big], \mathbb{E}\Big[||\mathbf{B}^1||^p_{\mathcal{Q}_1^{\gamma_\varepsilon}}\Big] \leq C_{\varepsilon,p} < \infty,$$
$$\lim_{N\to\infty}\mathbb{E}\Big[||\mathbf{B}^1_N - \mathbf{B}^1||^p_{\mathcal{Q}_1^{\gamma_\varepsilon}}\Big] = 0.$$

Here, $C_{\varepsilon,p}$ depends only on ε, p, d, and the various constants in Assumptions (A)–(E); i.e., α, β_B, C_B, etc.

Remark 3.16. By Proposition 3.15, upon replacing $\mathbf{A}^1, \mathbf{B}^1$ by suitable modifications, we may assume that $||\mathbf{B}^1||_{\mathcal{Q}_1^{(1/2)(d-1-\alpha)+\varepsilon}} < \infty$ for all $\varepsilon > 0$, while still ensuring that $\mathbf{B}^1$ is a first nonlinear part of $\mathbf{A}^1$. Hereafter, we assume that this holds for $\mathbf{A}^1, \mathbf{B}^1$.

3.3. Proofs of Theorem 1.19 and Corollary 1.22

We can now prove Theorem 1.19 by combining Propositions 3.5 and 3.15 with Theorem 2.9.

Proof of Theorem 1.19. Let $\mathbf{A}^1$ be as constructed in Section 3.1, and let $\mathbf{A}^1$ be as constructed in Section 3.2. We want to apply Theorem 2.9 to $\mathbf{A}^1, \mathbf{B}^1$. First, note that by the assumption that $\alpha > d - 4/3$ in the statement of Theorem 1.19, we have that $(1/4)(d-\alpha) < 1/2$, $(1/2)(d-1-\alpha) < 1/4$, $(1/4)(d-\alpha) + (1/2)(d-1-\alpha) < 1/2$. Therefore, we may take $\varepsilon_0 > 0$ small enough such that defining $\gamma_1 := (1/4)(d-\alpha)+\varepsilon_0$, $\gamma_2 := (1/2)(d-1-\alpha)+\varepsilon_0$, we have that $\gamma_1 \in [0,1/2)$, $\gamma_2 \in [0,1/4)$, and $\gamma_1 + \gamma_2 < 1/2$. By Proposition 3.5 and Remark 3.6, we have that $||\mathbf{A}^1||_{\mathcal{Q}_1^{\gamma_1}} < \infty$, and similarly by Proposition 3.15 and Remark 3.16, we have that $||\mathbf{B}^1||_{\mathcal{Q}_1^{\gamma_2}} < \infty$.

Let $R = \max\Big\{||\mathbf{A}^1||_{\mathcal{Q}_1^{\gamma_1}}, ||\mathbf{B}^1||_{\mathcal{Q}_1^{\gamma_2}}\Big\}$, and let $T = \tau_{\gamma_1\gamma_2}(R)$, where $\tau_{\gamma_1\gamma_2}$ is as in Theorem 2.9. The fact that $\mathbb{E}[T^{-p}] < \infty$ for all $p \geq 1$ follows by Lemma 2.10 and Propositions 3.5 and 3.15. By Theorem 2.9, there exists $\mathbf{B} \in \mathcal{Q}^{\gamma_2}_{T,3R}$ such that $\mathbf{A}^1 + \mathbf{B}$ is in $C^\infty((0,T)\times\mathbb{T}^d, \mathfrak{g}^d)$, and moreover $\mathbf{A}^1 + \mathbf{B}$ is a solution to (ZDDS) on $(0,T)$. We can then define the process $\mathbf{A} = (\mathbf{A}(t,x), t \in (0,1], x \in \mathbb{T}^d)$ by:

$$\mathbf{A}(t,x) := \mathbb{1}(t < T)(\mathbf{A}^1(t,x) + \mathbf{B}(t,x)).$$

We now move on to the second part of the theorem. Let $\{\mathbf{A}^1_N\}_{N\geq 0}$, $\{\mathbf{B}^1_N\}_{N\geq 0}$ be as constructed in Sections 3.1 and 3.2, respectively. For $N \geq 0$, let $R_N = \max(||\mathbf{A}^1_N||_{\mathcal{Q}_1^{\gamma_1}}, ||\mathbf{B}^1_N||_{\mathcal{Q}_1^{\gamma_2}})$, and let $T_N = \tau_{\gamma_1\gamma_2}(R_N)$. For the same reasons as before, we have that $\sup_{N\geq 0}\mathbb{E}[T_N^{-p}] < \infty$ for all $p \geq 1$. Also, by Propositions 3.5 and 3.15, we have that $R_N \xrightarrow{P} R$ (here $\xrightarrow{P}$ denotes convergence in probability) and thus, since $\tau_{\gamma_1\gamma_2}$ is continuous, we obtain that $T_N \xrightarrow{P} T$. This implies that $T_N^{-1} \xrightarrow{P} T^{-1}$. The fact that $\mathbb{E}[|T_N^{-1} - T^{-1}|^p] \to 0$ for all $p \geq 1$ now follows by Vitali's ([31, (21.2) Theorem]) theorem (combined with the L^p-boundedness for any $p \geq 1$, which gives uniform integrability).

For each $N \geq 0$, we apply Theorem 2.9 and Lemma 2.11 to obtain $\mathbf{B}_N \in \mathcal{Q}^{\gamma_2}_{T_N,3R_N}$ such that $\mathbf{A}^1_N + \mathbf{B}_N$ is in $C^\infty([0,T_N) \times \mathbb{T}^d, \mathfrak{g}^d)$, and moreover it is the solution to (ZDDS) on $[0,T_N)$ with initial data $\mathbf{A}^0_N$ (recall that the solution is unique, by Lemma 1.3). We may thus define the process $\mathbf{A}_N = (\mathbf{A}_N(t,x), t \in [0,1], x \in \mathbb{T}^d)$ by

$$\mathbf{A}_N(t,x) := \mathbb{1}(t < T_N)(\mathbf{A}^1_N(t,x) + \mathbf{B}_N(t,x)).$$

It remains to show the last claims about convergence of $\mathbf{A}_N$ to $\mathbf{A}$. Let $p \geq 1$, $\delta \in (0,1)$, $\varepsilon > 0$. Note that by Proposition 3.5, $||\mathbf{A}^1_N - \mathbf{A}^1||_{\mathcal{Q}_1^{\gamma_1}} \xrightarrow{P} 0$, and that by Proposition 3.15, $||\mathbf{B}^1_N - \mathbf{B}^1||_{\mathcal{Q}_1^{\gamma_2}} \xrightarrow{P} 0$. Combining this with Theorem 2.9, we obtain that $||\mathbf{B}_N - \mathbf{B}||_{\mathcal{Q}^{\gamma_2}_{(1-\delta)T}} \xrightarrow{P} 0$. (This can be shown by using the fact that convergence to 0 in probability is equivalent to the property that for any subsequence, there is a further subsequence which converges to 0 a.s. The $(1-\delta)T$ comes from the fact that $\mathbf{B}_N$ is a solution to (ZDDS) on $[0,T_N)$, and T_N may be less than T. However, we know that $T_N \xrightarrow{P} T$.) We thus also obtain $||\mathbf{A}_N - \mathbf{A}||_{\mathcal{Q}^{\gamma_1}_{(1-\delta)T}} \xrightarrow{P} 0$, because the assumption that $\alpha > d - 4/3$ implies that $\gamma_1 > \gamma_2$. Now to show the last claims about convergence of $\mathbf{A}_N$ to $\mathbf{A}$, it suffices (by Vitali's theorem — [31, (21.2) Theorem]) to show that for any $p \geq 1$, the sequence $\left\{||\mathbf{A}_N - \mathbf{A}||^p_{\mathcal{Q}^{\gamma_2}_{(1-\delta)T}}\right\}_{N\geq 0}$ is L^2-bounded (and thus uniformly integrable). Fix $p \geq 1$. Since $\mathbf{A}_N(t) = 0$ if $t \geq T_N$, we have that

$$||\mathbf{A}_N - \mathbf{A}||^{2p}_{\mathcal{Q}^{\gamma_1}_{(1-\delta)T}} \leq C_p\Big(||\mathbf{A}_N||^{2p}_{\mathcal{Q}^{\gamma_1}_{T_N}} + ||\mathbf{A}||^{2p}_{\mathcal{Q}^{\gamma_1}_{T}}\Big).$$

We have that $\mathbf{A}_N = \mathbf{A}^1_N + \mathbf{B}_N$, where $\mathbf{A}^1_N, \mathbf{B}_N \in \mathcal{Q}^{\gamma_1}_{T_N,3R_N}$, and similarly for $\mathbf{A}$. From this, we obtain

$$||\mathbf{A}_N||^{2p}_{\mathcal{Q}^{\gamma_1}_{T_N}} \leq C_p R_N^{2p}, \quad ||\mathbf{A}||^{2p}_{\mathcal{Q}^{\gamma_1}_{T}} \leq C_p R^{2p}.$$

Now by Propositions 3.5 and 3.15, we have that $\sup_{N\geq 0}\mathbb{E}[R_N^{2p}] < \infty$, $\mathbb{E}[R^{2p}] < \infty$. The desired L^2-boundedness now follows. Thus we have shown the last claims for $\varepsilon = \varepsilon_0$, where ε_0 is a small enough quantity that we fixed at the beginning. The last claims for general $\varepsilon > 0$ then follow, because due to monotonicity in ε, it just suffices to show the claims for small enough $\varepsilon > 0$. □

We next turn to proving Corollary 1.22.

Lemma 3.17. *Let $d=3$, and let $\mathbf{A}^0$ be a 3D $\mathfrak{g}^3$-valued GFF. Then Assumptions (A)–(E) are satisfied, and moreover Assumption (D) is satisfied with $\alpha=2$.*

Before we prove Lemma 3.17, we note that Corollary 1.22 follows directly.

Proof of Corollary 1.22. This follows from Theorem 1.19 and Lemma 3.17. □

The rest of this section is devoted to the proof of Lemma 3.17. Thus, we assume in the rest of this section that $d=3$, and $\mathbf{A}^0$ is a 3D $\mathfrak{g}^3$-valued GFF. Assumptions (A) and (C)-(E) may be readily checked by using standard properties of Gaussian distributions. Thus, we will only prove that Assumption (B) holds. (See the complete version of this paper on arXiv for a full proof.)

Proof of Assumption (B). We will show the assumption with $\beta_{\mathrm{B}} = 1$. We will verify the assumption "coordinate-wise", i.e. for the processes $\mathbf{A}_j^{0,a}$. First, for any $\phi \in C^\infty(\mathbb{T}^3, \mathbb{R})$, $a \in [d_{\mathfrak{g}}]$, $j \in [3]$, we have that $(\mathbf{A}_j^{0,a}, \phi) \sim N(0, (\sigma_{\phi,j}^a)^2)$ (here we define $(\sigma_{\phi,j}^a)^2$ to be the variance of $(\mathbf{A}_j^{0,a}, \phi)$), and thus by the standard Gaussian tail bound, we have that

$$\mathbb{P}(|(\mathbf{A}_j^{0,a}, \phi)| > u) \leq 2 \exp(-u^2/(2(\sigma_{\phi,j}^a)^2)), \quad u \geq 0.$$

By splitting into cases $u \leq \sigma_{\phi,j}^a$ and $u \geq \sigma_{\phi,j}^a$, it then follows that

$$\mathbb{P}(|(\mathbf{A}_j^{0,a}, \phi)| > u) \leq 2 \exp(-u/(2\sigma_{\phi,j}^a)), \quad u \geq 0.$$

We next turn to concentration for quadratic forms. Fix $N \geq 0$. Observe that $(\mathbf{A}_{N,j}^{0,a}(x), a \in [d_{\mathfrak{g}}], j \in [3], x \in \mathbb{T}^3)$ is a mean 0 Gaussian process with smooth sample paths. Let K be as in Assumption (B). For notational simplicity, let $Q = (\mathbf{A}_N^0, K\mathbf{A}_N^0)$. Let $k \geq 1$. By approximating $\mathbb{T}^3$ by a lattice with spacing $1/k$, we may obtain a random variable Q_k, which is a Riemann sum approximation of Q (recall the definition of $(\mathbf{A}_N^0, K\mathbf{A}_N^0)$ in Eq. (1.2)). Moreover, Q_k is a quadratic form of a centered Gaussian vector. Also, we have that $\mathbb{E}(Q_k) = 0$, because $\mathbf{A}_N^0$ is a mean 0 process, $\mathbf{A}_{N,j_1}^{0,a_1}, \mathbf{A}_{N,j_2}^{0,a_2}$ are independent for $a_1 \neq a_2$, and the assumption that $K((a,j_1,x),(a,j_2,y)) = 0$. Thus by Lemma B.1, we have that for all $k \geq 1$,

$$\mathbb{P}(|Q_k| > u) \leq 2e^{3/2} \exp(-u/(2(\mathbb{E}(Q_k^2))^{1/2})), \quad u \geq 0.$$

Now since $\mathbf{A}_N^0$ has smooth sample paths, we have that $Q_k \to Q$. It then follows that (by, e.g., Fatou's lemma)

$$\mathbb{P}(|Q| > u) \leq \liminf_k \mathbb{P}(|Q_k| > u).$$

Thus to finish, it suffices to show that $\limsup_k \mathbb{E}(Q_k^2) \leq \mathbb{E}(Q^2)$ (note that $\mathbb{E}(Q) = 0$ as well, for the same reasons why $\mathbb{E}(Q_k) = 0$). Actually, we have that $\mathbb{E}[(Q_k - Q)^2] \to 0$, because $Q_k \to Q$, and the sequence $\{(Q_k - Q)^2\}_{k \geq 1}$ is uniformly integrable. The uniform integrability follows because $\sup_{k\geq 1}\mathbb{E}(Q_k^4), \mathbb{E}(Q^4) < \infty$, which itself can be seen from the fact that $\mathbf{A}_N^0$ is a Gaussian process such that $\sup_{x\in\mathbb{T}^3}\mathbb{E}[|\mathbf{A}_N^0(x)|^2] < \infty$. □

3.4 A Related result

As mentioned in Section 1, the results of this paper will be applied in [23]. In particular, we will use Corollary 1.22 in [23]. However, we will not directly use Theorem 1.19. Instead, we now give a related result that will be more suited for the purposes of [23]. First, suppose that $\mathbf{A}^0 = (\mathbf{A}^0(x), x \in \mathbb{T}^d)$ is now a $\mathfrak{g}^d$-valued stochastic process with smooth sample paths. Note that this naturally induces a random $\mathfrak{g}^d$-valued distribution by defining $(\mathbf{A}^0, \phi) := \int_{\mathbb{T}^d}\mathbf{A}^0(x)\phi(x)dx$ for all $\phi \in C^\infty(\mathbb{T}^d, \mathbb{R})$. We say that $\mathbf{A}^0$ satisfies some given assumption if the corresponding random $\mathfrak{g}^d$-valued distribution satisfies the assumption.

