Defying Gravity and Gadget Numerosity: The Complexity of the Hanano Puzzle*

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Abstract. Using the notion of visibility representations, our paper establishes a new property of instances of the Nondeterministic Constraint Logic (NCL) problem (a PSPACE-complete problem that is very convenient to prove the PSPACE-hardness of reversible games with pushing blocks). Direct use of this property introduces an explosion in the number of gadgets needed to show PSPACE-hardness, but we show how to bring that number from 32 down to only three in general, and down to two in a specific case! We propose it as a step towards a broader and more general framework for studying games with irreversible gravity, and use this connection to guide an indirect polynomial-time many-one reduction from the NCL problem to the Hanano Puzzle—which is NP-hard—to prove it is in fact PSPACE-complete.

Keywords: Computational complexity · Irreversible games · Hardness of games · Minimizing gadgets

1 Introduction

The application of complexity theory to the study of games has allowed us to understand the hardness of many popular games. Many games that are limited to a single player are NP-complete (with respect to many-one polynomial-time reductions, which is what we will always refer to when using the terms "-hard" and "-complete"), while two-player games are typically PSPACE-complete [8]. However, the moment the board layout becomes dynamic or the number of moves becomes unbounded, the complexity of a one-player game can jump to being PSPACE-complete [8]. Surprisingly, the presence of irreversible gravity, which limits the number of moves possible, can yield complex games [8].

The Hanano Puzzle is a one-player game with a dynamic board, unbounded moves, and gravity developed by video game creator Qrostar [10]. Liu and Yang recently proved that the language version of the Hanano Puzzle is NP-hard [9]. In their paper, they ask if the problem is NP-complete and leave the question open. We pinpoint the problem's complexity by proving Hanano Puzzle's language version to be PSPACE-complete. We do so by providing an indirect

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reduction from the Nondeterministic Constraint Logic (NCL) problem (a known PSPACE-complete problem [8]). One of the major challenges of the reduction is overcoming the effects of gravity. We define a method that leverages graph-theoretic techniques to circumvent unwanted effects of gravity, thereby making reductions from NCL easier. To our knowledge, this method (of abstracting away the "harmful" effects of gravity) is new in this area, which makes our study interesting in this sense. We are also able to significantly reduce the number of gadgets that we need to build by constructing "base gadgets" from which other gadgets can be built. This design is entirely independent of the Hanano Puzzle, and so we believe it might have applications to other similar games.

2 Preliminaries

A simple planar graph is one that is loop-free, has no multi-edges (i.e., for each pair of vertices, there is at most one edge between the pair of vertices), and is planar (i.e., can be drawn on a piece of paper so that its edges only intersect at their common endpoints). Given a graph $G = (V, E)$, a *visibility representation* Γ for G maps every vertex $v \in V$ to a vertical vertex segment $\Gamma(v)$ and every edge $(u, v) \in E$ to a horizontal edge segment $\Gamma(u, v)$ such that each horizontal edge segment $\Gamma(u, v)$ has its respective endpoints lying on the vertical vertex segments $\Gamma(u)$ and $\Gamma(v)$, and no other segment intersections or overlaps occur [12].[1]

2.1 The Hanano Puzzle

This section expands on definitions by Liu and Yang [9]. The Hanano Puzzle comprises different levels. A level of the game is an $n \times m$ grid (with $n, m > 0$) that contains only the following components: immovable gray blocks, movable gray blocks, (movable) colored blocks, colored flowers, and empty spaces. Each colored block/flower can be red, blue, or yellow. Each flower is immovable and is affixed to some block. If that block is movable, then whenever it moves, the affixed flower moves with the block (see Figure 1d). Gray blocks can be of arbitrary shape and size, while all other components are 1×1 objects. In our gadgets, we try to the best of our ability to minimize the number of sides of each movable gray block. A block can slide (see Figure 1a) left or right, one step at a time. For a slide to occur, the space that the block will occupy after the slide must either be empty or be occupied by part of the block that is sliding. Two adjacent blocks of width one can also be swapped in one step (see Figure 1b) the positions of the two blocks can be swapped without moving any other component of the grid.

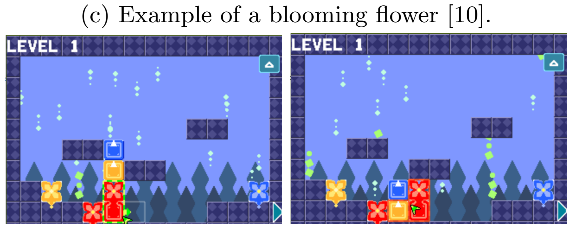
Figure 1 shows screenshots of the game that show sample game moves. Note that the checkered cells are what we call "immovable gray blocks." Movable gray blocks are not depicted in these figures. Because this is a game with gravity, after the player makes a move, every movable block that is not directly supported will

[1] This definition deviates slightly from the standard one in that the standard definition, vertices are mapped to horizontal segments, and edges are mapped to vertical segments. For our purposes, both definitions are equivalent.

