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Elliptic quintics on cubic fourfolds, O'Grady 10, and Lagrangian fibrations

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For a smooth cubic fourfold Y, we study the moduli space M of semistable objects of Mukai vector $2\lambda_1 + 2\lambda_2$ in the Kuznetsov component of Y. We show that with a certain choice of stability conditions, M admits a symplectic resolution $\widetilde{M}$, which is a smooth projective hyperkähler manifold, deformation equivalent to the 10-dimensional examples constructed by O'Grady. As applications, we show that a birational model of $\widetilde{M}$ provides a hyperkähler compactification of the twisted family of intermediate Jacobians associated to Y. This generalizes the previous result of Voisin [58] in the very general case. We also prove that $\widetilde{M}$ is the MRC quotient of the main component of the Hilbert scheme of quintic elliptic curves in Y, confirming a conjecture of Castravet.

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1. Introduction

Moduli spaces of stable sheaves on a K3 surface provide the major examples of projective hyperkähler manifolds. These examples are deformation equivalent to Hilbert schemes of points on a K3 surface, by the seminal work of Mukai [50] and the contribution of many other authors, including Beauville [10], O'Grady [53], Yoshioka [59,60]. In [54], O'Grady considered the case when the moduli space contains also strictly semistable sheaves. In particular, he constructed a symplectic resolution of the singular moduli space of semistable torsion-free sheaves on a K3 surface with rank 2, trivial first Chern class and second Chern class equal to 4. This construction provides a new example of a hyperkähler manifold of dimension 10, not deformation equivalent to the previous construction. O'Grady's result was generalized by Lehn and Sorger in [46] to moduli spaces of semistable sheaves on a K3 surface having Mukai vector of the form $v = 2v_0$ with $v_0^2 = 2$. In addition, they showed that the symplectic resolution of the moduli space can be obtained by blowing up the singular locus with the reduced scheme structure.

In this paper we investigate the analogous situation of O'Grady's example, in the case of moduli spaces of semistable complexes in the noncommutative K3 surface associated to a smooth cubic fourfold. By [38], the bounded derived category of a cubic fourfold Y has a semiorthogonal decomposition of the form

$$\mathrm{D}^{\mathrm{b}}(Y) = \langle \mathrm{Ku}(Y), \mathcal{O}_Y, \mathcal{O}_Y(H), \mathcal{O}_Y(2H)\rangle,$$

where $H \subset Y$ is a hyperplane section and $\mathrm{Ku}(Y)$ is a triangulated subcategory of K3 type, in the sense that it has the same Serre functor and Hochschild homology as the derived category of a K3 surface [36, Corollary 4.3], [40, Proposition 4.1]. We call this category $\mathrm{Ku}(Y)$ the *Kuznetsov component* of Y. One reason to study $\mathrm{Ku}(Y)$ is related to the birational geometry of Y. For instance, there is a folklore conjecture [38, Conjecture 1.1] that Y is rational if and only if $\mathrm{Ku}(Y)$ is equivalent to the derived category of a K3 surface.

Another interest in studying $\mathrm{Ku}(Y)$ is to generalize Mukai's construction to this noncommutative K3 surface. Bayer, Lahoz, Macrì and Stellari construct Bridgeland stability conditions on $\mathrm{Ku}(Y)$ in [13] (see Section 2 for a review of the construction). We denote by $\mathrm{Stab}^{\dagger}(\mathrm{Ku}(Y))$ the connected component of the stability manifold containing these stability conditions. In a second paper [12], joint also with Nuer and Perry, they develop the theory of families of stability conditions, which allows studying the properties of moduli spaces of stable objects in $\mathrm{Ku}(Y)$ by deforming to cubic fourfolds whose Kuznetsov components are equivalent to the derived category of a K3 surface. As a consequence, they produced infinite series of unirational, locally complete families of smooth polarized hyperkähler manifolds, deformation equivalent to Hilbert schemes of points on a K3 surface. These hyperkähler manifolds are given as moduli spaces of stable objects in $\mathrm{Ku}(Y)$ of primitive Mukai vector. It is worth to point out that the hyperkähler manifolds constructed from some Hilbert schemes of rational curves of low degree in Y can be interpreted as moduli spaces of stable objects in $\mathrm{Ku}(Y)$. Indeed, we gave in [45] a description of the Fano variety of lines in Y [9] and, when Y does not contain a plane, of the hyperkähler 8-fold constructed in [43] using twisted cubic curves in Y, as moduli spaces of stable objects in $\mathrm{Ku}(Y)$ with primitive Mukai vector.

In analogy to the case of K3 surfaces, the Mukai lattice of $\mathrm{Ku}(Y)$ has been defined in [7] and carries a weight two Hodge structure induced from that on the cohomology of Y. We denote by $H^*_{\mathrm{alg}}(\mathrm{Ku}(Y), \mathbb{Z})$ the sublattice of integral $(1,1)$ classes in the Mukai lattice of $\mathrm{Ku}(Y)$ (see Section 3.1).

Consider now a vector $v = 2v_0 \in H^*_{\mathrm{alg}}(\mathrm{Ku}(Y), \mathbb{Z})$ such that v_0 is primitive with $v_0^2 = 2$. Let τ be a stability condition in $\mathrm{Stab}^{\dagger}(\mathrm{Ku}(Y))$ which is generic with respect to v, in other words, the strictly τ-semistable objects with Mukai vector v are (S-equivalent to) direct sums of τ-stable objects with Mukai vector v_0. Let M be the moduli space of τ-semistable objects with Mukai vector v. The first result of this paper is the following.

Theorem 1.1 *(Theorem 3.1). The moduli space M has a symplectic resolution $\widetilde{M}$, which is a 10-dimensional smooth projective hyperkähler manifold, deformation equivalent to the O'Grady's example constructed in [54].*

In the second part we explain two main applications, which make a connection between the derived categorical viewpoint of Theorem 1.1 and the classical construction of hyperkähler manifolds from Y. Recall that by [7], the algebraic Mukai lattice of $\mathrm{Ku}(Y)$ contains two classes λ_1 and λ_2 spanning an A_2-lattice. Motivated by classical geometric constructions (as it will be clear later), we consider the case $v_0 = \lambda_1 + \lambda_2$, $v = 2v_0$ and we analyze the objects in $M := M_\sigma(v)$ where σ is a stability condition as constructed in [13]. It is not difficult to see that by [45] the strictly semistable locus of M is identified with the symmetric square of the Fano variety of lines in Y, up to a perturbation of the stability condition (see Remark 4.1). On the other hand, stable objects are harder to describe. If X is a smooth hyperplane section of Y, in other words, X is a smooth cubic threefold, then the moduli space M_{inst} parametrizing rank 2 instanton sheaves on X has been described by [21]. In particular, stable sheaves in M_{inst} belong to one of the following classes: rank 2 stable vector bundles constructed from non-degenerate elliptic quintics in X, rank 2 stable torsion free sheaves associated to smooth conics in X. Moreover, the strictly semistable objects in M_{inst} are direct sums of two ideal sheaves of lines in X (see Section 4.1 for a review). By [11,21] the moduli space M_{inst} is birational to the translate $J^2(X)$ of the intermediate Jacobian, which parametrizes 1-cycles of degree 2 on X.

Denote by σ a stability condition constructed in [13]. A key result for our applications is the following theorem, which provides a description of an open subset of the stable locus of $M := M_\sigma(2\lambda_1 + 2\lambda_2)$.

Theorem 1.2 *(Theorem 5.19). Let X be a smooth hyperplane section of Y. Then the projection in* $\mathrm{Ku}(Y)$ *of the stable rank* 2 *instanton sheaves associated to non degenerate* *quintic elliptic curves and smooth conics in X are σ-stable objects with Mukai vector $2\lambda_1 + 2\lambda_2$.*

We apply Theorem 1.2 to show that, up to a perturbation of the stability condition σ in $\mathrm{Stab}^\dagger(\mathrm{Ku}(Y))$ (see Section 6), the sympletic resolution $\widetilde{M}$, given by Theorem 1.1 has a connection to a classical construction of Jacobian fibration associated to Y. Consider the $(\mathbb{P}^5)^\vee$-family of cubic threefolds obtained as hyperplane sections of Y and let $\mathbb{P}_0$ be the locus parametrizing the smooth hyperplane sections. Consider the twisted family of intermediate Jacobians $p : J \to \mathbb{P}_0$, whose fibers are the twisted intermediate Jacobians of the smooth cubic threefolds parametrized by $\mathbb{P}_0$. It is known that there exists a holomorphic symplectic form on J by [20]. However, it remained a long standing question whether J can be compactified to a hyperkähler manifold $\bar{J}$ and a Lagrangian fibration $\bar{J} \to (\mathbb{P}^5)^\vee$ extending p. This has been recently proved for very general cubic fourfolds in the beautiful works [47] for the untwisted family and [58] by Voisin for J. We mention that in the recent preprint [57], Saccà extended the result for the untwisted family in [47] to all smooth cubic fourfolds. The same argument applies to the twisted family and extends Voisin's result to all smooth cubic fourfolds (see [57, Remark 1.10]).

Our main result is the following construction of a hyperkähler compactification of J for every cubic fourfold Y, obtained combining Theorems 1.1, 1.2 and some techniques in birational geometry of hyperkähler varieties.

Theorem 1.3 *(Propositions 6.1, 6.7). There exists a hyperkähler manifold N birational to $\widetilde{M}$, which admits a Lagrangian fibration structure compactifying the twisted intermediate Jacobian family $J \to \mathbb{P}_0$.*

It is worth to note that N and $\widetilde{M}$ are birational, but not isomorphic if Y is very general. In Example 6.8 we describe an explicit flop between them, involving the locus of stable objects in $\mathrm{Ku}(Y)$ coming from the projection of instanton sheaves associated to smooth conics in Y. In Remark 6.9, we explain how N is related to the compactification constructed by Voisin [58].

The next application arises from the following conjecture of Castravet. Note that the original conjecture involves rational quartics, but it can be equivalently stated for elliptic quintics by residuality (see Remark 7.3).

Conjecture 1.4 *([19, page 416]). Let $\mathcal{C}$ be the connected component of the Hilbert scheme $\mathrm{Hilb}^{5m}(Y)$ containing elliptic quintics in Y. Then the maximally rationally connected quotient of $\mathcal{C}$ is birationally equivalent to the twisted intermediate Jacobian of Y.*

Using Theorems 1.1, 1.2 and 1.3 we are able to prove Conjecture 1.4.

Proposition 1.5 *(Propositions 7.1, 7.2). The projection functor (see Definition 4.3) induces a rational map $\mathcal{C} \dashrightarrow M$ which is the maximally rationally connected fibration of*

$\mathcal{C}$. The maximally rationally connected quotient of $\mathcal{C}$ is birational to the twisted family J of intermediate Jacobians of Y.

Plan of the paper. In Section 2 we review some definitions and results about (weak) stability conditions on triangulated categories and semiorthogonal decompositions. Moreover, we recall the construction of stability conditions on the Kuznetsov component $\mathrm{Ku}(Y)$ of a cubic fourfold Y as in [13].

Section 3 is devoted to the proof of Theorem 1.1. For an element v_0 with square 2 in the algebraic Mukai lattice of $\mathrm{Ku}(Y)$, consider a stability condition τ on $\mathrm{Ku}(Y)$ which is $2v_0$-generic. We show that the blow-up $\widetilde{M}$ of the singular locus of the moduli space $M := M_\tau(2v_0)$ with the reduced scheme structure is a symplectic resolution, by describing the local structure of M at the worst singularity, as done in [46] for singular moduli spaces on K3 surfaces.

In Section 4 we compute the projection in the Kuznetsov component of some objects related to elliptic quintics and smooth conic curves in a cubic fourfold. We explain their relation with stable instanton sheaves on smooth hyperplane sections of Y, which were previously studied in [21].

Section 5 deals with the proof of Theorem 1.2. We show that the objects in the Kuznetsov component, constructed out of elliptic quintics and conics in Y, are σ-stable, where σ is any stability condition as constructed in [13]. Moreover, they describe an open subset of the moduli space $M_\sigma(2\lambda_1+2\lambda_2)$. Recall that σ is induced on $\mathrm{Ku}(Y)$ from the restriction of (a tilt of) a weak stability condition $\sigma_{\alpha,-1}$ on the bounded derived category of coherent $\mathcal{B}_0$-modules on $\mathbb{P}^3$, depending on a real parameter $\alpha > 0$. Here $\mathcal{B}_0$ is the even part of the sheaf of Clifford algebras associated to the conic fibration on $\mathbb{P}^3$ obtained by blowing-up a line in Y (see Section 2.3, Proposition and Definition 2.15). In Section 5.2 we compute the expression of our objects as complexes of $\mathcal{B}_0$-modules on $\mathbb{P}^3$. Then in Sections 5.4 and 5.5 we show they are $\sigma_{\alpha,-1}$-stable for α sufficiently large. Finally in Section 5.6 we show they are $\sigma_{\alpha,-1}$-stable for every α, proving there are no walls for stability.

In Section 6 we prove Theorem 1.3. Fix a stability condition σ_0 which is generic with respect to $2\lambda_1+2\lambda_2$ and with the same stable objects as σ. Applying Theorems 1.1 and 1.2 to the moduli space $M := M_{\sigma_0}(2\lambda_1+2\lambda_2)$, we consider the open subvariety M_0 of the symplectic resolution $\widetilde{M}$ of M consisting of stable objects associated to elliptic quintics in Y, with support on smooth hyperplane sections of Y. We define a line bundle $\mathcal{L}$ on $\widetilde{M}$ inducing a rational map from $\widetilde{M}$ to $(\mathbb{P}^5)^\vee$, which is defined on M_0 by sending the object to its support. Using some results in birational geometry of hyperkähler varieties, we show that there is a birational model N of $\widetilde{M}$ and a semiample line bundle $\mathcal{L}'$ on N, such that a multiple of $\mathcal{L}'$ induces a Lagrangian fibration structure on N which is a compactification of the twisted intermediate Jacobian J over $\mathbb{P}_0$.

We conclude with Section 7 where Conjecture 1.4 is proved as a consequence of Theorems 1.1, 1.2 and 1.3.

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2. Preliminaries on stability conditions on $\mathrm{Ku}(Y)$

In this section we review the definitions of (weak) stability conditions on a triangulated category, semiorthogonal decompositions, and Kuznetsov component $\mathrm{Ku}(Y)$ of a cubic fourfold Y. Then we recall the construction of stability conditions on $\mathrm{Ku}(Y)$ due to [13] and some useful properties. The new contributions are an easier expression for the central charge of these stability conditions in Proposition 2.15 and Lemma 2.16 which makes more clear how to check the stability of objects in $\mathrm{Ku}(Y)$.

2.1. (Weak) stability conditions

It is in general a difficult task to construct stability conditions on a triangulated category. In the case of the Kuznetsov component $\mathrm{Ku}(Y)$ of cubic fourfolds, it is proved in [13] that such stability conditions can be induced by 'restricting' certain weak stability conditions, which can be constructed via the tilting heart technique. In this section, we briefly recall the notion of weak stability conditions following the summary in [13, Section 2].

Let $\mathcal{T}$ be a $\mathbb{C}$-linear triangulated category. We denote by $\mathrm{K}_{\mathrm{num}}(\mathcal{T})$ the numerical Grothendieck group of $\mathcal{T}$. Let Λ be a finite rank lattice with a surjective homomorphism $v : \mathrm{K}_{\mathrm{num}}(\mathcal{T}) \twoheadrightarrow \Lambda$.

