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Rigidity of Mappings Between Degenerate and Indefinite Hyperbolic Spaces

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Abstract
In this paper, we prove rigidity results for holomorphic mappings between possibly degenerate and indefinite hyperbolic spaces.

Keywords Degenerate and indefinite hyperbolic spaces · Isometric mappings · Holomorphic proper mappings

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1 Introduction

Write $\mathbb{B}^n = \{z \in \mathbb{C}^n : \|z\| < 1\}$ for the n-dimensional complex unit ball, and $\partial\mathbb{B}^n$ for the unit sphere in $\mathbb{C}^n$. A classical rigidity result of Poincaré [22] states that a nonconstant holomorphic map sending an open connected piece of $\partial\mathbb{B}^2$ into $\partial\mathbb{B}^2$ is linear fractional and furthermore must extend to an automorphism of $\mathbb{B}^2$. This work is the starting point of numerous far reaching rigidity type results in several complex variables, including Tanaka [24], Chern-Moser [6], Alexander [1, 2], and so on. In particular, Alexander [2] showed that any holomorphic proper self-map of $\mathbb{B}^n$, $n \geq 2$, must be an automorphism. Since the work of Webster [25], much attention has been paid to the mapping problem for proper holomorphic maps between complex balls of different dimensions. In this scenario, the rigidity of a proper holomorphic map F

Dedicated to the memory of Nessim Sibony.

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from $\mathbb{B}^n$ to $\mathbb{B}^N$ fails dramatically in its full generality if $N > n$. However, rigidities can still be expected when the codimension is small and a certain boundary regularity of F is assumed. For more related research on this matter, the readers are referred to the work in [7–11, 14–16, 23, 27], etc. In 2005, Baouendi and the first author [3] discovered a rigidity phenomenon of different flavor for holomorphic mappings between generalized balls. Recall the (nondegenerate) generalized complex unit ball is defined as the following domain in $\mathbb{P}^n$:

$$\mathbb{B}_l^n = \{[z_0, \ldots, z_n] \in \mathbb{P}^n : |z_0|^2 + \cdots + |z_l|^2 > |z_{l+1}|^2 + \cdots + |z_n|^2\}.$$

The integer l with $0 \le l \le n-1$ is often called the signature of the generalized ball. When $l = 0$, the generalized ball is reduced to the standard unit ball $\mathbb{B}^n$. The boundary $\partial\mathbb{B}_l^n$ of $\mathbb{B}_l^n$ is not strongly pseudoconvex when $l \ge 1$, but it is still Levi-nondegenerate. It follows from [3] that, under some natural side-preserving assumptions, every holomorphic map sending an open piece of $\partial\mathbb{B}_l^n$ to $\partial\mathbb{B}_l^N$ must extends to a linear map from $\mathbb{P}^n$ to $\mathbb{P}^N$ for any $l \ge 1$ and $N \ge n$. For further investigation on mapping problems between generalized balls, see [5, 12, 13, 17, 18, 20, 26, 28] and references therein. In general, the complexity of mappings between two generalized balls $\mathbb{B}_l^n$ and $\mathbb{B}_{l'}^N$ heavily hinges on the signature difference $l' - l$. Recently, Ng-Zhu [21] and Gao-Ng [13] considered holomorphic mappings between degenerate generalized balls, whose boundaries are Levi-degenerate hypersurfaces. They developed a more algebraic approach to study holomorphic maps between generalized balls and extended a number of well-known rigidity results to the degenerate settings (cf. Theorem 1.1 in [13]).

Denote by $\mathbb{N}$ the set of positive integers, and by $\mathbb{Z}^{\ge 0}$ the set of non-negative integers. Let $m = r + s + t$ with $r, s \in \mathbb{N}, t \in \mathbb{Z}^{\ge 0}$. Write

$$[Z] = [z, \xi, \eta] = [z_1, \ldots, z_r, \xi_1, \ldots, \xi_s, \eta_1, \ldots, \eta_t]$$

for the homogeneous coordinates for $\mathbb{P}^{m-1}$. Here z, ξ and η denote the z_i, ξ_j, η_k coordinates, respectively. The possibly degenerate generalized complex unit ball is defined as the following domain in $\mathbb{P}^n$:

$$\mathbb{B}^{r,s,t} = \left\{ [Z] \in \mathbb{P}^{m-1} : \sum_{i=1}^{r} |z_i|^2 - \sum_{j=1}^{s} |\xi_j|^2 > 0 \right\}.$$

The generalized ball $\mathbb{B}^{r,s,t}$ possesses a canonical indefinite metric $\omega_{\mathbb{B}^{r,s,t}}$ that is invariant under the action of its automorphism group (see Proposition 2.3 in Sect. 2.1):

$$\omega_{\mathbb{B}^{r,s,t}} = -\sqrt{-1}\partial\bar{\partial} \log \left(\sum_{i=1}^{r} |z_i|^2 - \sum_{j=1}^{s} |\xi_j|^2 \right).$$

The generalized ball equipped with the metric $\omega_{\mathbb{B}^{r,s,t}}$ is called a generalized hyperbolic space form. When $r = 1, s \geq 1$ and $t = 0$, it is reduced to the standard hyperbolic space form (up to a normalization). We say $(\mathbb{B}^{r,s,t}, \omega_{\mathbb{B}^{r,s,t}})$ is an indefinite hyperbolic space if $r \geq 2$, and we say it is a degenerate hyperbolic space if $t \geq 1$. In the special case of $t = 0$, we will simply denote $\mathbb{B}^{r,s,0}$ by $\mathbb{B}^{r,s}$. The two notions of $\mathbb{B}^n_l$ and $\mathbb{B}^{r,s,t}$ are related by $\mathbb{B}^n_l = \mathbb{B}^{l+1,n-l}$.