(a) Example of a slide [10].

(b) Example of a swap [10].

(c) Example of a blooming flower [10].



(d) Example of effects of gravity [10].

Fig. 1: Screenshots of the Hanano Puzzle (reproduced with permission from Qrostar [11]).

fall (see Figure 1d). This can be viewed as happening in a single step. Each colored block contains an arrow, pointing either up, down, left, or right. If a colored block touches (by sharing a side; touching corners have no effect) a flower of the same color, a flower will bloom from the side of the colored block indicated by the arrow (see Figure 1c), and the new flower will stay affixed/attached to that block. We will sometimes say that the block has bloomed when this happens. If the blooming side is in contact with a block, the blooming flower attempts to "force" its way out by pushing against the surface in contact with the blooming side. This may result in that block in contact with the blooming side to be shifted, or in the blooming block to be shifted. If no shift is possible, then the flower does not bloom. A block can only bloom once and that action cannot be undone. Additionally, if the new flower is in contact with a different block of the same color, chain bloomings can occur within the same step. To solve (complete) a level, one must make every colored block bloom. Formally, we determine the complexity of HANANO = $\{H \mid H$ is a solvable level of the Hanano Puzzle$\}$.

2.2 Nondeterministic Constraint Logic (NCL)

The notions introduced in this section are from Hearn and Demaine [8]. An NCL graph is a directed graph consisting of edges of weights one or two (respectively called red and blue edges) that connect vertices while subject to the constraint that the sum of weights of edges into each vertex is at least two (aka the minimum inflow requirement/constraint). The only operation allowed on an NCL graph is flipping the direction of an edge such that the new graph is still an NCL graph. Given an NCL graph G and an edge e in the graph, deciding if there is a sequence of edge flips that eventually flip e is PSPACE-complete. It turns out that the problem remains PSPACE-complete even if the NCL graph is a planar AND/OR NCL graph, i.e., it is simple planar, each vertex is connected to exactly three edges, and each vertex is either an AND vertex (i.e., one incident edge is blue and the other two are red) or an OR vertex (i.e., all incident edges are blue). In this paper, we will tacitly assume that our NCL graphs are planar AND/OR NCL graphs. For readability and accessibility purposes, in addition to being colored, the blue edges in this paper will be solid and the red edges will be dashed. Typically, to show that NCL reduces to a problem A, it suffices to construct an AND gadget and an OR gadget. However, because of the effects of gravity, we will see in Section 3 that the gadgets need to "interact" properly.

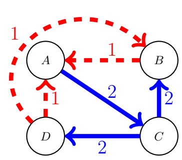


Fig. 2: An NCL graph.

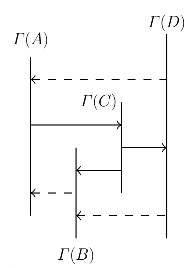


Fig. 3: A visibility representation of Figure 2.

3 PSPACE-Completeness

The HANANO is clearly in PSPACE, since the configure of the game at any point in time can be stored in polynomial space.

3.1 Defying Gravity with Structure

Part of the difficulty in devising a correct reduction is the fact that in NCL, every action is fully reversible while in HANANO, due to gravity and blooms, some moves are irreversible, so we cannot give direct gadgets. Liu and Yang [9] did not encounter this issue as their reduction from CIRCUIT-SAT leveraged the fact that in a boolean circuit, bits only need to move in one direction once. The technique behind their construction is very similar to that used by Friedman [6] to show the NP-hardness of a simple game (Cubic) with gravity. We thus need to be careful in our construction to make sure we do not prematurely make irreversible moves. Additionally, we shall build into our gadgets the constraints of the NCL game to help simulate NCL using HANANO.

Given an NCL graph, each node in the graph will be simulated using gadgets. To help identify the colored blocks and flowers of the gadgets in proofs, we label those items with special text. E.g., the label "B2", indicates the second blue block in the gadget, whereas the label "BF1" indicates the first blue flower. Blue blocks will represent both red and blue edges. It's important to note that *all our blocks bloom upwards*. Next to each flower will be a boldfaced white line to indicate where the flower is attached. Additionally, our gadgets will contain some grid lines to help the reader better gauge the distances. The presence of a block in a gadget will indicate that the edge represented by the block is directed into the node represented by the gadget. Blocks will only be allowed to move between gadgets by following the constraints imposed by NCL. This means that those constraints must be encoded within the gadgets, using the rules of the Hanano Puzzle. This is the first challenge. The second challenge is to overcome the nonreversibility induced by gravity. To help ensure that most block moves are reversible, the effects of gravity must be circumvented. Luckily, this is possible due to the planarity of NCL graphs. We shall start by addressing the second challenge. The first challenge will be resolved by designing the gadgets. One such way is by having blooms "force" a certain setting of the game when the inflow constraints are violated.