Definition 2.1. The *heart of a bounded t-structure* is a full subcategory $\mathcal{A}$ of $\mathcal{T}$ such that

(a) for any objects E and F in $\mathcal{A}$ and negative integer n, we have $\mathrm{Hom}(E, F[n]) = 0$;
(b) for every E in $\mathcal{T}$, there exists a sequence of morphisms

$$0 = E_0 \xrightarrow{\phi_1} E_1 \xrightarrow{\phi_2} \dots \xrightarrow{\phi_{m-1}} E_{m-1} \xrightarrow{\phi_m} E_m = E$$

such that the cone of ϕ_i is of the form $A_i[k_i]$, for some sequence $k_1 > k_2 > \cdots > k_m$ of integers and objects A_i in $\mathcal{A}$.

Recall that the heart of a bounded t-structure is an abelian category by [8].

Definition 2.2. Let $\mathcal{A}$ be an abelian category. A group homomorphism $Z : \mathrm{K}_{\mathrm{num}}(\mathcal{A}) \to \mathbb{C}$ is a *weak stability function* (resp. a *stability function*) on $\mathcal{A}$ if, for $E \in \mathcal{A}$, we have $\Im Z(E) \geq 0$, and in the case that $\Im Z(E) = 0$, we have $\Re Z(E) \leq 0$ (resp. $\Re Z(E) < 0$ when $E \neq 0$).

For every object E in $\mathcal{A}$, its slope with respect to Z is given by

$$\mu_Z(E) = \begin{cases} -\frac{\Re Z(E)}{\Im Z(E)} & \text{if } \Im Z(E) > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

An object E in $\mathcal{A}$ is *semistable* (resp. *stable*) with respect to Z if for every proper subobject F of E in $\mathcal{A}$, we have $\mu_Z(F) \leq \mu_Z(E)$ (resp. $\mu_Z(F) < \mu_Z(E/F)$).

Definition 2.3. A *weak stability condition* (with respect to Λ) on $\mathcal{T}$ is a pair $\sigma = (\mathcal{A}, Z)$, where $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{T}$ and Z is a group homomorphism from Λ to $\mathbb{C}$, satisfying the following properties:

(a) The composition $\mathrm{K}_{\mathrm{num}}(\mathcal{A}) = \mathrm{K}_{\mathrm{num}}(\mathcal{T}) \xrightarrow{v} \Lambda \xrightarrow{Z} \mathbb{C}$ is a weak stability function on $\mathcal{A}$.[1] We say that an object E in $\mathcal{A}[k]$ is σ-(semi)stable if $E[-k]$ is (semi)stable with respect to Z.
(b) Every object of $\mathcal{A}$ has a Harder–Narasimhan filtration with σ-semistable factors.
(c) There exists a quadratic form Q on $\Lambda \otimes \mathbb{R}$ such that the restriction of Q to $\ker Z$ is negative definite and $Q(E) \geq 0$ for all σ-semistable objects E in $\mathcal{A}$.

If Z is a stability function, then σ is a *stability condition* introduced by Bridgeland in [18]. In this situation, we will usually call Z the *central charge* of the stability condition. If the lattice Λ is the numerical Grothendieck group $\mathrm{K}_{\mathrm{num}}(\mathcal{T})$ and v is the identity map, then σ is called a *full numerical stability condition*.

Remark 2.4. There is usually a natural choice of the lattice $v : \mathrm{K}_{\mathrm{num}}(\mathcal{T}) \to \Lambda$ in each of triangulated categories considered in this paper.

[1] We will write $Z(-)$ instead of $Z(v(-))$ for simplicity.

2.2. Semiorthogonal decompositions and Kuznetsov components

Definition 2.5. Let $\mathcal{T}$ be a triangulated category. A *semiorthogonal decomposition*

$$\mathcal{T} = \langle \mathcal{D}_1, \dots, \mathcal{D}_m \rangle$$

is a sequence of full triangulated subcategories $\mathcal{D}_1, \dots, \mathcal{D}_m$ of $\mathcal{T}$ such that:

(a) $\mathrm{Hom}(F, G) = 0$, for any objects F in $\mathcal{D}_i$, G in $\mathcal{D}_j$ and $i > j$;
(b) For any object F in $\mathcal{D}$, there is a unique sequence of morphisms

$$0 = F_m \to F_{m-1} \to \cdots \to F_1 \to F_0 = F,$$

with factors $\mathsf{pr}_i(F) := \mathrm{Cone}(F_i \to F_{i-1}) \in \mathcal{D}_i$ for $1 \leq i \leq m$.

The subcategories $\mathcal{D}_i$ are called the *components* of the decomposition. We also have the functor pr_i from $\mathcal{T}$ to $\mathcal{D}_i$.

Definition 2.6. An object E in $\mathcal{T}$ is *exceptional* if $\mathrm{Hom}(E, E[p]) = 0$ for all integers $p \neq 0$, and $\mathrm{Hom}(E, E) \cong \mathbb{C}$.

A sequence of objects $\{E_1, \dots, E_m\}$ in $\mathcal{T}$ is an *exceptional collection* if E_i is an exceptional object for all i, and $\mathrm{Hom}(E_i, E_j[p]) = 0$ for all p and all $i > j$.

By [17], an exceptional collection $\{E_1, \dots, E_m\}$ in $\mathcal{T}$ provides a semiorthogonal decomposition

$$\mathcal{T} = \langle \mathcal{D}, E_1, \dots, E_m \rangle. \tag{2.1}$$

Here by abuse of notation, we write E_i also for the full triangulated subcategory of $\mathcal{T}$ generated by E_i. The full subcategory $\mathcal{D} := \langle E_1, \dots, E_m \rangle^{\perp}$ consists of objects

$$\{G \in \mathrm{Obj}(\mathcal{T}) |\, \mathrm{Hom}(E_i, G[p]) = 0 \text{ for all } p \text{ and } i\}. \tag{2.2}$$

Let Y be a smooth cubic fourfold. Denote by H a hyperplane section of Y. There is an exceptional collection $\{\mathcal{O}_Y, \mathcal{O}_Y(H), \mathcal{O}_Y(2H)\}$. The bounded derived category of coherent sheaves on Y admits a semiorthogonal decomposition of the form

$$\mathrm{D}^{\mathrm{b}}(Y) = \langle \mathrm{Ku}(Y), \mathcal{O}_Y, \mathcal{O}_Y(H), \mathcal{O}_Y(2H) \rangle. \tag{2.3}$$

The subcategory $\mathrm{Ku}(Y)$ is studied in details in [38], and it is now commonly referred to as the Kuznetsov component.

This projection functor in Definition 2.5 can be expressed by compositions of *left (right) mutation* functors, depending on the explicit semiorthogonal decomposition.

Definition 2.7. Let E be an exceptional object in $\mathcal{T}$. The left (resp. right) mutation functors L_E (resp. R_E) are defined as follows:

$$\mathsf{L}_E(F) := \mathrm{Cone}\left(\bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}(E[p], F) \otimes E[p] \xrightarrow{\mathsf{ev}} F\right);$$

$$\mathsf{R}_E(F) := \mathrm{Cone}\left(F \xrightarrow{\mathsf{ev}^\vee} \bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}(F, E[p])^\vee \otimes E[p]\right)[-1].$$

Remark 2.8. Note that since the Serre functor in $\mathrm{D^b}(Y)$ is given by

$$\mathsf{S}_Y(-) = - \otimes \mathcal{O}_Y(-3H)[4],$$

the Kuznetsov component of Y can be also given by ${}^\perp\langle \mathcal{O}_Y(-2H), \mathcal{O}_Y(-H)\rangle \cap \mathcal{O}_Y^\perp$. Namely, it appears in the semiorthogonal decomposition

$$\langle \mathcal{O}_Y(-2H), \mathcal{O}_Y(-H), \mathrm{Ku}(Y), \mathcal{O}_Y\rangle. \tag{2.4}$$

Note that the object $\mathsf{L}_E(F)$ (resp. $\mathsf{R}_E(F)$) is in $E^\perp$ (resp. ${}^\perp E$). We denote by

$$\mathsf{pr} : \mathrm{D^b}(Y) \to \mathrm{Ku}(Y)$$

the functor in Definition 2.5(b) to the component $\mathrm{Ku}(Y)$ with respect to the decomposition (2.4). In particular, it is given as the composition of mutations:

$$\mathsf{pr} = \mathsf{R}_{\mathcal{O}_Y(-H)}\mathsf{R}_{\mathcal{O}_Y(-2H)}\mathsf{L}_{\mathcal{O}_Y} = \mathsf{L}_{\mathcal{O}_Y}\mathsf{R}_{\mathcal{O}_Y(-H)}\mathsf{R}_{\mathcal{O}_Y(-2H)}. \tag{2.5}$$

We will use the functor pr to produce objects in $\mathrm{Ku}(Y)$ in Section 4. Note that the functor pr in our paper is different from the more standard projection functor $\mathsf{L}_{\mathcal{O}_Y}\mathsf{L}_{\mathcal{O}_Y(H)}\mathsf{L}_{\mathcal{O}_Y(2H)}$ with respect to the decomposition (2.3), which is also the left adjoint functor of the natural embedding of $\mathrm{Ku}(Y)$.

2.3. Kuznetsov components of $\mathrm{D^b}(Y)$ *and* $\mathrm{D^b}(\mathbb{P}^3, \mathcal{B}_0)$

It is usually a highly non-trivial task to construct stability conditions on the Kuznetsov component. The only technique so far is to restrict weak stability conditions on the whole derived category to its Kuznetsov component. In the cubic fourfold case, such weak stability conditions on $\mathrm{D^b}(Y)$ require a Bogomolov type inequality involving the third Chern character. Unfortunately, such inequality is not known yet for any cubic fourfold.

To avoid this technical difficulty, the idea in [13] is to embed $\mathrm{Ku}(Y)$ as a component in a bounded derived category of lower dimension. More precisely, the key observation in

[13, Section 7] is that $\mathrm{Ku}(Y)$ is equivalent to the Kuznetsov component of $\mathrm{D^b}(\mathbb{P}^3, \mathcal{B}_0)$. We briefly summarize the construction of this equivalence in this section.

Let $L \subset Y \subset \mathbb{P}^5$ be a line which is not on any plane in Y, and we denote by

$$\rho_L : \tilde{Y} \to Y$$

the blow-up of L in Y.

The projection from L to a disjoint $\mathbb{P}^3$ (in $\mathbb{P}^5$) equips $\tilde{Y}$ with a natural conic fibration structure

$$\pi : \tilde{Y} \to \mathbb{P}^3.$$

There is a rank three vector bundle $\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(-1)$ on $\mathbb{P}^3$ such that $\tilde{Y}$ embeds into the $\mathbb{P}^2$-bundle $\mathbb{P}_{\mathbb{P}^3}(\mathcal{F})$ as the zero locus of a section

$$s_{\tilde{Y}} \in \mathrm{H}^0(\mathbb{P}^3, \mathrm{Sym}^2\mathcal{F}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1)) \cong \mathrm{H}^0(\mathbb{P}_{\mathbb{P}^3}(\mathcal{F}), \mathcal{O}_{\mathbb{P}_{\mathbb{P}^3}(\mathcal{F})}(2) \otimes q^*\mathcal{O}_{\mathbb{P}^3}(1)).$$

We have the following diagram of morphisms:

$$\begin{tikzcd} \tilde{Y} \arrow[r, "\alpha"] \arrow[d, "\rho_L"'] \arrow[rrd, "\pi"] & \mathrm{Bl}_L\mathbb{P}^5 \arrow[r, equal] \arrow[d] & \mathbb{P}_{\mathbb{P}^3}(\mathcal{F}) \arrow[d, "q"] \\ Y \arrow[r] & \mathbb{P}^5 & \mathbb{P}^3. \end{tikzcd} \tag{2.6}$$

By [37, Section 3], we have an associated sheaf of Clifford algebras of π over $\mathbb{P}^3$. Denote its even part (resp. odd part) by $\mathcal{B}_0$ (resp. $\mathcal{B}_1$). By [37, (12)], as a sheaf on $\mathbb{P}^3$, the even part $\mathcal{B}_0$ is a rank four vector bundle:

$$\mathcal{O}_{\mathbb{P}^3} \oplus \left(\bigwedge^2 \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^3}(-1)\right) \cong \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 2}. \tag{2.7}$$

As for its algebra structure, the structure sheaf is central. The other relations are determined by

$$e_i \wedge e_k \cdot e_k \wedge e_j = s_{\tilde{Y}}(e_k \otimes e_k) e_i \wedge e_j, \quad e_i \wedge e_k \cdot e_i \wedge e_k = s_{\tilde{Y}}(e_i \otimes e_i) s_{\tilde{Y}}(e_k \otimes e_k), \tag{2.8}$$

for an orthogonal basis (e_1, e_2, e_3) of $\mathcal{F}$ and $i \neq j \neq k \neq i$.

Definition 2.9. We denote by $\mathrm{Coh}(\mathbb{P}^3, \mathcal{B}_0)$ the category of coherent sheaves on $\mathbb{P}^3$ with a right $\mathcal{B}_0$-module structure, and denote its bounded derived category by $\mathrm{D^b}(\mathbb{P}^3, \mathcal{B}_0)$. The natural forgetful functor is denoted by $\mathrm{Forg} : \mathrm{D^b}(\mathbb{P}^3, \mathcal{B}_0) \to \mathrm{D^b}(\mathbb{P}^3)$.

In particular, we have

$$\mathrm{Hom}_{\mathrm{D^b}(\mathbb{P}^3, \mathcal{B}_0)}(\mathcal{B}_0, \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{D^b}(\mathbb{P}^3)}(\mathcal{O}_{\mathbb{P}^3}, \mathrm{Forg}(\mathcal{G})) \tag{2.9}$$

for every $\mathcal{G} \in \mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$.

By [37, (14)], the odd part $\mathcal{B}_1$ as a coherent sheaf is

$$\mathrm{Forg}(\mathcal{B}_1) = \mathrm{Forg}(\mathcal{F} \oplus (\textstyle\bigwedge^3 \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^3}(-1))) \cong \mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2). \tag{2.10}$$

As in [37, (15)], we define the following $\mathcal{B}_0$-bimodules for $j \in \mathbb{Z}$:

$$\mathcal{B}_{2j} := \mathcal{B}_0 \otimes \mathcal{O}_{\mathbb{P}^3}(j) \quad \text{and} \quad \mathcal{B}_{2j+1} := \mathcal{B}_1 \otimes \mathcal{O}_{\mathbb{P}^3}(j). \tag{2.11}$$

The Serre functor on $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$ (see [13, page 28]) is explicitly given as

$$\mathsf{S}_{\mathcal{B}_0}(-) = (-) \otimes_{\mathcal{B}_0} \mathcal{B}_{-3}[3]. \tag{2.12}$$

By [37, Lemma 3.8 and Corollary 3.9] and a direct computation using (2.9) and (2.12), the ordered set $\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3\}$ is an exceptional collection in $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$. By (2.1), there is a semiorthogonal decomposition of the form

$$\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0) = \langle \mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0), \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \rangle. \tag{2.13}$$

One of the key observations in [13, Section 7] to construct stability conditions on $\mathrm{Ku}(Y)$ is as follows.