We continue to introduce more notations which will be needed to formulate our results. Denote by $\mathcal{F}(\mathbb{B}^{r,s,t}, \mathbb{B}^{R,S,T})$ the set of all holomorphic maps F satisfying the following three conditions:

(a) F is a holomorphic map from an open set $U \subset \mathbb{P}^{r+s+t-1}$, depending on F, into $\mathbb{P}^{R+S+T-1}$;

(b) $U \cap \partial\mathbb{B}^{r,s,t} \neq \emptyset$ and $U \cap \mathbb{B}^{r,s,t}$ is connected;

(c) $F(U \cap \mathbb{B}^{r,s,t}) \subseteq \mathbb{B}^{R,S,T}$, $F(U \cap \partial\mathbb{B}^{r,s,t}) \subseteq \partial\mathbb{B}^{R,S,T}$.

If $\mathcal{F}(\mathbb{B}^{r,s,t}, \mathbb{B}^{R,S,T}) \neq \emptyset$, then we must have $R \geq r$ and $S \geq s$ (see Remark 2.7. See also Proposition 3.9 in [13]). Let $F \in \mathcal{F}(\mathbb{B}^{r,s,t}, \mathbb{B}^{R,S,T})$. We say F is an isometry if it preserves the (possibly indefinite and degenerate) hyperbolic metrics: $F^*(\omega_{\mathbb{B}^{r,s,t}}) = \omega_{\mathbb{B}^{R,S,T}}$ on $U \cap \mathbb{B}^{r,s,t}$. We recall Theorem 1.1 in [18], which can be formulated as follows in the above terminology.

Theorem 1 [18] *Fix $R, S, r, s \in \mathbb{N}$ with $R \geq r \geq 2$ and $S \geq s \geq 2$. Assume one of the following conditions holds:*

$$\begin{aligned}
&(1)\quad R < 2r-1, R < r+s-1;\\
&(2)\quad R < 2r-1, S < r+s-1;\\
&(3)\quad S < 2s-1, R < r+s-1;\\
&(4)\quad S < 2s-1, S < r+s-1.
\end{aligned}$$

Then every $F \in \mathcal{F}(\mathbb{B}^{r,s}, \mathbb{B}^{R,S})$ is isometric.

Likewise, Theorem 2 in [26] has the following formulation.

Theorem 2 [26] *Let $R, r, s \in \mathbb{N}$. Assume $r + s \geq 5$ and $R \leq 2r + 2s - 3$. Write $S = 2r+2s-R-2$. If $R \neq 2r-1$ and $R \neq r+s-1$, then every $F \in \mathcal{F}(\mathbb{B}^{r,s}, \mathbb{B}^{R,S})$ is isometric.*

We are now in a position to introduce the main result of this paper.

Theorem 1.1 *Let $R, S, r, s \in \mathbb{N}$, and $t, T \in \mathbb{Z}^{\geq 0}$. Then each $F \in \mathcal{F}(\mathbb{B}^{r,s,t}, \mathbb{B}^{R,S,T})$ is an isometry if and only if each $G \in \mathcal{F}(\mathbb{B}^{r,s}, \mathbb{B}^{R,S})$ is isometry.*

Remark 1.2 (1) Let $r, s \in \mathbb{N}$ and $r + s \geq 3$. It is well known that every $F \in \mathcal{F}(\mathbb{B}^{r,s}, \mathbb{B}^{r,s})$ is isometric (cf. [3]). Then by Theorem 1.1, every $G \in \mathcal{F}(\mathbb{B}^{r,s,t}, \mathbb{B}^{r,s,T})$ is isometric for $t, T \in \mathbb{Z}^{\geq 0}$.

(2) We immediately obtain a consequence of Theorem 1.1 as follows. Let $R, S, r, s \in \mathbb{N}$, and $t_1, t_2, T_1, T_2 \in \mathbb{Z}^{\geq 0}$. Then every $F \in \mathcal{F}(\mathbb{B}^{r,s,t_1}, \mathbb{B}^{R,S,T_1})$ is isometric if and only if every $G \in \mathcal{F}(\mathbb{B}^{r,s,t_2}, \mathbb{B}^{R,S,T_2})$ is isometric.

We combine Theorems 1 and 1.1 to get the following:

Proposition 1.3 *Let $R, S, r, s \in \mathbb{N}$ and $t, T \in \mathbb{Z}^{\geq 0}$. Assume $R \geq r \geq 2$ and $S \geq s \geq 2$ and one of the conditions (1)–(4) in Theorem 1 holds. Then every $F \in \mathcal{F}(\mathbb{B}^{r,s,t}, \mathbb{B}^{R,S,T})$ is isometric.*

As in [18] (see Remark 1.5 there), the above proposition is optimal in the sense that if none of the conditions (1)–(4) holds, then the conclusion fails. Theorems 2 and 1.1 yield the following result.

Proposition 1.4 *Let $R, r, s \in \mathbb{N}$ and $t, T \in \mathbb{Z}^{\geq 0}$. Assume $r + s \geq 5$ and $R \leq 2r + 2s - 3$. Assume $R \neq 2r - 1$ and $R \neq r + s - 1$. Write $S = 2r + 2s - R - 2$. Then every $F \in \mathcal{F}(\mathbb{B}^{r,s,t}, \mathbb{B}^{R,S,T})$ is isometric.*

As elucidated in [26] (see Remark 1.2 there), Proposition 1.4 is also optimal in the sense that if $R = 2r - 1$ or $R = r + s - 1$, the conclusion fails. As applications of Propositions 1.3 and 1.4, we have the following corollaries. In particular, Corollary 1.5 extends Corollary 1.3 in [18]. For a holomorphic rational map F from $\mathbb{P}^{r+s+t-1}$ to $\mathbb{P}^{R+S+T-1}$ with $I \subseteq \mathbb{P}^{r+s+t-1}$ its set of indeterminacy, we say F is a rational proper map from $\mathbb{B}^{r,s,t}$ to $\mathbb{B}^{R,S,T}$, if F maps from $\mathbb{B}^{r,s,t} \setminus I$ to $\mathbb{B}^{R,S,T}$, and maps $\partial\mathbb{B}^{r,s,t} \setminus I$ to $\partial\mathbb{B}^{R,S,T}$.