Theorem 1 ([13,12]). *A graph admits a visibility representation if and only if it is planar. Furthermore, a visibility representation for a planar graph can be constructed in linear time.*[2]

Since our NCL graphs are planar, we can compute the visibility representation of an NCL graph in linear time. This has the advantage that if we're trying to reduce to a game of sliding blocks, and we wish to represent the direction of the edges by using blocks that are sliding from one gadget (where each vertex is represented using a gadget) to another, then by having all the edges be horizontal, we remove the danger of having gravity make an "edge flip" irreversible. Implicit in this, is that our gadgets will need to be "size-adaptable," i.e., as we will see based on the length of the segment to which a vertex is mapped, a gadget's height may need to change. Our constructions will have the property that they can be internally padded so as to make gadgets artificially long without affecting their correctness. Thus those "edge flips" in the game of interest

[2] In an earlier version of this paper, we independently proved a weaker version of this theorem. Our result established the "if" direction (the "only if" direction is trivial), but our polynomial-time algorithm did not run in linear time.

(here, the Hanano Puzzle) are fully reversible, to the extent that is required to be compatible with NCL's reversibility.

3.2 Gadgets and Schemas

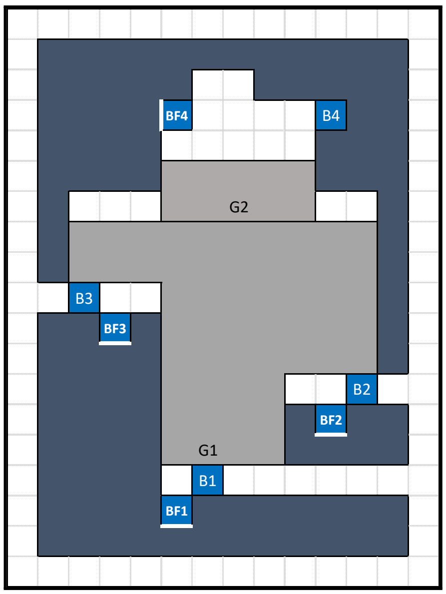
Each vertex in the NCL graph will be represented using a gadget and those gadgets will be connected using tunnels that will represent the edges. For each of these tunnels, there will be a blue block and that block will be placed in the gadget representing the vertex to which the edge is incident. Thus, flipping edges will be represented by moving blocks from one gadget to another. For gadgets to interact properly, we must ensure that a block can only travel through its designated tunnel and that the minimum inflow requirement is always met, i.e., for each gadget, there is always either one blue block representing a blue edge or two blue blocks each representing a red edge in the gadget, at any point in time. Since each vertex in the NCL graph is connected to exactly three edges, each gadget will have three entry points that can each lie either on the left or the right of the gadget. Consider the following notation to represent a gadget: $x_1x_2x_3|y_1y_2y_3$, where for each $i \in \{1,2,3\}$, $\{x_i, y_i\} \in \{\{R,\cdot\},\{B,\cdot\}\}$, and the list $[x_1, x_2, x_3, y_1, y_2, y_3]$ contains either exactly three Bs, or exactly one B and two Rs. (The last condition simply captures the idea that each vertex is either an AND vertex or an OR vertex.) For example, $R\cdot\cdot|\cdot BR$ means that the top entry point on the left side of the gadget is for a red edge's blue block, the middle on the right side is for a blue edge's blue block, and the bottom on the right side is for a red edge's blue block. The remaining entry points are considered blocked off, i.e., not entry points. It's easy to see that due to this structure enforced, the number of gadgets needed to show the reduction goes up from two to *32* (8 OR gadgets + 24 AND gadgets).[3] However, we will show how, by giving only three gadgets, we can derive all the remaining gadgets (i.e., show their existence). The approach will in fact not rely on properties of the Hanano Puzzle, and thus will be reusable for other purposes. However, if one restricts their attention to HANANO, then we will argue that one of the gadgets need not be constructed. Let us first look at how to construct the OR gadgets.

Lemma 1. *The gadget in Figure 4a satisfies the same constraints as an* NCL OR *vertex.*

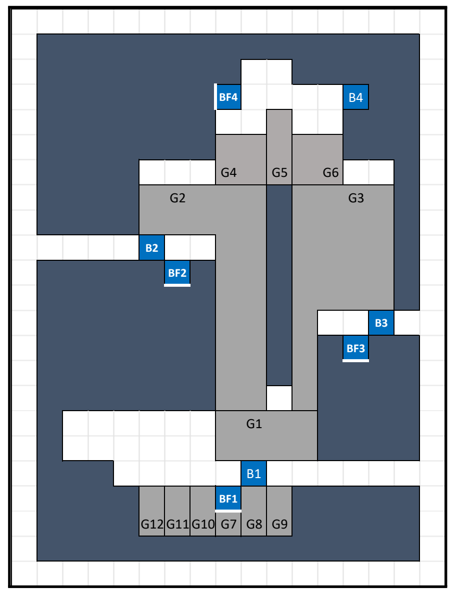
Proof. First notice that each movable gray block has very limited movement. G2 can only move up by one "unit," and G1 can either move up or move down by one unit. Thus for any blue block Bx, the only flower that it can reach in that gadget is BFx. Now, the only way for B4 to bloom is if B4 is in contact with

[3] One could try to argue that given one OR gadget and one AND gadget, it suffices to "just place the tunnels on the correct side" to obtain the remaining 30 gadgets, but that approach does not take into account the structure of the gadget. This suggested approach certainly works for our OR gadgets (see Figure 4a), but not for our AND gadgets (see Figure 4b). However, we do show how to derive all the OR/AND gadgets in a way that is independent of the structure of the underlying gadgets.

