Revisit to a non-degeneracy property for extremal mappings

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Dedicated to the memory of Professor Jia-Ru Yu

Abstract

We extend an earlier result obtained by author in [Hua2].

1 Introduction

Let D be a bounded domain in $\mathbb{C}^{n+1}$ with $n \geq 1$. An extremal map ϕ of D is a holomorphic map from the unit disk $\Delta := \{\xi \in \mathbb{C} : \ |\xi| < 1\}$ into D such that the Kobayashi metric of D at $\phi(0)$ along the direction $\phi'(0)$ is realized by ϕ. We say ϕ is a complex geodesic if it realizes the Kobayashi distance between any two points in $\phi(\Delta)$ ([Ab2]).

Fundamental work on extremal maps and complex geodesics has been done in the earlier 1980s by Lempert [Lem1-2], Poletsky [Pol], Abate[Ab], etc. Among many other things, Lempert showed that extremal maps of a bounded strongly convex domain with a $C^{2,\alpha}$-smooth boundary are complex geodesics. Poletsky [Pol] showed that extremal maps of a bounded pseudo-convex domain with a reasonably smooth boundary, say C^1-smooth, are almost proper. Both Lempert [Lem1] and Poletsky [Pol] derived the Euler-Lagrange equations for extremal maps in their considerations. In his paper in the earlier 1990s [Hua1] [Hua2], motivated by

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the Abate-Vensentini problem on establishing the Wolff-Denjoy iteration theory for bounded contractible strongly pseudo-convex domains, the author proved two localization theorems for extremal maps of pseudoconvex domains near a C^3-smooth strongly pseudoconvex point, which was fundamentally used to resolving the Abate-Vensentini conjecture in [Hua2]. These localization results have also found many other applications later, e.g., in solving the homogeneous Monge-Ampére equstions with a prescribed boundary singularity. (See [BP][BPT][HW].)

In this note, we extend the non-degeneracy property obtained in [Hua2] to a strongly psuedoconvex point with only $C^{2,\alpha}$-regularity for $\alpha \in (0, 1]$.

Before stating our result, we need to introduce some notations. Let D be a bounded domain with a C^2-smooth boundary near $p \in \partial D$. For any point $z \in D$ near p, we have a unique boundary point of D near p, denoted by $\pi(z)$ such that $|z-\pi(z)|$ is precisely the distance from z to ∂D. Then for any $v \in T_z^{(1,0)}D$, we use v_{tan} and v_{nor} to denote the tangential and complex normal component of v at $\pi(z)$, respectively. Namely, $v = v_{tan} + v_{nor}$ with $v_{tan} \in T_{\pi(z)}^{(1,0)}\partial D$ and $v_{nor} \perp v_{tan}$.

Theorem 1.1. *Let $D \subset \mathbb{C}^{n+1}$ be a bounded domain with p a $C^{2,\alpha}$-strongly pseudoconvex boundary point of D, where $\alpha \in (0, 1]$. Then there are two positive numbers $\varepsilon(p)$ and $\delta(p)$ with $0 < \epsilon(p) < \delta(p) << 1$ such that for any $p^* \in \partial\Omega$ with $|p^* - p| < \epsilon(p)$ and for any extremal disk φ of D with $|\varphi(\xi) - p| < \delta(p)$ for any $\xi \in \Delta$ it holds that*

$$|(\varphi'(\xi))_{nor}| \leq C\eta^{\alpha}(\varphi, p^*)|(\varphi'(\xi))_{tan}| \quad \forall\ \xi \in \overline{\Delta}.$$

Here $\eta(\varphi, p^*) := \max_{\xi\in\overline{\Delta}} |\varphi(\xi) - p^*|$, C is a constant independent of $\xi \in \Delta, p^*$ and φ.

Theorem 1.1 was proved in [Hua2] when $\alpha = 1$. Our proof here is very similar to that in [Hua2]. If we make the φ in Theorem 1.1 sufficiently small such that $|\varphi(\xi) - p| < \epsilon(p)$ for $\xi \in \Delta$ and apply Theorem 1.1 with $p^* = \varphi(1)$, noticing that we now have $\eta(\varphi, \varphi(1)) \approx \operatorname{diam}(\varphi) := \max_{\xi_1,\xi_2\in\Delta} |\varphi(\xi_1) - \varphi(\xi_2)|$, we arrive at the following:

Corollary 1.2. *Let $D \subset \mathbb{C}^{n+1}$ be a bounded domain with p being a $C^{2,\alpha}$-strongly pseudoconvex boundary point of D, where $\alpha \in (0, 1]$. Then there is a small positive number $\varepsilon(p)$ such that for any extremal map φ of D with $|\varphi(\xi) - p| < \epsilon(p)$ for any $\xi \in D$*

$$|(\varphi'(\xi))_{nor}| \leq C\mathrm{diam}^{\alpha}(\varphi)(\varphi'(\xi))_{\mathrm{tan}}| \quad \forall\ \xi \in \overline{\Delta}.$$

Here C is a constant independent of $\xi \in \Delta$ and φ.