Corollary 1.5 *Let $R, S, r, s \in \mathbb{N}$ and $t, T \in \mathbb{Z}^{\geq 0}$. Assume $R \geq r \geq 2$ and $S \geq s \geq 2$ and one of the conditions (1)–(4) in Theorem 1 holds. Let F be a rational proper map from $\mathbb{B}^{r,s,t}$ to $\mathbb{B}^{R,S,T}$. Then F extends to a linear map from $\mathbb{P}^{r+s+t-1}$ to $\mathbb{P}^{R+S+T-1}$.*

Corollary 1.6 *Let R, S, T, r, s, t be as in Proposition 1.4. Assume in addition that $r \geq 2$. Let F be a rational proper map from $\mathbb{B}^{r,s,t}$ to $\mathbb{B}^{R,S,T}$. Then F extends to a linear map from $\mathbb{P}^{r+s+t-1}$ to $\mathbb{P}^{R+S+T-1}$.*

Remark 1.7 Note if $r \geq 2$, then every proper holomorphic map from $\mathbb{B}^{r,s,t}$ to $\mathbb{B}^{R,S,T}$ extends to a rational map from $\mathbb{P}^{r+s+t-1}$ to $\mathbb{P}^{R+S+T-1}$. (This fact follows from the same proof as in Proposition 3.2 of [20]). Consequently, Corollaries 1.5 and 1.6 still hold if we assume F to be a proper holomorphic from $\mathbb{B}^{r,s,t}$ to $\mathbb{B}^{R,S,T}$.

We should refer the readers to [13] (cf. Theorem 1.1 there) for many related results. The paper is organized as follows. In Sect. 2.1, we discuss the automorphism group of possibly degenerate and indefinite hyperbolic spaces. We describe isometric maps between degenerate and indefinite hyperbolic spaces in Sect. 2.2. Then in Sect. 2.3, we give the proofs of Theorem 1.1 and Corollaries 1.5 and 1.6.

2 Proof of the Main Theorem and Corollaries

2.1 Automorphisms of Degenerate Hyperbolic Spaces

For $r, s \in \mathbb{N}$ and $t \in \mathbb{Z}^{\geq 0}$ with $m = r + s + t$, write $E(r, s, t)$ for the $m \times m$ diagonal matrix, where its first r diagonal elements equal -1, the next s diagonal elements

equal 1 and the rest equal 0. When $t = 0$, we will simply write $E(r, s)$ for $E(r, s, 0)$. We define the generalized unitary group $U(r, s, t)$ as follows.

$$U(r, s, t) = \{\mathcal{X} \in GL(m, \mathbb{C}) : \mathcal{X}E(r, s, t)\overline{\mathcal{X}}^t = E(r, s, t)\}.$$

Recall the element $\mathcal{Y} \in GL(m, \mathbb{C})$ naturally acts on $\mathbb{P}^{m-1}$ by sending $[Z] \in \mathbb{P}^{m-1}$ to $[Z\mathcal{Y}]$. It is clear that every element $\mathcal{X}$ in $U(r, s, t)$ gives a (biholomorphic) automorphism of $\mathbb{B}^{r,s,t}$. Moreover, such an automorphism preserves the metric $\omega_{\mathbb{B}^{r,s,t}}$.

Write $\mathbf{0}_{k\times l}$ for the $k \times l$ zero matrix and denote by $M(n, m; \mathbb{C})$ the space of $n \times m$ matrices with complex entries. For $\mathcal{X} \in GL(m, \mathbb{C})$. Then one can readily verify that $\mathcal{X} \in U(r, s, t)$ if and only if

$$\mathcal{X} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathbf{0}_{t\times(r+s)} & \mathcal{C} \end{pmatrix},$$

where $\mathcal{A} \in M(r+s, r+s; \mathbb{C})$, $\mathcal{B} \in M(r+s, t; \mathbb{C})$, $\mathcal{C} \in M(t, t; \mathbb{C})$ and $\mathcal{A}E(r, s)\overline{\mathcal{A}}^{\mathsf{T}} = E(r, s)$, $\det \mathcal{C} \neq 0$.

Define an equivalence relation in $U(r, s, t)$ by setting $\mathcal{X} \sim \mathcal{Y}$ if $\mathcal{X} = e^{i\theta}\mathcal{Y}$ for some $\theta \in \mathbb{R}$. Set $PU(s, u, t) := U(r, s, t)/\sim$ be the quotient group of $U(r, s, t)$ by this equivalence relation. Equivalently, $PU(r, s, t)$ is the quotient group of $U(r, s, t)$ by the normal subgroup $\{e^{i\theta}I_m : \theta \in \mathbb{R}\}$, where I_m denotes the $m \times m$ identity matrix. Then $PU(r, s, t)$ gives a subgroup of the automorphism group $\mathrm{Aut}(\mathbb{B}^{r,s,t})$ of $\mathbb{B}^{r,s,t}$.

One can also verify that $PU(r, s, t)$ acts transitively on $\mathbb{B}^{r,s,t}$. When $t = 0$, the group also acts transitively on $\partial\mathbb{B}^{r,s,0}$. When $t > 0$, however, $PU(r, s, t)$ does not act transitively on $\partial\mathbb{B}^{r,s,t}$. Indeed, in this case $\partial\mathbb{B}^{r,s,t}$ decomposes into two orbits under the action of $PU(r, s, t)$:

$$M_1 = \{[z, \xi, \eta] \in \partial\mathbb{B}^{r,s,t} : z \neq 0\};$$
$$M_2 = \{[z, \xi, \eta] \in \partial\mathbb{B}^{r,s,t} : z = \xi = 0\}.$$

Here in the above homogeneous coordinates, z, ξ, and η have r, s, and t components, respectively. Note if $t = 0$, M_2 is empty.

When $t = 0$, $\mathrm{Aut}(\mathbb{B}^{r,s,0})$ equals to $PU(r, s, 0)$ (cf. [3]). Now assume $t \geq 1$. If in addition $r \geq 2$, $\mathrm{Aut}(\mathbb{B}^{r,s,t})$ also equals to $PU(r, s, t)$. (See Proposition 2.4.) However, if $t \geq 1$ and $r = 1$, then the automorphism of $\mathbb{B}^{1,s,t}$ might not be linear or even rational, as elucidated by the following examples. In any case, the automorphism of $\mathbb{B}^{r,s,t}$ always preserves the metric $\omega_{\mathbb{B}^{r,s,t}}$. See Proposition 2.3.