Proposition 2.10 *([13, Lemma 7.6 and Proposition 7.7]). The Kuznetsov component* $\mathrm{Ku}(Y)$ *is equivalent to* $\mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0)$.

Remark 2.11. The functor of this equivalence is given explicitly as

$$\Psi \circ \rho_L^* : \mathrm{Ku}(Y) \hookrightarrow \mathrm{D}^{\mathrm{b}}(Y) \xrightarrow{\rho_L^*} \mathrm{D}^{\mathrm{b}}(\tilde{Y}) \xrightarrow{\Psi} \mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0) \supset \mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0), \tag{2.14}$$

where the functor Ψ is defined by

$$\Psi(-) = \pi_*(- \otimes \mathcal{E}[1]).$$

Here $\mathcal{E}$ is a sheaf of right $\pi^*\mathcal{B}_0$-modules on $\tilde{Y}$ defined in (5.2). We will only make essential use of this functor in Section 5.

2.4. Weak stability conditions on $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$

We first review the notion of t-structure by tilting. Let $\sigma = (\mathcal{A}, Z)$ be a weak stability condition on $\mathcal{T}$, and $t \in \mathbb{R}$. We can form the following subcategories of A:

$$\mathcal{A}_\sigma^{>t} := \{E \mid \text{every Harder–Narasimhan factor } F \text{ of } E \text{ has } \mu_Z(F) > t\};$$
$$\mathcal{A}_\sigma^{\leq t} := \{E \mid \text{every Harder–Narasimhan factor } F \text{ of } E \text{ has } \mu_Z(F) \leq t\}.$$

It follows from the existence of Harder–Narasimhan filtrations that $(\mathcal{A}_\sigma^{>\mu}, \mathcal{A}_\sigma^{\leq\mu})$ forms a *torsion pair* in $\mathcal{A}$ in the sense of [24]. In particular, we can obtain a new heart of a bounded t-structure by tilting.

Proposition and Definition 2.12 *([24]). Given a weak stability condition $\sigma = (\mathcal{A}, Z)$ on $\mathcal{T}$ and a choice of slope $t \in \mathbb{R}$, there exists a heart of a bounded t-structure defined by:*

$$\mathcal{A}_\sigma^t := \langle \mathcal{A}_\sigma^{>t}, \mathcal{A}_\sigma^{\leq t}[1] \rangle_{extension\ closure}.$$

For an object F in $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$, we define its modified Chern character as

$$\mathrm{ch}_{\mathcal{B}_0}(F) = \mathrm{ch}(\mathrm{Forg}(F))(1 - \frac{11}{32} l), \tag{2.15}$$

where l denotes the class of a line in $\mathbb{P}^3$. Expand the formula, we have $\mathrm{ch}_{\mathcal{B}_0,1}(F) = \mathrm{ch}_1(\mathrm{Forg}(F))$ and $\mathrm{ch}_{\mathcal{B}_0,2}(F) = \mathrm{ch}_2(\mathrm{Forg}(F)) - \frac{11}{32}\,\mathrm{rk}(F) l$.

For every $\beta \in \mathbb{R}$, we define the twisted Chern character as

$$\mathrm{ch}_{\mathcal{B}_0}^\beta = e^{-\beta h}\,\mathrm{ch}_{\mathcal{B}_0} = (\mathrm{rk}, \mathrm{ch}_{\mathcal{B}_0,1} - \mathrm{rk}\,\beta h, \mathrm{ch}_{\mathcal{B}_0,2} - \beta h \cdot \mathrm{ch}_{\mathcal{B}_0,1} + \mathrm{rk}\,\frac{\beta^2}{2} h^2, \dots),$$

where h denotes the class of a plane in $\mathbb{P}^3$.

We fix the lattice Λ of $\mathrm{K}_{\mathrm{num}}(\mathbb{P}^3, \mathcal{B}_0)$ in Definition 2.3 as:

$$v = \mathrm{ch}_{\mathcal{B}_0, \leq 2} : \mathrm{K}_{\mathrm{num}}(\mathbb{P}^3, \mathcal{B}_0) \to \Lambda$$

where $\Lambda = \{(\mathrm{rk}(F), l \cdot \mathrm{ch}_1(\mathrm{Forg}(F)), h \cdot \mathrm{ch}_2(\mathrm{Forg}(F)) - \frac{11}{32}\,\mathrm{rk}(F)) \colon F \in \mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)\}$. To simplify the notation, we will also write ch_1 (resp. ch_2) for $l \cdot \mathrm{ch}_1$ (resp. $h \cdot \mathrm{ch}_2$) in the future when there is no ambiguity.

The *discriminant* of an object F in $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$ is defined as follows:

$$\Delta_{\mathcal{B}_0}(F) := (\mathrm{ch}_{\mathcal{B}_0,1}(F))^2 - 2\,\mathrm{rk}(F)\,\mathrm{ch}_{\mathcal{B}_0,2}(F) = (\mathrm{ch}_{\mathcal{B}_0,1}^\beta(F))^2 - 2\,\mathrm{rk}(F)\,\mathrm{ch}_{\mathcal{B}_0,2}^\beta(F). \tag{2.16}$$

Setting $Z_{\mathrm{slope}} = i\,\mathrm{rk} - \mathrm{ch}_{\mathcal{B}_0,1}$, then the classical slope stability

$$\sigma_{\mathrm{slope}} = (\mathrm{Coh}(\mathbb{P}^3, \mathcal{B}_0), Z_{\mathrm{slope}})$$

is a weak stability condition in the sense of Definition 2.3. For $\beta \in \mathbb{R}$, as in Proposition and Definition 2.12, we have the heart of bounded t-structure

$$\mathrm{Coh}^\beta(\mathbb{P}^3, \mathcal{B}_0) := \langle \mathrm{Coh}_{\sigma_{\mathrm{slope}}}^{>\beta}(\mathbb{P}^3, \mathcal{B}_0), \mathrm{Coh}_{\sigma_{\mathrm{slope}}}^{\leq\beta}(\mathbb{P}^3, \mathcal{B}_0)[1] \rangle.$$

With this notation, we can state the following result.

Proposition and Definition 2.13 *([13, Proposition 9.3]). Given $\alpha > 0$ and $\beta \in \mathbb{R}$, the pair $\sigma_{\alpha,\beta} = (\mathrm{Coh}^{\beta}(\mathbb{P}^3, \mathcal{B}_0), Z_{\alpha,\beta})$ with*

$$Z_{\alpha,\beta}(F) = i\,\mathrm{ch}^{\beta}_{\mathcal{B}_0,1}(F) + \frac{1}{2}\alpha^2\,\mathrm{ch}^{\beta}_{\mathcal{B}_0,0}(F) - \mathrm{ch}^{\beta}_{\mathcal{B}_0,2}(F)$$

defines a weak stability condition on $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$. These stability conditions vary continuously as $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$ varies. The quadratic form in Definition 2.3 can be taken as $\Delta_{\mathcal{B}_0}$. In particular, if an object F is $\sigma_{\alpha,\beta}$-semistable for some $\alpha > 0, \beta \in \mathbb{R}$, then we have

$$\Delta_{\mathcal{B}_0}(F) \geq 0.$$

Remark 2.14.

(i) By ‘vary continuously’, we mean that if an object F is $\sigma_{\alpha_0,\beta_0}$-stable for some $\alpha_0 > 0$ and $\beta_0 \in \mathbb{R}$, then F is $\sigma_{\alpha,\beta}$-stable for (α, β) in an open neighborhood of (α_0, β_0).
(ii) For all $j \in \mathbb{Z}$, the object $\mathcal{B}_j$ is $\sigma_{\alpha,\beta}$-stable for every $\alpha > 0$ and $\beta \in \mathbb{R}$ by [13, Remark 9.4].

2.5. Stability conditions on $\mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0)$ *and* $\mathrm{Ku}(Y)$

The weak stability conditions in Proposition 2.13 do not restrict to stability conditions on $\mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0)$ directly. We need to modify them by one more tilting.

Fix some $0 < \alpha < \frac{1}{4}$ and $\beta = -1$. Consider the tilting of $\sigma_{\alpha,-1}$ with respect to the slope value 0 as in Proposition and Definition 2.12. We refurbish the main result for the stability conditions on $\mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0)$ in [13, Theorem 1.2] as follows.

Proposition and Definition 2.15. *Let $\alpha \in \mathbb{R}$ with $0 < \alpha < \frac{1}{4}$. The pair*

$$\sigma_{\alpha} := \left(\left(\mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)\right)^{0}_{\sigma_{\alpha,-1}} \bigcap \mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0), Z = \mathrm{ch}^{-1}_{\mathcal{B}_0,1} - i\,\mathrm{rk} \right) \tag{2.17}$$

is a stability condition on $\mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0)$ with respect to the natural rank-2 lattice

$$\mathrm{ch}_{\mathcal{B}_0,\leq 1} : \mathrm{K}_{\mathrm{num}}(\mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0)) \to \Lambda = \{(\mathrm{rk}(F), \mathrm{ch}_1(\mathrm{Forg}(F)) \colon F \in \mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0)\}.$$

Moreover, the stability condition σ_{α} does not depend on the choice of α.

Proof. By Proposition 2.13 and [13, Proposition 2.15, Proposition 5.1 and Proof of Theorem 1.2], if we replace the central charge in the pair (2.17) by $-iZ_{\alpha,-1}$, then that will be a stability condition on $\mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0)$. We only need to check that the central charge in (2.17) induces the same slope function as that induced by $-iZ_{\alpha,-1}$.

For any object E in $\mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0)$, by (2.2) and (2.13), we have

$$\chi_{\mathrm{D^b}(\mathbb{P}^3,\mathcal{B}_0)}(\mathcal{B}_1, E) = \chi_{\mathrm{D^b}(\mathbb{P}^3,\mathcal{B}_0)}(\mathcal{B}_3, E) = 0. \tag{2.18}$$

Recall that the Riemann–Roch formula for $\mathbb{P}^3$ is given as

$$\chi_{\mathbb{P}^3}(F) = \mathrm{ch}_3(F) + 2\,\mathrm{ch}_2(F) + \frac{11}{6}\,\mathrm{ch}_1(F) + \mathrm{rk}(F) \tag{2.19}$$

for every object F in $\mathrm{D^b}(\mathbb{P}^3)$. Recall that to simplify the notation, we write ch_1 (resp. ch_2) for $l \cdot \mathrm{ch}_1$ (resp. $h \cdot \mathrm{ch}_2$) here.

Denote $G = \mathrm{Forg}(E \otimes_{\mathcal{B}_0} \mathcal{B}_{-1})$. By (2.9), (2.18) and [37, Corollary 3.9], we have

$$0 = \chi_{\mathrm{D^b}(\mathbb{P}^3,\mathcal{B}_0)}(\mathcal{B}_1, E) = \chi_{\mathbb{P}^3}(G) = \mathrm{ch}_3(G) + 2\,\mathrm{ch}_2(G) + \frac{11}{6}\,\mathrm{ch}_1(G) + \mathrm{rk}(G), \tag{2.20}$$

$$0 = \chi_{\mathrm{D^b}(\mathbb{P}^3,\mathcal{B}_0)}(\mathcal{B}_3, E) = \chi_{\mathbb{P}^3}(G(-H)) = \mathrm{ch}_3(G) + \mathrm{ch}_2(G) + \frac{1}{3}\,\mathrm{ch}_1(G). \tag{2.21}$$

By [37, Corollary 3.9], for every $\mathcal{B}_i$ we have

$$\mathrm{ch}_2(\mathcal{B}_i \otimes_{\mathcal{B}_0} \mathcal{B}_{-1}) = \mathrm{ch}_2(\mathcal{B}_{i-1}) = \mathrm{ch}_2(\mathrm{Forg}(\mathcal{B}_i)) - \frac{1}{2}\,\mathrm{ch}_1(\mathrm{Forg}(\mathcal{B}_i)) + \frac{1}{8}\,\mathrm{rk}(\mathcal{B}_i)$$

Note that the Chern characters of $\mathrm{Forg}(\mathcal{B}_i)$ can be computed using (2.7), (2.10) and (2.11). By [15, Proposition 2.12] and restricting to a general hyperplane, the character $\mathrm{ch}_{\leq 2}(E)$ is spanned by $\mathrm{ch}_{\leq 2}(\mathcal{B}_i)$'s. As the sheaf $\mathcal{B}_{-1}$ is a flat $\mathcal{B}_0$-module, the operation $- \otimes_{\mathcal{B}_0} \mathcal{B}_{-1}$ is linear on $\mathrm{ch}_{\leq 2}$, so we have

$$\begin{aligned} \mathrm{ch}_2(G) &= \mathrm{ch}_2(\mathrm{Forg}(E)) - \frac{1}{2}\,\mathrm{ch}_1(\mathrm{Forg}(E)) + \frac{1}{8}\,\mathrm{rk}(E) \text{ and} \\ \mathrm{ch}_1(G) &= \mathrm{ch}_1(\mathrm{Forg}(E)) - \frac{1}{2}\,\mathrm{rk}(E). \end{aligned}$$

By subtracting the equations (2.20) and (2.21), we have

$$\mathrm{ch}_2(\mathrm{Forg}(E)) = -\,\mathrm{ch}_1(\mathrm{Forg}(E)) - \frac{3}{8}\,\mathrm{rk}(E). \tag{2.22}$$

Note that

$$\begin{aligned} \Im(-iZ_{\alpha,-1}(E)) &= \mathrm{ch}_2^{-1}(\mathrm{Forg}\,E) - \frac{11}{32}\,\mathrm{rk}(E) - \frac{1}{2}\alpha^2\,\mathrm{rk}(E) \\ &= \mathrm{ch}_2(\mathrm{Forg}\,E) + \mathrm{ch}_1(\mathrm{Forg}\,E) + \frac{1}{2}\,\mathrm{rk}(E) - \frac{11}{32}\,\mathrm{rk}(E) - \frac{1}{2}\alpha^2\,\mathrm{rk}(E) \\ &= -\,(\frac{7}{32} + \frac{1}{2}\alpha^2)\,\mathrm{rk}(E) \qquad \text{by (2.22).} \end{aligned}$$

We have

$$Z(E) = \mathrm{ch}^{-1}_{\mathcal{B}_0,1}(E) - i\,\mathrm{rk}(E) = \Re(-iZ_{\alpha,-1}(E)) + \frac{i}{\frac{7}{32}+\frac{1}{2}\alpha^2}\Im(-iZ_{\alpha,-1}(E)).$$

Therefore, the slope function induced by $-iZ_{\alpha,-1}$ is the same up to a constant scalar as that of $Z = \mathrm{ch}^{-1}_{\mathcal{B}_0,1} - i\,\mathrm{rk}$. So the pair (2.17) is a stability condition on $\mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0)$. Note that the stability conditions σ_α's vary continuously when α varies. Since they have the same central charge, all of them are the same stability condition. □

It is worth pointing out that to check the stability of an object in $\mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0)$, we usually only need to work in the heart $\mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$. The following simple lemma makes this more precise.

Lemma 2.16. *Let E be a $\sigma_{\alpha,-1}$-stable object such that*

(a) *E is an object in $\left(\mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)\right)^0_{\sigma_{\alpha,-1}} \bigcap \mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0)$;*
(b) $\mathrm{Hom}_{\mathcal{B}_0}(T, E) = 0$ *for every* $T \in \mathrm{Coh}(\mathbb{P}^3, \mathcal{B}_0)$ *of zero dimensional support.*

Then E is σ_α-stable.