2 Proof of Theorem 1.1

We basically follow the argument presented in [Hua2]. We include enough details to facilitate a reader's reading.

Let D be a bounded domain in $\mathbb{C}^{n+1}$ with $p \in \partial D$ a $C^{2,\alpha}$-smooth boundary. Here, $0 < \alpha \leq 1$. Namely, there is an open neighborhood U of $p \in \mathbb{C}^{n+1}$ and a $C^{2,\alpha}$-smooth function ρ over $\overline{U}$ such that $D \cap U = \{\rho < 0\}$ and $d\rho|_{\partial D \cap U} \neq 0$. For instance, when $\alpha = 1$, we require any second derivative of ρ is Lipschitz continuous over $\overline{U}$. Since there is a strongly convex subdomain of Ω (after a biholomorphic change of coordinates) which shares a piece of boundary near p with D, by the monotonicity property of Kobayashi metrics we conclude that there exists a small neighborhood U_p of p in $\mathbb{C}^{n+1}$ such that any extremal map φ of D with $\varphi(\Delta) \subset U_p$ is a complex geodesic of Ω. Moreover φ satisfies the Euler-Langrange equation in the sense of Lempert-Polestky. Namely, there exist a $p(\xi) \in C^{1,\alpha^-}(\partial\Delta)$ with $p(\xi) > 0$ such that $\widetilde{\varphi} = \xi p(\xi)\overline{\nu(\varphi(\xi))}$ extends to a holomorphic map over Δ with $\widetilde{\varphi} \in C^{1,\alpha^-}(\overline{\Delta})$. Here $\nu(q)$ is the outward unit vector of $\partial\Omega$ at q; $C^{1,\alpha^-}(\overline{\Delta}) = C^{1,\alpha}(\overline{\Delta})$ for $\alpha \in (0,1)$ and $C^{1,1^-}(\overline{\Delta}) = \bigcap_{\alpha\in(0,1)} C^{1,\alpha}(\overline{\Delta})$. Now, shrinking U_p if needed, for any $p^* \approx p$, we have a quadratic holomorphic polynomial change of coordinates $\Psi(\cdot; p^*)$ defined over U_p that maps p to 0 and $M_{p^*} = \Psi(\partial\Omega \cap U_p; p^*)$ near 0 is defined by an equation of the form:

$$\rho_{p^*} = \overline{z}_{n+1} + z_{n+1} + \sum_{j=1}^{n} |z_j|^2 + o(|z|^2). \tag{2.1}$$

Here ρ_{p^*} depends $C^{2,\alpha}$-smoothly on $p^* \approx p$.

The unit outer normal vector in the normal coordinates as in (2.1) is given by $\nu_{p^*} = (\nu_1, \cdots, \nu_{n+1})$ with $\nu_j = \frac{\frac{\partial \rho_{p^*}}{\partial \overline{z}_j}}{|\nabla_z \rho|}$ along M for $j = 1, \cdots, n+1$. Hence, $\nu_j = z_j + o(|z|),\ j = 1, \cdots, n,\ \nu_{n+1} = 1 + o(|z|)$. The Webster surface W of M near $p = 0$ is given by

$$W = \{w = (z, \omega) \in \mathbb{C}^{2n+1} : z \in M, z \simeq 0\},$$

and $\omega = (\frac{\overline{\nu}_1(z)}{\overline{\nu}_{n+1}(z)}, \cdots, \frac{\overline{\nu}_n(z)}{\overline{\nu}_{n+1}}(z))$ or $w = (z', iy_{n+1}, \overline{z}') + o(|z|)$. Then T_pW at $p = 0$ is spanned by $T_{1,r}, \cdots, T_{n,r}, T_{n+1}, T_{1,i}, \cdots, T_{n,i}$ where

$$T_{j,r} = (0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0), T_{j,i} = (0, \cdots, 0, i, \cdots, -i, 0, \cdots, 0).$$

Write

$$A_0 = \begin{pmatrix} T_{1,r} \\ \vdots \\ T_{n+1} \\ \vdots \\ T_{n,i} \end{pmatrix} = \begin{pmatrix} I_n & 0 & I_n \\ 0 & i & 0 \\ iI_n & 0 & -iI_n \end{pmatrix}. \tag{2.2}$$