Example 2.1 Let $s \geq 1, t \geq 1$. Write $[Z] = [z, \xi, \eta]$ for the homogeneous coordinates of $\mathbb{P}^{s+t}$, where z is a scalar, $\xi = (\xi_1, \ldots, \xi_s)$ has s components and $\eta = (\eta_1, \ldots, \eta_t)$ has t components. Let Φ be the following rational map from $\mathbb{P}^{s+t}$ to $\mathbb{P}^{s+t}$.

$$\Phi : [Z] \to [z^2, z\xi_1, \ldots, z\xi_s, z\eta_1 - \xi_1^2, z\eta_2, \ldots, z\eta_t].$$

One can readily verify that Φ is an automorphism of $\mathbb{B}^{1,s,t}$.

Example 2.2 Note $\mathbb{B}^{1,s,t}$ can be naturally identified with $\mathbb{B}^s \times \mathbb{C}^t$. Write ξ and η for the coordinates of $\mathbb{C}^s$ and $\mathbb{C}^t$, respectively. Let $f(\xi)$ be any holomorphic map from $\mathbb{C}^s$ to $\mathbb{C}^t$. Then the map $\Phi : (\xi, \eta) \to (\xi, \eta - f(\xi))$ is an automorphism of $\mathbb{B}^s \times \mathbb{C}^t$.

The following propositions will not be used to prove the other results in the paper. We include them anyway to describe the automorphisms of $\mathbb{B}^{r,s,t}$.

Proposition 2.3 *Let $r, s \in \mathbb{N}, t \in \mathbb{Z}^{\geq 0}$. Then every automorphism of $\mathbb{B}^{r,s,t}$ must preserve the metric $\omega_{\mathbb{B}^{r,s,t}}$.*

Proof First assume $r = 1$. Note $\mathbb{B}^{1,s,t}$ can be naturally identified with $\mathbb{B}^s \times \mathbb{C}^t$. Fix an automorphism Ψ of $\mathbb{B}^s \times \mathbb{C}^t$. Write z and ξ for the coordinates of $\mathbb{C}^s$ and $\mathbb{C}^t$, respectively. Write $\Psi = (G, H)$, where G has s components and H has t components. By Liouville's theorem, G only depends on z. Then by considering the inverse of Ψ, one can readily verify that G is an automorphism of $\mathbb{B}^s$. Consequently, Ψ preserves the metric $\omega_{\mathbb{B}^{1,s,t}}$.

Next let $r \geq 2$, and thus $r + s \geq 3$. By the extension theorem of Ivashkovich [19], every automorphism of $\mathbb{B}^{r,s,t}$ can be regarded as an element in $\mathcal{F}(\mathbb{B}^{r,s,t}, \mathbb{B}^{r,s,t})$. Then the conclusion follows from Remark 1.2. □

Proposition 2.4 *Let $r \geq 2, s \geq 1$ and $t \geq 0$. Then every automorphism of $\mathbb{B}^{r,s,t}$ is linear, and can be further identified with an element in $PU(r, s, t)$. Consequently,* $\mathrm{Aut}(\mathbb{B}^{r,s,t}) = PU(r, s, t)$.

Proof Let Ψ be an automorphism of $\mathbb{B}^{r,s,t}$. First by Proposition 2.3, Ψ is isometric with respect to the metric $\omega_{\mathbb{B}^{r,s,t}}$. Secondly by the same proof of Proposition 3.2 in [20], every automorphism of $\mathbb{B}^{r,s,t}$ must be rational. Then the linearity of Ψ follows from Lemma 2.11 (see Sect. 2.3). Write $\Psi([Z]) = [Z\mathcal{Y}]$ for some $m \times m$ matrix $\mathcal{Y}$. Since Ψ is a biholomorphism of $\mathbb{B}^{r,s,t}$, one can readily verify that $\mathcal{Y} \in GL(m, \mathbb{C})$. Finally, since Ψ maps $\mathbb{B}^{r,s,t}$ to $\mathbb{B}^{r,s,t}$, and maps $\partial\mathbb{B}^{r,s,t}$ to $\partial\mathbb{B}^{r,s,t}$, we must have $\mathcal{Y}E(r, s, t)\overline{\mathcal{Y}}^t = \lambda E(r, s, t)$ for some $\lambda \in \mathbb{R}^+$. By scaling $\mathcal{Y}$, we can assume $\lambda = 1$. Hence Ψ can be identified with an element in $PU(r, s, t)$. □

2.2 Description of Isometric Maps

In this subsection, we give a description of isometric maps between two generalized hyperbolic spaces which are possibly degenerate and indefinite. The result will be used in the proofs of Corollaries 1.5 and 1.6.

Theorem 2.5 *Let $r, s, R, S \in \mathbb{N}$ and $t, T \in \mathbb{Z}^{\geq 0}$. Let F be a holomorphic map from an open connected subset Ω of $\mathbb{B}^{r,s,t}$ to $\mathbb{B}^{R,S,T}$. Assume Ω is contained in the affine cell $U_0 = \{[z_0, \ldots z_{r+s+t-1}] \in \mathbb{P}^{r+s+t-1} : z_0 \neq 0\}$ and $F(\Omega)$ is contained in the affine cell $V_0 = \{[w_0, \ldots, w_{R+S+T-1}] \in \mathbb{P}^{R+S+T-1} : w_0 \neq 0\}$. Then the following are equivalent:*

(a) *F is isometric with respect to $\omega_{\mathbb{B}^{r,s,t}}$ to $\omega_{\mathbb{B}^{R,S,T}}$.*

(b) *After composing with appropriate elements in $PU(r, s, t)$ and $PU(R, S, T)$ from the right and the left, respectively, F locally (shrinking Ω if needed) equals to*

the following map in the standard affine coordinates on U_0 and V_0:

$$\zeta = (\zeta_1, \ldots, \zeta_{r+s+t-1}) \to (\zeta_1, \ldots, \zeta_{r-1}, \phi, \zeta_r, \ldots, \zeta_{r+s-1}, \psi, h).$$

Here ϕ, ψ and h are holomorphic maps in ζ with $R-r$, $S-s$ and T components, respectively, and satisfy $\|\phi\| \equiv \|\psi\|$.