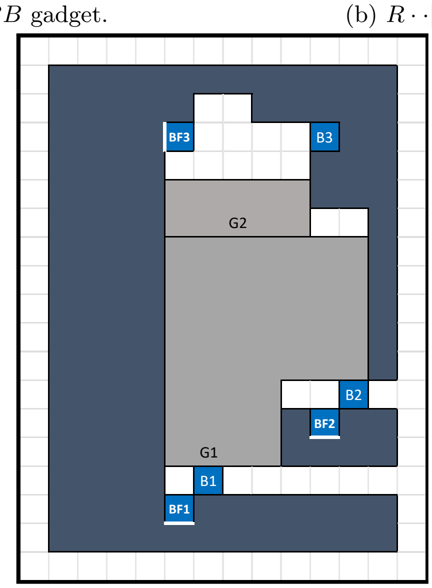
BF4, and it must be on BF4's right side (the only other exposed side of BF4 is the bottom side, but if B4 is directly under BF4 it will not have enough room



(a) $B\cdot\cdot|\cdot BB$ gadget.



(b) $R\cdot\cdot|\cdot RB$ gadget.



(c) Red bend gadget.

Fig. 4: Our three gadgets: an OR gadget ($B\cdot\cdot|\cdot BB$), an AND gadget ($R\cdot\cdot|\cdot RB$), and a red bend gadget.

to bloom). Thus B4 can bloom iff G2 moves up by one unit. This can happen exactly if G1 moves up by one unit, which can happen iff one of B1, B2, or B3 blooms. Finally, notice that if B1, B2, and B3 all leave the gadget, then G1 and G2 both drop by one unit with no possibility of returning to their original configuration, thus making it impossible to bloom B4. We conclude by noting that we could have merged G1 and G2 into a single block, but opted not to as we sought to minimize the number of sides on each movable gray block. □

We will show how to construct certain gadgets from other gadgets, by essentially chaining certain gadgets together. For our convenience we define a "constrained blue edge terminator" gadget, that will allow us to force an edge from pointing out of a gadget, without connecting the edge to other nodes. This allows us to simplify the design of our gadgets. We state the following proposition in a general form, i.e., its proof will not depend on the Hanano Puzzle's properties.

Proposition 1. *The constrained blue edge terminator gadget can be constructed using any gadget that satisfies the same constraints as an* NCL OR *vertex.*

Proof. We want the constrained blue edge terminator to be a gadget that, when attached to a tunnel that represents a blue edge, will force the block that represents the edge's orientation to be inside itself (i.e., inside the constrained blue edge terminator) so as to not violate the inflow constraints.

Fix a gadget that satisfies the same constraints as an NCL OR vertex. There must exist a configuration of the gadget that corresponds to the NCL OR with one (blue) edge pointing into the vertex and the remaining edges (i.e., two blue edges) pointing out of the vertex. Now, block off the tunnels that correspond to the two edges pointing of the vertex. This configuration of the gadget correctly constraints the represented blue edge's orientation. □

A natural question to ask is whether $B\cdot\cdot|\cdot BB$ is special, or whether this result can be achieved using any of the other OR gadgets, and we answer in the positive that indeed, any of the eight OR gadgets suffices. We first note that it suffices to consider the gadgets for $B\cdot\cdot|\cdot BB$, $\cdot B\cdot|B\cdot B$, $\cdot\cdot B|BB\cdot$, and $\cdots|BBB$, as the remaining ones can be obtained via vertical symmetry. In an abuse of notation, we will sometimes use the shorthand for a gadget to refer to the gadget that is obtained from it via vertical symmetry in our schemas. Edges that are connected to the constrained blue edge terminator have a direction assigned and point to a $\oslash$ to indicate the termination. The remaining edges have no direction, indicating that they can be assigned in any way that satisfies the minimum inflow constraints.