For $\xi \in \partial\Delta$, we have

$$\Phi = (\varphi(\xi), \varphi^*(\xi)) = \left(\varphi(\xi), \frac{\overline{v}_1(\varphi)}{\overline{v}_{n+1}(\varphi)}, \cdots, \frac{\overline{v}_n(\varphi)}{\overline{v}_{n+1}(\varphi)}\right).$$

Replacing φ by $\varphi \circ \sigma$ for a suitable $\sigma \in \mathrm{Aut}(\Delta)$ such that φ^* has only a simple pole at 0. Write $\varphi^* = \frac{\varphi^{**}}{\xi}$. Then

$$\varphi^{**}(0) = \frac{1}{2\pi}\int_0^{2\pi} \varphi^*(e^{i\theta})d\theta = O(\|\varphi\|_{C^0}).$$

where $\|\varphi\|_{C^0} := \sup_\Delta |\varphi(\xi)| = \sup_{\xi\in\partial\Delta} |\varphi(\xi)|$. Write $W^* = W \cdot A_0^{-1} = \{w \cdot A_0^{-1} : w \in W\}$ which is a $C^{1,\alpha}$-regular totally real submaifold in $\mathbb{C}^{2n+1}$ near the origin. Then W^* is defined near 0 by $Y = H(X)$ with $X + iY \in \mathbb{C}^{2n+1}$ where $H(0) = 0, d_0H = 0$ and H is $C^{1,\alpha}$-smooth near 0.

Consider the following Riemann-Hilbert problem:

$$\begin{cases} \mathrm{Im}\{Q(\mathrm{X}) \cdot (I + i\frac{\partial H}{\partial X})^{-1}(X)\} = 0, \\ \mathrm{Re}(Q(\mathrm{X})(0)) = I_{2n+1}. \end{cases} \tag{2.3}$$

where $\mathrm{X} \in C^{\frac{1}{2}}(\partial\Delta)$, $Q(\mathrm{X})(\cdot)$ is a $(2n+1)\times(2n+1)$ holomorphic matrix in Δ depending on X belonging to the Hardy space $H^4(\Delta)$, which thus has L^4-integrable boundary value. Write $\mathrm{X} = (\mathrm{X}_1, \cdots, \mathrm{X}_{2n+1})$ and $H = (h_1, \cdots, h_{2n+1})$. Then

$$\frac{\partial H}{\partial X} = \begin{pmatrix} \frac{\partial h_1}{\partial \mathrm{X}_1} & \cdots & \frac{\partial h_{2n+1}}{\partial \mathrm{X}_1} \\ \cdots & \cdots & \cdots \\ \frac{\partial h_1}{\partial \mathrm{X}_{2n+1}} & \cdots & \frac{\partial h_{2n+1}}{\partial X_{2n+1}} \end{pmatrix}.$$

For $0 < \varepsilon \ll 1$, we write

$$\mathcal{B}_\varepsilon = \{X(\xi) \in C^{\frac{1}{2}}(\partial\Delta) : \|X\|_{C^{\frac{1}{2}}} < \varepsilon\}.$$

For $X_0^* \in \mathbb{R}^{2n+1}$ with $|X_0^*| \leq \frac{1}{2}\epsilon$, we write $\mathcal{B}_\varepsilon(X_0^*) = X_0^* + \mathcal{B}_\varepsilon$. In what follows, for each $X \in \mathcal{B}_\varepsilon(X_0^*)$, we write $X = X_0^* + \hat{X}$ with $\hat{X} \in \mathcal{B}_\varepsilon$.

Write $Q = q_1 + iq_2$, $\left(I + i\frac{\partial H}{\partial X}\right)^{-1} \circ X = e_1(X) + ie_2(X)$. Then $\|e_1(X) - I\|_{C^{\frac{\alpha}{2}}}, \|e_2(X)\|_{C^{\frac{\alpha}{2}}} \lesssim C\epsilon^\alpha$ for $X \in \mathcal{B}_\varepsilon(X_0^*)$ with $\varepsilon \ll 1$. Here and in what follows, C stands for a constant, independent of ξ, X_0^* and X that may be different in different contexts. Write S for the standard Hilbert transform with $u + iS(u)$ having a holomorphic extension to an element in the Hardy space $H^4(\Delta)$ whose imaginary part has 0 value at 0 for any $u \in L^4(\partial\Delta)$. Then $q_1 = -S(q_2) + I$. Notice that S is a bounded self-operator acting both on $L^4(\partial\Delta)$ and $C^{\frac{\alpha}{2}}(\partial\Delta)$. Hence (2.3) is reduced to

$$q_2(X) - S(q_2(X))h(X) = h(X). \tag{2.4}$$

Here $h(X) = -(e_2 \cdot e_1^{-1}) \circ X$. Notice that the solution of (2.4) is $h(X_0^*)$ when $X = X_0^*$, for the Hilbert transform maps a constant function to 0. Now for $X \in \mathcal{B}_\varepsilon(X_0^*)$ with $0 < \varepsilon \ll 1$, we can also solve uniquely (2.4) to get q_2 and thus Q in the $C^{\frac{\alpha}{2}}(\partial\Delta)$-space with

$$\|q_2(X) - h(X_0^*)\|_{C^{\frac{\alpha}{2}}} \leq C\|\hat{X}\|^\alpha_{C^{\frac{1}{2}}}.$$