In the above, $\|\cdot\|$ denotes the usual Euclidean norm, and the standard affine coordinates on U_0 are given by $\zeta_j = \frac{z_j}{z_0}$, $1 \le j \le r+s-1$, and likewise for the standard affine coordinates on V_0. We remark that if a map F as in (a) exists, then we must have $R \ge r$ and $S \ge s$. The proof of Theorem 2.5 is analogous to that of Theorem 2.1 in [17]. For the self-containedness, we sketch a proof here.

Proof of Theorem 2.5 It is easy to see (b) implies (a). It remains to show (a) implies (b). Let $F : \Omega \to \mathbb{B}^{R,S,T}$ be as (a). Write $p_0 = [1, 0, \ldots, 0] \in U_0$. By composing F with elements in $PU(r, s, t)$ and $PU(R, S, T)$, and shrinking Ω if necessary, we can assume that $p_0 \in \Omega$, $F(p_0) = [1, 0, \ldots, 0] \in V_0$, and $F(\Omega) \subset V_0$. To keep notions simple, we still denote the map by $F(\zeta) = (F_1(\zeta), \ldots, F_{R+S+T-1}(\zeta))$ in the standard affine coordinates of U_0 and V_0. By the isometry assumption, we have

$$\partial\bar\partial \log\left(1 + \sum_{i=1}^{R-1} |F_i|^2 - \sum_{i=R}^{R+S-1} |F_i|^2\right) = \partial\bar\partial \log\left(1 + \sum_{i=1}^{r-1} |\zeta_i|^2 - \sum_{i=r}^{r+s-1} |\zeta_i|^2\right).$$

Since now $F(0) = 0$, by a standard reduction, we get

$$\sum_{i=1}^{R-1} |F_i|^2 - \sum_{i=R}^{R+S-1} |F_i|^2 = \sum_{i=1}^{r-1} |\zeta_i|^2 - \sum_{i=r}^{r+s-1} |\zeta_i|^2.$$

By Proposition 2.2 in [17], $R \ge r$ and $S \ge s$. Moreover, there exists a matrix $\mathcal{A} \in GL(R+S-1; \mathbb{C})$ and two holomorphic maps ϕ, ψ with $R-r$ and $S-s$ components, respectively, such that

(1): $\mathcal{A}E(R-1, S)\overline{\mathcal{A}}^{\mathsf{T}} = E(R-1, S)$;
(2): $(F_1, \ldots, F_{R+S-1})\mathcal{A} = (\zeta_1, \ldots, \zeta_{r-1}, \phi, \zeta_r \ldots, \zeta_{r+s-1}, \psi)$;
(3): $\|\phi\| \equiv \|\psi\|$.

Write I_T for the $T \times T$ identity matrix. Set $\mathcal{X}$ to be the $(R+S+T)\times(R+S+T)$ block diagonal matrix diag $(1, \mathcal{A}, I_T)$. Then $\mathcal{X}E(R, S, T)\overline{\mathcal{X}}^{\mathsf{T}} = E(R, S, T)$. Therefore, $\mathcal{X}$ can be identified with an element in $PU(R, S, T)$. The composition of F with $\mathcal{X}$ has the desired form in (b) under standard affine coordinates. This finished the proof of Theorem 2.5. □

2.3 Proof of Theorem 1.1 and Corollaries

In this subsection, we prove Theorem 1.1, and Corollaries 1.5 and 1.6. We start with the proof of Theorem 1.1.

Proof of Theorem 1.1 We first observe the following fact.

Proposition 2.6 *Let $R, S, r, s \in \mathbb{N}$ and $t, T \in \mathbb{Z}^{\geq 0}$. Then every $H \in \mathcal{F}(\mathbb{B}^{r,s,t}, \mathbb{B}^{R,S})$ is isometric if and only if every $F \in \mathcal{F}(\mathbb{B}^{r,s,t}, \mathbb{B}^{R,S,T})$ is isometric.*

Proof Fix $H \in \mathcal{F}(\mathbb{B}^{r,s,t}, \mathbb{B}^{R,S})$. By shrinking the domain U of H if necessary, we assume U is contained in some affine cell of $\mathbb{P}^{r+s+t-1}$ and H is given by

$$H(w) = [H_1(w), \ldots, H_{R+S}(w)] \text{ on } U.$$

Here w is some affine coordinates of $\mathbb{P}^{r+s+t-1}$ on U, and the right-hand side of the above equation is in the homogeneous coordinates of $\mathbb{P}^{R+S-1}$. Define a map $\widetilde{H} : U \to \mathbb{P}^{R+S+T-1}$ by

$$\widetilde{H}(w) = [H_1(w), \ldots, H_{R+S}(w), 0, \ldots, 0] \text{ on } U,$$

where the right-hand side is in the homogeneous coordinates of $\mathbb{P}^{R+S+T-1}$. Since $H \in \mathcal{F}(\mathbb{B}^{r,s,t}, \mathbb{B}^{R,S})$, it is clear that $\widetilde{H} \in \mathcal{F}(\mathbb{B}^{r,s,t}, \mathbb{B}^{R,S,T})$. Furthermore, H is isometric if and only if $\widetilde{H}$ is so.

Conversely, fix $F \in \mathcal{F}(\mathbb{B}^{r,s,t}, \mathbb{B}^{R,S,T})$. By shrinking the domain V of F, we assume V is contained in some affine cell of $\mathbb{P}^{r+s+t-1}$ and F is given by

$$F(w) = [F_1(w), \ldots, F_{R+S+T}(w)] \text{ on } V.$$

Here w is some affine coordinates of $\mathbb{P}^{r+s+t-1}$ on V, and the right-hand side is in the homogeneous coordinates of $\mathbb{P}^{R+S+T-1}$. By shrinking V if necessary and dropping the last T components in the above, we obtain a new (well-defined) map $\widetilde{F}$ from V to $\mathbb{P}^{R+S-1}$. It is clear that $\widetilde{F} \in \mathcal{F}(\mathbb{B}^{r,s,t}, \mathbb{B}^{R,S})$. Furthermore, F is isometric if and only if $\widetilde{F}$ is so.