Proof. We only need to show that E is stable with respect to the weak stability condition $\left(\left(\mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)\right)^0_{\sigma_{\alpha,-1}}, -iZ_{\alpha,-1}\right)$. Denote $\mathcal{A} = \mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$. Let F be a non-zero proper subobject of E in $\left(\mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)\right)^0_{\sigma_{\alpha,-1}}$, then we have the exact sequence of objects in $\mathcal{A}$:

$$0 \to \mathcal{H}^{-1}_{\mathcal{A}}(F) \to \mathcal{H}^{-1}_{\mathcal{A}}(E) \to \mathcal{H}^{-1}_{\mathcal{A}}(E/F) \to \mathcal{H}^{0}_{\mathcal{A}}(F) \to \mathcal{H}^{0}_{\mathcal{A}}(E) \to \mathcal{H}^{0}_{\mathcal{A}}(E/F) \to 0. \quad (2.23)$$

Since E is $\sigma_{\alpha,-1}$-stable, either $\mathcal{H}^{-1}_{\mathcal{A}}(E)$ or $\mathcal{H}^{0}_{\mathcal{A}}(E) = 0$. When $\mathcal{H}^{0}_{\mathcal{A}}(E) \neq 0$, we have $\mathcal{H}^{-1}_{\mathcal{A}}(F) = \mathcal{H}^{-1}_{\mathcal{A}}(E) = 0$. Since $E = \mathcal{H}^{0}_{\mathcal{A}}(E)$ is $\sigma_{\alpha,-1}$-stable, we have

$$\mu_{Z_{\alpha,-1}}(\mathcal{H}^{0}_{\mathcal{A}}(E/F)) \geq \mu_{Z_{\alpha,-1}}(E) > 0. \quad (2.24)$$

By condition (b), we have $Z_{\alpha,-1}(F) \neq 0$. Since $\mu_{Z_{\alpha,-1}}(\mathcal{H}^{-1}_{\mathcal{A}}(E/F)) < 0$ or $\mathcal{H}^{-1}_{\mathcal{A}}(E/F) = 0$, it is clear that

$$\mu_{Z_{\alpha,-1}}(E) \geq \mu_{Z_{\alpha,-1}}(F) > 0. \quad (2.25)$$

The equality in (2.25) can hold only when $\mathcal{H}^{-1}_{\mathcal{A}}(E/F) = 0$ and $Z_{\alpha,-1}(E/F) = 0$. In particular, $E/F = \mathcal{H}^{0}_{\mathcal{A}}(E/F)$ in this case. By Definition 2.2, we always have

$$\mu_{Z_{\alpha,-1}}(E/F)) > \mu_{Z_{\alpha,-1}}(F) > 0.$$

Therefore,

$$\mu_{-iZ_{\alpha,-1}}(E/F) > \mu_{-iZ_{\alpha,-1}}(F) > 0.$$

By Definition 2.2, the object E is $\mu_{-iZ_{\alpha,-1}}$-stable. A similar argument also holds for the case when $\mathcal{H}^{-1}_{\mathcal{A}}(E) \neq 0$. $\square$

Another issue is that the Clifford structure and the embedding of $\mathrm{Ku}(Y)$ in $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$ depend on the choice of the line L to blow up, see Remark 2.11. However, for the induced stability conditions on the Kuznetsov component, we have the following result.

Proposition 2.17 *([45, Proposition 2.6]). If σ is a stability condition as defined in (2.17), then the induced stability condition $(\Psi \circ \rho_L^*)^{-1}\sigma$ on $\mathrm{Ku}(Y)$ is independent of the choice of L.*

Remark 2.18. As we are only interested in the stability of objects in $\mathrm{Ku}(Y)$, we will omit L in all the morphisms and functors that rely on L in what follows. For simplicity, we will also write σ instead of $(\Psi \circ \rho_L^*)^{-1}\sigma$ for the stability condition on $\mathrm{Ku}(Y)$.

The stability condition σ is also a full numerical stability condition. The whole connected component $\mathrm{Stab}^\dagger(\mathrm{Ku}(Y))$ containing σ is described in [12, Theorem 29.1].

3. Symplectic resolution of the moduli space $M_\tau(2v_0)$ with $v_0^2 = 2$

This section is devoted to the proof of Theorem 1.1. After recalling the definition of algebraic Mukai lattice of $\mathrm{Ku}(Y)$ and stating the main result, we describe the local structure of the moduli space $M_\tau(2v_0)$ at the worst singular points. This is used to construct the symplectic resolution $\widetilde{M}$ by blowing up the singular locus with the reduced scheme structure as in [46]. Finally, we obtain the projectivity and the deformation class of $\widetilde{M}$ by specializing to Kuznetsov components equivalent to the bounded derived category of a K3 surface.

3.1. Algebraic Mukai lattice of $\mathrm{Ku}(Y)$

Let Y be a cubic fourfold over $\mathbb{C}$ and $\mathrm{Ku}(Y)$ be its Kuznetsov component. The *algebraic Mukai lattice* $H^*_{\mathrm{alg}}(\mathrm{Ku}(Y), \mathbb{Z})$ of $\mathrm{Ku}(Y)$ is introduced in [13, Proposition and Definition 9.5]. It consists of algebraic cohomology classes of Y which are orthogonal to the classes of $\mathcal{O}_Y$, $\mathcal{O}_Y(H)$, $\mathcal{O}_Y(2H)$ with respect to the Euler pairing.

As for an alternative description, the algebraic Mukai lattice is $\mathrm{K}_{\mathrm{num}}(\mathrm{Ku}(Y))$ equipped with a Mukai pairing:

$$([E],[F]) := -\chi(E,F) = -\chi(F,E) \tag{3.1}$$

for objects E and F in $\mathrm{Ku}(Y)$. The signature of the Mukai pairing is $(2,\rho)$, where $0 \leq \rho \leq 20$.

We will be only interested in a sub-lattice in $\mathrm{K}_{\mathrm{num}}(\mathrm{Ku}(Y))$ generated by two special classes

$$\lambda_i := [\mathsf{L}_{\mathcal{O}_Y}\mathsf{L}_{\mathcal{O}_Y(H)}\mathsf{L}_{\mathcal{O}_Y(2H)}(\mathcal{O}_L(iH))] \text{ for } i = 1 \text{ and } 2, \tag{3.2}$$

where L is a line on Y and $\mathsf{L}_{\mathcal{O}_Y}\mathsf{L}_{\mathcal{O}_Y(H)}\mathsf{L}_{\mathcal{O}_Y(2H)} : \mathrm{D^b}(Y) \to \mathrm{Ku}(Y)$ is the projection functor with respect to the semiorthogonal decomposition $\langle \mathrm{Ku}(Y), \mathcal{O}_Y, \mathcal{O}_Y(H), \mathcal{O}_Y(2H)\rangle$ of $\mathrm{D^b}(Y)$. The Mukai pairing of them can be computed as:

$$(\lambda_1, \lambda_1) = (\lambda_2, \lambda_2) = 2, (\lambda_1, \lambda_2) = -1. \tag{3.3}$$

In particular, when Y is a very general cubic fourfold, the Mukai lattice is spanned by λ_1 and λ_2.

3.2. The main theorem

Fix a primitive element v_0 in the algebraic Mukai lattice of $\mathrm{Ku}(Y)$ such that $(v_0, v_0) = 2$ and set $v := 2v_0$. Let $\mathrm{Stab}^\dagger(\mathrm{Ku}(Y))$ be the connected component of full numerical stability conditions on $\mathrm{Ku}(Y)$ containing σ. By [12, Theorem 21.24] (which makes use of the main result in [1]), for every $\tau \in \mathrm{Stab}^\dagger(\mathrm{Ku}(Y))$, the moduli stack $\mathcal{M}_\tau(v)$ parametrizing τ-semistable objects in $\mathrm{Ku}(Y)$ admits a good moduli space $M_\tau(v)$, which is a proper algebraic space.

In this section, we fix a stability condition $\tau \in \mathrm{Stab}^\dagger(\mathrm{Ku}(Y))$ which is *generic* with respect to v. In other words, the strictly τ-semistable objects in $M_\tau(v)$ are S-equivalent to the direct sum of two τ-stable objects with Mukai vector v_0. Note that v-generic stability conditions exist as $\mathrm{Stab}^\dagger(\mathrm{Ku}(Y))$ is a connected component of full numerical stability conditions. Also note that this stability condition may be different as that in Remark 2.18, which was denoted by σ, when Y is not very general. Set $M := M_\tau(v)$, then by [12, Theorem 29.2 and Remark 29.3], M is an irreducible proper algebraic space, and there is a holomorphic symplectic form on the smooth locus of M. The aim of this section is to prove the following result.

Theorem 3.1. *The moduli space M has a symplectic resolution $\widetilde{M}$, which is a 10-dimensional smooth projective hyperkähler manifold, deformation equivalent to the O'-Grady's example constructed in [54].*

The construction of the symplectic resolution is done in [46] in the case of the moduli space of semistable sheaves having Mukai vector $2v_0$ with $(v_0, v_0) = 2$ over a polarized K3 surface. A large part of their argument applies to our more general setup without much change. For this reason, we only sketch the proof, referring to [46] for a complete discussion. The main difference is that in the case of moduli of sheaves, the moduli are constructed as a GIT quotient. To study the local structure, it is enough to take an étale slice. In our case, we instead use the result on étale slice of algebraic stacks [2], and we give the details for this part of the proof.

The strategy is to study the local structure of the moduli space at the worst singularity and prove that its normal cone is isomorphic to an affine model obtained as a nilpotent orbit in the symplectic Lie algebra $\mathfrak{sp}(4)$. It turns out that the singularity is formally isomorphic to its normal cone. Since the singularity at the generic point of the singular locus of M is of type A_1, one can conclude that the blow up $\widetilde{M}$ of M at its singular locus endowed with the reduced scheme structure is a symplectic resolution of M. The other properties of $\widetilde{M}$ (projectivity, deformation type) will be obtained by degeneration to the locus of cubic fourfolds with Kuznetsov component equivalent to the bounded derived category of a K3 surface, as in [12].

3.3. Local structure of M

We have the following possibilities for $E \in M$:

(1) E is τ-stable. Its automorphism group is $\mathrm{Aut}(E) \cong \mathbb{C}^*$.
(2) E is S-equivalent to $F \oplus F'$ with non-isomorphic $F, F' \in M_\tau(v_0)$. In this case, we have $\mathrm{Aut}(E) \cong \mathbb{C}^* \times \mathbb{C}^*$.
(3) E is S-equivalent to $F^{\oplus 2}$ for $F \in M_\tau(v_0)$. Then, $\mathrm{Aut}(E) \cong \mathrm{GL}(2, \mathbb{C})$.

In this section, we investigate the structure of M in a formal neighborhood of a semistable point as in item (3).

Let E be a τ-semistable object in M. As in [46], the first ingredient for the proof is the description of the infinitesimal deformation of E. In the case of polystable sheaves on a K3 surface a good summary of the results is provided in [6, Sections 2 and 4], which we follow in our case. The deformation theory for perfect complexes in the derived category has been studied in [42]. In our setting, we consider the functor

$$\mathrm{Def}_E : \mathrm{Art} \to \mathrm{Sets}$$

from the category of local Artinian $\mathbb{C}$-algebras to the category of sets, which assigns to an object A in Art, the set $\mathrm{Def}_E(A)$ of equivalence classes of deformations of E to $Y_A := Y \times \operatorname{Spec} A$. Explicitly, objects in $\mathrm{Def}_E(A)$ are equivalence classes of pairs (E_A, φ), where E_A is a complex on Y_A together with an isomorphism $\varphi : E_A \otimes_A^{\mathbb{L}} \mathbb{C} \cong E$ (see [42, Definition 3.2.1]). Two pairs (E_A, φ) and (E'_A, φ') are equivalent if there is an isomorphism $\psi : E_A \cong E'_A$ such that $\varphi' \circ \psi = \varphi$.

Note that by [42, Lemma 3.2.4], E_A is an object in $\mathrm{D^b}(Y_A)$. By base change and the definition of E_A, if p is the closed point of $\operatorname{Spec} A$, then

$$\mathbf{R}\mathcal{H}om(\mathcal{O}_{Y_A}(iH), E_A)_p \cong \mathbf{R}\mathrm{Hom}(\mathcal{O}_{Y_A}(iH)_p, E_{A\,p}) \cong \mathbf{R}\mathrm{Hom}(\mathcal{O}_Y(iH), E) = 0$$

for $i = 0, 1, 2$. So the property of being in $\mathrm{Ku}(Y)$ is an open condition, and we may assume E_A is an object in $\mathrm{Ku}(Y_A) := \langle \mathcal{O}_{Y_A}, \mathcal{O}_{Y_A}(H), \mathcal{O}_{Y_A}(2H)\rangle^\perp$, where $\mathcal{O}_{Y_A}(H)$ is the trivial deformation of $\mathcal{O}_Y(H)$ to Y_A. By [42, Theorem 3.1.1 and Proposition 3.5.1], the functor Def_E is a deformation functor and its tangent space $\mathrm{Def}_E(\mathbb{C}[\epsilon])$ is $\mathrm{Ext}^1(E, E)$, where $\mathbb{C}[\epsilon] := \mathbb{C}[t]/(t^2)$.

As proved in [35], the definition of the trace map requires an additional step. Denote by ϵ_E the *linkage class* of E (see [35, Proposition 3.1]) and consider the composition

$$\mathrm{tr} : \mathrm{Ext}^2(E,E) \xrightarrow{\epsilon_E \circ -} \mathrm{Ext}^4(E, E \otimes \Omega_Y^4) \xrightarrow{\mathrm{Tr}} H^4(\Omega_Y^4) \cong \mathbb{C},$$

where the first map is given by the composition with ϵ_E and the second map is the usual trace map. We set $\mathrm{Ext}^2(E,E)_0 := \ker(\mathrm{tr})$, which is the obstructions space.

As noted at the beginning of the Appendix in [46], every polystable sheaf E on a smooth projective surface admits an injective resolution which is equivariant with respect to the canonical action of the automorphism group $\mathrm{Aut}(E)$ of E. The same argument applies to a polystable object $E \in M$. In fact, every $E \in \mathrm{D^b}(Y)$ has an injective resolution (see for instance [29, Proposition 2.35]). If E is stable, then $\mathrm{Aut}(E) \cong \mathbb{C}^*$, thus any injective resolution is $\mathrm{Aut}(E)$-equivariant. If $E = F \oplus F'$, where F and F' are non-isomorphic τ-stable objects, then consider two injective resolutions $F \to I$, $F' \to I'$ and define the injective resolution $E \to I \oplus I'$ which is $\mathrm{Aut}(E) \cong \mathbb{C}^* \times \mathbb{C}^*$-equivariant. Similarly, if $E = F^{\oplus 2}$, then $E \to I^{\oplus 2}$ is an injective resolution which is equivariant with respect to $\mathrm{Aut}(E) \cong \mathrm{GL}(2, \mathbb{C})$.