Proposition 3.18. *Let $\mathbf{A}^0 = (\mathbf{A}^0(x), x \in \mathbb{T}^d)$ be a $\mathfrak{g}^d$-valued stochastic process with smooth sample paths. Suppose that it satisfies Assumptions (A)–(E). Moreover, suppose that Assumption (D) is satisfied with $\alpha \in (\max\{d-4/3, d/2\}, d)$. Then there exists a $\mathfrak{g}^d$-valued stochastic process $\mathbf{A} = (\mathbf{A}(t,x), t \in [0,1), x \in \mathbb{T}^d)$, and a random variable $T \in (0,1]$, such that the following hold. The function $(t,x) \mapsto \mathbf{A}(t,x)$ is in $C^\infty([0,T) \times \mathbb{T}^d, \mathfrak{g}^d)$, and moreover it is the solution to (ZDDS) on $[0,T)$ with initial data $\mathbf{A}(0) = \mathbf{A}^0$. Also, $\mathbb{E}[T^{-p}] \leq C_p < \infty$ for all $p \geq 1$, where C_p depends only on p, d, and the various constants in Assumptions (A)–(E); i.e., α, β_B, C_B, etc. Finally, for any $k \in \{0,1\}$, $p \geq 1$, $\varepsilon > 0$, we have that*

$$\mathbb{E}\left[\sup_{t\in(0,T)} t^{p((k/2)+(1/4)(d-\alpha)+\varepsilon)}||\mathbf{A}(t)||_{C^k}^p\right] \leq C_{p,\varepsilon}.$$

Here $C_{p,\varepsilon}$ depends only on p, ε, d, and the various constants in Assumptions (A)–(E); i.e., α, β_B, C_B, etc.

This proposition gives bounds on $\mathbf{A}$, as opposed to Theorem 1.19, which only gives that $\mathbf{A}$ exists, and that $\mathbf{A}_N$ converges to $\mathbf{A}$ in a suitable sense. Also, note that in contrast to Theorem 1.19, we take sup over $t \in (0,T)$ as opposed to sup over $t \in (0,(1-\delta)T)$ in the final two inequalities. This is because we are only bounding $\mathbf{A}$, which we know exists on $(0,T)$, as opposed to $\mathbf{A}_N - \mathbf{A}$. Recall that $\mathbf{A}_N$ is a solution to (ZDDS) only on $[0,T_N)$, and it may be the case that $T_N < T$ (but on the other hand, we do have that $T_N > (1-\delta)T$ with probability tending to 1 as $N \to \infty$). Before we prove Proposition 3.18, we need the following natural lemma, whose proof is omitted (the complete version of this paper on arXiv contains a proof).

Lemma 3.19. *Let $\mathbf{A}^0 = (\mathbf{A}^0(x), x \in \mathbb{T}^d)$ be as in Proposition 3.18. Let $\mathbf{A}^1, \mathbf{B}^1$ be constructed using $\mathbf{A}^0$ as in Sections 3.1 and 3.2. Then a.s., for all $t \in (0,1], x \in \mathbb{T}^d$, we have that $\mathbf{A}^1(t,x) = (e^{t\Delta}\mathbf{A}^0)(x)$, $\mathbf{B}^1(t,x) = (\rho(\mathbf{A}^1)(t))(x)$.*

Proof of Proposition 3.18. We slightly modify the proof of Theorem 1.19. As in that proof, take $\varepsilon_0 > 0$ small enough such that defining $\gamma_1 := (1/4)(d-\alpha) + \varepsilon_0$, $\gamma_2 := (1/2)(d-1-\alpha) + \varepsilon_0$, we have that $\gamma_1 \in [0,1/2)$, $\gamma_2 \in [0,1/4)$, and $\gamma_1 + \gamma_2 < 1/2$. Let

$\mathbf{A}^1, \mathbf{B}^1$ be constructed using $\mathbf{A}^0$ as in Sections 3.1 and 3.2. By Lemma 3.19, after a suitable modification of $\mathbf{A}^1, \mathbf{B}^1$, we have that $\mathbf{A}^1(t) = e^{t\Delta}\mathbf{A}^0$, $\mathbf{B}^1 = \rho(\mathbf{A}^1)$. Then by arguing as in the proof of Theorem 1.19, we can obtain a stochastic process $\mathbf{A} = (\mathbf{A}(t,x), t \in [0,1), x \in \mathbb{T}^d)$ by letting

$$\mathbf{A}(t,x) := \mathbb{1}(t < T)(\mathbf{A}^1(t,x) + \mathbf{B}(t,x)),$$

such that $\mathbf{A}$ is the solution to (ZDDS) on $[0,T)$ with initial data $\mathbf{A}(0) = \mathbf{A}^0$.

Next, by Propositions 3.5 and 3.15, we have that $\mathbb{E}[R^p] \leq C_{p,\varepsilon_0}$. By Lemma 2.10, we have that $\mathbb{E}[T^{-p}] \leq C_p + C_p\mathbb{E}[R^{4p/(1-2\max(\gamma_1,\gamma_2))}]$. From this, we obtain $\mathbb{E}[T^{-p}] \leq C_p$ for all $p \geq 1$, where C_p depends only on p, d, and the various constants in Assumptions (A)–(E).

For the final two inequalities, note that since $\mathbf{B} \in \mathcal{Q}^{\gamma_2}_{T,3R}$ and $\mathbf{A}^1 \in \mathcal{Q}^{\gamma_1}_{T,R}$, we have that $||\mathbf{A}||_{\mathcal{Q}^{\gamma_1}_T} \leq 4R$ (here we use that $\gamma_1 > \gamma_2$, which follows by the assumption that $\alpha > d - 4/3$). Thus recalling that $\mathbb{E}[R^p] \leq C_{p,\varepsilon_0}$, we have shown the last two inequalities for $\varepsilon = \varepsilon_0$, where ε_0 is a small enough quantity that we fixed at the beginning. The last two inequalities for general $\varepsilon > 0$ then follow, because due to the monotonicity in ε, it just suffices to show the inequalities for small enough $\varepsilon > 0$. □

4. Technical proofs

We first show some general results which will be needed for both the linear and nonlinear parts. Many of the proofs of this section are omitted; see the complete version of this paper on arXiv for the full arguments. Recall the covariance function $\rho : (\mathbb{T}^d)^2 \to L(\mathfrak{g}^d, \mathfrak{g}^d)$ from Assumption (C). For notational simplicity, let $\tau : \mathbb{T}^d \to \mathbb{R}$ be defined by $\tau(x) := \mathrm{Tr}(\rho(x, 0))$. Since ρ is integrable and translation invariant by Assumption (C), we have that τ is also integrable. We will denote the Fourier coefficients of τ by $\widehat{\tau}(n)$ for $n \in \mathbb{Z}^d$.

The following lemma shows that the translation invariance assumption leads to the Fourier coefficients being uncorrelated. The proof is a short computation, and thus it is omitted.

Lemma 4.1. *Suppose that Assumption (C) holds. For any $n^1, n^2 \in \mathbb{Z}^d$, we have that*

$$\mathbb{E}\Big[\langle \widehat{\mathbf{A}}^0(n^1), \widehat{\mathbf{A}}^0(n^2)\rangle\Big] = \mathbb{1}(n^1 = n^2)\widehat{\tau}(n^1).$$

Consequently, $\mathbb{E}[|\widehat{\mathbf{A}}^0(n)|^2] = \widehat{\tau}(n) \geq 0$. Additionally, for any $a_1, a_2 \in [d_\mathfrak{g}]$, $j_1, j_2 \in [d]$, $n^1, n^2 \in \mathbb{Z}^d$, we have that

$$|\mathbb{E}\Big[\widehat{\mathbf{A}}^{0,a_1}_{j_1}(n^1)\overline{\widehat{\mathbf{A}}^{0,a_2}_{j_2}(n^2)}\Big]| \leq \mathbb{1}(n^1 = n^2)\widehat{\tau}(n^1).$$

The following few lemmas will be needed in Sections 4.1 and 4.2. Most proofs are fairly standard and are omitted.

Lemma 4.2. *Let $\alpha \in (0, d)$. For $t \in (0, 1]$, we have that*

$$||e^{t\Delta} G_0^\alpha||_{C^0} \le C_{d,\alpha} t^{-(1/2)(d-\alpha)}.$$

Here $C_{d,\alpha}$ depends only on d, α.

Lemma 4.3. *Suppose that Assumption (D) holds for some $\alpha \in (0, d)$. For any $k \ge 0, t \in (0, 1]$, we have that*

$$||e^{t\Delta}\tau||_{C^k} \le C_k t^{-(d+k-\alpha)/2}.$$

Here, C_k depends only on k, d, α, and the constant C_{D} from Assumption (D).

Definition 4.4. For $t \in (0, 1]$, define the metric d_t on $(t/2, t] \times \mathbb{T}^d$ by

$$d_t((r, x), (s, y)) := \frac{|r-s|}{t} + \min\left\{\frac{d_{\mathbb{T}^d}(x, y)}{\sqrt{t}}, 1\right\}.$$

For $\varepsilon > 0$, let $N_{t,\varepsilon}$ be the minimum number of ε-balls needed to cover the metric space $((t/2, t] \times \mathbb{T}^d, d_t)$.

Lemma 4.5. *For any $t \in (0, 1]$, $\varepsilon > 0$, we have that*

$$N_{t,\varepsilon} \le Ct^{-d/2}\varepsilon^{-(d+1)}.$$

If $\varepsilon \ge 3/2$, then we have that $N_{t,\varepsilon} = 1$.

Proof. First, note that the diameter of $((t/2, t] \times \mathbb{T}^d, d_t)$ is at most 3/2, and thus the second claim follows. For the first claim, note that the metric space $(t/2, t]$ equipped with Euclidean distance may be covered by $O(\varepsilon^{-1})$ balls of radius $(\varepsilon/2)t$. Let $\{x_i\}_{i\in[n]} \subseteq (t/2, t]$ be such a cover. The metric space $(\mathbb{T}^d, d_{\mathbb{T}^d})$ may be covered by $O((t^{1/2}\varepsilon)^{-d})$ balls of radius $\sqrt{t}(\varepsilon/2)$. Let $\{y_j\}_{j\in[m]} \subseteq \mathbb{T}^d$ be such a cover. We then have that $\{(x_i, y_j)\}_{i\in[n], j\in[m]}$ is an ε-cover of $((t/2, t] \times \mathbb{T}^d, d_t)$, and additionally $mn = O(t^{-d/2}\varepsilon^{-(d+1)})$. The desired result now follows. □

Lemma 4.6. *For any $t \in (0, 1]$, $\beta > 0$, we have that*

$$\int_0^\infty (\log N_{t,\varepsilon})^\beta d\varepsilon \le C_{\beta,d}(1 + |\log t|^\beta).$$

Here $C_{\beta,d}$ depends only on β, d.

Proof. By Lemma 4.5, we may bound (using that $\beta \neq 0$)

$$\int_0^\infty (\log N_{t,\varepsilon})^\beta d\varepsilon \le \int_0^{3/2} (\log (Ct^{-d/2}\varepsilon^{-(d+1)}))^\beta d\varepsilon.$$

Now, note that

$$(\log (Ct^{-d/2}\varepsilon^{-(d+1)}))^\beta \le C_{\beta,d} + C_{\beta,d}|\log t|^\beta + C_{\beta,d}|\log \varepsilon|^\beta.$$

The desired result now follows by noting that (here using that $\beta \geq 0$)

$$\int_0^{3/2} |\log \varepsilon|^{\beta} d\varepsilon \leq C_\beta < \infty.$$

□

In what follows, given a (possibly vector-valued) random variable X, we will write $||X||_{L^2}$ as a shorthand for the L^2 norm of X, i.e., $||X||_{L^2} := (\mathbb{E}(|X|^2))^{1/2}$.