The conclusion in the proposition then follows from the above observations. □

Remark 2.7 We remark that if $\mathcal{F}(\mathbb{B}^{r,s,t}, \mathbb{B}^{R,S,T}) \neq \emptyset$, then we must have $R \geq r$ and $S \geq s$. Indeed, assume $F \in \mathcal{F}(\mathbb{B}^{r,s,t}, \mathbb{B}^{R,S,T})$. Then as in the proof of Proposition 2.6, we can construct a map $\widetilde{F} \in \mathcal{F}(\mathbb{B}^{r,s,t}, \mathbb{B}^{R,S})$. Furthermore, we can find a local holomorphic embedding $I : \mathbb{P}^{r+s-1} \to \mathbb{P}^{r+s+t-1}$ sending an open piece of $\partial\mathbb{B}^{r,s}$ to $\partial\mathbb{B}^{r,s,t}$, such that the composition $\hat{F} = \widetilde{F} \circ I \in \mathcal{F}(\mathbb{B}^{r,s}, \mathbb{B}^{R,S})$. By the existence of such a map $\hat{F}$, we apply Theorem 1.1 of [4] (or Lemma 4.1 in [3]) and a standard CR geometric argument (cf. Lemma 2.1 in [3]) to obtain $R \geq r$ and $S \geq s$.

To establish Theorem 1.1, by Proposition 2.6, it suffices to prove that, for $R, S, r, s, t \in \mathbb{N}$, every $F \in \mathcal{F}(\mathbb{B}^{r,s,t}, \mathbb{B}^{R,S})$ is isometric if and only if every $G \in \mathcal{F}(\mathbb{B}^{r,s}, \mathbb{B}^{R,S})$ is isometric. We note that the forward direction is easy:

Proposition 2.8 *Let $R, S, r, s, t \in \mathbb{N}$. If every $H \in \mathcal{F}(\mathbb{B}^{r,s,t}, \mathbb{B}^{R,S})$ is isometric, then every $G \in \mathcal{F}(\mathbb{B}^{r,s}, \mathbb{B}^{R,S})$ is isometric.*

Proof Write $[W] = [W_1, \ldots, W_{r+s}]$ for the homogeneous coordinates of $\mathbb{P}^{r+s-1}$. Write U_0 for the affine cell $\{[W] \in \mathbb{P}^{r+s-1} : W_1 \neq 0\}$. Fix $G \in \mathcal{F}(\mathbb{B}^{r,s}, \mathbb{B}^{R,S})$. By shrinking the domain $V \subseteq \mathbb{P}^{r+s-1}$ of G if necessary, we assume that $V \subseteq U_0$, and

that we still have $V \cap \partial\mathbb{B}^{r,s} \neq \emptyset$ and $V \cap \mathbb{B}^{r,s}$ is connected. Therefore, every point $[W]$ in V can be written in the form $[1, w]$ with w the affine coordinates of $\mathbb{P}^{r+s-1}$ on U_0. Assume

$$G([1, w]) = [G_1(w), \ldots, G_{R+S}(w)] \text{ on } V,$$

where the right-hand side is in the homogeneous coordinates of $\mathbb{P}^{R+S-1}$. We define an open subset $\widetilde{V}$ of $\mathbb{P}^{r+s+t-1}$ to be

$$\widetilde{V} = \{[1, w, \eta] \in \mathbb{P}^{r+s+t-1} : [1, w] \in V, \eta \in \mathbb{C}^t\}.$$

By the assumption on V , we have $\widetilde{V} \cap \partial\mathbb{B}^{r,s,t} \neq \emptyset$ and $\widetilde{V} \cap \mathbb{B}^{r,s,t}$ is connected. Define a map $\widetilde{G}$ from $\widetilde{V}$ to $\mathbb{P}^{R+S-1}$ by

$$\widetilde{G}([1, w, \eta]) = [G_1(w), \ldots, G_{R+S}(w)].$$

It is clear that $\widetilde{G} \in \mathcal{F}(\mathbb{B}^{r,s,t}, \mathbb{B}^{R,S})$. By assumption, $\widetilde{G}$ is isometric. But since the functions G'_js only depend on w, this implies G is also isometric. This finishes the proof. □

It remains to show the converse of Proposition 2.8.

Proposition 2.9 *Let $R, S, r, s, t \in \mathbb{N}$. If every $G \in \mathcal{F}(\mathbb{B}^{r,s}, \mathbb{B}^{R,S})$ is isometric, then every $H \in \mathcal{F}(\mathbb{B}^{r,s,t}, \mathbb{B}^{R,S})$ is isometric.*

Proof of Proposition 2.9 Write $[W] = [W_1, \ldots, W_{r+s+t}]$ for the homogeneous coordinates of $\mathbb{P}^{r+s+t-1}$. Write U_0 for the affine cell $\{[W] : W_1 \neq 0\}$ of $\mathbb{P}^{r+s+t-1}$. Write $p_0 = [1, \mathbf{0}_{r-1}, 1, \mathbf{0}_{s+t-1}] \in \partial\mathbb{B}^{r,s,t} \cap U_0$, where $\mathbf{0}_k$ denotes the k-dimensional zero row vector. Similarly, write V_0 for the affine cell of $\mathbb{P}^{r+s-1}$ consisting of points whose first component in homogeneous coordinates is nonzero.

Fix $H \in \mathcal{F}(\mathbb{B}^{r,s,t}, \mathbb{B}^{R,S})$. By shrinking the domain $\Omega \subseteq \mathbb{P}^{r+s+t-1}$ of H and composing H with an appropriate element in $PU(r, s, t)$, we can assume $p_0 \in \Omega$ and $\Omega \subseteq U_0$. Denote by $w = (z, \xi, \eta) \in \mathbb{C}^{r-1} \times \mathbb{C}^s \times \mathbb{C}^t$ the standard affine coordinates on $U_0 \approx \mathbb{C}^{r+s+t-1}$. That is, w is identified with the point $[1, w] \in U_0$. In the affine coordinates, $p_0 = (\mathbf{0}_{r-1}, 1, \mathbf{0}_{s+t-1})$.