Then the argument in [46, Appendix] allows to construct a formal map

$$\kappa = \kappa_2 + \kappa_3 + \cdots : \mathrm{Ext}^1(E,E) \to \mathrm{Ext}^2(E,E)_0$$

known as the Kuranishi map, defined inductively on the order, with the following properties:

(1) The map κ is equivariant with respect to the conjugation action of $\mathrm{Aut}(E)$.
(2) The second order term $\kappa_2 : \mathrm{Ext}^1(E,E) \to \mathrm{Ext}^2(E,E)_0$ is given by the Yoneda product $\kappa_2(e) = e \smile e$ for $e \in \mathrm{Ext}^1(E,E)$.
(3) By [56] there exists an $\mathrm{Aut}(E)$-equivariant formal deformation $(\widehat{E}, \widehat{\varphi})$ of E having the versality property, parametrized by the formal scheme $D_\kappa := \kappa^{-1}(0)$.

Denote by $A := \mathbb{C}[\mathrm{Ext}^1(E,E)]$ the polynomial ring on $\mathrm{Ext}^1(E,E)$. Let $\widehat{A}$ be the completion of the ring A with respect to the maximal ideal $\mathfrak{m}$ of polynomial functions vanishing at 0. The Kuranishi map can be also written dually as

$$\kappa^* : \mathrm{Ext}^2(E,E)_0^* \to \mathfrak{m}^2\widehat{A}.$$

If $\mathfrak{a} \subset \widehat{A}$ is the ideal generated by the image of κ^*, then by definition we have

$$D_\kappa = \mathrm{Spf}(\widehat{A}/\mathfrak{a}) = \mathrm{colim}_n \, \mathrm{Spec}((\widehat{A}/\mathfrak{a})/\mathfrak{m}^n),$$

where $\mathfrak{m}$ is the maximal ideal of $\widehat{A}/\mathfrak{a}$ by abuse of notation.

On the other hand, the object E defines a closed point x in the moduli stack $\mathcal{M} := \mathcal{M}_\tau(v)$ and the S-equivalence class of E determines a point $\pi(x) \in M$, where $\pi : \mathcal{M} \to M$

is the good moduli space. The stabilizer G_x of x is identified with $\mathrm{Aut}(E)$. If $\mathfrak{n}$ is the maximal ideal defining the inclusion of the residual gerbe $BG_x \hookrightarrow \mathcal{M}$, we denote by $\mathcal{M}_n \hookrightarrow \mathcal{M}$ the n-th nilpotent thickening of $\mathcal{M}$ at x defined by $\mathfrak{n}^{n+1}$ for $n \geq 0$. By [2, Theorem 4.16] there exists the coherent completion of $\mathcal{M}$ at x, which is a complete local stack $(\widehat{\mathcal{M}}_x, \widehat{x})$ and a morphism $(\widehat{\mathcal{M}}_x, \widehat{x}) \to (\mathcal{M}, x)$ inducing isomorphisms on the n-th nilpotent thickenings of $\widehat{x}$ and x. Moreover, since $\mathcal{M}$ has a good moduli space M, by [2, Theorem 4.16(3)] we have $\widehat{\mathcal{M}}_x = \mathcal{M} \times_M \mathrm{Spec}(\widehat{\mathcal{O}}_{M,\pi(x)})$ and $\widehat{\mathcal{M}}_x \to \mathrm{Spec}(\widehat{\mathcal{O}}_{M,\pi(x)})$ is a good moduli space. Note that $\widehat{\mathcal{M}}_x \to \mathcal{M}$ is formally versal at x, i.e. for every commutative diagram

$$\begin{tikzcd}
(\widehat{\mathcal{M}}_x)_0 \arrow[r, hook] & \mathcal{Z} \arrow[r] \arrow[d, hook] & \widehat{\mathcal{M}}_x \arrow[d] \\
& \mathcal{Z}' \arrow[r] \arrow[ur, dashed] & \mathcal{M}
\end{tikzcd}$$

where $\mathcal{Z} \hookrightarrow \mathcal{Z}'$ is an inclusion of local artinian stacks, there is a lift $\mathcal{Z}' \to \widehat{\mathcal{M}}_x$ filling the above diagram (see [2, Definition A.13]).

The next lemma is a generalization of a well-known result for moduli spaces of sheaves on a K3 surface (see [30, Section 2.6] or [46, Proposition 4.1(3)]). In that case the proof relies on the description of the moduli space as a GIT quotient of an open subset of a Quot scheme and on the Luna slice Theorem. In our case of moduli spaces of complexes, we apply the results in [2], which among other things imply that the stack $\mathcal{M}$ is étale-locally a GIT quotient.

Lemma 3.2. *Assume $E = F \oplus F$ where F is τ-stable of Mukai vector v_0. Adopt the notation of $\pi(x)$, A and $\mathfrak{a}$ as above, then*

$$\widehat{\mathcal{O}}_{M,\pi(x)} \cong (\widehat{A}/\mathfrak{a})^{\mathrm{Aut}(E)} \cong \widehat{A}^{\mathrm{Aut}(E)}/(\mathfrak{a} \cap \widehat{A}^{\mathrm{Aut}(E)}).$$

Proof. Consider the quotient stack $\mathcal{T} := [\mathrm{Spec}(\mathrm{Sym}^\bullet(T_{\mathcal{M},x}))/G_x]$, where $T_{\mathcal{M},x}$ is the tangent space to $\mathcal{M}$ at x. By definition $T_{\mathcal{M},x} = \mathrm{Def}_E(\mathbb{C}[\epsilon]) \cong \mathrm{Ext}^1(E, E)$, so in this case

$$\mathcal{T} = [\mathrm{Spec}\, A/\, \mathrm{Aut}(E)] \to T := \mathrm{Spec}\, A \,/\!\!/\, \mathrm{Aut}(E) = \mathrm{Spec}\, A^{\mathrm{Aut}(E)},$$

which is a good moduli space. We denote by $\mathcal{T}_n$ the n-th thickening of $\mathcal{T}$ at the point 0. As computed in the proof of [2, Theorem 1.1], since $G_x = \mathrm{Aut}(E)$ is linearly reductive and smooth, the isomorphisms $\mathcal{M}_0 = BG_x \cong \mathcal{T}_0$ and $\mathcal{M}_1 \cong \mathcal{T}_1$ lift to closed immersions $\mathcal{M}_n \hookrightarrow \mathcal{T}_n$ which effectivize to a closed immersion $\widehat{\mathcal{M}}_x \hookrightarrow \widehat{\mathcal{T}}$, where $\widehat{\mathcal{T}} := \mathcal{T} \times_T \mathrm{Spec}\, \widehat{\mathcal{O}}_{T,0}$. Note that $\widehat{\mathcal{O}}_{T,0} \cong ((A^{\mathrm{Aut}(E)})_{\mathfrak{m}})\widehat{\ } \cong (\widehat{A^{\mathrm{Aut}(E)}})_{\mathfrak{m}} \cong \widehat{A^{\mathrm{Aut}(E)}} \cong \widehat{A}^{\mathrm{Aut}(E)}$, as localization and completion with respect to a maximal ideal commute.

On the other hand, note that G_x acts on the quotient $\widehat{A}/\mathfrak{a}$. Indeed, as κ is G_x-equivariant, we have $G_x(\mathfrak{a}) \subset \mathfrak{a}$, so the action on the quotient is well-defined.

We claim that $(\widehat{A}/\mathfrak{a})^{G_x}$ is a complete local ring. In order to prove this, we firstly show that there is an isomorphism of rings $(\widehat{A}/\mathfrak{a})^{G_x} \cong \widehat{A}^{G_x}/(\mathfrak{a} \cap \widehat{A}^{G_x})$. Indeed, note that the surjection $\widehat{A} \twoheadrightarrow (\widehat{A}/\mathfrak{a})$ induces surjections $\widehat{A}/\mathfrak{m}^n \twoheadrightarrow (\widehat{A}/\mathfrak{a})/\mathfrak{m}^n$ for every n. Now recall that G_x is linearly reductive, so every surjection $B \twoheadrightarrow C$ of G_x-rings induces a surjection $B^{G_x} \twoheadrightarrow C^{G_x}$ on the invariant rings. As a consequence, we have the surjections $(\widehat{A}/\mathfrak{m}^n)^{G_x} \twoheadrightarrow ((\widehat{A}/\mathfrak{a})/\mathfrak{m}^n)^{G_x}$ for every n. Passing to the completions, it follows that there is a surjection $\widehat{A}^{G_x} \twoheadrightarrow (\widehat{A}/\mathfrak{a})^{G_x}$. An easy computation shows that this surjection induces a surjection $\widehat{A}^{G_x}/(\mathfrak{a} \cap \widehat{A}^{G_x}) \twoheadrightarrow (\widehat{A}/\mathfrak{a})^{G_x}$, which is injective.

Now note that $\widehat{A}^{G_x}/(\mathfrak{a} \cap \widehat{A}^{G_x})$ is a local ring. Indeed, $\widehat{A}^{G_x} \cong \widehat{A^{G_x}}$ is a local ring, and the quotient of a local ring is a local ring. Moreover, by [46, Equation (4.7)] we have the explicit description of $\widehat{A}^{G_x}/(\mathfrak{a} \cap \widehat{A}^{G_x})$ as the quotient of the ring of formal power series, i.e.

$$\widehat{A}^{G_x}/(\mathfrak{a} \cap \widehat{A}^{G_x}) \cong \mathbb{C}[\![X_1, \ldots, X_4, Y_{11}, Y_{12}, \ldots, Y_{44}]\!]/I,$$

where I is an ideal of $\mathbb{C}[\![X_1, \ldots, X_4, Y_{11}, Y_{12}, \ldots, Y_{44}]\!]$ (for the precise definition see [46, Section 4, page 762]). Indeed, the same computation as in [46] with respect to a fixed symplectic basis on $V := \operatorname{Ext}^1(F, F)$ can be performed using $G_x \cong \mathrm{GL}_2$, $\operatorname{Ext}^1(E, E) \cong \mathfrak{gl}_2 \otimes V$, $\operatorname{Ext}^2(E, E)_0 \cong \mathfrak{sl}_2$ and the description of the generators of A^{SL_2} in terms of the traces of the coordinate functions on A. Since the quotient of a Noetherian complete local ring is complete, we deduce the desired properties for $(\widehat{A}/\mathfrak{a})^{G_x}$.

Define the stack $\mathcal{K} := [\operatorname{Spec}(\widehat{A}/\mathfrak{a})/G_x]$, whose good moduli space is $\mathcal{K} \to \operatorname{Spec}(\widehat{A}/\mathfrak{a})^{G_x}$ (see [3, Example 8.3]). By the above computation and [2, Theorem 1.3] we have that $\mathcal{K}$ is coherently complete along x. Let $\mathcal{K}_n$ be the n-th thickening of $\mathcal{K}$ at 0. The G_x-equivariant versal family $(\widehat{E}, \widehat{\varphi})$ constructed out of κ defines a collection of equivariant compatible objects $(E_n, \varphi_n) \in \operatorname{Def}_E((\widehat{A}/\mathfrak{a})/\mathfrak{m}^{n+1})$ for every n. Equivalently we have the compatible collection $\mathcal{K}_n \to \mathcal{M}$. Since $\mathcal{K}$ is coherently complete, by [2, Corollary 2.6] these morphisms effectivize to $\mathcal{K} \to \mathcal{M}$. Also $\mathcal{K}$ satisfies the versality property as $(\widehat{E}, \widehat{\varphi})$ does.

Now note that $\mathcal{K}_0 \cong \mathcal{T}_0$ and $\mathcal{K}_1 \cong \mathcal{T}_1$ as $\mathfrak{a} \subset \mathfrak{m}^2\widehat{A}$. Thus we have the commutative diagram

$$\begin{tikzcd}
\mathcal{M}_1 \arrow[r, hook] \arrow[d, "\cong"] & \widehat{\mathcal{M}}_x \arrow[d] \\
\mathcal{K}_1 \arrow[d, hook] & \mathcal{M} \\
\mathcal{K} \arrow[ur] \arrow[uur, dashed, "f"] &
\end{tikzcd} .$$

By the universal property of $\widehat{\mathcal{M}}_x$, there exists a lifting $f : \mathcal{K} \to \widehat{\mathcal{M}}_x$ filling the above diagram and inducing a collection of morphisms $f_n : \mathcal{K}_n \to \widehat{\mathcal{M}}_x$ for every $n \geq 0$. Since

$\mathcal{K}_1 \cong \mathcal{M}_1 \hookrightarrow \widehat{\mathcal{M}}_x$ is a closed immersion, by [2, Proposition A.8] we have that f_n and f are closed immersions. So we have the commutative diagram

$$\begin{tikzcd} \mathcal{K} \arrow[r, "\mathrm{id}"] \arrow[d, hook, "f"'] & \mathcal{K} \arrow[d] \\ \widehat{\mathcal{M}}_x \arrow[r] \arrow[ur, dashed, "g"] & \mathcal{M}. \end{tikzcd}$$

The versality property of $\mathcal{K}$ implies that there exists a lifting $g : \widehat{\mathcal{M}}_x \to \mathcal{K}$ filling the above diagram. This implies that f is an isomorphism.

Now consider the maps $A\colon \mathcal{K} \cong^f \widehat{\mathcal{M}}_x \to \mathrm{Spec}(\widehat{\mathcal{O}}_{M,\pi(x)})$ and $B\colon \widehat{\mathcal{M}}_x \cong^{f^{-1}} \mathcal{K} \to \mathrm{Spec}(\widehat{A}/\mathfrak{a})^{G_x}$. By [3, Theorem 6.6] we have the bijections

$$\mathrm{Hom}(\mathcal{K}, \mathrm{Spec}(\widehat{\mathcal{O}}_{M,\pi(x)})) \cong \mathrm{Hom}(\mathrm{Spec}(\widehat{A}/\mathfrak{a})^{G_x}, \mathrm{Spec}(\widehat{\mathcal{O}}_{M,\pi(x)}))$$

and

$$\mathrm{Hom}(\widehat{\mathcal{M}}_x, \mathrm{Spec}(\widehat{A}/\mathfrak{a})^{G_x}) \cong \mathrm{Hom}(\mathrm{Spec}(\widehat{\mathcal{O}}_{M,\pi(x)}), \mathrm{Spec}(\widehat{A}/\mathfrak{a})^{G_x}).$$

Thus A and B factor through $a\colon \mathrm{Spec}(\widehat{A}/\mathfrak{a})^{G_x} \to \mathrm{Spec}(\widehat{\mathcal{O}}_{M,\pi(x)})$ and $b\colon \mathrm{Spec}(\widehat{\mathcal{O}}_{M,\pi(x)}) \to \mathrm{Spec}(\widehat{A}/\mathfrak{a})^{G_x}$, respectively. By construction, we have that b is the inverse of a. This implies an isomorphism $(\widehat{A}/\mathfrak{a})^{\mathrm{Aut}(E)} \cong \widehat{\mathcal{O}}_{M,\pi(x)}$ of local rings as we wanted. $\square$

Note that since the tangent space to the moduli space at a polystable object E is identified with $\mathrm{Ext}^1(E,E)^{\mathrm{Aut}(E)}$, the singular part M^{sing} of M corresponds to the locus where the dimension of the tangent space jumps, i.e. the locus of polystable objects. In particular, we have a stratification

$$\Delta \subset M^{\mathrm{sing}} \subset M,$$

where

$$\Delta \cong M_\tau(v_0) \quad \text{and} \quad M^{\mathrm{sing}} \cong \mathrm{Sym}^2(M_\tau(v_0)).$$

3.4. Affine model

The affine model Z for the local structure at the worst singularities of M is described in [46, Section 2]. Here we recall the definition for sake of completeness referring to [46] for the details.