4.1. Linear part

As in Section 3.1, throughout this section, let $\mathbf{A}^0$ be a random $\mathfrak{g}^d$-valued distribution satisfying Assumptions (A), (B), (C), and (D). We just assume that Assumption (D) holds for some $\alpha \in (0, d)$ — that is, for this section, we do not need the restriction that $\alpha > \max\{d - 4/3, d/2\}$ which appears in Theorem 1.19. These assumptions hold, even if they are not explicitly stated in the various lemmas or propositions. We first show the following result. Recall the definition of $\mathbf{A}^1_N$ from Definition 3.3.

Lemma 4.7. *We have that a.s., for all $t > 0$, $k \geq 0$,*

$$\sum_{n \in \mathbb{Z}^d} (1 + |n|)^k e^{-4\pi^2 |n|^2 t} |\widehat{\mathbf{A}}^0(n)| < \infty.$$

Consequently, we have that a.s., for all $t \in (0, 1]$, $x \in \mathbb{T}^d$,

$$\lim_{N \to \infty} \mathbf{A}^1_N(t, x) = \sum_{n \in \mathbb{Z}^d} e^{-4\pi^2 |n|^2 t} \widehat{\mathbf{A}}^0(n) e_n(x).$$

We also have that for all $t \in (0, 1]$, $x \in \mathbb{T}^d$,

$$\mathbf{A}^1(t, x) \stackrel{a.s.}{=} \sum_{n \in \mathbb{Z}^d} e^{-4\pi^2 |n|^2 t} \widehat{\mathbf{A}}^0(n) e_n(x).$$

Proof. It suffices (by monotonicity) to prove the a.s. result for fixed $t > 0$. Toward this end, note that by the Cauchy–Schwarz inequality,

$$\sum_{n \in \mathbb{Z}^d} (1 + |n|)^k e^{-4\pi^2 |n|^2 t} |\widehat{\mathbf{A}}^0(n)| \leq C \bigg(\sum_{n \in \mathbb{Z}^d} e^{-4\pi^2 |n|^2 t} |\widehat{\mathbf{A}}^0(n)|^2 \bigg)^{1/2},$$

for some finite constant C. To finish, observe that (using Lemmas 4.1 and 4.3)

$$\mathbb{E}\left[\sum_{n \in \mathbb{Z}^d} e^{-4\pi^2 |n|^2 t} |\widehat{\mathbf{A}}^0(n)|^2 \right] = \sum_{n \in \mathbb{Z}^d} e^{-4\pi^2 |n|^2 t} \widehat{\tau}(n) = (e^{t\Delta} \tau)(0) < \infty.$$

The a.s. convergence follows immediately from the first claim. The a.s. equality follows from the a.s. convergence and the fact that $\mathbf{A}^1_N(t, x) \xrightarrow{L^2} \mathbf{A}^1(t, x)$ (recall Remark 3.4). □

Proof of Lemma 3.2. Let E be the event that for all $t > 0$, $k \geq 0$, we have that

$$\sum_{n \in \mathbb{Z}^d} (1 + |n|)^k e^{-4\pi^2 |n|^2 t} |\widehat{\mathbf{A}}^0(n)| < \infty.$$

By Lemma 4.7, we have that $\mathbb{P}(E) = 1$. The desired modification is obtained by setting $\mathbf{A}^1$ to be identically 0 off the event E. □

Remark 4.8. From Lemma 4.7, we can also ensure that (after a suitable modification) for all $k \geq 0$, $t_0 \in (0, 1]$,

$$\sup_{t \in [t_0, 1]} ||\mathbf{A}^1(t)||_{C^k} < \infty, \tag{4.1}$$

$$\lim_{N \to \infty} \sup_{t \in [t_0, 1]} ||\mathbf{A}^1_N(t) - \mathbf{A}^1(t)||_{C^k} = 0. \tag{4.2}$$

We next begin to work toward the proof of Proposition 3.5.

Lemma 4.9. *For any $k \in \{0, 1, 2\}$, $l_1, ..., l_k \in [d]$, $t \in (0, 1], x \in \mathbb{T}^d$, we have that*

$$\mathbb{E}\Big[|\partial_{l_1} \cdots \partial_{l_k} \mathbf{A}^1(t, x)|^2\Big] \leq Ct^{-(1/2)(d-\alpha)-k}. \tag{4.3}$$

The above inequalities are also true with $\mathbf{A}^1$ replaced by $\mathbf{A}^1_N$ for any $N \geq 0$. Here, C depends only on d and the constants α, C_{D} from Assumption (D).

Proof. Let $N \geq 0$. We have that for some constant C,

$$\partial_{l_1} \cdots \partial_{l_k} \mathbf{A}^1_N(t, x) = C \sum_{\substack{n \in \mathbb{Z}^d \\ |n|_\infty \leq N}} n_{l_1} \cdots n_{l_k} e^{-4\pi^2 |n|^2 t} \widehat{\mathbf{A}}^0(n) e_n(x).$$

Thus by Lemma 4.1, we obtain

$$\mathbb{E}\Big[|\partial_{l_1} \cdots \partial_{l_k} \mathbf{A}^1_N(t, x)|^2\Big] \leq C \sum_{\substack{n \in \mathbb{Z}^d \\ |n|_\infty \leq N}} |n|^{2k} e^{-8\pi^2 |n|^2 t} \widehat{\tau}(n) \leq C||e^{2t\Delta}\tau||_{C^{2k}}.$$

By Lemma 4.3, the right hand side is bounded by $Ct^{-(1/2)(d-\alpha)-k}$, where the constant C is uniform in N. To finish, we use Fatou's lemma, combined with Eq. (4.2). □

Let $t \in (0, 1]$, $x \in \mathbb{T}^d$. For $N, M \geq 0$, let

$$\mathbf{D}^1_N(t, x) := \mathbf{A}^1_N(t, x) - \mathbf{A}^1(t, x), \qquad \mathbf{D}^1_{N,M}(t, x) := \mathbf{A}^1_N(t, x) - \mathbf{A}^1_M(t, x).$$

Lemma 4.10. *There exists a sequence $\{\delta^{4.10}_N\}_{N \geq 0}$ of non-increasing functions $\delta^{4.10}_N : (0, 1] \to [0, 1]$ such that for any $t \in (0, 1]$, $\lim_{N \to \infty} \delta^{4.10}_N(t) = 0$, and for any $k \in \{0, 1, 2\}$, $l_1, ..., l_k \in [d]$, $N \geq 0$, $t \in (0, 1]$, $x \in \mathbb{T}^d$,*

$$\mathbb{E}\left[|\partial_{l_1}\cdots\partial_{l_k}\mathbf{D}_N^1(t,x)|^2\right] \leq Ct^{-(1/2)(d-\alpha)-k}\delta_N^{4.10}(t).$$

Here, C depends only on d and the constants α, C_D *from Assumption (D).*

Proof. It suffices to show the inequalities with $\mathbf{D}_N^1$ replaced by $\mathbf{D}_{N,M}^1$ for any $M \geq N$, since $\mathbf{D}_{N,M}^1 \to \mathbf{D}_N^1$ as $M \to \infty$ (recall (4.2)). For $N \geq 0$, $j \geq 0$, $t \in (0,1]$, define

$$S(N,j,t) := \sum_{\substack{n\in\mathbb{Z}^d \\ |n|_\infty > N}} |n|^j e^{-8\pi^2|n|^2 t}\widehat{\tau}(n).$$

Arguing as in the proof of Lemma 4.9, we can obtain that

$$\sup_{M\geq N}\mathbb{E}\left[|\partial_{l_1}\cdots\partial_{l_k}\mathbf{D}_{N,M}^1(t,x)|^2\right] \leq C_k S(N,2k,t).$$

Observe that for $m \in \{0,2,4\}$, we have that $S(N,m,t) \leq S(0,m,t)$, and moreover, from the proof of Lemma 4.9, we have that $S(0,m,t) \leq Ct^{-(1/2)(d+m-\alpha)}$. We may thus define

$$\delta_N^{4.10}(t_0) := \sup_{t\in[t_0,1]} \max_{m\in\{0,2,4\}} \frac{S(N,m,t_0)}{\max\{1, S(0,m,t_0)\}}, \quad t_0 \in (0,1].$$

This ensures that $\delta_N^{4.10}$ is non-increasing, and that it maps into $[0,1]$. To finish, we need to show that $\delta_N^{4.10}(t_0) \to 0$. To show this, it suffices to show that for $m \in \{0,2,4\}$, we have that

$$\lim_{N\to\infty}\sup_{t\in[t_0,1]} S(N,m,t_0) = 0.$$

Fix $m \in \{0,2,4\}$. Note that for any $N \geq 0$, $S(N,m,t)$ is non-increasing in t (this follows since $\widehat{\tau}(n) \geq 0$ for all $n \in \mathbb{Z}^d$ — recall Lemma 4.1), and thus it just suffices to show that $S(N,m,t_0) \to 0$. This follows by the fact that $S(0,m,t_0) < \infty$, combined with the definition of $S(N,m,t_0)$ and dominated convergence. □

In what follows, recall the definition of d_t in Definition 4.4.

Lemma 4.11 . *For any* $t \in (0,1]$, *and any* $r,s \in (t/2,t]$, $x,y \in \mathbb{T}^d$, *we have that*

$$||\mathbf{A}^1(r,x) - \mathbf{A}^1(s,y)||_{L^2} \leq Ct^{-(1/4)(d-\alpha)}d_t((r,x),(s,y)).$$

The above inequality is also true with $\mathbf{A}^1$ *replaced by* $\mathbf{A}_N^1$ *for any* $N \geq 0$. *Here, C depends only on d and the constants* α, C_D *from Assumption (D).*

Proof. We will show the result for $\mathbf{A}^1$. The proof for $\mathbf{A}_N^1$ for $N \geq 0$ will be essentially the same. We will show that

$$||\mathbf{A}^1(r,x) - \mathbf{A}^1(r,y)||_{L^2} \leq Ct^{-(1/4)(d-\alpha)}\min\left\{\frac{d_{\mathbb{T}^d}(x,y)}{\sqrt{t}}, 1\right\},$$
$$||\mathbf{A}^1(r,y) - \mathbf{A}^1(s,y)||_{L^2} \leq Ct^{-(1/4)(d-\alpha)}\frac{|r-s|}{t}.$$

Let $\ell : [0,1] \to \mathbb{T}^d$ be a geodesic from y to x, so that ℓ has constant speed $|\ell'| = d_{\mathbb{T}^d}(x,y)$. Then

$$\mathbf{A}^1(r,x) - \mathbf{A}^1(r,y) = \int_0^1 \nabla \mathbf{A}^1(r,\ell(u)) \cdot \ell'(u) du.$$

We thus have

$$\begin{aligned} |\mathbf{A}^1(r,x) - \mathbf{A}^1(r,y)|^2 &\leq \int_0^1 |\nabla \mathbf{A}^1(r,\ell(u)) \cdot \ell\prime(u)|^2 du \\ &\leq d_{\mathbb{T}^d}(x,y)^2 \int_0^1 |\nabla \mathbf{A}^1(r,\ell(u))|^2 du. \end{aligned}$$

Taking expectations and applying (4.3) gives

$$||\mathbf{A}^1(r,x) - \mathbf{A}^1(r,y)||_{L^2} \leq Ct^{-(1/4)(d-\alpha)} \frac{d_{\mathbb{T}^d}(x,y)}{t^{1/2}}.$$

Combining this with (4.3), we obtain the first desired inequality.

For the second inequality, assume without loss of generality that $s < r$, and note that

$$\mathbf{A}^1(r,y) - \mathbf{A}^1(s,y) = \int_s^r \partial_u \mathbf{A}^1(u,y) du = \int_s^r \Delta \mathbf{A}^1(u,y) du.$$

We thus obtain

$$|\mathbf{A}^1(r,y) - \mathbf{A}^1(s,y)|^2 \leq |r-s| \int_s^r |\Delta \mathbf{A}^1(u,y)|^2 du.$$

Applying (4.3), we obtain the second desired inequality. □

The following result will allow us to show the convergence of $\mathbf{A}^1_N$ to $\mathbf{A}^1$ (recall the statement of Proposition 3.5).

Lemma 4.12. *There is a sequence $\{\delta_N^{4.12}\}_{N \geq 0}$ of functions $\delta_N^{4.12} : (0,1] \to [0,1]$, such that the following hold. For any $t \in (0,1]$, we have that $\lim_{N\to\infty} \delta_N^{4.12}(t) = 0$. Also, for any $N \geq 0$, $t \in (0,1]$, $r,s \in (t/2,t]$, $x,y \in \mathbb{T}^d$, we have that*

$$\begin{aligned} ||\mathbf{D}^1_N(r,x)||_{L^2} &\leq Ct^{-(1/4)(d-\alpha)} \delta_N^{4.12}(t), \\ ||\mathbf{D}^1_N(r,x) - \mathbf{D}^1_N(s,y)||_{L^2} &\leq Ct^{-(1/4)(d-\alpha)} d_t((r,x),(s,y)) \delta_N^{4.12}(t). \end{aligned}$$

Here, C depends only on d and the constants α, C_D from Assumption (D).