This is because $\|h(\hat{X})\|_{C^{\frac{\alpha}{2}}} \lesssim \|\hat{X}\|^\alpha_{C^{\frac{1}{2}}}$. If $X \in C^{\frac{1}{2}}(\partial\Delta)$ only with $\|X\|_{C^0} \ll 1$ we can still solve (2.4) in the $L^4(\partial\Delta)$-space to get $q_2(X)$ with the estimate $\|q_2(X) - q_2(X_0^*)\|_{L^4} \leq C\|\hat{X}\|^\alpha_{C^0}$. By the uniqueness, when $\epsilon << 1$, these two solutions are the same. We can similarly consider the Riemann-Hilbert problem:

$$\begin{cases} \mathrm{Im}\{(I + i\frac{\partial H}{\partial X})\widehat{Q}(X)\} = 0 \\ \mathrm{Re}(\widehat{Q}(X)(0)) = I_{2n+1}. \end{cases} \tag{2.5}$$

We similarly solve it in the $H^4(\Delta)$-space when $X \in C^{\frac{1}{2}}(\Delta)$ with $\|X\|_{C^0} \ll 1$. We can also solve it in the $C^{\frac{\alpha}{2}}(\overline{\Delta})$-space when $X \in \mathcal{B}_\varepsilon(X_0^*)$ with $0 < \varepsilon \ll 1$. Notice that $\mathrm{Im}(\mathrm{Q(X)}\widehat{\mathrm{Q}}(\mathrm{X})) = 0$. Since $Q(X)\widehat{Q}(X) \in H^2(\Delta)$. Then by applying the reflection principle, one easily sees that it is a constant invertible matrix depending on X. Hence, we have

$$Q^{-1}(X) = C^{*-1}(X)\widehat{Q}(X)$$

with $C^* = Q(X)(0)\widehat{Q}(X)(0) \approx I$ for $X \in C^{\frac{1}{2}}(\partial\Delta)$ and $\|X\|_{C^0} << 1$. Indeed, $|C^* - I| \lesssim C\|X\|^\alpha_{C^0}$ when $\|X\|_{C^0} \ll 1$. Summarizing the above, we have

Proposition 2.1. *Let $X \in C^{\frac{1}{2}}(\partial\Delta)$. When $\|X\|_{C^0}$ is sufficiently small, (2.3) can be uniquely solved with $Q(X) \in \mathcal{H}^4(\Delta)$,*

$$\|Q(X) - (I + ih(X_0^*))\|_{L^4(\partial\Delta)} \le C\|\hat{X}\|_{C^0}^{\alpha}, \|Q^{-1}(X) - (I + ih(X_0^*))^{-1}\|_{L^4(\partial\Delta)} \le C\|\hat{X}\|_{C^0}^{\alpha}.$$

When $X \in \mathcal{B}_\varepsilon(X_0^*)$ with $0 < \varepsilon \ll 1$, (2.3) can be solved uniquely with $Q \in C^{\frac{\alpha}{2}}(\overline{\Delta})$ with

$$\|Q(X) - (I + ih(X_0^*))\|_{C^{\frac{\alpha}{2}}}, \|Q^{-1}(X) - (I + ih(X_0^*))^{-1}\|_{C^{\frac{\alpha}{2}}} \le C\varepsilon^{\alpha}.$$

These two solutions are the same for $\epsilon << 1$.

Now, back to our extremal map φ, we write

$$\Phi^* = \Phi \cdot A_0^{-1} = X + iY$$

along $\partial\Delta$. Notice that $X, Y \in C^{1,\alpha}(\partial\Delta)$ with $\|X\|_{C^0(\overline{\Delta})} << 1$. Then since Φ^* is attached to W^*, we get $Y = H(X)$ along $\partial\Delta$ and we have

$$\frac{d\Phi^*}{d\theta} = \frac{dX}{d\theta} + i\frac{dY}{d\theta} = \frac{dX}{d\theta}(I_{2n+1} + i\frac{\partial H}{\partial X})$$

or

$$\mathrm{Im}\left\{\frac{d\Phi^*}{d\theta}(I_{2n+1} + i\frac{\partial H}{\partial X})^{-1}\right\} = \mathrm{Im}\frac{dX}{d\theta} = 0.$$