Let V be a 4-dimensional $\mathbb{C}$-vector space with a symplectic form ω. Denote by $\mathfrak{sp}(V)$ the associated symplectic Lie algebra which has dimension 10. Consider the set $Z \subset \mathfrak{sp}(v)$ defined as

$$Z = \{B \in \mathfrak{sp}(V) | B^2 = 0\}.$$

The ideal $I_0 \subset \mathbb{C}[\mathfrak{sp}(V)]$ defining Z is generated by the coefficients of the matrix B^2, which are linearly dependent, and Z has dimension 6. The singular locus Z_{sing} of Z is

$$Z_{\mathrm{sing}} = \{B \in Z | \operatorname{rk}(B) \leq 1\}$$

and it is defined by the ideal L_0 generated by the 2×2-minors of B. Note that Z_{sing} has dimension 4 and is singular in the origin.

On the other hand, consider the Grassmannian G parametrizing maximal isotropic subspaces $U \subset V$. Define the incidence subvariety

$$\widetilde{Z} = \{(B, U) \in Z \times G | B(U) = 0\} \subset Z \times G.$$

The canonical projection $\pi : \widetilde{Z} \to G$ to the second factor is identified with the canonical projection $T^*G \to G$. Moreover, the first projection $\sigma : \widetilde{Z} \to Z$ is a semi-small resolution. Indeed, over matrices $B \in Z_{\mathrm{sing}}$ with $\operatorname{rk}(B) = 1$, the fiber of σ is $\mathbb{P}((\ker B/\operatorname{im} B)^*) \cong \mathbb{P}^1$, while over $B = 0$ the fiber is G.

Proposition 3.3 *([46], Théorème 2.1). The resolution $\sigma : \tilde{Z} \to Z$ is isomorphic to the blow-up of Z along $Z^{sing} \subset Z$.*

A key property of Z is that its singularity is rigid with respect to deformations, meaning that a deformation of Z which does not change the singularities of Z around the origin cannot change the singularity at the origin [46, Théorème 3.1].

3.5. Symplectic resolution of M

The main result of this section is the following.

Theorem 3.4. *The blow-up $\widetilde{M}$ of the singular locus of M with the structure of reduced algebraic space is a symplectic resolution of M.*

Here M is an irreducible proper algebraic space by [12, Theorem 29.2 and Remark 29.3]. For the definition of blow-up of an algebraic space and reduced algebraic space consult Stacks Project, Sections 69.17 and 64.7, respectively. The argument is due to [46] and we summarize it for the interested reader.

Theorem 3.4 is a consequence of the following result.

Proposition 3.5 *([46, Théorème 4.5]). Let $E := F^{\oplus 2}$ where $F \in M_\tau(v_0)$. Then there is an isomorphism of germs of analytic spaces*

$$(M, [E]) \cong (\mathbb{C}^4 \times Z, 0).$$

Proof. We use the notation introduced in Sections 3.3 and 3.4. By Lemma 3.2 we have the isomorphism $\widehat{\mathcal{O}}_{M,\pi(x)} \cong \widehat{A}^{\mathrm{Aut}(E)}/(\mathfrak{a} \cap \widehat{A}^{\mathrm{Aut}(E)})$.

Set $V = \mathrm{Ext}^1(F,F)$; then $\mathrm{Ext}^1(E,E) \cong \mathfrak{gl}_2 \otimes V \cong \mathfrak{gl}_2^{\oplus 4}$ and $\mathrm{Ext}^2(E,E)_0 \cong \mathfrak{sl}_2$. By [46, Proposition 4.3], we have $\widehat{A}^{\mathrm{Aut}(E)}/(\mathfrak{a} \cap \widehat{A}^{\mathrm{Aut}(E)}) \cong \widehat{R}/I$, where $R = \mathbb{C}[\mathbb{C}^4 \times \mathfrak{sp}(4)]$, $\widehat{R}$ is the completion of R at 0 and I is an ideal of $\widehat{R}$. Moreover, by [46, Proposition 4.3(3)] the ideal I_0 corresponds to the locus of strictly semistable objects via the isomorphism above and by [46, Lemma 4.4] the ideal of initial terms of I satisfies $\mathrm{in}(I) = I_0 R$.

In order to prove $(M,[E]) \cong (\mathbb{C}^4 \times Z, 0)$ by Artin's Theorem [5, Corollary 1.6] and the above observations, it is enough to show $\widehat{R}/I \cong \widehat{R}/I_0\widehat{R}$. By the computation in [46, Section 5] the deformation of $\widehat{R}/I$ towards its normal cone is trivial. This implies the statement. $\square$

Proof of Theorem 3.4. The same computation as in [54, (2.2.4), Claim (1.8.8)] shows that the singularity of a point in $M^{\mathrm{sing}} \setminus \Delta$ is of type A_1 transversally to M^{sing}. Thus the blow-up of $M \setminus \Delta$ in $M^{\mathrm{sing}} \setminus \Delta$ is a resolution of these singularities and the symplectic form over the smooth part of M extends to the exceptional divisor of this blow-up. By Propositions 3.5 and 3.3 applied to the points in Δ, we have that the blow-up σ of M in M^{sing} is a resolution of singularities. Note that the fiber of σ over a point in Δ is a 3-dimensional quadric. Thus the symplectic form extends to M by Hartog's Theorem. $\square$

Remark 3.6. Note that the moduli space M is normal, as it is locally described by Z.

3.6. Relative version

In order to complete the proof of Theorem 3.1, we need to apply the theory introduced in [12] about families of stability conditions and relative moduli spaces.

Recall that given a family of cubic fourfolds $\mathcal{Y} \to S$ over a smooth quasi-projective variety S with relative ample class $\mathcal{O}_{\mathcal{Y}}(1)$, by [12, Lemma 30.1] there exists an admissible subcategory $\mathrm{Ku}(\mathcal{Y}) \subset \mathrm{D}^{\mathrm{b}}(\mathcal{Y})$ which defines a family of Kuznetsov components over S, obtained from the relative exceptional collection $\mathcal{O}_{\mathcal{Y}}, \mathcal{O}_{\mathcal{Y}}(1), \mathcal{O}_{\mathcal{Y}}(2)$. For every point $s \in S$ the base change category $\mathrm{Ku}(\mathcal{Y})_s$ to $\mathrm{Spec}(\kappa(s))$ is the right orthogonal to the exceptional collection $\mathcal{O}_{\mathcal{Y}_s}, \mathcal{O}_{\mathcal{Y}_s}(1), \mathcal{O}_{\mathcal{Y}_s}(2)$. In particular, $\mathrm{Ku}(\mathcal{Y})_s$ is the Kuznetsov component $\mathrm{Ku}(\mathcal{Y}_s)$ of $\mathcal{Y}_s$ for every $s \in S$. A stability condition on $\mathrm{Ku}(\mathcal{Y})$ is a collection $\underline{\tau} = (\tau_s)_{s\in S}$ of stability conditions τ_s on $\mathrm{Ku}(\mathcal{Y}_s)$ for $s \in S$, satisfying the compatibility conditions of [12, Definitions 20.5 and 21.15].

The next result is the relative version of Theorem 3.4 over a 1-dimensional base and is the generalization of [12, Corollary 32.1] to the case of a non-primitive Mukai vector.

Proposition 3.7. *Let Y be a cubic fourfold, let $v = 2v_0$ be a Mukai vector in $H^*_{\mathrm{alg}}(\mathrm{Ku}(Y),\mathbb{Z})$ with $(v_0,v_0) = 2$ and let $\tau \in \mathrm{Stab}^\dagger(\mathrm{Ku}(Y))$ be v-generic. Let Y' be another cubic fourfold such that there is smooth family of cubic fourfolds over a connected quasi-projective base with fibers Y and Y' along which v_0 remains a Hodge class.*

Then there exist a family $g : \mathcal{Y} \to C$ of cubic fourfolds over a smooth connected quasi-projective curve, complex points $0, 1 \in C(\mathbb{C})$ and a stability condition $\underline{\tau}$ on $\mathrm{Ku}(\mathcal{Y})$ over C such that:

(1) *$\mathcal{Y}_0 = Y$ and $\mathcal{Y}_1 = Y'$.*
(2) *v_0 is a primitive vector in $H^*_{\mathrm{alg}}(\mathrm{Ku}(\mathcal{Y}_c), \mathbb{Z})$ for all $c \in C$.*
(3) *τ_c is v-generic for all $c \in C$ and τ_0 is a small deformation of τ so that $M_{\tau_0}(v) = M_\tau(v)$.*
(4) *There exist an algebraic space $M_{\underline{\tau}}(v)$ and a proper morphism $M_{\underline{\tau}}(v) \to C$ such that every fiber is the connected component containing the singular locus $\mathrm{Sym}^2(M_{\tau_c}(v_0))$ of the good moduli space $M_{\tau_c}(v)$ of semistable objects in $\mathrm{Ku}(\mathcal{Y}_c)$.*
(5) *There exist an algebraic space $\widetilde{M_{\underline{\tau}}(v)}$ and a proper morphism $\widetilde{M_{\underline{\tau}}(v)} \to C$ making $\widetilde{M_{\underline{\tau}}(v)}$ a relative symplectic resolution of $M_{\underline{\tau}}(v)$: its fiber over any point $c \in C$ is a symplectic resolution of the fiber of $M_{\underline{\tau}}(v)$ over c, obtained by blowing up the singular locus $\mathrm{Sym}^2(M_{\tau_c}(v_0))$.*

Proof. Properties (1)-(3) are a consequence of the assumptions and [12, Proposition 30.8] (in loc. cit. the authors assume v is primitive, but the same proof adapts to the non-primitive case).

In order to prove (4), note that by [12, Theorem 21.24(3)] the moduli stack $\mathcal{M}_{\underline{\tau}}(v)$, parametrizing $\underline{\tau}$-semistable objects in $\mathrm{Ku}(\mathcal{Y})$, admits a good moduli space $M_{\underline{\tau}}(v)$ which is a proper algebraic space over C. By Remark 3.6, the fiber $M_{\tau_c}(v)$ is normal for every $c \in C$. Thus $M_{\tau_c}(v)$ is a finite disjoint union of normal irreducible components. Denote by $M_{\tau_c}(v)'$ the irreducible component of $M_{\tau_c}(v)$ containing $\mathrm{Sym}^2(M_{\tau_c}(v_0))$. Then consider the irreducible component of $M_{\underline{\tau}}(v)$ with fiber $M_{\tau_c}(v)'$ at a point $c \in C$. By abuse of notation, we denote this component by $M_{\underline{\tau}}(v)$ and this defines the proper algebraic space of item (4).

Part (5) follows from the fact that the relative symmetric product $\mathrm{Sym}^2(M_{\underline{\tau}}(v_0))$ over C is proper over C and satisfies $\mathrm{Sym}^2(M_{\underline{\tau}}(v_0))_c = \mathrm{Sym}^2(M_{\tau_c}(v_0))$ for every $c \in C$, by [12, Corollary 32.1] applied to v_0. Thus we define $\widetilde{M_{\underline{\tau}}(v)}$ as the blow up of $M_{\underline{\tau}}(v)$ in $\mathrm{Sym}^2(M_{\underline{\tau}}(v_0))$ and we have that $\widetilde{M_{\underline{\tau}}(v)}_c$ is the blow up of $M_{\tau_c}(v)'$ in $\mathrm{Sym}^2(M_{\tau_c}(v_0))$. This implies (5). □

3.7. Proof of Theorem 3.1

By Theorem 3.4, we know $\widetilde{M}$ is smooth, connected, proper and symplectic of dimension 10. In this paragraph, we end the proof of Theorem 3.1. In particular, we show that M and $\widetilde{M}$ are projective, by proving that they carry an ample divisor, and that $\widetilde{M}$ is deformation equivalent to the O'Grady's 10-dimensional example.

Consider the irreducible component $M' \subset M$ containing M^{sing}. By abuse of notation, we still denote by $\widetilde{M}$ the blow up of M' in the reduced singular locus. By Proposition 3.7(3) we have that M' is a limit (in the sense of [55, Definition 1.12]) of moduli spaces M_n of semistable objects in the derived category of a K3 surface with Mukai vector v with respect to a v-generic stability condition which is geometric. Indeed, it is enough to choose a curve C in the moduli space of cubic fourfolds, such that its intersection with the loci of cubic fourfolds having Kuznetsov component equivalent to the bounded derived category of a K3 surface is dense in C. Such a choice of C is possible since the locus of cubic fourfolds with Kuznetsov component equivalent to the bounded derived category of a K3 surface is a countable union of divisors in the quasi-projective moduli space of cubic fourfolds [7], [12, Corollary 29.7], thus it is dense in the moduli space of cubic fourfolds. By [12, Proposition 32.4] and [52, Proposition 2.2, Corollary 3.16], the moduli space M_n admits a symplectic resolution $\widetilde{M}_n$ which is deformation equivalent to the irreducible holomorphic symplectic manifold constructed by O'Grady in [54]. Then by Proposition 3.7(4) the blow-up $\widetilde{M}$ is the limit of the smooth irreducible holomorphic symplectic varieties $\widetilde{M}_n$.

By the same argument used in [12, version 1, page 125], there is a non-degenerate quadratic form q defined over $H^2(\widetilde{M}, \mathbb{Z})$, which is the Beauville–Bogomolov–Fujiki form. By [55, Theorem 1.14], there is a bimeromorphic map $f : \widetilde{M} \dashrightarrow \widetilde{M}''$, where $\widetilde{M}''$ is a projective irreducible holomorphic symplectic manifold. Moreover, the bimeromorphic map f induces an isometry $H^2(\widetilde{M}, \mathbb{Z}) \cong H^2(\widetilde{M}'', \mathbb{Z})$ respecting the Beauville–Bogomolov–Fujiki forms, arguing as in [22, Section 27.1].

Now denote by l the divisor class on M' constructed in [14]. By [14, Theorem 1.1], the class l is strictly nef, i.e. $l \cdot C > 0$ for every curve $C \subset M'$. On the other hand, if $\tilde{l}$ is the pullback via the blow-up σ of l, then $q(\tilde{l}) > 0$. Indeed, the same statement is true for the desingularized moduli spaces of semistable objects on K3 surfaces, and the divisor class l behaves well with respect to deformations by [12, Theorem 21.25].

Let $\tilde{l}''$ be the line bundle of $\widetilde{M}''$ such that $\tilde{l} = f^*\tilde{l}''$; note that $q(\tilde{l}'') = q(\tilde{l}) > 0$. By [27, Corollary 3.10] [28], $\tilde{l}''$ is big. Since f is an isomorphism out of codimension 2, it follows $\tilde{l}$ is big too. Since $\widetilde{M}$ has trivial canonical bundle, the Base Point Free Theorem (see [34], or [4] for algebraic spaces) implies that $m\tilde{l}$ is globally generated for a certain integer $m \gg 0$. Since by Theorem 3.4, the moduli space M has rational singularities, we deduce that also ml is globally generated. Together with the fact that ml is strictly nef, we conclude that ml is ample. This implies the projectivity of M', and then of $\widetilde{M}$.