Proof. Let $\{\delta_N^{4.10}\}_{N \geq 0}$ be the sequence of functions from Lemma 4.10. For $t \in (0,1]$, let $\delta_N^{4.12}(t) := (\delta_N^{4.10}(t/2))^{1/2}$. The first inequality follows by Lemma 4.10 and the fact that $\delta_N^{4.10}$ is non-increasing. The second inequality follows by the same argument as in the proof of Lemma 4.11, where we use Lemma 4.10 in place of Lemma 4.9. In the course of the argument, we also use that $\delta_N^{4.10}$ is non-increasing, so that the value of this function at $u \in (t/2,t]$ is bounded by its value at $t/2$. □

By combining Lemmas 4.11 and 4.12 with Assumption (B), we obtain the following result.

Lemma 4.13. *For $t \in (0,1]$, $r, s \in (t/2, t]$, $x, y \in \mathbb{T}^d$, we have that for $u \geq 0$,*

$$\begin{aligned}\mathbb{P}(|\mathbf{A}^1(r,x)| > u) &\leq C \exp\left(-(u/t^{-(1/4)(d-\alpha)})^{\beta_B}\right),\\ \mathbb{P}(|\mathbf{A}^1(r,x) - \mathbf{A}^1(s,y)| > u) &\leq \\ & C \exp\left(-(u/(t^{-(1/4)(d-\alpha)} d_t((r,x),(s,y))))^{\beta_B}/C\right).\end{aligned}$$

The above inequalities are also true with $\mathbf{A}^1$ replaced by $\mathbf{A}^1_N$ for any $N \geq 0$. Additionally, let $\{\delta_N^{4.12}\}_{N \geq 0}$ be the sequence of functions from Lemma 4.12. Then for any $N \geq 0$, we have that for $u \geq 0$,

$$\begin{aligned}\mathbb{P}(|\mathbf{D}^1_N(r,x)| > u) &\leq C \exp\left(-(u/(t^{-(1/4)(d-\alpha)} \delta_N^{4.12}(t)))^{\beta_B}/C\right),\\ \mathbb{P}(|\mathbf{D}^1_N(r,x) - \mathbf{D}^1_N(s,y)| > u) &\leq \\ & C \exp\left(-(u/(t^{-(1/4)(d-\alpha)} d_t((r,x),(s,y)) \delta_N^{4.12}(t)))^{\beta_B}/C\right).\end{aligned}$$

Here, C depends only on d and the constants $\beta_B, C_B, \alpha, C_D$ from Assumptions (B) and (D).

Proof. We will prove the first two inequalities for $\mathbf{A}^1$. The proof for $\mathbf{A}^1_N$ for $N \geq 0$ will be essentially the same. Note that (recall (1.7) for the definition of Φ)

$$\mathbf{A}^1(r,x) \overset{a.s.}{=} (\mathbf{A}^0, \Phi(r, x - \cdot)).$$

The first inequality now follows by Assumption (B) and the fact that

$$||\mathbf{A}^1(r,x)||_{L^2} \leq C t^{-(1/4)(d-\alpha)}$$

(which holds by Lemma 4.9). Similarly, note that

$$\begin{aligned}\mathbf{A}^1(r,x) - \mathbf{A}^1(s,y) &\overset{a.s.}{=} (\mathbf{A}^0, \Phi(r, x - \cdot)) - (\mathbf{A}^0, \Phi(s, y - \cdot))\\ &\overset{a.s.}{=} (\mathbf{A}^0, \Phi(r, x - \cdot) - \Phi(s, y - \cdot)).\end{aligned}$$

The second inequality now follows by Assumption (B) and Lemma 4.11. For the last two inequalities, note that (recall Definition 3.3)

$$\mathbf{A}^1_N(r,x) \overset{a.s.}{=} (F_N \mathbf{A}^0, \Phi(r, x - \cdot)) = (\mathbf{A}^0, (F_N \Phi)(r, x - \cdot)).$$

We then proceed as before, using Lemma 4.12 instead of Lemmas 4.9 and 4.11. □

The tail bounds from Lemma 4.13 allow us to obtain following result.

Lemma 4.14. *For any $p \geq 1$, $t_0 \in (0,1]$, we have that*

$$\mathbb{E}\left[\sup_{\substack{t \in (t_0/2, t_0]\\ x \in \mathbb{T}^d}} |\mathbf{A}^1(t,x)|^p\right] \leq C t^{-p(1/4)(d-\alpha)} (1 + |\log t_0|^{1/\beta_B})^p.$$

The above also holds with $\mathbf{A}^1$ replaced by $\mathbf{A}^1_N$ for any $N \geq 0$. Additionally, let $\{\delta_N^{4.12}\}_{N \geq 0}$ be the sequence of functions from Lemma 4.12. Then for any $N \geq 0$, we have that

$$\mathbb{E}\left[\sup_{\substack{t\in(t_0/2,t_0]\\ x\in\mathbb{T}^d}} |\mathbf{A}^1_N(t,x) - \mathbf{A}^1(t,x)|^p\right] \leq$$
$$Ct^{-p(1/4)(d-\alpha)}(1+|\log t_0|^{1/\beta_{\mathrm{B}}})^p(\delta_N^{4.12}(t_0))^p.$$

Here, C depends only on d, p and the constants $\beta_{\mathrm{B}}, C_{\mathrm{B}}, \alpha, C_{\mathrm{D}}$ from Assumptions (B) and (D).

Proof. Fix $p \geq 1$. Define the stochastic process $(X_{t,x}, (t,x) \in (t_0/2, t_0] \times \mathbb{T}^d)$ by

$$X_{t,x} := (C^{-1/\beta_{\mathrm{B}}})t^{(1/4)(d-\alpha)}\mathbf{A}^1(t,x),$$

where C is the constant from Lemma 4.13. Then by Lemma 4.13, we have that for $(r,x),(s,y) \in (t_0/2,t_0] \times \mathbb{T}^d$, $u \geq 0$,

$$\mathbb{P}(|X_{r,x}| > u) \leq C\exp(-u^{\beta_{\mathrm{B}}}),$$
$$\mathbb{P}(|X_{r,x} - X_{s,y}| > ud_t((r,x),(s,y))) \leq C\exp(-u^{\beta_{\mathrm{B}}}).$$

By the first inequality and [32, Lemma A.2], we have that

$$\sup_{(r,x)\in(t_0/2,t_0]\times\mathbb{T}^d} \mathbb{E}\big[|X_{r,x}|^p\big] \leq C.$$

Using this bound, together with the second inequality in the previous display and the tail bound from Lemma 4.6, we can now apply Theorem A.3 to obtain the first desired result. The second desired result follows similarly. □

We can now finally prove Proposition 3.5.

Proof of Proposition 3.5. First, observe that for $t \in (0,1]$, we have that

$$||\mathbf{A}^1(t)||_{C^1} = ||e^{(t/2)\Delta}\mathbf{A}^1(t/2)||_{C^1} \leq Ct^{-1/2}||\mathbf{A}^1(t/2)||_{C^0}.$$

It follows that

$$\sup_{t\in(0,1]} t^{(1/2)+\gamma_\varepsilon}||\mathbf{A}^1(t)||_{C^1} \leq C\sup_{t\in(0,1]} t^{\gamma_\varepsilon}||\mathbf{A}^1(t)||_{C^0}.$$

The same thing holds with $\mathbf{A}^1$ replaced by $\mathbf{A}^1_N$ for any $N \geq 0$. Thus for the first desired result, it suffices to show that

$$\sup_{N\geq 0}\mathbb{E}\left[\sup_{t\in(0,1]} t^{p\gamma_\varepsilon}||\mathbf{A}^1_N(t)||^p_{C^0}\right], \quad \mathbb{E}\left[\sup_{t\in(0,1]} t^{p\gamma_\varepsilon}||\mathbf{A}^1(t)||^p_{C^0}\right] < \infty.$$

Similarly, for the second desired result, it suffices to show that

$$\lim_{N\to\infty}\mathbb{E}\left[\sup_{t\in(0,1]} t^{p\gamma_\varepsilon}||\mathbf{A}^1_N(t) - \mathbf{A}^1(t)||^p_{C^0}\right] = 0.$$

The first result follows by combining Lemma 4.14 with Lemma A.1. The second result follows by combining Lemma 4.14 with Lemma A.2. □

4.2. Nonlinear part

As in Section 3.2, we assume throughout this section that $\mathbf{A}^0$ is a random $\mathfrak{g}^d$-valued distribution satisfying Assumptions (A)-(E). For this section, we just assume that Assumption (D) holds with $\alpha \in (d/2, d)$. These assumptions hold, even if they are not explicitly stated in the various lemmas, corollaries, or propositions. For space reasons, we will omit proofs of many of the results in this section. For full proofs, please see the complete version of this paper on arXiv.

For many of the arguments in this section, we will work with the scalar quantities $\mathbf{A}^{2,a_0}_{N,j_0}$ instead of the vector quantity $\mathbf{A}^2_N$. Accordingly, recall the definitions of I and d in Definition 2.14, and recall that by Lemma 2.16, $\mathbf{A}^{2,a_0}_{N,j_0}$ has the following explicit form (in the following, $m = (n^1, n^2)$, $a = (a_0, a_1, a_2)$, $j = (j_0, j_1, j_2)$):

$$\mathbf{A}^{2,a_0}_{N,j_0}(t_0, x_0) = \sum_{\substack{a_1,a_2 \in [d_\mathfrak{g}] \\ j_1,j_2 \in [d]}} \sum_{\substack{n^1,n^2 \in \mathbb{Z}^d \\ |n^1|_\infty, |n^2|_\infty \leq N}} I(m, t_0) d(m, a, j) \widehat{\mathbf{A}}^{0,a_1}_{N,j_1}(n^1) \widehat{\mathbf{A}}^{0,a_2}_{N,j_2}(n^2) e_{n^1+n^2}(x_0). \tag{4.4}$$

Definition 4.15. For $N \geq 0$, $k \geq 0$, $t \geq 0$, define

$$S(N, k, t) := \sum_{\substack{n^1,n^2 \in \mathbb{Z}^d \\ \max\{|n^1|_\infty, |n^2|_\infty\} \geq N}} |n^1 + n^2|^k I((n^1, n^2), t)^2 (|n^1|^2 + |n^2|^2) \widehat{\tau}(n^1) \widehat{\tau}(n^2).$$

We now state the following technical lemma, which is one of the key intermediate steps for proving the results of Section 3.2. The proof is omitted.

Lemma 4.16. *For all $k \geq 0$, there exists a sequence $\left\{\delta^{4.16}_{N,k}\right\}_{N \geq 0}$ of non-increasing functions $\delta^{4.16}_{N,k} : (0, 1] \to [0, 1]$ such that the following hold. For any $t \in (0, 1]$, we have that $\lim_{N\to\infty} \delta^{4.16}_{N,k}(t) = 0$. Also, for any $N \geq 0$, $t \in (0, 1]$, we have that*

$$S(N, k, t) \leq C_k t^{-(d-1+(k/2)-\alpha)} \delta^{4.16}_{N,k}(t).$$

Here, C_k depends only on k, d, and the constants α, C_{D} from Assumption (D).

We proceed to use Lemma 4.16 to obtain moment bounds on $\mathbf{A}^2_N$.

Lemma 4.17. *For any $k \in \{0, 1, 2, 3\}$, $l_1, \ldots, l_k \in [d]$, $N \geq 0$, $t \in (0, 1]$, $x \in \mathbb{T}^d$, we have that*

$$\mathbb{E}\left[|\partial_{l_1 \cdots l_k} \mathbf{A}^2_N(t, x)|^2\right] \leq C t^{-(d+k-1-\alpha)}.$$

Here, C depends only on d and the constants $\alpha, C_{\mathrm{D}}, C_{\mathrm{E}}$ from Assumptions (D), (E).