Hence

$$\mathrm{Im}\left\{\frac{d\Phi^*}{d\theta} \cdot Q^{-1}(X) \cdot Q(X)(I_{2n+1} + i\frac{\partial H}{\partial X})^{-1}\right\} = 0$$

or

$$\mathrm{Im}\left\{\frac{d\Phi^*}{d\theta} \cdot Q^{-1}(X)\right\} = 0$$

along $\partial\Delta$. Since $\frac{d\Phi^*}{d\theta} = \frac{d\Phi^*}{d\xi} \cdot i\xi$ and $Q, Q^{-1} \in H^4(\Delta)$, we obtain by the reflection principle that

$$\xi\frac{d\Phi^*}{d\xi}Q^{-1}(X)(\xi) = \frac{\alpha(X)}{\xi} - \overline{\alpha(X)}\xi + i\beta(X). \tag{2.6}$$

Here,

$$\alpha(X) = \lim_{\xi\to 0}\xi^2\frac{d\Phi^*}{d\xi}Q^{-1}(X)(\xi) = (0, -\varphi^{**}(0))A_0^{-1}Q^{-1}(X)(0).$$

Write $R(X)(\xi) = Q(X)(\xi)(I_{2n+1} + i\frac{\partial H}{\partial X})^{-1}$, $\gamma = -\varphi^{**}(0) = O(|X_0^*| + \|\hat{X}\|)$. Notice that

$$\frac{d\Phi^*}{d\theta} Q^{-1}(X)(\xi) = \frac{dX}{d\theta} R^{-1}(X).$$

We get the following equation for X from (2.6)

$$\frac{dX}{d\theta} = i(\frac{\alpha(X)}{\xi} - \overline{\alpha(X)}\xi + i\beta(X))R(X)(\xi), \xi = e^{i\theta}. \tag{2.7}$$

Note that

$$\int_0^{2\pi} R(X) d\theta = 2\pi + O(\|\hat{X}\|_{C^0}^{\alpha})$$

and β is determined by $\int_0^{2\pi} \frac{dX}{d\theta} d\theta = 0$ or

$$\beta = \frac{i\int_0^{2\pi} (\frac{\alpha(X)}{\xi} - \overline{\alpha(X)}\xi)R(X)(\xi)d\theta}{\int_0^{2\pi} R(X)(e^{i\theta})d\theta}.$$

By the formula for α and β, we have

$$|\alpha(X) - \alpha(X_0^*)|, |\beta(X) - \beta(X_0^*)| \leq C\|\hat{X}\|_{C^0}.$$

Denote $X(1) = X_0$. Then (2.7) can be written as

$$X(e^{i\theta}) = i\int_0^{\theta} (\frac{\alpha(X)}{\xi} - \overline{\alpha(X)}\xi + i\beta(X))R(X)(e^{i\theta})d\theta + X_0 \tag{2.8}$$

with $|\alpha(X)|, |\beta(X)| \leq C\|X\|_{C^0}$, $\|R(X) - I\|_{L^2(\partial\Delta)}| \lesssim \|R(X) - I\|_{L^4(\partial\Delta)} \leq C\|X\|_{C^0}^{\alpha}$. By the Hölder inequality, we get

$$\begin{aligned}
|\hat{X}(e^{i\theta_1}) - \hat{X}(e^{i\theta_2})| &\leq \int_{\theta_1}^{\theta_2} |(\frac{\alpha(X)}{\xi} - \overline{\alpha(X)}\xi + i\beta(X))R(X)(\xi)|d\theta \\
&\leq C\|X\|_{C^0} \int_{\theta_1}^{\theta_2} |R(X)|d\theta \\
&\leq C\|X\|_{C^0(\partial\Delta)}|\theta_2 - \theta_1|^{\frac{1}{2}}\|R(X)\|_{L^2(\partial\Delta)} \\
&\leq C\|X\|_{C^0}^{\alpha}|\theta_2 - \theta_1|^{\frac{1}{2}}.
\end{aligned}$$