Write $\Delta(0, \epsilon)$ for the open disk in $\mathbb{C}$ centered at 0 with radius $\epsilon > 0$. Write $\Delta^k(0, \epsilon) = \Delta(0, \epsilon) \times \cdots \times \Delta(0, \epsilon)$ for the polydisk in $\mathbb{C}^k$. To make the argument simpler, we further shrink Ω to be a polydisk centered at p_0 of the form $\{w \in U_0 : w - p_0 \in \Delta^{r+s+t-1}(0, \epsilon)\}$ for some small $\epsilon > 0$ (one can easily verify that $\Omega \cap \mathbb{B}^{r,s,t}$ is still connected).

Recall (z, ξ, η) denotes the standard affine coordinates on U_0. Fix a point $p = (\widetilde{z}, \widetilde{\xi}, \widetilde{\eta}) \in \Omega \cap \mathbb{B}^{r,s,t} \subseteq U_0$. Write $z = (z_1, \ldots, z_{r-1})$ and correspondingly $\widetilde{z} = (\widetilde{z}_1, \ldots, \widetilde{z}_{r-1})$. Similar notations apply to $\xi, \widetilde{\xi}$ as well as $\eta, \widetilde{\eta}$. Next we fix row vectors $\lambda_1, \ldots, \lambda_{r-1}, \mu_1, \ldots, \mu_s \in \mathbb{C}^t$ and denote by (z, ξ) the standard affine coordinates on V_0. We define a canonical embedding L (depending on p and

$\lambda_1, \ldots, \lambda_{r-1}, \mu_1, \ldots, \mu_s)$ from $V_0 \approx \mathbb{C}^{r+s-1}$ to $U_0 \approx \mathbb{C}^{r+s+t-1}$ as follows:

$$L(z,\xi) = \left(z, \xi, \widetilde{\eta} + \sum_{i=1}^{r-1} (z_i - \widetilde{z}_i)\lambda_i + \sum_{j=1}^{s} (\xi_j - \widetilde{\xi}_j)\mu_j \right). \tag{2.1}$$

It is clear that $L(V_0 \cap \mathbb{B}^{r,s}) \subseteq \mathbb{B}^{r,s,t}$ and $L(V_0 \cap \partial\mathbb{B}^{r,s}) \subseteq \partial\mathbb{B}^{r,s,t}$. Moreover, L preserves the metric: $L^*(\omega_{\mathbb{B}^{r,s,t}}) = \omega_{\mathbb{B}^{r,s}}$ on $V_0 \cap \mathbb{B}^{r,s}$.

We truncate p to get $q \in V_0 : q = (\widetilde{z}, \widetilde{\xi})$. It is clear that $q \in \mathbb{B}^{r,s}$ and $L(q) = p$. Similarly we truncate p_0 to get $q_0 = (\mathbf{0}_{r-1}, 1, \mathbf{0}_{s-1}) \in V_0 \cap \partial\mathbb{B}^{r,s}$. Set $\mathcal{N}$ to be the projection of Ω :

$$\mathcal{N} := \left\{ (z,\xi) \in V_0 : (z,\xi) - q_0 \in \Delta^{r+s-1}(0,\epsilon) \right\}.$$

One can readily verify that $q \in \mathcal{N} \cap \mathbb{B}^{r,s}$ and $\mathcal{N} \cap \mathbb{B}^{r,s}$ are connected. Furthermore, if we choose vectors $\lambda_1, \ldots, \lambda_{r-1}, \mu_1, \ldots, \mu_s \in \mathbb{C}^t$ with sufficiently small norm, then we have $L(\mathcal{N}) \subset \Omega$. Consequently, $H \circ L$ is well defined on $\mathcal{N}$ and thus gives an element in $\mathcal{F}(\mathbb{B}^{r,s}, \mathbb{B}^{R,S})$. By the assumption of Proposition 2.9, $H \circ L$ is isometric. This yields $(H \circ L)^*(\omega_{\mathbb{B}^{R,S}}) = \omega_{\mathbb{B}^{r,s}} = L^*(\omega_{\mathbb{B}^{r,s,t}})$ on $\mathcal{N} \cap \mathbb{B}^{r,s}$. This further implies

$$L^*\left(H^*(\omega_{\mathbb{B}^{R,S}}) - \omega_{\mathbb{B}^{r,s,t}}\right) = 0 \text{ on } \mathcal{N} \cap \mathbb{B}^{r,s}. \tag{2.2}$$

We pause to prove the following lemma.

Lemma 2.10 *The Hermitian* $(1,1)$*-form* $\omega := H^*(\omega_{\mathbb{B}^{R,S}}) - \omega_{\mathbb{B}^{r,s,t}}$ *equals* 0 *at* p.

Proof of Lemma 2.10 To make notations simple, we also write the coordinates $w = (z, \xi, \eta)$ as $(w_1, \ldots, w_{r+s+t-1})$. That is, we identify w_i with z_j, ξ_k and η_l accordingly. Write $n = r+s-1$, $m = r+s+t-1$ and write $\omega = \sqrt{-1}\sum_{1\le i,j\le m} g_{i\bar{j}}(w) dw_i \wedge d\overline{w}_j$ in $\Omega \cap \mathbb{B}^{r,s,t}$ with $\mathcal{G}(w) := (g_{i\bar{j}}(w))_{1\le i,j\le m}$ a Hermitian matrix-valued real analytic function. By (2.2), in particular $L^*(\omega) = 0$ at q. Recall $L(q) = p$. A standard calculation of the pull-back form $L^*(\omega)$ at q yields:

$$\mathcal{L}\mathcal{G}(p)\overline{\mathcal{L}}^{\mathsf{T}} = 0. \tag{2.3}$$

Here $\mathcal{L}$ is the complex Jacobian matrix of L, which by (2.1) has the following expression.

$$\mathcal{L} = (I_n, \mathcal{A}), \text{ where the } n \times t \text{ matrix } \mathcal{A} = \begin{pmatrix} \lambda_1 \\ \cdots \\ \lambda_{r-1} \\ \mu_1 \\ \cdots \\ \mu_s \end{pmatrix}. \tag{2.4}$$

By the discussion preceding to Lemma 2.10, (2.3) holds for $\mathcal{L}$ as in (2.4) if we choose vectors $\lambda_1, \ldots, \lambda_{r-1}, \mu_1, \ldots, \mu_s \in \mathbb{C}^t$ with sufficiently small norm. We then see (2.3)

holds for any choices of vectors $\lambda_1, \ldots, \lambda_{r-1}, \mu_1, \ldots, \mu_s \in \mathbb{C}^t$ by the analyticity of the left-hand side of (2.3).