Proof. We will work with the scalar quantities $\mathbf{A}^{2,a_0}_{N,j_0}$. Fix $a = (a_0, a_1, a_2)$, $j = (j_0, j_1, j_2)$. Define

$$\mathbf{B}(t,x) := \sum_{\substack{n^1,n^2\in\mathbb{Z}^d \\ |n_1|_\infty,|n_2|_\infty\le N}} e_{n^1+n^2}(x) I(m,t) d(m,a,j) \widehat{\mathbf{A}}^{0,a_1}_{j_1}(n^1)\widehat{\mathbf{A}}^{0,a_2}_{j_2}(n^2).$$

We first look at the $k=0$ case. For this, it suffices to show that $\mathbb{E}[|\mathbf{B}(t,x)|^2] \le Ct^{-(d-1-\alpha)}$. Toward this end, note that

$$\mathbb{E}\left[|\mathbf{B}(t,x)|^2\right] = \sum_{\substack{n^1,n^2\in\mathbb{Z}^d \\ k^1,k^2\in\mathbb{Z}^d \\ |n^i|_\infty,|k^i|_\infty\le N \\ i=1,2}} I((n^1,n^2),t) I((k^1,k^2),t) d((n^1,n^2),a,j) \times \overline{d((k^1,k^2),a,j)} e_{n^1+n^1-(k^1+k^2)}(x) \times \mathbb{E}\left[\widehat{\mathbf{A}}^{0,a_1}_{j_1}(n^1)\widehat{\mathbf{A}}^{0,a_2}_{j_2}(n^2)\overline{\widehat{\mathbf{A}}^{0,a_1}_{j_1}(k^1)\widehat{\mathbf{A}}^{0,a_2}_{j_2}(k^2)}\right].$$

We have that (recall Remark 2.15)

$$|d((n^1,n^2),a,j)| \le C(|n^1|+|n^2|).$$

Also by Remark 2.15, we can assume $a_1 \ne a_2$, since otherwise $d((n^1,n^2),a,j)=0$. Thus by Assumption (E), we have that

$$|\mathbb{E}\left[\widehat{\mathbf{A}}^{0,a_1}_{j_1}(n^1)\ \widehat{\mathbf{A}}^{0,a_2}_{j_2}(n^2)\overline{\widehat{\mathbf{A}}^{0,a_1}_{j_1}(k^1)\widehat{\mathbf{A}}^{0,a_2}_{j_2}(k^2)}\right]| \le C_{\mathrm{E}}(|E_1E_2|+|E_3E_4|)$$

where

$$E_1 := \mathbb{E}\left[\widehat{\mathbf{A}}^{0,a_1}_{j_1}(n^1)\overline{\widehat{\mathbf{A}}^{0,a_1}_{j_1}(k^1)}\right],\quad E_2 := \mathbb{E}\left[\widehat{\mathbf{A}}^{0,a_2}_{j_2}(n^2)\overline{\widehat{\mathbf{A}}^{0,a_2}_{j_2}(k^2)}\right],$$
$$E_3 := \mathbb{E}\left[\widehat{\mathbf{A}}^{0,a_1}_{j_1}(n^1)\overline{\widehat{\mathbf{A}}^{0,a_2}_{j_2}(k^2)}\right],\quad E_4 := \mathbb{E}\left[\widehat{\mathbf{A}}^{0,a_2}_{j_2}(n^2)\overline{\widehat{\mathbf{A}}^{0,a_1}_{j_1}(k^1)}\right].$$

By Lemma 4.1, we have that

$$|E_1| \le \mathbb{1}(n^1=k^1)\widehat{\tau}(n^1),\quad |E_2| \le \mathbb{1}(n^2=k^2)\widehat{\tau}(n^2),$$
$$|E_3| \le \mathbb{1}(n^1=k^2)\widehat{\tau}(n^1),\quad |E_4| \le \mathbb{1}(n^2=k^1)\widehat{\tau}(n^2).$$

Combining these bounds, we see that it suffices to bound (note that $I((n^1,n^2),t) = I((n^2,n^1),t)$)

$$\sum_{n^1,n^2\in\mathbb{Z}^d} I((n^1,n^2),t)^2(|n^1|^2+|n^2|^2)\widehat{\tau}(n^1)\widehat{\tau}(n^2).$$

The $k=0$ case now follows by Lemma 4.16 with $N=0, k=0$.

For the case $k\in\{1,2,3\}$, note that following the same steps as before, we may reduce to bounding

$$\sum_{n^1,n^2\in\mathbb{Z}^d} |n^1+n^2|^{2k} I((n^1,n^2),t)^2(|n^1|^2+|n^2|^2)\widehat{\tau}(n^1)\widehat{\tau}(n^2).$$

By Lemma 4.16, this is bounded by $Ct^{-(d-1+k-\alpha)}$, as desired. □

Definition 4.18. For $N, M \geq 0$, $t \in (0,1]$, $x \in \mathbb{T}^d$, let

$$\mathbf{D}^2_{N,M}(t,x) := \mathbf{A}^2_N(t,x) - \mathbf{A}^2_M(t,x).$$

Lemma 4.19. *There exists a sequence $\{\delta^{4.19}_N\}_{N\geq 0}$ of non-increasing functions $\delta^{4.19}_N : (0,1] \to [0,1]$ such that the following hold. For any $t \in (0,1]$, we have that $\lim_{N\to\infty} \delta^{4.19}_N(t) = 0$. Also, for any $k \in \{0,1,2,3\}$, $l_1, ..., l_k \in [d]$, $M \geq N \geq 0$, $t \in (0,1]$, $x \in \mathbb{T}^d$, we have that*

$$\mathbb{E}\left[|\partial_{l_1\cdots l_k}\mathbf{D}^2_{N,M}(t,x)|^2\right] \leq Ct^{-(d+k-1-\alpha)}\delta^{4.19}_N(t).$$

Here, C depends only on d and the constants α, C_D, C_E from Assumptions (D), (E).

Proof. For $m \geq 0$, let $\left\{\delta^{4.16}_{N,m}\right\}_{N\geq 0}$ be the sequence of functions from Lemma 4.16. Define

$$\delta^{4.19}_N(t) := \max_{m\in\{0,2,4,6\}} \delta^{4.16}_{N,m}(t), \quad t \in (0,1].$$

We first look at the $k=0$ case. We will work with the scalar quantities $\mathbf{D}^{2,a_0}_{N,M,j_0}$. By arguing as in the proof of Lemma 4.17, we may bound

$$\begin{aligned}&\mathbb{E}\left[|\mathbf{D}^{2,a_0}_{N,Mj_0}(t,x)|^2\right]\\ &\leq C \sum_{\substack{a_1,a_2\in[d_{\mathfrak{g}}]\\ j_1,j_2\in[d]}} \sum_{\tau_1,\tau_2\in\left\{\tau^{a_1}_{j_1},\tau^{a_2}_{j_2}\right\}} \sum_{\substack{n^1,n^2\in\mathbb{Z}^d\\ N<\max\left\{|n^1|_\infty,|n^2|_\infty\right\}\leq M}} I((n^1,n^2),t)^2 \times\\ &\qquad (|n^1|^2+|n^2|^2)\widehat{\tau}(n^1)\widehat{\tau}(n^2).\end{aligned}$$

Note that we may obtain a further upper bound by replacing the sum over $n^1, n^2 \in \mathbb{Z}^d$ such that $N < \max\{|n^1|_\infty, |n^2|_\infty\} \leq M$ by a sum over $n^1, n^2 \in \mathbb{Z}^d$ such that $\max\{|n^1|_\infty, |n^2|_\infty\} > N$. The $k=0$ case now follows by Lemma 4.16. The case $k \in \{1,2,3\}$ may be argued similarly. □

Proof of Lemma 3.10. This is now a direct consequence of Lemma 4.19. □

We next prove various technical lemmas which will help in obtaining moment bounds on quantities such as $\mathbf{A}^2_N(t,x) - \mathbf{A}^2_N(s,y)$.

Definition 4.20. Let $N \geq 0$, $a_0 \in [d_{\mathfrak{g}}]$, $j_0 \in [d]$, $t \in (0,1]$, $x \in \mathbb{T}^d$. Define

$$\begin{aligned}\mathbf{F}^{a_0}_{N,j_0}(t,x) := \sum_{\substack{a_1,a_2\in[d_{\mathfrak{g}}]\\ j_1,j_2\in[d]}} \sum_{\substack{n^1,n^2\in\mathbb{Z}^d\\ |n_1|_\infty,|n_2|_\infty\leq N}} d(m,a,j) \times\\ e^{-4\pi^2|n^1|^2t}\widehat{\mathbf{A}}^{0,a_1}_{j_1}(n^1)e^{-4\pi^2|n^2|^2t}\widehat{\mathbf{A}}^{0,a_2}_{j_2}(n^2)e_{n^1+n^2}(x),\end{aligned}$$

where $m = (n_1, n_2)$. Using the collection of $\mathbb{R}$-valued process $(\mathbf{F}^a_{N,j}, a \in [d_{\mathfrak{g}}], j \in [d])$ (as well as the relation (1.1), we may define the $\mathfrak{g}^d$-valued process $\mathbf{F}_N = (\mathbf{F}_N(t,x), t \in (0,1], x \in \mathbb{T}^d)$.

By Lemma 2.16, we have that for any $N \geq 0$,

$$\partial_t \mathbf{A}_N^2(t,x) = \Delta \mathbf{A}_N^2(t,x) + \mathbf{F}_N(t,x), \quad t \in (0,1], x \in \mathbb{T}^d. \tag{4.5}$$

Lemma 4.21. *For any $N \geq 0$, $t \in (0,1]$, $x \in \mathbb{T}^d$, $l \in [d]$, we have that*

$$\mathbb{E}\left[|\mathbf{F}_N(t,x)|^2\right] \leq Ct^{-(d+1-\alpha)}, \quad \mathbb{E}\left[|\partial_l \mathbf{F}_N(t,x)|^2\right] \leq Ct^{-(d+2-\alpha)}.$$

Here, C depends only on d and the constants $\alpha, C_{\mathrm{D}}, C_{\mathrm{E}}$ from Assumptions (D), (E).

Proof. We will work with the scalar quantities $\mathbf{F}_{N,j_0}^{a_0}$. Fix $a = (a_0, a_1, a_2)$, $j = (j_0, j_1, j_2)$. Define

$$\mathbf{B}(t,x) := \sum_{\substack{n^1,n^2 \in \mathbb{Z}^d \\ |n_1|_\infty, |n_2|_\infty \leq N}} e_{n^1+n^2}(x) d(m,a,j) \times e^{-4\pi^2|n^1|^2 t} \widehat{\mathbf{A}}_{j_1}^{0,a_1}(n^1) e^{-4\pi^2|n^2|^2 t} \widehat{\mathbf{A}}_{j_2}^{0,a_2}(n^2),$$

where $m = (n^1, n^2)$, as usual. For the first inequality, it suffices to show that $\mathbb{E}[|\mathbf{B}(t,x)|^2] \leq Ct^{-(d+1-\alpha)}$. Toward this end, we have that

$$\mathbb{E}\left[|\mathbf{B}(t,x)|^2\right] = \sum_{\substack{n^1,n^2 \in \mathbb{Z}^d \\ k^1,k^2 \in \mathbb{Z}^d \\ |n^i|_\infty, |k^i|_\infty \leq N \\ i=1,2}} e_{n^1+n^2-(k^1+k^2)}(x) d((n^1,n^2),a,j) \overline{d((k^1,k^2),a,j)} \times$$
$$e^{-4\pi^2(|n^1|^2+|n^2|^2+|k^1|^2+|k^2|^2)t} \mathbb{E}\left[\widehat{\mathbf{A}}_{j_1}^{0,a_1}(n^1) \widehat{\mathbf{A}}_{j_2}^{0,a_2}(n^2) \overline{\widehat{\mathbf{A}}_{j_1}^{0,a_1}(k^1) \widehat{\mathbf{A}}_{j_2}^{0,a_2}(k^2)}\right].$$

Arguing as in the proof of Lemma 4.17, we may reduce to bounding

$$\sum_{n^1,n^2 \in \mathbb{Z}^d} (|n^1|^2 + |n^2|^2) e^{-8\pi^2|n^1|^2 t} \widehat{\tau}(n^1) e^{-8\pi^2|n^2|^2 t} \widehat{\tau}(n^2),$$

Using that $|n|^2 e^{-4\pi^2|n|^2 t} \leq \sup_{x \geq 0} x e^{-4\pi^2 xt} \leq Ct^{-1}$, we obtain the further upper bound

$$Ct^{-1} \sum_{n^1 \in \mathbb{Z}^d} e^{-4\pi^1|n^1|^2 t} \widehat{\tau}(n^1) \sum_{n^2 \in \mathbb{Z}^d} e^{-4\pi^2|n^2|^2 t} \widehat{\tau}(n^2).$$

Observe that the above is equal to

$$Ct^{-1}(e^{t\Delta}\tau)(0)(e^{t\Delta}\tau)(0).$$

Using that $||e^{t\Delta}\tau||_{C^0}, ||e^{t\Delta}\tau||_{C^0} \leq Ct^{-(1/2)(d-\alpha)}$ (by Lemma 4.3), the first desired result now follows.

For the second inequality, by arguing as before, we may reduce to bounding

$$\sum_{n^1,n^2 \in \mathbb{Z}^d} |n^1 + n^2|^2 (|n^1|^2 + |n^2|^2) e^{-8\pi^2|n^1|^2 t} \widehat{\tau}(n^1) e^{-8\pi^2|n^2|^2 t} \widehat{\tau}(n^2).$$

We may bound

$$|n^1 + n^2|^2(|n^1|^2 + |n^2|^2) \leq C(|n^1|^4 + |n^2|^4),$$

and so arguing as before, we obtain the further upper bound

$$Ct^{-2}(e^{t\Delta}\tau)(0)(e^{t\Delta}\tau)(0),$$

which is bounded by $Ct^{-2}t^{-(d-\alpha)}$, as desired. □

Lemma 4.22. *There exists a sequence $\{\delta_N^{4.22}\}_{N\geq 0}$ of non-increasing functions $\delta_N^{4.22} : (0,1] \to [0,1]$, such that the following hold. For any $t \in (0,1]$, we have that $\lim_{N\to\infty}\delta_N^{4.22}(t) = 0$. Also, for any $M \geq N \geq 0$, $t \in (0,1]$, $x \in \mathbb{T}^d$, we have that*

$$\mathbb{E}\left[|\mathbf{F}_N(t,x) - \mathbf{F}_M(t,x)|^2\right] \leq Ct^{-(d+1-\alpha)}\delta_N^{4.22}(t).$$

Additionally, for any $l \in [d]$, we have that

$$\mathbb{E}\left[|\partial_l\mathbf{F}_N(t,x) - \partial_l\mathbf{F}_M(t,x)|^2\right] \leq Ct^{-(d+2-\alpha)}\delta_N^{4.22}(t).$$

Here, C depends only on d and the constants α, C_D, C_E from Assumptions (D), (E).