Also, $|X(e^{i\theta})| \leq |X_0| + \|X\|_{C^0} \leq 2\|X\|_{C^0(\partial\Delta)}$. We thus get

$$\|\hat{X}\|_{C^{\frac{1}{2}}(\partial\Delta)} \leq C\|\hat{X}\|_{C^0}^{\alpha}$$

with C independent of X when $\|X\|_{C^0} \ll 1$. Hence, for any $0 < \varepsilon \ll 1$, when $\|X\|_{C^0}$ is sufficiently small, $X \in \mathcal{B}_\varepsilon$. Thus by Proposition 2.1,

$$\|Q(X) - I\|_{C^{\frac{\alpha}{2}}} \leq C\|X\|^{\alpha}_{\frac{1}{2}}, \ \|Q^{-1}(X) - I\|_{C^{\frac{\alpha}{2}}} \leq C\|X\|^{\alpha}_{\frac{1}{2}}, \ \|R(X) - I\|_{C^{\frac{\alpha}{2}}} \leq C\|X\|^{\alpha}_{\frac{1}{2}}.$$

With these estimates under our disposal, we can now complete the proof of Theorem 1.1 as follows:

Proof. Let B_0 be the $(2n+1) \times (n+1)$ matrix, formed by the first $(n+1)$-columns of A_0. Namely,

$$B_0 = \begin{pmatrix} I_n & 0 \\ 0 & i \\ iI_n & 0 \end{pmatrix}.$$

Then $\varphi(\xi) = \Phi^* B_0$ and thus for $\xi \neq 0$,

$$\varphi'(\xi) = \Phi^{*'}_\xi \cdot B = \left(\frac{a(X)}{\xi^2} - \overline{a(X)} + i\beta(X)\frac{1}{\xi} \right) \cdot Q(X)(\xi) \cdot B.$$

Write $Q_1(X) = Q(X) - Q(X)(0)$, $Q_2(X) = Q_1(X) - Q'_\xi(X)(0)\xi$. Then

$$\varphi'_\xi = \left(\frac{\alpha}{\xi^2} Q_2(X)(\xi) - \overline{\alpha(X)} Q(X)(\xi) + i\beta \frac{Q_1(X)}{\xi} \right) \cdot B$$

Since $X \in \mathcal{B}_\varepsilon$ with $\varepsilon \ll 1$ after making $\|\varphi\|_{C^0} \ll 1$, by the Cauchy formula, we have

$$\left\| \frac{Q_2}{\xi^2} \right\|_{C^0(\overline{\Delta})}, \ \left\| \frac{Q_1}{\xi} \right\|_{C^0(\overline{\Delta})} = O(\|X\|^{\alpha}_{C^0})$$

as $\|X\|_{C^0} \to 0$. Notice that

$$a(X) = (0, \gamma)A_0^{-1} + O(|\gamma| \|X\|^{\alpha}_{C^0}), \ \beta = O(|\gamma|), \ \gamma = \varphi^{**}(0) \neq 0.$$

Hence, we get

$$\frac{1}{|\gamma|}\varphi'(\xi) = -\overline{\left(0, \frac{\gamma}{|\gamma|}\right) A_0^{-1}} \times \mathcal{B} + O(\|X\|^{\alpha}_{C^0}).$$

Write $\frac{\gamma}{|\gamma|} = (b_1, \cdots, b_n)$ with $\sum |b_j|^2 = 1$. Also notice that

$$A_0^{-1} = \begin{pmatrix} \frac{1}{2}I_n & 0 & -\frac{i}{2}I_n \\ 0 & -i & 0 \\ \frac{1}{2}I_n & 0 & \frac{iI_n}{2}, \end{pmatrix}$$

we get

$$\begin{aligned} \frac{1}{|\gamma|}\varphi'(\xi) &= -\frac{\overline{(0,\gamma)A_0^{-1}}}{|\gamma|} \times B_0 + O(\|\hat{X}\|^{\alpha}_{C^0}) \\ &= -(0,b)\begin{pmatrix} \frac{1}{2}I_n & 0 & \frac{i}{2}I_n \\ 0 & i & 0 \\ \frac{1}{2}I_n & 0 & \frac{-iI_n}{2} \end{pmatrix}\begin{pmatrix} I_n & 0 \\ 0 & i \\ iI_n & 0 \end{pmatrix} + O(\|X\|^{\alpha}_{C^0(\partial\Delta)}) \\ &= -(b,0) + O(\|\hat{X}\|^{\alpha}_{C^0(\partial\Delta)}). \end{aligned}$$

or $\varphi'_{n+1}(\xi) = O(|\gamma|\|X\|^{\alpha}_{C^0(\partial\Delta)})$, $(\varphi'_1(\xi), \cdots, \varphi'_n(\xi)) = -b|\gamma| + O(\|X\|^{\alpha}_{C^0}|\gamma|)$. We see that

$$\frac{|\varphi'_{n+1}(\xi)|}{|(\varphi'_1(\xi), \cdots, \varphi'_n(\xi))|} \leq C\|\hat{X}\|^{\alpha}_{C^0} \quad \text{for any } \xi \in \Delta$$

with C independent of φ. Notice that $\|\hat{X}\|_{C^0(\partial\Delta)} \approx \eta(\varphi, p^*) = \max_{\xi} |\varphi(\xi) - p^*|$.