Lemma 2. *For each gadget G in the following list, the remaining gadgets in that same list can be constructed from G: 1. $B\cdot\cdot|\cdot BB$, 2. $\cdot B\cdot|B\cdot B$, 3. $\cdot\cdot B|BB\cdot$, and 4. $\cdots|BBB$.*

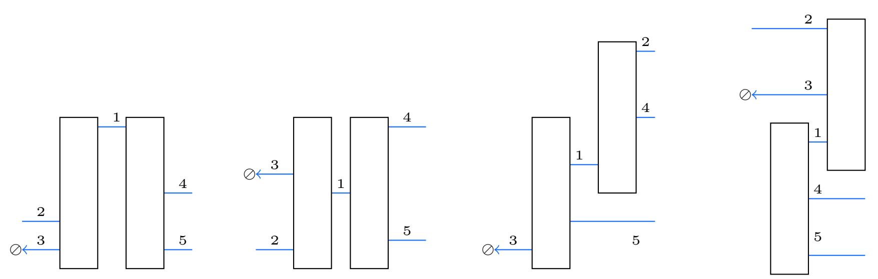
Proof. We prove this lemma by showing how to construct 2 from 1, how to construct 3 from 2, how to construct 4 from 3, and finally how to construct 1 from 4. In each case, we will have a gadget that satisfies the same constraints as an NCL OR vertex, so we tacitly appeal to Proposition 1 to, for "free," have a constrained blue edge terminator. We construct in Figure 5 schematic diagrams to aid in our proof.

From 1 to 2. Figure 5a depicts the construction. The edges of the $\cdot B \cdot | B \cdot B$ gadget are 2, 4, and 5. If those edges all point out (i.e., 2 points left and the other two point right), then the minimum inflow constraint is violated as 3 cannot be flipped and 1 can only point into one the two gadgets. Thus one of 2, 4, or 5 must always be pointed inwards, and this gadgets satisfies the same constraints and as NCL OR vertex.

From 2 to 3. Figure 5b depicts the construction. The edges of the $\cdot B \cdot | B \cdot B$ gadget are 2, 4, and 5. If those edges all point out (i.e., 2 points left and the other two point right), then the minimum inflow constraint is violated as 3 cannot be flipped and 1 can only point into one the two gadgets. Thus one of 2, 4, or 5 must always be pointed inwards, and this gadgets satisfies the same constraints and as NCL OR vertex.

From 3 to 4. Figure 5c depicts the construction. The edges of the $\cdot B \cdot | B \cdot B$ gadget are 2, 4, and 5. If those edges all point out (i.e., they all point right), then the minimum inflow constraint is violated as 3 cannot be flipped and 1 can only point into one the two gadgets. Thus one of 2, 4, or 5 must always be pointed inwards, and this gadgets satisfies the same constraints and as NCL OR vertex.

From 4 to 1. Figure 5d depicts the construction. The edges of the $\cdot B \cdot | B \cdot B$ gadget are 2, 4, and 5. If those edges all point out (i.e., 2 points left and the other two point right), then the minimum inflow constraint is violated as 3 cannot be flipped and 1 can only point into one the two gadgets. Thus one of 2, 4, or 5 must always be pointed inwards, and this gadgets satisfies the same constraints and as NCL OR vertex. □



(a) $\cdot B \cdot | B \cdot B$. (b) $\cdot \cdot B | BB \cdot$. (c) $\cdots | BBB$. (d) $B \cdot \cdot | \cdot BB$.

Fig. 5: Schemas showing the "equivalence" of the OR gadgets.

The AND gadgets are trickier, but we can construct all of them using two gadgets: a gadget that satisfies the same constraints as an NCL AND vertex, and a "red bend" gadget. The red bend gadget will allow us to, in some sense, reorient the direction of a tunnel that corresponds to a red edge (hence the "red" in the name even though all our blocks are blue; in the context of HANANO, the red bend gadget also works on "blue edges" but that does not necessarily hold in general). Lemma 4 gives an analogous result to Lemma 2.

Lemma 3. *The gadget in Figure 4b satisfies the same constraints as an* NCL AND *vertex, with B1 representing the blue edge, and with B3 and B4 representing the red edges.*

Proof. Let us first describe the gadget before arguing its correctness.

G4 and G6 can only move up by 1 "unit," and G2 and G3 can each either move up or move down by one unit. Thus each colored block, can only reach one flower. If both B2 and B3 exit the gadget, then B1 must remain to support G1 (which in turn supports G2 and G3), as otherwise, G2 and G3 will drop by 1 unit and G4 and G6 will never be able to move up. Similarly, if B1 is to exit, all the gray blocks must remain supported. This is only possible if G1 is stowed to the left and B2 and B3 remain in the gadget to, respectively, support G3 and G2. Additionally, the area underneath B1 is made up of multiple movable gray blocks for a simple reason. B1 must be able to move horizontally without blooming (either to exit the gadget or two carry G1 to "stow" it). By this setting, we can move the location of BF1 (by swapping G7 with an adjacent movable block of width one; BF1 is affixed to G7 and the two will move as if they were one 2×1 block) to make sure it is always on the left of B1.

Now, the only way for B4 to bloom is if B4 is in contact with BF4, and it must be on BF4's right side (the only other exposed side of BF4 is the bottom side, but if B4 is directly under BF4 it will not have enough room to bloom). Thus B4 can bloom if and only if G4 and G6 move up by one unit. This can happen exactly if both G2 and G3 move up by one unit. There are two ways this can happen: Either both B2 and B3 bloom, or B1 blooms (thus pushing G1 up by one unit). Thus B4 blooms if and only if either B2 and B3 bloom in the gadget, or B1 blooms in the gadgets. □

We now define an important property of the red bend gadget that is used in the proof of Lemma 4. Intuitively, the results means that the inflow constraint on red bend gadgets is one (or two under the "blue bend" interpretation).