Recall $M(n, m; \mathbb{C})$ denotes the space of $n \times m$ matrices with complex entries. Now for a generic element $\mathcal{Y} \in M(n, m; \mathbb{C}) \approx \mathbb{C}^{nm}$, there exists some $\mathcal{Z} \in GL(n, \mathbb{C})$ such that $\mathcal{Y} = \mathcal{Z}\mathcal{L}$ for some matrix $\mathcal{L} \in M(n, m; \mathbb{C})$ in the form of (2.4). Consequently,

$$\mathcal{Y}\mathcal{G}(p)\overline{\mathcal{Y}}^{\mathsf{T}} = \mathcal{Z}\mathcal{L}\mathcal{G}(p)\overline{\mathcal{L}}^{\mathsf{T}}\overline{\mathcal{Z}}^{\mathsf{T}} = 0.$$

It then follows again from the analyticity that $\mathcal{Y}\mathcal{G}(p)\overline{\mathcal{Y}}^{\mathsf{T}} = 0$ for every $\mathcal{Y} \in M(n, m; \mathbb{C})$. This yields $\mathcal{G}(p) = 0$ and therefore $\omega = 0$ at p. This proves the lemma. □

By Lemma 2.10, since p can be any point in $\Omega \cap \mathbb{B}^{r,s,t}$, we obtain $\omega = 0$ in $\Omega \cap \mathbb{B}^{r,s,t}$. This proves H is isometric and we thus establish Proposition 2.9. □

Theorem 1.1 now follows from Propositions 2.6, 2.8 and 2.9. □

We finally prove Corollaries 1.5 and 1.6. For that, we establish the following lemma.

Lemma 2.11 *Let $R, S, r, s \in \mathbb{N}$, and $t, T \in \mathbb{Z}^{\geq 0}$. Let F be a rational proper map from $\mathbb{P}^{r+s+t-1}$ to $\mathbb{P}^{R+S+T-1}$ with I its set of indeterminacy. If $r \geq 2$ and F is an isometric map from $(\mathbb{B}^{r,s,t} \setminus I, \omega_{\mathbb{B}^{r,s,t}})$ to $(\mathbb{B}^{R,S,T}, \omega_{\mathbb{B}^{R,S,T}})$, then F is a linear map from $\mathbb{P}^{r+s+t-1}$ to $\mathbb{P}^{R+S+T-1}$.*

Proof of Lemma 2.11 The proof is very similar to that of Lemma 2.5 in [18]. For convenience of the readers, we sketch a proof here. Recall $[Z] = [z, \xi, \eta]$ denotes the homogeneous coordinates on $\mathbb{P}^{r+s+t-1}$, where z, ξ, η have r, s, t components, respectively.

By Theorem 2.5, we conclude, by composing elements in $PU(r, s, t)$ and $PU(R, S, T)$, F equals to the map: $[Z] = [z, \xi, \eta] \to [z, \phi, \xi, \psi, h]$, where ϕ, ψ, and h are holomorphic maps in Z with $R - r$, $S - s$, and T components, respectively, and satisfy $\|\phi\| = \|\varphi\|$ at points where they are defined. Moreover, by the rationality assumption, ϕ, ψ, and h are rational maps in $Z = (z, \xi, \eta)$.

Thus we can write $\phi = \frac{p_1}{q}, \psi = \frac{p_2}{q}, h = \frac{p_3}{q}$. Here p_1, p_2, ad p_3 are polynomial maps in Z, such that their nonzero components are all homogeneous polynomials with the same degree. And $q \neq 0$ is also a homogeneous polynomial maps in Z. They satisfy the following conditions:

(A) $\|p_1(Z)\| = \|p_2(Z)\|, \quad \forall Z \in \mathbb{C}^{r+s+t}$;

(B) p_1, p_2, p_3 and q have only trivial common factors;

(C) For each $1 \leq i \leq 3$, $\deg q = \deg p_i - 1$ if p_i is not identically zero.

Write $z = (z_1, \ldots, z_r)$ and $\xi = (\xi_1, \ldots, \xi_s)$, and rewrite F as

$$F([Z]) = [z_1 q, \ldots, z_r q, p_1, \xi_1 q, \ldots, \xi_s q, p_2, p_3]. \tag{2.5}$$

Note the set of indeterminacy I of F satisfies

$$I \subseteq \{[Z] \in \mathbb{P}^{r+s+t-1} : p_1(Z) = p_2(Z) = p_3(Z) = q(Z) = 0\} \cup \{[Z] \in \mathbb{P}^{r+s+t-1} : z = 0, \xi = 0\}.$$

Note I is of codimension at least 2 in $\mathbb{P}^{r+s+t-1}$. We claim q is a constant function. Otherwise, since $r \geq 2$, we can find a point $[Z^*] = [z_0^*, z_1^*, 0, \ldots, 0] \in \mathbb{B}^{r,s,t}$ such that $q(Z^*) = 0$. Since I is of codimension at least 2, we can find a point $[\tilde{Z}] \in \mathbb{B}^{r,s,t}$ close to $[Z^*]$ such that $q(\tilde{Z}) = 0$, and $[\tilde{Z}] \notin I$. By Eq. (2.5), $F([\tilde{Z}]) \in \partial\mathbb{B}^{R,S,T}$. This contradicts the definition of rational proper maps from $\mathbb{B}^{r,s,t}$ to $\mathbb{B}^{R,S,T}$. Hence q must be constant. Consequently, either $\deg p_i = 1$, or p_i is identically zero for each $1 \leq i \leq 3$. Therefore, F is linear. □

Proof of Corollaries 1.5 and 1.6 Corollary 1.5 follows from Proposition 1.3 and Lemma 2.11. Similarly, Corollary 1.6 follows from Proposition 1.4 and Lemma 2.11. □

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Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

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