Proof. We will work with the scalar quantities $\mathbf{F}_{N,j_0}^{a_0}$. For $N \geq 0$, $t \in (0,1]$, define

$$G_N(t) := \sum_{\substack{a_1,a_2\in[d_\mathfrak{g}] \\ j_1,j_2\in[d]}} \sum_{\substack{n^1,n^2\in\mathbb{Z}^d \\ \max\{|n^1|_\infty,|n^2|_\infty\}\geq N}} e^{-4\pi^2|n^1|^2t}\widehat{\tau}(n^1)e^{-4\pi^2|n^2|^2t}\widehat{\tau}(n^2).$$

By arguing as in the proof of Lemma 4.21, we may obtain

$$\begin{aligned}\mathbb{E}\left[|\mathbf{F}_{N,j_0}^{a_0}(t,x) - \mathbf{F}_{M,j_0}^{a_0}(t,x)|^2\right] &\leq Ct^{-1}G_N(t),\\ \mathbb{E}\left[|\partial_l\mathbf{F}_{N,j_0}^{a_0}(t,x) - \partial_l\mathbf{F}_{M,j_0}^{a_0}(t,x)|^2\right] &\leq Ct^{-2}G_N(t).\end{aligned}$$

Observe that $G_N(t) \leq G_0(t)$ for all $t \in (0,1]$, and from the proof of Lemma 4.21, we have that $G_0(t) \leq Ct^{-(d-\alpha)}$. Thus we may define

$$\delta_N^{4.22}(t_0) := \sup_{t\in[t_0,1]} \frac{G_N(t)}{\max\{1, G_0(t)\}}, \quad t_0 \in (0,1].$$

This ensures that $\delta_N^{4.22}$ maps into $[0,1]$, and that it is non-increasing. It remains to show that $\lim_{N\to\infty}\delta_N^{4.22}(t_0) = 0$ for all $t_0 \in (0,1]$. Fix $t_0 \in (0,1]$, $a_1, a_2 \in [d_\mathfrak{g}]$, $j_1, j_2 \in [d]$. Since $\max\{|n^1|_\infty, |n^2|_\infty\} \geq N$ implies that at least one of $|n^1|_\infty, |n^2|_\infty$ is at least N, it suffices to show that

$$\lim_{N\to\infty}\sup_{t\in[t_0,1]}\sum_{\substack{n^1,n^2\in\mathbb{Z}^d \\ |n^1|_\infty>N}} e^{-4\pi^2|n^1|^2t}\widehat{\tau}(n^1)e^{-4\pi^2|n^2|^2t}\widehat{\tau}(n^2) = 0.$$

Note that without the limit, the left hand side above can be bounded by

$$\sum_{\substack{n^1 \in \mathbb{Z}^d \\ |n^1|_\infty > N}} e^{-4\pi^2|n^1|^2 t_0} \widehat{\tau}(n^1) \sum_{n^2 \in \mathbb{Z}^d} e^{-4\pi^2|n^2|^2 t_0} \widehat{\tau}(n^2).$$

Note that the second sum is $(e^{t_0 \Delta}\tau)(0)$, which is finite by Lemma 4.3 (and the fact that $t_0 > 0$). For the same reason, we have that $\sum_{n^1 \in \mathbb{Z}^d} e^{-4\pi^2|n^1|^2 t_0} \widehat{\tau}(n^1) < \infty$. The desired result now follows by dominated convergence. □

We next use the previous technical lemmas to control the C^0 norm of $\mathbf{A}^2_N(t)$, culminating in Proposition 4.30 below. After we control the C^0 norm, we will then move on to controlling the C^1 norm. In the following, recall the definition of d_t from Definition 4.4. The proof of the next lemma will be omitted, as it is very similar to the proof of Lemma 4.11, where we use Lemmas 4.17 and 4.21 in place of Lemma 4.10, and Eq. (4.5) in place of the heat equation $\partial_u \mathbf{A}^1 = \Delta \mathbf{A}^1$.

Lemma 4.23. *For any $N \geq 0$, the following holds. For $t \in (0, 1]$, $r, s \in (t/2, t]$, $x, y \in \mathbb{T}^d$, we have that*

$$\begin{aligned} ||\mathbf{A}^2_N(t, x)||_{L^2} &\leq Ct^{-(1/2)(d-1-\alpha)}, \\ ||\mathbf{A}^2_N(r, x) - \mathbf{A}^2_N(s, y)||_{L^2} &\leq Ct^{-(1/2)(d-1-\alpha)} d_t((r, x), (s, y)). \end{aligned}$$

Here, C depends only on d and the constants $\alpha, C_{\mathrm{D}}, C_{\mathrm{E}}$ from Assumptions (D), (E).

In the following, recall the definition of $\mathbf{D}^2_{N,M}$ from Definition 4.18. The proofs of the following few lemmas will be omitted, as they are all very similar to the proofs of the analogous lemmas in Section 4.1.

Lemma 4.24. *There is a sequence $\{\delta^{4.24}_N\}_{N \geq 0}$ of functions $\delta^{4.24}_N : (0, 1] \to [0, 1]$ such that the following hold. For any $t \in (0, 1]$, we have that $\lim_{N \to \infty} \delta^{4.24}_N(t) = 0$. Also, for any $M \geq N \geq 0$, $t \in (0, 1]$, $r, s \in (t/2, t]$, $x, y \in \mathbb{T}^d$, we have that*

$$\begin{aligned} ||\mathbf{D}^2_{N,M}(r, x)||_{L^2} &\leq Ct^{-(1/2)(d-1-\alpha)} \delta^{4.24}_N(t), \\ ||\mathbf{D}^2_{N,M}(r, x) - \mathbf{D}^2_{N,M}(s, y)||_{L^2} &\leq Ct^{-(1/2)(d-1-\alpha)} d_t((r, x), (s, y)) \delta^{4.24}_N(t). \end{aligned}$$

Here, C depends only on d and the constants $\alpha, C_{\mathrm{D}}, C_{\mathrm{E}}$ from Assumptions (D), (E).

Definition 4.25. For $N \geq 0$, $t \in (0, 1]$, $x \in \mathbb{T}^d$, let

$$\mathbf{D}^2_N(t, x) := \mathbf{A}^2_N(t, x) - \mathbf{A}^2(t, x).$$

The following result is a direct consequence of Lemmas 4.23 and 4.24 and Definition 3.11.

Corollary 4.26. *Let $\{\delta^{4.24}_N\}_{N \geq 0}$ be the sequence of functions from Lemma 4.24. Then for any $N \geq 0$, $t \in (0, 1]$, $r \in (t/2, t]$, $x \in \mathbb{T}^d$, we have that*

$$||\mathbf{D}^2_N(r, x)||_{L^2} \leq Ct^{-(1/2)(d-1-\alpha)} \delta^{4.24}_N(t).$$

Here, C depends only on d and the constants $\alpha, C_{\mathrm{D}}, C_{\mathrm{E}}$ from Assumptions (D), (E).

We can now use Assumption (B), Lemma 2.17, and the various moment estimates to obtain tail bounds for $\mathbf{A}^2$ and related quantities.

Lemma 4.27. *For any $t \in (0,1]$, $r,s \in (t/2,t]$, $x,y \in \mathbb{T}^d$, we have that for $u \geq 0$,*

$$\mathbb{P}(|\mathbf{A}^2(r,x)| > u) \leq C \exp\left(-(u/t^{-(1/2)(d-1-\alpha)})^{\beta_B}/C\right),$$
$$\mathbb{P}(|\mathbf{A}^2(r,x) - \mathbf{A}^2(s,y)| > u) \leq C \exp\left(-(u/(t^{-(1/2)(d-1-\alpha)} d_t((r,x),(s,y))))^{\beta_B}/C\right).$$

The above inequalities also hold with $\mathbf{A}^2$ replaced by $\mathbf{A}^2_N$ for any $N \geq 0$. Additionally, let $\{\delta_N^{4.24}\}_{N \geq 0}$ be the sequence of functions from Lemma 4.24. Then for any $N \geq 0$, we have that for $u \geq 0$,

$$\mathbb{P}(|\mathbf{D}^2_N(r,x)| > u) \leq C \exp\left(-(u/(t^{-(1/2)(d-1-\alpha)} \delta_N^{4.24}(t)))^{\beta_B}/C\right),$$
$$\mathbb{P}(|\mathbf{D}^2_N(r,x) - \mathbf{D}^2_N(s,y)| > u) \leq C \exp\left(-(u/(t^{-(1/2)(d-1-\alpha)} d_t((r,x),(s,y)) \delta_N^{4.24}(t)))^{\beta_B}/C\right).$$

Here, C depends only on d and the constants $\beta_B, C_B, \alpha, C_D, C_E$ from Assumptions (B), (D) and (E).

Lemma 4.27 may be used to obtain the following result.

Lemma 4.28. *The process $\mathbf{A}^2 = (\mathbf{A}^2(t,x), t \in (0,1], x \in \mathbb{T}^d)$ has a continuous modification.*

Hereafter, we assume that (after a suitable modification) the process $\mathbf{A}^2 = (\mathbf{A}^2, t \in (0,1], x \in \mathbb{T}^d)$ has continuous sample paths. The following lemma is the analogue of Lemma 4.14. We omit the proof, as it is very similar to the proof of that lemma (where we use Lemma 4.27 instead of Lemma 4.13).

Lemma 4.29. *For any $p \geq 1$, $t_0 \in (0,1]$, we have that*

$$\mathbb{E}\left[\sup_{(t,x) \in (t_0/2,t_0] \times \mathbb{T}^d} |\mathbf{A}^2(t,x)|^p\right] \leq C t_0^{-p(1/2)(d-1-\alpha)} (1 + |\log t_0|^{1/\beta_B})^p.$$

The above inequality also holds with $\mathbf{A}^2$ replaced by $\mathbf{A}^2_N$ for any $N \geq 0$. Additionally, let $\{\delta_N^{4.24}\}_{N \geq 0}$ be the sequence of functions from Lemma 4.24. Then for any $N \geq 0$, we have that

$$\mathbb{E}\left[\sup_{(t,x) \in (t_0/2,t_0] \times \mathbb{T}^d} |\mathbf{D}^2_N(t,x)|\right] \leq C t_0^{-p(1/2)(d-1-\alpha)} (1 + |\log t_0|^{1/\beta_B})^p (\delta_N^{4.24}(t_0))^p.$$

Here, C depends only on d, p, and the constants $\beta_B, C_B, \alpha, C_D, C_E$ from Assumptions (B), (D) and (E).

Proposition 4.30. *For $\varepsilon > 0$, let $\gamma_\varepsilon := (1/2)(d-1-\alpha) + \varepsilon$. For any $\varepsilon > 0$, $p \geq 1$, we have that*

$$\sup_{N\geq 0}\mathbb{E}\left[\sup_{t\in(0,1]} t^{p\gamma_\varepsilon}||\mathbf{A}^2_N(t)||^p_{C^0}\right],\quad \mathbb{E}\left[\sup_{t\in(0,1]} t^{p\gamma_\varepsilon}||\mathbf{A}^2(t)||^p_{C^0}\right]\leq C_{\varepsilon,p}<\infty.$$

Moreover, we have that

$$\lim_{N\to\infty}\mathbb{E}\left[\sup_{t\in(0,1]} t^{p\gamma_\varepsilon}||\mathbf{A}^2_N(t)-\mathbf{A}^2(t)||^p_{C^0}\right]=0.$$

Here the constant $C_{\varepsilon,p}$ depends only on $\varepsilon, p, d,$ and the constants $\beta_{\mathrm{B}}, C_{\mathrm{B}}, \alpha, C_{\mathrm{D}}, C_{\mathrm{E}}$ from Assumptions (B), (D) and (E).

Proof. The first result follows by combining Lemma 4.29 with Lemma A.1. The second result follows by combining Lemma 4.29 with Lemma A.2. □

As previously mentioned, having controlled the C^0 norm, we now move on to controlling the C^1 norm. We first show the following preliminary result, which will also allow us to prove Lemmas 3.12 and 3.14.