Proposition 2. *The red bend gadget in Figure 4c is solvable if and only if either B1 or B2 blooms while supporting G1.*

Proof. The design of the gadget is simply a restricted/modified version of that in Figure 4a, so we omit the description of the gadget.

$\Longrightarrow$: Suppose the gadget is solvable. Then B3 must come in contact with BF3. This is only possible if both G1 and G2 move up by one unit. For this to happen, either B1 or B2 must bloom while supporting G1.

$\Longleftarrow$: Suppose that either B1 or B2 blooms while supporting G1. In both cases, G1 moves up by one unit, thus also pushing G2 up by one unit, hence allowing B3 to come in contact with BF3 to bloom. □

Earlier, we mentioned that when our attention is focus on HANANO, we only need two gadgets. This is indeed possible because all the blocks that we use are of the same color, and so we can define the red bend gadget to be a restricted version of the OR gadget. Consider the gadget in Figure 4a. If we place a constrained blue edge terminator at the tunnel for B3, then the resulting gadget is essentially the red bend gadget.

Lemma 4. *For each gadget G in the following list, the remaining gadgets in that same list can be constructed from G, the red bend gadget, and any* OR *gadget: 1.* $R\cdot\cdot|\cdot RB$, *2.* $\cdot R\cdot|R\cdot B$, *3.* $\cdot\cdot\cdot|RRB$, *4.* $\cdot\cdot B|RR\cdot$, *5.* $\cdot\cdot\cdot|BRR$, *6.* $\cdot\cdot R|BR\cdot$, *7.* $B\cdot\cdot|\cdot RR$, *8.* $\cdot R\cdot|B\cdot R$, *9.* $\cdot\cdot\cdot|RBR$, *10.* $R\cdot\cdot|\cdot BR$, *11.* $\cdot B\cdot|R\cdot R$, *and 12.* $\cdot\cdot R|RB\cdot$.

Proof. We represent our red bend gadgets using a vertex with exactly two red edges on the same side of the gadget. The structure of this proof resembles that of Lemma 2.

From 1 to 2. Figure 6a depicts the construction. If edge 5 points right, then edge 1 points right and edge 4 points left. Thus edge two must point left, leaving edge 3 to point right. If edge 5 points left, then edge 4 is free to point in either direction. In that case, we can fix edge 1 to point left and edge 2 to point right, thus leaving edge 3 to point in any direction. Therefore, this gadget satisfies the same constraints as an NCL AND vertex.

From 2 to 3. Figure 6b depicts the construction. If 4 points right, then 2 must point right and 3 must point left. Thus 1 must point left. If 4 points left, we can fix 2 to point left, leaving 1 and 3 free to point in either direction. Therefore, this gadget satisfies the same constraints as an NCL AND vertex.

From 3 to 4. Figure 6c depicts the construction. If 4 points left, then 3 must point right, forcing both 1 and 2 to point left. If 4 points right, we can fix 3 to point left, thus leaving 1 and 2 free to point in either direction. Therefore, this gadget satisfies the same constraints as an NCL AND vertex.

From 4 to 5. Figure 6d depicts the construction. If 4 points right, then 3 must point left, forcing both 1 and 2 to point left. If 4 points left, we can fix 3 to point right, thus leaving 1 and 2 free to point in either direction. Therefore, this gadget satisfies the same constraints as an NCL AND vertex.

From 5 to 6. Figure 6e depicts the construction. If 1 points right, then 2 and 3 must point left, and so 4 must point left. If 1 points left, then we can fix 3 to point right, leaving 2 and 4 free to point in either direction. Therefore, this gadget satisfies the same constraints as an NCL AND vertex.

From 6 to 7. Figure 6f depicts the construction. If 1 points right, then 2 must point left and 3 must point right, and so 4 must point right. If 1 points left, then we can fix 2 to point right, leaving 3 and 4 free to point in either direction. Therefore, this gadget satisfies the same constraints as an NCL AND vertex.

From 7 to 8. Figure 6g depicts the construction. If 1 points right, then 2 and 3 must point right too, and so 4 must point left. If 1 points left, then we can fix 3 to point left, leaving 2 and 4 free to point in either direction. Therefore, this gadget satisfies the same constraints as an NCL AND vertex.

From 8 to 9. Figure 6h depicts the construction. If 1 points right, then 2 must point right and 3 must point left, and so 4 must point left. If 1 points left, then we can fix 2 to point left, leaving 3 and 4 free to point in either direction. Therefore, this gadget satisfies the same constraints as an NCL AND vertex.

From 9 to 10. Figure 6i depicts the construction. If 1 points right, then 2 and 3 must point left, and so 4 must point right. If 1 points left, then we can fix 3 to point right, leaving 2 and 4 free to point in either direction. Therefore, this gadget satisfies the same constraints as an NCL AND vertex.