Next for any vector $\chi(\in \mathbb{C}^{n+1}) = (\chi', \chi_{n+1}) \neq 0$ and for any $z \in \partial\Omega$ with $|z| < k\eta(\varphi)$ for a fixed positive integer k, the orthogonal projection of χ to the orthogonal complement of $T_z^{(1,0)}\partial\Omega$ is denoted and given by $\chi_{z,nor} = \chi\cdot\overline{\nu(z)}\nu(z)$. Here $\nu(z) = \frac{\frac{\partial\rho}{\partial\bar{z}}}{|\nabla_z\rho|} = (0, \cdots, 0, 1) + O(|z|)$. The orthogonal projection of χ to $T_z^{(1,0)}\partial\Omega$ is denoted and given by $\chi_{z,tan} = \chi - \chi_{z,nor}$. Now assume that $|\chi_{n+1}| \leq C\eta(\varphi)^{\alpha}\|\chi'\|$. Then $|\chi_{z,nor}| = |\chi_{n+1}| + O(|z||\chi|)$ and $|\chi_{z,tan}| = |\chi'| + O(|\chi||z|) = |\chi'|(1 + O(|z|)$. Hence $\frac{|\chi_{z,nor}|}{|\chi_{z,tan}|} = \frac{|\chi_{n+1}|}{|\chi'|} + O(|z|) \lesssim \eta(\varphi)^{\alpha}$. We thus derive that for any $z \in \partial D$ with $|z| < k\eta(\varphi)$ and for any $\xi \in \Delta$

$$|(\varphi'(\xi))_{z,nor}| \leq C_k\eta^{\alpha}(\varphi)|(\varphi'(\xi))_{z,tan}| \tag{2.9}$$

in the normal coordinate (2.1). In particular, letting $k = 1$, we arrive

$$|(\varphi'(\xi))_{nor}| \leq C\eta^{\alpha}(\varphi)|(\varphi'(\xi))_{tan}|.$$

To verify the estimate in (2.9) is also valid after a holomorphic change of coordinates. Let $w = F(z)$ be a biholomorphic map from a neighborhood U_p of $p = 0$ to a neighborhood of

$\widetilde{p} \in \mathbb{C}^{n+1}$ sending $\Omega \cap U_p$ to $\widetilde{\Omega} \cap \widetilde{U}$ and $F(p) = \widetilde{p}$. Let φ be a proper holomorphic embedding from Δ into $\Omega \cap U_p$ with $\varphi(\Delta) \approx p$ such that for any $z \in \partial\Omega$ with $|z-p| \leq k\eta(\varphi)$, $|(\varphi'(\xi))_{z,nor}| \leq C_k \eta^\alpha(\varphi)|(\varphi'(\xi))_{z,tan}|$. Let $\psi = F \circ \varphi$. Then $\psi'(\xi) = \varphi'(\xi)\frac{\partial w}{\partial z}$. Write the orthogonal decomposition $\varphi'(\xi) = (\varphi'(\xi))_{z,tan} + (\varphi'(\xi))_{z,nor}$. Then $F_*(\varphi'(\xi)) = (\varphi'(\xi))_{z,tan}\frac{\partial w}{\partial z}(\varphi(\xi)) + (\varphi'(\xi))_{z,nor}\frac{\partial w}{\partial z}(\varphi(\xi)) = (\varphi'(\xi))_{z,tan}\frac{\partial w}{\partial z}(z) + (\varphi'(\xi))_{z,nor}\frac{\partial w}{\partial z}(z) + |\varphi'(\xi)| \cdot O(|z-p|)$. By the Hopf lemma, we can write $(\varphi'(\xi))_{z,nor}\frac{\partial w}{\partial z}(z) = a_{w,tan} + a_{w,nor}$ with $a_{w,tan} \in T_w^{(1,0)}\partial\widetilde{D}$ and $a_{w,nor} \neq 0$ along the normal direction of $\partial\widetilde{D}$ at $w = F(z)$. Notice that $|a_{w,nor}|, |a_{w,tan}| \leq C_k|(\varphi'(\xi))_{z,nor}| \leq C_k\eta^\alpha(\varphi)|(\varphi'(\xi))_{z,tan}|$.