Lemma 4.31. *For all $t_0, t_1 \in (0,1]$, $t_0 < t_1$, $x \in \mathbb{T}^d$, we have that a.s.,*

$$\begin{aligned}\mathbf{A}^2(t_1,x) &= (e^{(t_1-t_0)\Delta}\mathbf{A}^2(t_0))(x)\\ &\quad+\int_0^{t_1-t_0}(e^{(t_1-t_0-s)\Delta}X^{(2)}(\mathbf{A}^1(t_0+s)))(x)ds.\end{aligned}\tag{4.6}$$

Proof. Note that the result is true if we replace $\mathbf{A}^2, \mathbf{A}^1$ by $\mathbf{A}^2_N, \mathbf{A}^1_N$, since $\mathbf{A}^2_N=\rho(\mathbf{A}^1_N)$ by definition (recall Definition 3.9). Taking $N\to\infty$, we have (recall Definition 3.11) that $\mathbf{A}^2_N(t_1,x)\xrightarrow{L^2}\mathbf{A}^2(t_1,x)$. To finish, it suffices to show that

$$(e^{(t_1-t_0)\Delta}\mathbf{A}^2_N(t_0))(x)\xrightarrow{P}(e^{(t_1-t_0)\Delta}\mathbf{A}^2(t_0))(x),$$

and that

$$\begin{aligned}&\int_0^{t_1-t_0}(e^{(t_1-t_0-s)\Delta}\ X^{(2)}(\mathbf{A}^1_N(t_0+s)))(x)ds\xrightarrow{P}\\ &\qquad\qquad\int_0^{t_1-t_0}(e^{(t_1-t_0-s)\Delta}X^{(2)}(\mathbf{A}^1(t_0+s)))(x)ds.\end{aligned}$$

Note that by Proposition 4.30, $\mathbb{E}[||\mathbf{A}^2_N(t_0)-\mathbf{A}^2(t_0)||_{C^0}]\to 0$. The first claim now follows since

$$||e^{(t_1-t_0)\Delta}(\mathbf{A}^2_N(t_0)-\mathbf{A}^2(t_0))||_{C^0}\ \leq\ ||\mathbf{A}^2_N(t_0)-\mathbf{A}^2(t_0)||_{C^0}.$$

For the second claim, define $\widetilde{\mathbf{A}}^1_N(t):=\mathbf{A}^1_N(t+t_0)$ for $t\in[0,1-t_0]$, and analogously for $\widetilde{\mathbf{A}}^1$. Since $\mathbf{A}^1_N(0)$ is smooth, we have that $\mathbf{A}^1_N\in\mathcal{P}^1_1$ for all $N\geq 0$, and thus $\widetilde{\mathbf{A}}^1_N\in\mathcal{P}^1_{1-t_0}$ for all $N\geq 0$. We also have that $\widetilde{\mathbf{A}}^1\in\mathcal{P}^1_{1-t_0}$, which follows since $\mathbf{A}^1$ is a solution to the heat equation (by Lemma 3.2). Now by (4.2), we have that $||\widetilde{\mathbf{A}}^1_N-\widetilde{\mathbf{A}}^1||_{\mathcal{P}^1_{1-t_0}}\to 0$. The second claim follows by Lemma 2.12. □

Corollary 4.32. *On an event of probability 1, we have that (4.6) holds for all $t_0, t_1 \in (0,1]$, $t_0 < t_1$, $x \in \mathbb{T}^d$.*

Proof. Let E be the event that (4.6) holds for all t_0, t_1 in a countable dense subset of $(0,1]$, and all x in a countable dense subset of $\mathbb{T}^d$. By Lemma 4.31, $\mathbb{P}(E) = 1$. Note that $\mathbf{A}^2$ has continuous sample paths (recall just after Lemma 4.28), and that $\sup_{t \in [t_0,1]} ||\mathbf{A}^1(t)||_{C^1} < \infty$ for all $t_0 \in (0,1]$ (recall (4.1)). The latter implies that for $t_0 \in (0,1]$, if we define $\widetilde{\mathbf{A}}^1(t) := \mathbf{A}^1(t_0 + t)$, then $\widetilde{\mathbf{A}}^1 \in \mathcal{P}^1_{1-t_0}$, and thus by Lemma 2.6, we have that $\rho^{(2)}(\widetilde{\mathbf{A}}^1) \in \mathcal{P}^1_{1-t_0}$ as well. By combining the previous few observations, we have that on the event E, the identity (4.6) extends by continuity to all $t_0, t_1 \in (0,1]$, $t_0 < t_1$, and $x \in \mathbb{T}^d$. □

Proof of Lemma 3.12. Let E be the probability 1 event given by Corollary 4.32. As in the proof of Corollary 4.32, on the event E, we have that for all $t_0 \in (0,1]$, $t_1 \in [t_0, 1]$,

$$\mathbf{A}^2(t_1) = e^{(t_1 - t_0)\Delta}\mathbf{A}^2(t_0) + \rho^{(2)}(\widetilde{\mathbf{A}}^1)(t_1 - t_0), \tag{4.7}$$

where $\widetilde{\mathbf{A}}^1(t) := \mathbf{A}^1(t_0 + t)$ for $t \in [0, 1-t_0]$. Moreover, as noted in that proof, we have that $\rho^{(2)}(\widetilde{\mathbf{A}}^1) \in \mathcal{P}^1_{1-t_0}$. Combining this with the fact that $\mathbf{A}^2(t) \in C^0(\mathbb{T}^d, \mathfrak{g}^d)$ for all $t \in (0,1]$ (so that $s \mapsto e^{s\Delta}\mathbf{A}^2(t)$ is a continuous function $(0,\infty) \to C^1(\mathbb{T}^d, \mathfrak{g}^d)$ for all $t \in (0,1]$), we obtain that on E, the map $t \mapsto \mathbf{A}^2(t)$ is a continuous function from $(0,1] \to C^1(\mathbb{T}^d, \mathfrak{g}^d)$. We can modify $\mathbf{A}^2$ to be identically 0 off E. □

Proof of Lemma 3.14. This follows by combining Corollary 4.32 with (3.1). □

We now turn to getting bounds on $\nabla \mathbf{A}^2(t)$. We will omit the proofs of the following few results, as they are all very similar to the proofs from Section 4.1. As done after Lemma 3.12, we will assume that (after a suitable modification) $t \mapsto \mathbf{A}^2(t)$ is a continuous function from $(0,1] \to C^1(\mathbb{T}^d, \mathfrak{g}^d)$. Even more, from the the proof of Lemma 3.12, we may assume that (4.7) holds for all $t_0 \in (0,1]$, $t_1 \in [t_0, 1]$. The next lemma is the analogue of Lemma 4.23.

Lemma 4.33. *For any $N \geq 0$, the following holds. For $l \in [d]$, $t \in (0,1]$, $r, s \in (t/2, t]$, $x, y \in \mathbb{T}^d$, we have that*

$$\begin{aligned} ||\partial_l \mathbf{A}^2_N(t,x)||_{L^2} &\leq Ct^{-(1/2)(d-\alpha)}, \\ ||\partial_l \mathbf{A}^2_N(r,x) - \partial_l \mathbf{A}^2_N(s,y)||_{L^2} &\leq Ct^{-(1/2)(d-\alpha)} d_t((r,x),(s,y)). \end{aligned}$$

Here, C depends only on d and the constants $\alpha, C_{\mathrm{D}}, C_{\mathrm{E}}$ from Assumptions (D), (E).

The next lemma is the analogue of Lemma 4.24.

Lemma 4.34. *There is a sequence $\{\delta_N^{4.34}\}_{N \geq 0}$ of maps $\delta_N^{4.34} : (0,1] \to [0,1]$ such that the following hold. For any $t \in (0,1]$, we have that $\lim_{N \to \infty} \delta_N^{4.34}(t) = 0$. Also, for any $M \geq N \geq 0$, $l \in [d]$, $t \in (0,1]$, $r, s \in (t/2, t]$, $x, y \in \mathbb{T}^d$, we have that*

$$||\partial_l \mathbf{D}^2_{N,M}(r,x)||_{L^2} \leq Ct^{-(1/2)(d-\alpha)} \delta_N^{4.34}(t),$$
$$||\partial_l \mathbf{D}^2_{N,M}(r,x) - \partial_l \mathbf{D}^2_{N,M}(s,y)||_{L^2} \leq Ct^{-(1/2)(d-\alpha)} d_t((r,x),(s,y)) \delta_N^{4.34}(t).$$

Here, C depends only on d and the constants $\alpha, C_{\mathrm{D}}, C_{\mathrm{E}}$ from Assumptions (D), (E).

Lemma 4.35. *For any $l \in [d]$, we have that $\partial_l \mathbf{A}^2_N(t,x) \xrightarrow{L^2} \partial_l \mathbf{A}^2(t,x)$.*

Proof. By Lemma 4.34, the sequence $\{\partial_l \mathbf{A}^2_N(t,x)\}_{N\geq 0}$ is Cauchy in L^2, and thus it converges in L^2 to some random variable, call it Y. To finish, it suffices to also show that $\partial_l \mathbf{A}^2_N(t,x) \xrightarrow{P} \partial_l \mathbf{A}^2(t,x)$. Toward this end, let $t_0 \in (0,t)$, and let $\widetilde{\mathbf{A}}^1(t) := \mathbf{A}^1(t_0+t)$ for $t \in [0, 1-t_0]$. We have that (by (4.7))

$$\partial_l \mathbf{A}^2(t,x) = (\partial_l e^{(t-t_0)\Delta}\ \mathbf{A}^2(t_0))(x) + (\partial_l \rho^{(2)}(\widetilde{\mathbf{A}}^1)(t-t_0))(x).$$

By construction, the above identity is also true with $\mathbf{A}^2, \mathbf{A}^1$ replaced by $\mathbf{A}^2_N, \mathbf{A}^1_N$ for any $N \geq 0$. Thus it suffices to show that

$$(\partial_l e^{(t-t_0)\Delta} \mathbf{A}^2_N(t_0))(x) \xrightarrow{P} (\partial_l e^{(t-t_0)\Delta} \mathbf{A}^2(t_0))(x),$$
$$||\rho^{(2)}(\widetilde{\mathbf{A}}^1_N)(t-t_0) - \rho^{(2)}(\widetilde{\mathbf{A}}^1)(t-t_0)||_{C^1} \xrightarrow{P} 0.$$

(Here $\widetilde{\mathbf{A}}^1_N(t) := \mathbf{A}^1_N(t_0+t)$ for $t \in [0, 1-t_0]$.) Note that by Proposition 4.30, $\mathbb{E}[||\mathbf{A}^2_N(t_0) - \mathbf{A}^2(t_0)||_{C^0}] \to 0$. We have that

$$||\partial_l e^{(t-t_0)\Delta} \mathbf{A}^2_N(t_0) - \partial_l e^{(t-t_0)\Delta} \mathbf{A}^2(t_0)||_{C^0} \leq ||e^{(t-t_0)\Delta} \mathbf{A}^2_N(t_0) - e^{(t-t_0)\Delta} \mathbf{A}^2(t_0)||_{C^1}.$$

We can then obtain the further upper bound

$$C(t-t_0)^{-1/2} ||\mathbf{A}^{2,a}_{N,j}(t_0) - \mathbf{A}^{2,a}_j(t_0)||_{C^0}.$$

The first claim follows. For the second claim, note that by (4.2), $||\widetilde{\mathbf{A}}^1_N - \widetilde{\mathbf{A}}^1||_{\mathcal{P}^1_{1-t_0}} \to 0$. The second claim then follows by Lemma 2.12. □

The following proposition is the analogue of Proposition 4.30. It can be proven by first proving the analogues of Lemmas 4.27 and 4.29 for $\nabla \mathbf{A}^2$ (by using the various estimates on $\nabla \mathbf{A}^2$ that we have shown). The proofs are omitted.

Proposition 4.36. *For $\varepsilon > 0$, let $\gamma_\varepsilon := (1/2)(d-1-\alpha)+\varepsilon$. For any $\varepsilon > 0, p \geq 1$, we have that*

$$\sup_{N \leq \infty} \mathbb{E}\left[\sup_{t\in(0,1]} t^{p((1/2)+\gamma_\varepsilon)} ||\nabla \mathbf{A}^2_N(t)||^p_{C^0}\right] \leq C_{\varepsilon,p} < \infty,$$

where we use the notation $\nabla \mathbf{A}^2_\infty := \nabla \mathbf{A}^2$. Moreover, we have that

$$\lim_{N\to\infty} \mathbb{E}\left[\sup_{t\in(0,1]} t^{p((1/2)+\gamma_\varepsilon)} ||\nabla \mathbf{A}^2_N(t) - \nabla \mathbf{A}^2(t)||^p_{C^0}\right] = 0.$$

Here, $C_{\varepsilon,p}$ depends only on ε, p, d, and the constants $\beta_{\mathrm{B}}, C_{\mathrm{B}}, \alpha, C_{\mathrm{D}}, C_{\mathrm{E}}$ from Assumptions (B), (D) and (E).

Proof of Proposition 3.15. The first two claims follow by combining Propositions 4.30 and 4.36. The final two claims follow by combining the first two claims with Lemma 3.8. □

Acknowledgements

We are grateful to Nelia Charalambous, Persi Diaconis, Len Gross, and Phil Sosoe for various helpful comments and discussions. We are also grateful to the referees for many helpful and detailed comments.

Funding

S. Cao was partially supported by NSF grant DMS RTG 1501767 and S. Chatterjee was partially supported by NSF grants DMS 1855484 and DMS 2113242.