From 10 to 11. Figure 6j depicts the construction. If 1 points right, then 2 must point left and 3 must point right, and so 4 must point right. If 1 points left, then we can fix 2 to point right, leaving 3 and 4 free to point in either direction. Therefore, this gadget satisfies the same constraints as an NCL AND vertex.

From 11 to 12. Figure 6k depicts the construction. If 1 points right, then 2 and 3 must point right, and so 4 must point left. If 1 points left, then we can fix 3 to point left, leaving 2 and 4 free to point in either direction. Therefore, this gadget satisfies the same constraints as an NCL AND vertex.

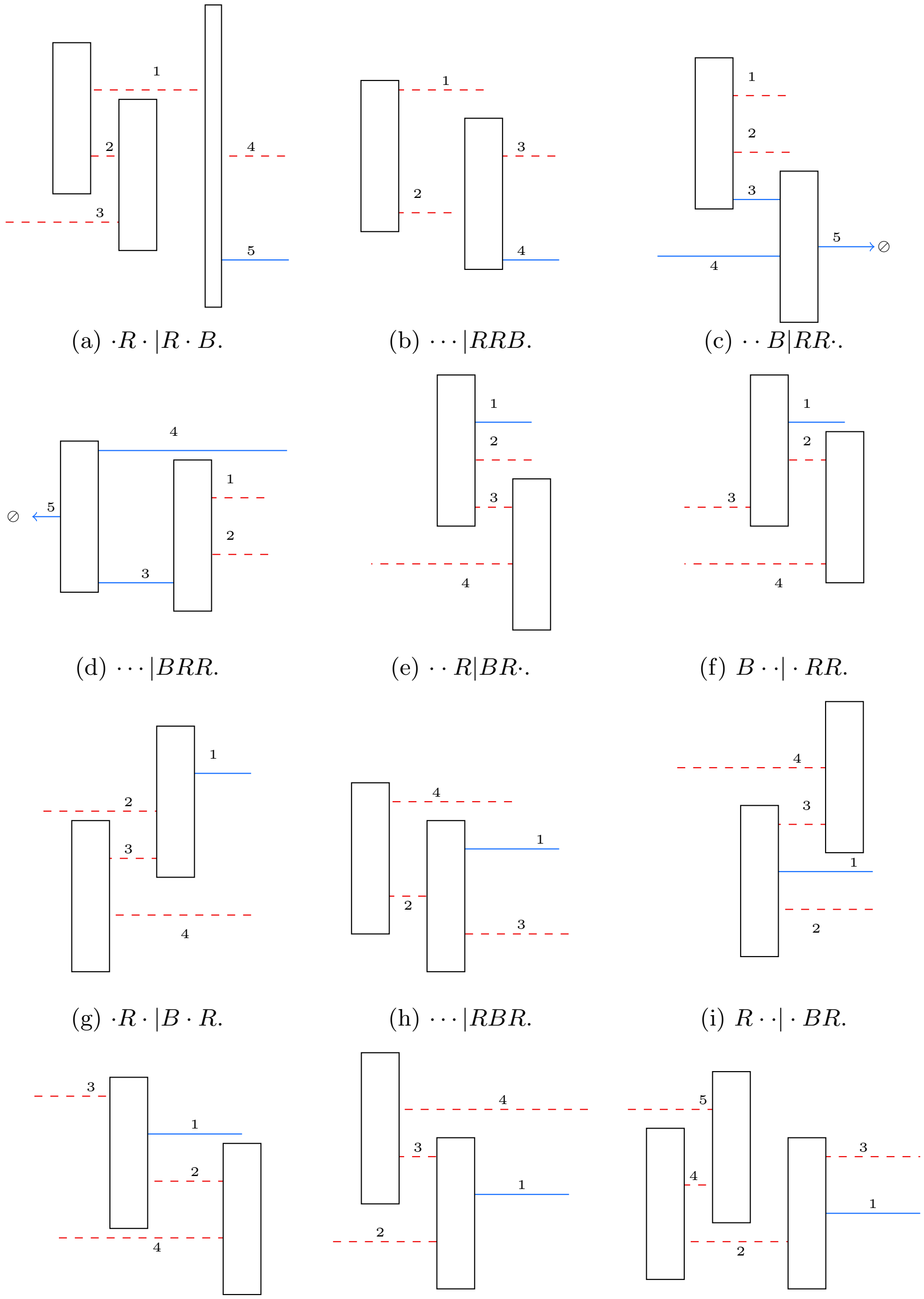
From 12 to 1. Figure 6l depicts the construction. If 1 points right, then 2 must point right and 3 must point left, and so 4 must point left, and 5 must point right. If 1 points left, then we can fix 2 to point left and fix 4 to point right, leaving 3 and 5 free to point in either direction. Therefore, this gadget satisfies the same constraints as an NCL AND vertex. □

3.3 Main Result

Theorem 2. HANANO *is* PSPACE*-complete even if (1) all flowers and colored blocks have the same color, and (2) colored blocks can only bloom upwards.*

Proof. Since HANANO $\in$ PSPACE, it suffices to show that NCL $\leq_m^p$ HANANO. Consider the clearly-polynomial-time-computable function that we describe in the next paragraph. We assume without loss of generality that the input is an NCL graph and a valid target edge, as we can easily detect in polynomial time if is not and map to a fixed element that is not in HANANO.