Finally note that since M' is normal and projective, we can apply the same argument in [33, Theorem 4.4] to deduce that $M' = M$, namely that M is irreducible, as explained in Lemma 3.8 below. The deformation type of $\widetilde{M}$ is obtained by degeneration to the loci of cubic fourfolds with associated K3 surface. This ends the proof of Theorem 3.1.

Lemma 3.8. *The moduli space M and its symplectic resolution $\widetilde{M}$ are irreducible.*

Proof. Since M is normal, we have that M is a finite disjoint union of normal irreducible components. The singular locus of M is $\mathrm{Sym}^2(M_\tau(v_0))$ which is connected, so it is contained in only one component of M which we denote by M'. The symplectic resolution

$\widetilde{M}$ is then a finite disjoint union of irreducible components where one is the blowup $\widetilde{M'}$ of M' in $\mathrm{Sym}^2(M_\tau(v_0))$.

Assume that M' has a universal family $\mathcal{U}$; let $\widetilde{\mathcal{U}}$ be the pullback of $\mathcal{U}$ to $\widetilde{M'}$. Assume there exists a point in $\widetilde{M}$ which is not in $\widetilde{M'}$. Note that this point determines the (S-equivalence) class of a τ-stable object G. Fix a τ-stable object $F \in M'$ which defines a point of $\widetilde{M'}$. Consider the projections $p\colon \widetilde{M'} \times Y \to \widetilde{M'}$ and $q\colon \widetilde{M'} \times Y \to Y$. We consider the following objects of $\mathrm{D^b}(\widetilde{M'})$: $p_* \mathcal{H}om(q^*F, \widetilde{\mathcal{U}})$ and $p_* \mathcal{H}om(q^*G, \widetilde{\mathcal{U}})$. Arguing as in the proof of [33, Theorem 4.1, item 3] (see also [13, Proposition A.7]), since $\widetilde{M'}$ and Y are smooth and projective, it is possible to show that $p_* \mathcal{H}om(q^*G, \widetilde{\mathcal{U}})[-1]$ is a locally free sheaf on $\widetilde{M'}$, while $p_* \mathcal{H}om(q^*F, \widetilde{\mathcal{U}})$ is quasi-isomorphic to a complex of locally free sheaves on $\widetilde{M'}$ supported in the degrees $0, 1, 2$. On the other hand, the numerical classes of F and G are equal, then by Grothendieck-Riemann-Roch the same is true for the relative objects $p_* \mathcal{H}om(q^*F, \widetilde{\mathcal{U}})$ and $p_* \mathcal{H}om(q^*G, \widetilde{\mathcal{U}})$. This leads to a contradiction with the previous computation as explained in [33]. We deduce that $\widetilde{M}$ is irreducible and so is M.

In general, we only have the existence of a quasi-universal family. In this case, it is enough to use the construction in [33, Lemma 4.2] and argue as before to conclude the proof of the statement. □

4. Stable objects in the moduli space $M_\sigma(2\lambda_1 + 2\lambda_2)$

In this section, we introduce the objects which form an open subset of $M_\sigma(2\lambda_1 + 2\lambda_2)$. After recalling the definition of instanton sheaves on a smooth cubic threefold from [21], we compute the projection of the push-forward of the stable instanton sheaves to $\mathrm{Ku}(Y)$.

Remark 4.1. Comparing with σ-stable objects, strictly semistable objects are easier to describe. Note that we may vary the stability condition σ to σ_0 in $\mathrm{Stab}^\dagger(\mathrm{Ku}(Y))$ such that

(a) $M^s_\sigma(2\lambda_1 + 2\lambda_2) = M^s_{\sigma_0}(2\lambda_1 + 2\lambda_2)$ and $M^s_\sigma(\lambda_1 + \lambda_2) = M^s_{\sigma_0}(\lambda_1 + \lambda_2)$;
(b) σ_0 is generic with respect to $2\lambda_1 + 2\lambda_2$.

By condition (b), as the character $\lambda_1 + \lambda_2$ is primitive, the Jordan–Hölder factors of strictly σ_0-semistable objects are all with character $\lambda_1 + \lambda_2$. By [45, Theorem 1.1], such a factor is always of the form

$$P_\ell := \mathsf{pr}(\mathcal{O}_\ell[-1]) = \mathrm{Cone}(\mathcal{I}_\ell[-1] \xrightarrow{\mathsf{ev}} \mathcal{O}_Y(-H)[1]), \tag{4.1}$$

where ℓ is a line in $Y \subset \mathbb{P}^5$, $\mathcal{I}_\ell$ denotes the ideal sheaf of ℓ. Let $F(Y)$ be the Fano variety of lines on Y; then the strictly semistable locus in $M_{\sigma_0}(2\lambda_1 + 2\lambda_2)$ is isomorphic to $\mathrm{Sym}^2 F(Y)$.

4.1. Moduli space of semistable instanton sheaves on a smooth cubic threefold

Recall the definition of λ_1 and λ_2 in (3.2). By a direct computation, their Chern characters are

$$\mathrm{ch}(\lambda_1) = (3, -H, -\frac{H^2}{2}, \frac{H^3}{6}, \frac{3}{8}) \text{ and } \mathrm{ch}(\lambda_2) = (-3, 2H, 0, -\frac{H^3}{3}, 0). \tag{4.2}$$

In particular, we have

$$\mathrm{ch}(2\lambda_1 + 2\lambda_2) = (0, 2H, -H^2, -\frac{H^3}{3}, \frac{3}{4}). \tag{4.3}$$

On the other hand, in [21] Druel studies the moduli space of semistable sheaves F on a smooth cubic threefold X with Chern classes (note that these are the classes on the threefold X)

$$\mathrm{rk}(F) = 2,\ c_1(F) = 0,\ c_2(F) = \frac{2H_X^2}{3} \text{ and } c_3(F) = 0.$$

We follow the definition in [44,39], and call such sheaves *rank* 2 *instanton sheaves* on cubic threefolds. Let $X = H \cap Y$ be a smooth cubic threefold and denote by $\iota : X \to Y$ the closed embedding. For such an instanton sheaf F, by a direct computation, we have

$$\mathrm{ch}(\iota_* F) = \mathrm{ch}(2\lambda_1 + 2\lambda_2).$$

We summarize the results about rank 2 instanton sheaves in [11,21] as follows.

Remark 4.2. Let X be a smooth cubic threefold. The moduli space M_{inst} of rank 2 instanton sheaves on X consists of the following objects, see [21, Theorem 3.5].

(i) F_Γ: For every stable rank 2 instanton bundle F, the zero locus of a non-zero section of $F(H)$ is a non-degenerate *elliptic quintic* curve Γ. Recall that a quintic elliptic curve is a *locally complete intersection* quintic curve with trivial canonical bundle and $h^0(\mathcal{O}_\Gamma) = 1$, and the curve is called non-degenerate if it spans $\mathbb{P}^4$. Conversely, for a generic section, the curve Γ is smooth.
For every non-degenerate quintic elliptic curve Γ, one can produce the vector bundle F_Γ by the Serre construction. For a more categorical description,

$$F_\Gamma := \mathrm{Cone}\left(\mathcal{I}_\Gamma(H)[-1] \xrightarrow{\mathsf{ev}} \mathcal{O}_X(-H)\right). \tag{4.4}$$

All these stable bundles form a dense affine open subset $M^{\mathrm{s}}_{\mathrm{inst}}$ in M_{inst}, see [11, Corollary 6.6].

(ii) F_C: Stable non-locally-free rank 2 instanton sheaves are one-to-one corresponding to smooth conic curves C. For each smooth conic C, one can define a stable sheaf F_C as the kernel of

$$\mathcal{O}_X \otimes \mathrm{Hom}(\mathcal{O}_X, \theta_C(H)) \xrightarrow{\mathrm{ev}} \theta_C(H). \tag{4.5}$$

Here we write θ_C for the theta-characteristic of C so that $\theta_C(H) \cong \mathcal{O}_{\mathbb{P}^1}(1)$ is a degree 1 line bundle on C.

The locus A in M_{inst} that parametrizes these sheaves is of dimension 4.

(iii) $\mathcal{I}_{\ell_1} \oplus \mathcal{I}_{\ell_2}$: Every strictly semistable rank 2 instanton sheaves is S-equivalent to this direct sum. Here ℓ_1 and ℓ_2 are lines (possibly the same) on X.

The locus B in M_{inst} that parametrizes these sheaves is isomorphic to $\mathrm{Sym}^2 F(X)$, where $F(X)$ stands for the Fano surface of lines on X.

The following properties of M_{inst} are summarized from [21, Section 4] and [11, Section 6]. Let $J^2(X)$ be the translate of the intermediate Jacobian which parametrizes 1-cycles of degree 2 on X. Consider the morphism

$$\mathfrak{c}_2 : M_{\mathrm{inst}} \to J^2(X) : F \mapsto \tilde{c}_2(F), \tag{4.6}$$

where $\tilde{c}_2(F)$ is the Abel–Jacobi invariant of $c_2(F)$ and where $c_2(F)$ is the second Chern class in the Chow group of 1-cycle Y.

(1) The moduli space M_{inst} is smooth and connected. The morphism $\mathfrak{c}_2$ induces an isomorphism of $M^{\mathrm{s}}_{\mathrm{inst}}$ onto its image in $J^2(X)$.
(2) The morphism $\mathfrak{c}_2$ contracts the locus A to $F_{\mathrm{conic}}(X) \subset J^2(X)$, where $F_{\mathrm{conic}}(X)$ is the image of the variety of conics and is isomorphic to $F(X)$. In particular, the morphism $\mathfrak{c}_2$ is isomorphic to the blowing up of $J^2(X)$ along $F_{\mathrm{conic}}(X)$.
(3) The morphism $\mathfrak{c}_2$ maps B onto an ample divisor which we denote by D_{F+F} in $J^2(X)$.

We will make use of these further details on M_{inst} in Section 6.

4.2. Formulas of E_Γ and E_C

The classification of semistable rank 2 instanton sheaves summarized in the previous section inspires us the construction of some objects in $\mathrm{Ku}(Y)$ with character $2\lambda_1 + 2\lambda_2$. Recall the definition of the projection functor $\mathsf{pr} = \mathsf{R}_{\mathcal{O}_Y(-H)}\mathsf{R}_{\mathcal{O}_Y(-2H)}\mathsf{L}_{\mathcal{O}_Y} = \mathsf{L}_{\mathcal{O}_Y}\mathsf{R}_{\mathcal{O}_Y(-H)}\mathsf{R}_{\mathcal{O}_Y(-2H)}$ as in (2.5).

Definition 4.3. Let Y be a smooth cubic fourfold. Let Γ be a quintic elliptic curve on Y. We define the object E_Γ as:

$$E_\Gamma := \mathsf{pr}(\mathcal{I}_\Gamma(H)). \tag{4.7}$$

Let C be a smooth conic curve on Y. We define the object E_C as:

$$E_C := \mathsf{pr}(\theta_C(H))[-1]. \tag{4.8}$$

We will need a more explicit expression of E_C, as computed in the following lemma.

Lemma 4.4. *The object E_C can be written as*

$$\mathrm{Cone}\left(\mathsf{L}_{\mathcal{O}_Y}(\theta_C(H))[-2] \xrightarrow{\mathsf{ev}} \mathcal{O}_Y(-H)[1] \otimes (\mathrm{Hom}(\mathsf{L}_{\mathcal{O}_Y}(\theta_C(H)), \mathcal{O}_Y(-H)[3]))^*\right). \tag{4.9}$$

Proof. Note that

$$\mathrm{Hom}_{\mathrm{D^b}(Y)}(\mathcal{O}_Y, \theta_C(H)[i]) = \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{O}_C, \theta_C(H)[i]) = \begin{cases} \mathbb{C}^2, & \text{when } i = 0; \\ 0, & \text{when } i \neq 0. \end{cases} \tag{4.10}$$

In particular, the object $\mathsf{L}_{\mathcal{O}_Y}(\theta_C(H))[-1]$ is a coherent sheaf on Y. Note that

$$\theta_C(H), \mathcal{O}_Y \in \mathcal{O}_Y(H)^{\perp} = {}^{\perp}\mathcal{O}_Y(-2H),$$

therefore the object $\mathsf{L}_{\mathcal{O}_Y}(\theta_C(H))$ is also in ${}^{\perp}\mathcal{O}_Y(-2H)$, in other words,

$$\mathsf{R}_{\mathcal{O}_Y(-2H)}\mathsf{L}_{\mathcal{O}_Y}(\theta_C(H)) = \mathsf{L}_{\mathcal{O}_Y}(\theta_C(H)).$$

Since $\mathcal{O}_Y \in {}^{\perp}\mathcal{O}_Y(-H)$, by Serre duality we have

$$\mathrm{Hom}_{\mathrm{D^b}(Y)}(\mathsf{L}_{\mathcal{O}_Y}(\theta_C(H)), \mathcal{O}_Y(-H)[i]) \cong \mathrm{Hom}_{\mathrm{D^b}(Y)}(\theta_C(H), \mathcal{O}_Y(-H)[i]) \tag{4.11}$$

$$\cong (\mathrm{Hom}_{\mathrm{D^b}(Y)}(\mathcal{O}_Y(-H), \theta_C(-2H)[4-i]))^* = \begin{cases} \mathbb{C}^2, & \text{when } i = 3; \\ 0, & \text{when } i \neq 3. \end{cases} \tag{4.12}$$

By Definition 2.7, the formula (4.9) for E_C holds. □

Proposition 4.5. *Let Y be a smooth cubic fourfold and X be a smooth hyperplane section of Y and $\iota : X \to Y$ be the embedding morphism. We have the following statements for objects of the form E_C and E_Γ.*

(1) *If C is a smooth conic contained in X, then*

$$E_C \cong \mathsf{pr}(\iota_* F_C),$$

where F_C is defined in (4.5).

(2) *If Γ is a non-degenerate quintic elliptic curve contained in X, then $E_\Gamma \cong \iota_* F_\Gamma$ which is defined in (4.4). In particular, the object E_Γ sits in the short exact sequence in* $\mathrm{Coh}(Y)$*:*

$$0 \to \mathcal{O}_X(-H) \to E_\Gamma \to \mathcal{I}_{\Gamma/X}(H) \to 0. \tag{4.13}$$

(3) *Let ℓ be a line on X; then $P_\ell \cong \mathsf{pr}(\mathcal{I}_{\ell/X})$.*
(4) *Both E_Γ and E_C are in $\mathrm{Ku}(Y)$ with character $2\lambda_1 + 2\lambda_2$.*

Remark 4.6. Note that the objects E_Γ are exactly $i_*\mathcal{E}$ in $P(Y)$ as that considered in [35, Theorem 7.3].