Hence
$$\frac{|F_*(\varphi'(\xi))_{w,nor}|}{|F_*(\varphi'(\xi))_{w,tan}|} = \frac{|a_{w,nor} + \varphi'(\xi)O(|z-p|)|}{|(\phi'(\xi))_{z,tan}\frac{\partial w}{\partial z}(z) + a_{w,tan} + \varphi'(\xi)O(|z-p|)|}.$$
Notice that $|(\varphi'(\xi))_{z,tan}\frac{\partial w}{\partial z}(z) + a_{w,tan} + \varphi'(\xi)O(|z-p|)| \lesssim C_k|(\varphi'(\xi))_{z,tan}|(1+\eta^\alpha(\varphi))$ and $\frac{|a_{w,nor}|}{|(\phi'(\xi))_{z,tan}|} \leq C_k\eta^\alpha(\varphi)$. We easily conclude that $\frac{|F_*(\phi'(\xi))_{w,nor}|}{|F_*(\phi'(\xi))_{w,tan}|} \leq C_k\eta^\alpha(\varphi)$. Now we choose k such that $\max_{\xi\in\Delta}|\psi(\xi) - \widetilde{p}| \leq \frac{k}{2}\eta(\varphi)$. Then we conclude that there is an $0 < \epsilon(\widetilde{p}) << 1$ such that when $|\psi - \widetilde{p}| < \epsilon(\widetilde{p})$ and for any $w \in \partial\widetilde{\Omega}$ with $|w - \widetilde{p}| < 2\eta(\psi)$, it holds
$$|(\psi'(\xi))_{z,nor}| \leq C\eta^\alpha(\psi)|(\psi'(\xi))_{z,tan}|. \tag{2.10}$$

This, in particular, completes the proof of our theorem. □

Remark A: Let $p^*(\approx p) \in \partial\Omega$. We choose a holomorphic change of coordinates which depends $C^{2,\alpha}$ on p^* such that p^* is mapped to the origin and the local defining function of Ω_{p^*} is defined by an equation ρ_{p^*} of the normal form as in (2.1) with ρ_{p^*} also depending $C^{2,\alpha}$ on p^*. Then we proceed the same way as above to trace the dependence on p^* for each quality to obtain the following statement:

There are small positive numbers δ, ϵ such that for any $p^* \approx p$ and any extremal disk φ of D such that when $|\varphi(\xi) - p^*| < \epsilon$ for each $\xi \in \Delta$ we have for any $z \in \partial D$ with $|p - z| < \delta$,
$$|(\varphi'(\xi))_{z,nor}| \leq C\eta^\alpha(\varphi, p^*)|(\varphi'(\xi))_{z,tan}| \tag{2.11}$$
with C independent of ϕ and p^*. Here $\eta(\varphi, p^*) = \max_{\xi\in\Delta}|\varphi(\xi) - p^*|$. Now, when the extremal disk ϕ is sufficiently close to p, by picking $p^* = \phi(1)$, since $\eta(\varphi, p^*) \approx \mathrm{diam}(\varphi) := \max_{\xi_1,\xi_2}|\phi(\xi_1) - \phi(\xi_2)|$, we see that in Theorem 1.1, we can replace $\eta(\varphi)$ by $\mathrm{diam}(\varphi)$. By using the Lebesque covering lemma, we then have the following:

Theorem 2.2. *Let $\Omega \subset \mathbb{C}^{n+1}$ be a bounded strongly pseudoconvex domain with a $C^{2,\alpha}$-smooth boundary, where $\alpha \in (0,1]$. Then there is a positive number ε with $0 < \varepsilon(p) \ll 1$, such that for any extremal map φ of D when* $\mathrm{diam}(\varphi) < \epsilon$ *it holds that*

$$|(\varphi'(\xi))_{nor}| \leq C\mathrm{diam}^{\alpha}(\varphi)|(\varphi'(\xi))_{\tan}|, \ \forall \ \xi \in \overline{\Delta}.$$

Here C is a constant independent of $\xi \in \Delta$ and φ.

However, I do not know the answer to the following conjecture, which asserts that an extremal disk wandering around the boundary should be a small disks:

Conjecture 2.3. *Let D be a bounded strongly pseudoconvex domain with a $C^{2,\alpha}$-smooth boundary. Then for any $\delta > 0$ there is a small number ϵ such that for any extremal disk φ of D, if* $\max_{\xi\in\Delta} \mathrm{dist}(\varphi(\xi), \partial \mathrm{D}) < \epsilon$, *then we must have* $\mathrm{diam}(\varphi) < \delta$.

Remark B: Once Theorem 1.1 is proved, the existence part of [Theorem 3, Hua2] also holds with the domain D being just assumed to be $C^{2,\alpha}$-smooth for $\alpha \in (0,1)$. However, the uniqueness part which was only discussed much later in [HW] only holds for the bounded strongly convex domain D being $C^{2,\alpha}$ smooth for $\alpha > 1/2$. [Theorem 1, Hua2] also holds for the domain D there to be $C^{2,\alpha}$-smooth.

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