Construct in polynomial time a visibility representation for the input NCL graph and construct a game grid based on the visibility representation, replacing each vertex of the graph by a suitable gadget, and replacing edges with the appropriate tunnels. The game grid will be polynomially larger than the visibility representation since the gadgets have constant-bounded size. We must ensure that the game is only solvable when the target edge $e = (u, v)$ is flipped. If the flower than blooms b is attached to an immovable gray block, replace that flower with an immovable gray block. Otherwise, the flower that blooms b must be attached to the top of a 1×1 movable gray block. Replace that gray block with a 2×1 movable gray block. There is now no flower in the gadget for v that can bloom b, so to bloom, b must move the gadget for u. If the game is solvable, then b must bloom, and so there will exist a sequence of block movements corresponding to edge flips, so the edge e can be flipped in G. If there is a sequence of edge



(a) $\cdot R \cdot |R \cdot B$. (b) $\cdots|RRB$. (c) $\cdot\cdot B|RR\cdot$.

(d) $\cdots|BRR$. (e) $\cdot\cdot R|BR\cdot$. (f) $B\cdot\cdot|\cdot RR$.

(g) $\cdot R \cdot |B \cdot R$. (h) $\cdots|RBR$. (i) $R\cdot\cdot|\cdot BR$.

(j) $\cdot B \cdot |R \cdot R$. (k) $\cdot\cdot R|RB\cdot$. (l) $R\cdot\cdot|\cdot RB$.

Fig. 6: Schemas showing the "equivalence" of the AND gadgets.

flips that eventually flips edge e in G, there is sequence of block movements that respect the inflow constraints and eventually see b move from the gadget representing v to the gadget representing u. Thus the colored block b (and all the other ones in the game) can bloom and the game is solvable. Finally note that all the colored blocks in our gadgets have their arrows pointing up, and that we only use blue blocks/flowers. □

4 Related Work

The literature on the complexity of games is rich and covers a variety of games. We refer readers to Appendix A of Hearn and Demaine's book [8], which contains an extensive survey of games whose complexity were known at their time of writing. The introduction of NCL [7] helped simplify the process of showing that many games with sliding blocks are PSPACE-complete by limiting the number of gadgets to simulate to two. The work on motion planning through doors [1] provides a framework to show the PSPACE-hardness of certain problems by simulating *one* gadget. However, that paper's contribution does not solve the major problem faced by classifying the Hanano Puzzle: circumventing certain effects of gravity. There are games with gravity that were studied prior to the introduction of NCL. For example, Friedman [6] proved Cubic to be NP-hard. Clickomania is another game with gravity that was studied before the introduction of NCL. This game is a one-player game with a bounded number of moves and it is in fact NP-complete [3]. Solving a level of Super Mario Brothers (SMB), which is another game with gravity, has also been proven to be PSPACE-complete [5]. However, the framework used in that proof does not rely on NCL, since SMB is not a game that involves pulling blocks. Another famous game with gravity is Tetris. While the "offline" version is NP-complete [4], in the general case, it is NP-hard [2]. On the other hand, Jelly-no-Puzzle, also by Qrostar, is known to be NP-hard is the general case [14]. Our paper uses NCL to study a game with sliding blocks and irreversible gravity, and extends this line of work by providing a framework to study such games using only three gadgets in general, and by having only two gadgets when focusing on HANANO.

5 Conclusion and Open Directions

After establishing the NP-hardness of HANANO, Liu and Yang stated as an open problem the task of determining whether HANANO $\in$ NP. It follows from our PSPACE-completeness result that HANANO $\notin$ NP unless NP = PSPACE.

Another contribution of this paper is the use of the visibility representations. Our Proposition 1 and our Lemmas 2 and 4 helped significantly reduce the number of gadgets needed. Since these proofs are independent of HANANO, we believe that they can be reused to analyze additional games.

By leveraging schemas and symmetry, we only needed to provide three gadgets (instead of 32 gadgets). If we focus our attention to HANANO, then we can derive the red bend gadget from $B \cdot \cdot | \cdot BB$. And so, an interesting direction would be to investigate whether the reduction could be carried out using only two gadgets (or even one) in the general case. (One might posit that by placing a constrained blue edge terminator on the blue edge for $\cdots|BRR$, we get what looks like a red bend gadget, but that omits the inflow constraint of two. Thus both red edges would need to face into the gadget, which is too strong of a requirement.) Finally, we mention that our movable gray blocks have up to six sides in two gadgets and up to eight sides in the third gadget. We followed the closest paper in the literature, which states that those movable gray blocks can have "any size or shape" [9]. However, it would be interesting if our result could be strengthened to only have movable gray blocks with exactly four sides.

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