ORCID

Sky Cao http://orcid.org/0000-0003-0345-0608
Sourav Chatterjee http://orcid.org/0000-0003-4460-209X

References

[1] Atiyah, M.F., Bott, R. (1983). The Yang-Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A.* 308(1505):523–615.
[2] Feehan, P. M. N. (2016). Global existence and convergence of solutions to gradient systems and applications to Yang–Mills gradient flow. *Preprint* Available at arXiv:1409.1525
[3] Charalambous, N., Gross, L. (2013). The Yang–Mills heat semigroup on three-manifolds with boundary. *Comm. Math. Phys.* 317(3):727–785.
[4] Charalambous, N., Gross, L. (2015). Neumann domination for the Yang–Mills heat equation. *J. Math. Phys.* 56(7):073505. 21pp.
[5] Charalambous, N., Gross, L. (2017). Initial behavior of solutions to the Yang–Mills heat equation. *J. Math. Anal. Appl.* 451(2):873–905.
[6] Gross, L. (2016). The Yang–Mills heat equation with finite action. *Preprint* Available at arXiv:1606.04151
[7] Gross, L. (2017). Stability of the Yang–Mills heat equation for finite action. *Preprint* Available at arXiv:1711.00114
[8] Oh, S.J., Tataru, D. (2017). The Threshold Theorem for the $(4+1)$-dimensional Yang–Mills equation: an overview of the proof. *Preprint* Available at arXiv:1709.09088
[9] Waldron, A. (2019). Long-time existence for Yang–Mills flow. *Invent. Math.* 217(3):1069–1147.
[10] Zwanziger, D. (1981). Covariant quantization of gauge fields without Gribov ambiguity. *Nuclear Phys. B.* 192(1):259–269.
[11] Deturck, D. M. (1983). Deforming metrics in the direction of their Ricci tensors. *J. Differential Geom.* 18(1):157–162.
[12] Donaldson, S. K. (1985). Anti self dual Yang–Mills connections over complex algebraic surfaces and stable vector bundles. *Proc. London Math. Soc.* (3) 50(1):1–26.
[13] Sadun, L. A. (1987). Continuum regularized Yang–Mills theory. Ph.D. thesis. p. 67. + pages. University of California, Berkeley.
[14] Chandra, A., Chevyrev, I., Hairer, M., Shen, H. (2020). Langevin dynamic for the 2D Yang–Mills measure. *Preprint* Available at arXiv:2006.04987

[15] Bourgain, J. (1996). Invariant measures for the 2D-defocusing nonlinear Schrödinger equation. *Comm. Math. Phys.* 176:421–445.
[16] Bourgain, J. (1999). *Global solutions of nonlinear Schrödinger equations*, American Mathematical Society Colloquium Publications 46, Providence, RI: American Mathematical Society.
[17] Deng, Y., Nahmod, A.R., Yue, H. (2019). Invariant Gibbs measures and global strong solutions for nonlinear Schrödinger equations in dimension two. *Preprint* Available at arXiv: 1910.08492
[18] Lebowitz, J., Rose, R., Speer, E. (1988). Statistical mechanics of the nonlinear Schrödinger equation. *J. Statist. Phys.* 50:657–687.
[19] Oh, T., Sosoe, P., Tolomeo, L. (2022). Optimal integrability threshold for Gibbs measures associated with focusing NLS on the torus. *Invent. Math.* 227:1323–1429.
[20] Burq, N., Tzvetkov, N. (2008). Random data Cauchy theory for supercritical wave equations I: Local theory. *Invent. Math.* 173:449–475.
[21] Burq, N., Tzvetkov, N. (2008). Random data Cauchy theory for supercritical wave equations II: a global existence result. *Invent. Math.* 173:477–496.
[22] Nahmod, A. R., Pavlovíc, N., Staffilani, G. (2013). Almost sure existence of global weak solutions for super-critical Navier-Stokes equations. *SIAM J. Math. Anal.* 45(6):3431–3452.
[23] Cao, S., Chatterjee, S. A state space for 3D Euclidean Yang-Mills theories. *Preprint* Available at arXiv:2111.12813
[24] Stein, E.M., Weiss, G. (1971). *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton Mathematical Series 32. Princeton, New Jersey: Princeton University Press (46:4102).
[25] Chandra, A., Chevyrev, I., Hairer, M., Shen, H. (2022). Stochastic quantisation of Yang–Mills–Higgs in 3D. *Preprint* Available at arXiv:2201.03487
[26] Da Prato, G., Debussche, A. (2002). Two-dimensional Navier-Stokes equations driven by a space-time white noise. *J. Funct. Anal.* 196(1):180–210.
[27] Da Prato, G., Debussche, A. (2003). Strong solutions to the stochastic quantization equations. *Ann. Probab.* 31(4):1900–1916.
[28] Berestycki, N., Powell, E. (2021). Gaussian free field, Liouville quantum gravity and Gaussian multiplicative chaos. *Preprint.* Available at https://homepage.univie.ac.at/nathanael.berestycki/Articles/master.pdf.
[29] Sheffield, S. (2007). Gaussian free fields for mathematicians. *Probab. Theory Related Fields.* 139:521–541.
[30] Werner, W., Powell, E. (2020). Lecture notes on the Gaussian free field. *Preprint* Available at arXiv:2004.04720
[31] Rogers, L. C. G., Williams, D. (1994). Diffusions, Markov processes, and martingales, 2nd ed. Wiley Series in Probability and Mathematical Statistics 1. Chichester: Wiley.
[32] Dirksen, S. (2015). Tail bounds via generic chaining. *Electron. J. Probab.* 20:1–29.

Appendix A: Suprema of stochastic processes

Let $(\mathbf{A}(t,x), t \in (0,1], x \in \mathbb{T}^d)$ be a $\mathfrak{g}^d$-valued stochastic process with continuous sample paths.

Lemma A.1. *Let $p \geq 1$. Suppose that there is some C_0, γ, β such that the following holds. For all $t_0 \in (0,1]$, we have that*

$$\mathbb{E}\left[\sup_{t\in(t_0/2,t_0],x\in\mathbb{T}^d}|\mathbf{A}(t,x)|^p\right] \leq C_0 t_0^{-p\gamma}(1+|\log t_0|^\beta)^p.$$

Then for any $\varepsilon > 0$, there is some non-increasing function $\delta : \mathbb{N} \to [0,\infty)$ depending only on $p, C_0, \beta, \varepsilon$ such that for any integer $k_0 \geq 0$, we have that

$$\mathbb{E}\left[\sup_{t\in(0,2^{-k_0}],x\in\mathbb{T}^d} t^{p(\gamma+\varepsilon)}|\mathbf{A}(t,x)|^p\right] \le \delta(k_0).$$

Moreover, $\delta(k_0) \to 0$ *as* $k_0 \to \infty$.

Proof. Let $k_0 \ge 0$. We may bound

$$\mathbb{E}\left[\sup_{t\in(0,2^{-k_0}],x\in\mathbb{T}^d} t^{p(\gamma+\varepsilon)}|\mathbf{A}(t,x)|^p\right] \le \sum_{k=k_0}^{\infty}\mathbb{E}\left[\sup_{t\in(2^{-(k+1)},2^{-k}],x\in\mathbb{T}^d} t^{p(\gamma+\varepsilon)}|\mathbf{A}(t,x)|^p\right],$$

which may be further bounded by

$$C_0\sum_{k=k_0}^{\infty}(2^{-k})^{p(\gamma+\varepsilon)}2^{pk\gamma}(1+(k\log 2)^{\beta})^p.$$

Thus we may set $\delta(k_0)$ to be the above. The fact that $\lim_{k_0\to\infty}\delta(k_0)=0$ follows because $\delta(0)<\infty$ combined with dominated convergence. □

Now suppose we have a sequence $\{(\mathbf{A}_N(t,x), t\in(0,1], x\in\mathbb{T}^d)\}_{N\ge 0}$ of $\mathfrak{g}^d$-valued stochastic processes with continuous sample paths.

Lemma A.2. *Let* $p \ge 1$. *Suppose there is a sequence* $\{\delta_N\}_{N\ge 0}$ *of functions* $\delta_N : (0,1] \to [0,1]$, *and* C_0, γ, β, *such that the following hold. For all* $t \in (0,1]$, *we have that* $\lim_{N\to\infty}\delta_N(t) = 0$. *Also, for all* $t_0 \in (0,1]$, $N \ge 0$, *we have that*

$$\mathbb{E}\left[\sup_{t\in(t_0/2,t_0],x\in\mathbb{T}^d}|\mathbf{A}_N(t,x)|^p\right] \le C_0 t_0^{-p\gamma}(1+|\log t_0|^{\beta})^p\delta_N(t_0).$$

Then for any $\varepsilon > 0$, *we have that*

$$\lim_{N\to\infty}\mathbb{E}\left[\sup_{t\in(0,1],x\in\mathbb{T}^d} t^{p(\gamma+\varepsilon)}|\mathbf{A}_N(t,x)|^p\right] = 0.$$

Proof. Fix $\varepsilon > 0$. Let $k_0 \ge 0$. For $N \ge 0$, we may bound

$$\mathbb{E}\left[\sup_{t\in(0,1],x\in\mathbb{T}^d} t^{p(\gamma+\varepsilon)}|\mathbf{A}_N(t,x)|^p\right] \le I_{1,k_0,N} + I_{2,k_0,N},$$

where

$$I_{1,k_0,N} := \mathbb{E}\left[\sup_{t\in(0,2^{-k_0}],x\in\mathbb{T}^d} t^{p(\gamma+\varepsilon)}|\mathbf{A}_N(t,x)|^p\right],$$
$$I_{2,k_0,N} := \mathbb{E}\left[\sup_{t\in(2^{-k_0},1],x\in\mathbb{T}^d} t^{p(\gamma+\varepsilon)}|\mathbf{A}_N(t,x)|^p\right].$$

By Lemma A.1, we have some function $\delta : \mathbb{N} \to [0,\infty)$ such that $\delta(k) \to 0$ as $k \to \infty$, and such that $\sup_{N\ge 0} I_{1,k_0,N} \le \delta(k_0)$. Next, observe that

$$\begin{aligned} I_{2,k_0,N} &\le \sum_{k=0}^{k_0-1}\mathbb{E}\left[\sup_{t\in(2^{-(k+1)},2^{-k}],x\in\mathbb{T}^d} t^{p(\gamma+\varepsilon)}|\mathbf{A}_N(t,x)|^p\right] \\ &\le C_0\sum_{k=0}^{k_0-1}2^{-pk(\gamma+\varepsilon)}(2^{-k})^{-p\gamma}(1+|k\log 2|^{\beta})^p\delta_N(2^{-k}). \end{aligned}$$

Since k_0 is finite, and δ_N converges pointwise to 0, we obtain that for any fixed k_0, $\lim_{N\to\infty} I_{2,k_0,N} = 0$. We thus obtain for any fixed k_0,

$$\limsup_{N\to\infty}\mathbb{E}\left[\sup_{t\in(0,1],x\in\mathbb{T}^d} t^{p(\gamma+\varepsilon)}|\mathbf{A}_N(t,x)|^p\right] \leq \delta(k_0).$$

Using that $\delta(k_0) \to 0$ as $k_0 \to \infty$, the desired result now follows. □

For $t_0 \in (0,1]$, let $T_{t_0} := (t_0/2, t_0] \times \mathbb{T}^d$. Recall the notation from Definition 4.4, in particular d_t, $N_{t,\varepsilon}$. The following theorem is an immediate consequence of [32, Theorem 3.2]. Thus, the proof is omitted. See the complete version of this paper on arXiv for the proof.

Theorem A.3. *Let $(V, |\cdot|)$ be a normed finite-dimensional vector space. Let $t_0 \in (0,1]$. Let $T_{t_0} = (t_0/2, t_0] \times \mathbb{T}^d$. Let $(X_{t,x}, (t,x) \in T_{t_0})$ be a V-valued stochastic process with continuous sample paths. Suppose for some constants $C_1 \geq 0$, $\beta > 0$, we have that for all $(r,x), (s,y) \in T_{t_0}$,*

$$\mathbb{P}(|X_{r,x} - X_{s,y}| > u d_{t_0}((r,x),(s,y))) \leq C_1 \exp(-u^\beta), \quad u \geq 0.$$

Then for any $p \geq 1$, we have that

$$\mathbb{E}\left[\sup_{(r,x)\in T_{t_0}} |X_{r,x}|^p\right] \leq C_2 \sup_{(r,x)\in T_{t_0}} \mathbb{E}\left[|X_{r,x}|^p\right] + C_2\left(\int_0^\infty (\log N_{t_0,\varepsilon})^{1/\beta} d\varepsilon\right)^p.$$

Here C_2 depends only on V, C_1, β, and p.

Appendix B: Concentration of Gaussian quadratic forms

The proof of the following lemma is a fairly standard Chernoff bound argument. Thus, it is omitted. See the complete version of this paper on arXiv for the proof.

Lemma B.1. *Let Q be a quadratic form in centered Gaussian random variables. That is, Q is of the form*

$$Q = X^T M X = \sum_{i,j=1}^n X_i M_{ij} X_j,$$

where $n \geq 1$, $X = (X_1, ..., X_n)$ is a mean 0 Gaussian random vector, and M is an $n \times n$ matrix. Then for any $u \geq 0$,

$$\mathbb{P}(|Q - \mathbb{E}Q| > u) \leq 2e^{3/2} \exp\left(-\frac{u}{2\sqrt{\mathrm{Var}(Q)}}\right).$$