Proof of Proposition 4.5. (1). When C is contained in a smooth cubic threefold X, note that F_C is stable on X, so we have $\mathrm{Hom}(\mathcal{O}_Y, \iota_* F_C) = 0$. Note that $\iota_* F_C = \mathrm{Cone}(\iota_*\mathcal{O}_X^{\oplus 2} \to \theta_C(H))[-1]$ and $\mathrm{Hom}(\mathcal{O}_Y, \iota_*\mathcal{O}_X[i]) = 0$ when $i \neq 0$. Together with (4.10), this implies that $\iota_* F_C \in \mathcal{O}_Y^\perp$. Since both $\mathcal{O}_C, \mathcal{O}_X \in \mathcal{O}_Y(H)^\perp$, we have

$$\mathsf{pr}(\iota_* F_C) = \mathsf{R}_{\mathcal{O}_Y(-H)}(\iota_* F_C).$$

By Serre duality

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D^b}(Y)}(\iota_*\mathcal{O}_X, \mathcal{O}_Y(-H)[i]) &= \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X(-2H)[4-i]) \\ &= \begin{cases} \mathbb{C}, & \text{when } i = 1; \\ 0, & \text{when } i \neq 1. \end{cases}\end{aligned} \tag{4.14}$$

The unique extension gives the obvious triangle $\iota_*\mathcal{O}_X \to \mathcal{O}_Y(-H)[1] \to \mathcal{O}_Y[1] \xrightarrow{+}$. Therefore, we have the natural commutative diagram of distinguished triangles:

$$\begin{tikzcd}
\iota_*\mathcal{O}_X^{\oplus 2}[-1] \arrow[r] \arrow[d, "\mathsf{ev}"] & \mathcal{O}_Y(-H)^{\oplus 2} \arrow[r] \arrow[d] & \mathcal{O}_Y^{\oplus 2} \arrow[r, "+"] \arrow[d, "\mathsf{ev}"] & {} \\
\theta_C(H)[-1] \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \theta_C(H) \arrow[r, "+"] \arrow[d] & {} \\
\iota_* F_C \arrow[r, "\mathsf{ev}"] & \mathcal{O}_Y(-H)[1]^{\oplus 2} \arrow[r] & \mathsf{L}_{\mathcal{O}_Y}(\theta_C(H)) \arrow[r, "+"] & {} .
\end{tikzcd}$$

The morphism ev at the bottom line is $\iota_* F_C \to \mathcal{O}_Y(-H)[1] \otimes (\mathrm{Hom}(\iota_* F_C, \mathcal{O}_Y(-H)[1]))^*$. By (4.12) and (4.14), the object $\mathsf{R}_{\mathcal{O}_Y(-H)}(\iota_* F_C)$ is

$$\mathrm{Cone}\Big(\iota_* F_C \xrightarrow{\mathsf{ev}} \mathcal{O}_Y(-H)^{\oplus 2}[1] \bigoplus \mathcal{O}_Y(-H)[2] \otimes (\mathrm{Hom}(\mathsf{L}_{\mathcal{O}_Y}(\theta_C(H)), \mathcal{O}_Y(-H)[2]))^*\Big) \\ \times [-1],$$

which is isomorphic to the object in (4.9).

(2). Consider the short exact sequence

$$0 \to \mathcal{I}_\Gamma(H) \to \mathcal{O}_Y(H) \to \mathcal{O}_\Gamma(H) \to 0.$$

As $h^1(\mathcal{O}_\Gamma(H)) = h^0(\mathcal{O}_\Gamma(-H)) = 0$ and $\chi(\mathcal{O}_\Gamma(mH)) = 5m$ as Γ is a non-degenerate quintic elliptic curve, we have $h^0(\mathcal{O}_\Gamma(H)) = 5$. Note that $h^0(\mathcal{O}_Y(H)) = 6$ and the induced map

$$H^0(\mathcal{O}_Y(H)) \to H^0(\mathcal{O}_\Gamma(H))$$

is surjective, since the linear span of Γ is a $\mathbb{P}^4$. We conclude that

$$\mathsf{L}_{\mathcal{O}_Y}(\mathcal{I}_\Gamma(H)) = \mathrm{Cone}\left(\mathcal{O}_Y \xrightarrow{\mathsf{ev}} \mathcal{I}_{\Gamma/Y}(H)\right) = \mathcal{I}_{\Gamma/X}(H).$$

Consider on the category $\mathcal{O}_Y^\perp$ the following semiorthogonal decomposition with two components:

$$\langle\langle \mathcal{O}_Y(-2H), \mathcal{O}_Y(-H)\rangle, \mathrm{Ku}(Y)\rangle. \tag{4.15}$$

Consider the expression of $\mathcal{I}_{\Gamma/X}(H)$ in Definition 2.5 (b):

$$0 = F_0 \to F_1 = \iota_* F_\Gamma \xrightarrow{\mathsf{ev}} F_2 = \mathcal{I}_{\Gamma/X}(H). \tag{4.16}$$

Here $\mathrm{Cone}(F_1 \to F_2)$ is $\mathcal{O}_X(-H)[1]$ by (4.4). Note that

$$\mathcal{O}_X(-H)[1] = \mathrm{Cone}(\mathcal{O}_Y(-2H) \to \mathcal{O}_Y(-H))[1] \in \langle \mathcal{O}_Y(-2H), \mathcal{O}_Y(-H)\rangle.$$

By [39, Lemma 3.1],

$$\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y(jH), \iota_* F_\Gamma[i]) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(jH), F_\Gamma[i]) = 0 \tag{4.17}$$

for $j = 0, 1, 2$ and all $i \in \mathbb{Z}$. Therefore, $\iota_* F_\Gamma \in \mathrm{Ku}(Y)$. By Remark 2.8, the functor $\mathsf{pr}_{\mathrm{Ku}(Y)}$ with respect to (4.15) maps $\mathsf{L}_{\mathcal{O}_Y}(\mathcal{I}_\Gamma(H))$ to $\iota_* F_\Gamma$.

(3). Since $\mathcal{I}_{\ell/X} \in \langle \mathcal{O}_Y, \mathcal{O}_Y(H)\rangle^\perp$, we have $\mathsf{pr}(\mathcal{I}_{\ell/X}) = \mathsf{R}_{\mathcal{O}_Y(-H)}(\mathcal{I}_{\ell/X})$. By the same argument as that for the conic case, the statement holds.

(4). By (4.10) and (4.9), the character of E_C is

$$\begin{aligned}\mathrm{ch}(E_C) &= \mathrm{ch}(\mathsf{L}_{\mathcal{O}_Y}(\theta_C(H))[-1]) - 2\,\mathrm{ch}(\mathcal{O}_Y(-H)) \\ &= 2\,\mathrm{ch}(\mathcal{O}_Y) - \mathrm{ch}(\theta_C(H)) - 2\,\mathrm{ch}(\mathcal{O}_Y(-H)) = \mathrm{ch}(2\lambda_1 + 2\lambda_2).\end{aligned}$$

The Chern character of E_Γ is $\mathrm{ch}(\iota_* F_\Gamma)$ which is the same as $\mathrm{ch}(2\lambda_1 + 2\lambda_2)$. $\square$

5. Stability of E_Γ and E_C

In this technical section, we prove Theorem 5.19, namely, Theorem 1.2 in the introduction. In particular, we study the essential image of the objects E_Γ and E_C defined in Section 4 via the equivalence between $\mathrm{Ku}(Y)$ and $\mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0)$ of Proposition 2.10. We show that these objects are stable with respect to tilt-stability $\sigma_{\alpha,-1}$ on $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$ by a wall-crossing argument.

For this purpose, we will inevitably work with details about the category $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$ and we will prove some additional properties used for the computation, which are also of independent interest. This is the only section where $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$ is involved. For readers not familiar with this setting, there is no harm to skip the whole section, since the only result that we will use in the rest of the paper is Theorem 5.19.

5.1. More on the equivalence between $\mathrm{Ku}(Y)$ *and* $\mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0)$

Recall that the construction of the stability condition σ on $\mathrm{Ku}(Y)$ is via pull-back of the stability condition induced on $\mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0)$. In particular, in order to prove the stability of an object E in $\mathrm{Ku}(Y)$, we need to show that $\Psi(\rho^* E)$ in $\mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0)$ is stable. In this section, we recall some properties of the functor Ψ which we will use in the next.

Recall from (2.6) and Remark 2.11 that $\rho : \tilde{Y} \to Y$ is the blow-up morphism. The functor Ψ is defined in [13, Section 7, page 32] by

$$\Psi(-) = \pi_*(- \otimes \mathcal{E}[1]) : \mathrm{D}^{\mathrm{b}}(\tilde{Y}) \to \mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0). \tag{5.1}$$

Here $\mathcal{E}$ is the sheaf of right $\pi^*\mathcal{B}_0$-modules on $\tilde{Y}$ defined by the short exact sequence of right $q^*\mathcal{B}_0$-modules

$$0 \to \mathcal{O}_{\mathbb{P}_{\mathbb{P}^3}(\mathcal{F})/\mathbb{P}^3}(-2) \otimes q^*\mathcal{B}_1 \xrightarrow{\delta_{-1,2}} \mathcal{O}_{\mathbb{P}_{\mathbb{P}^3}(\mathcal{F})/\mathbb{P}^3}(-1) \otimes q^*\mathcal{B}_2 \to \alpha_*\mathcal{E} \to 0, \tag{5.2}$$

where the morphism $\delta_{-1,2}$ is defined in [37, Section 3.1 and 3.4]. We would not use further details about $\delta_{-1,2}$ here, but the following fact will be important for us.

Lemma 5.1 *([37, Lemma 4.7]). The $\mathcal{O}_{\tilde{Y}}$-coherent sheaf* $\mathrm{Forg}(\mathcal{E})$ *is locally free with rank* 2.

By [37, Lemma 4.12] and [13, Proposition 7.7], the image of some objects under $\Psi \circ \rho^*$ is as follows:

$$\Psi(\rho^*\mathcal{O}_Y) = 0; \quad \Psi(\rho^*\mathcal{O}_Y(-H)) = \mathcal{B}_{-1}; \quad \Psi(\rho^*\mathcal{O}_Y(H)) = \mathcal{B}_2[1]. \tag{5.3}$$

By [37, Lemma 4.10], the functor Ψ has a left adjoint functor

$$\Phi(-) = \pi^*(-) \otimes_{\pi^*\mathcal{B}_0} \mathcal{E}', \tag{5.4}$$

where $\mathcal{E}'$ is a left $\pi^*\mathcal{B}_0$-module on $\tilde{Y}$ defined by the following short exact sequence of left $q^*\mathcal{B}_0$-modules:

$$0 \to \mathcal{O}_{\mathbb{P}_{\mathbb{P}^3}(\mathcal{F})/\mathbb{P}^3}(-2) \otimes q^*\mathcal{B}_0 \xrightarrow{\delta'_{-1,1}} \mathcal{O}_{\mathbb{P}_{\mathbb{P}^3}(\mathcal{F})/\mathbb{P}^3}(-1) \otimes q^*\mathcal{B}_1 \to \alpha_*\mathcal{E}' \to 0. \tag{5.5}$$

Remark 5.2. The rank of torsion-free $\mathcal{B}_0$-modules on $\mathbb{P}^3$ is always a multiple of 4 by [13, Remark 8.4]. The functor $\Psi\rho^*$ maps the characters λ_1 and λ_2 to the twisted Chern characters

$$\mathrm{ch}^{-1}_{\mathcal{B}_0,\leq 2}(\Psi\rho^*(\lambda_1)) = (4, 3, -\frac{7}{8}) \text{ and } \mathrm{ch}^{-1}_{\mathcal{B}_0,\leq 2}(\Psi\rho^*(\lambda_2)) = (-8, 0, \frac{7}{4}) \tag{5.6}$$

respectively, as computed in [13, Proof of Proposition 9.10]. In particular, for an object E in $\mathrm{Ku}(Y)$ with character $2\lambda_1 + 2\lambda_2$, the twisted Chern character of $\Psi\rho^*(E)$ is

$$\mathrm{ch}^{-1}_{\mathcal{B}_0,\leq 2}(\Psi\rho^*(E)) = (-8, 6, \frac{7}{4}). \tag{5.7}$$

5.2. Expression of $\Psi\rho^*(E_\Gamma)$

Let Γ be a non-degenerate smooth elliptic quintic contained in a smooth cubic threefold X (which is unique). By the formula (4.13), the object $\Psi\rho^*(E_\Gamma)$ sits in the distinguished triangle

$$\Psi\rho^*(\mathcal{O}_X(-H)) \to \Psi\rho^*(E_\Gamma) \to \Psi\rho^*(\mathcal{I}_{\Gamma/X}(H)) \xrightarrow{+} . \tag{5.8}$$

Recall that the morphism ρ is the blow-up along a line L on Y, and by Proposition 2.17, the choice of L does not affect the stability of an object in $\mathrm{Ku}(Y)$. As a consequence, we can choose L such that $\Psi\rho^*(E_\Gamma)$ has a more explicit and nicer description. More precisely, given Γ and X, we may choose the line L not contained in a plane on Y such that:

Condition 5.3.

(a) The line L intersects X at a point P;
(b) the point P is not on the secant variety of Γ (since X is smooth, the segment variety of Γ does not contain X);
(c) the point P is only on finitely many lines on X.

By condition (a), the restriction of ρ to $\rho^{-1}(X)$ is the blow-up $\tilde{X}$ of X in the point P. By condition (c), a plane containing L intersects with X at either three points (counting multiplicity) including P or a line through P. A fiber of $\pi|_{\tilde{X}} : \tilde{X} \to \mathbb{P}^3$ is either two points or a line.

By condition (b), the image $\pi\rho^{-1}(\Gamma)$ in $\mathbb{P}^3$ is isomorphic to Γ. By definition (5.1) and Lemma 5.1,

$$\Psi\rho^*(\mathcal{O}_\Gamma(H)) = \mathcal{T}_\Gamma[1] \tag{5.9}$$

for some torsion sheaf $\mathcal{T}_\Gamma$ supported on $\pi\rho^{-1}(\Gamma)$. By (5.3), we have

$$\Psi\rho^*(\mathcal{O}_X(H)) = \Psi\rho^*(\mathcal{O}_Y(H)) = \mathcal{B}_2[1]. \tag{5.10}$$

We deduce the distinguished triangle for one object in (5.8):

$$\mathcal{T}_\Gamma \to \Psi\rho^*(\mathcal{I}_{\Gamma/X}(H)) \to \mathcal{B}_2[1] \xrightarrow{+} . \tag{5.11}$$

In order to compute the other factor $\Psi\rho^*(\mathcal{O}_X(-H))$ in (5.8), we consider the sequence

$$0 \to \mathcal{O}_X(-H) \to \mathcal{O}_X \to \mathcal{O}_S \to 0,$$

where S is a smooth cubic surface not containing P. The object

$$\Psi\rho^*\mathcal{O}_S = \mathcal{T}_S[1],$$

where $\mathcal{T}_S$ is a torsion $\mathcal{B}_0$-module supported on $\pi\rho^{-1}(S)$. On the other hand, by (5.3), we have

$$\Psi\rho^*(\mathcal{O}_X) = \Psi\rho^*(\mathcal{O}_Y(-H)[1]) = \mathcal{B}_{-1}[1].$$

In conclusion, we have the distinguished triangle

$$\mathcal{T}_S \to \Psi\rho^*(\mathcal{O}_X(-H)) \to \mathcal{B}_{-1}[1] \xrightarrow{+} . \tag{5.12}$$

Putting everything together, we observe the following property of $\Psi\rho^*(E_\Gamma)$.

Lemma 5.4. *Let Γ be a non-degenerate smooth elliptic quintic spanning a smooth cubic threefold X. Then*

$$\mathrm{Hom}(\mathcal{B}_i[1], \Psi\rho^*(E_\Gamma)) = 0 \tag{5.13}$$

for every $i \geq 1$.

Proof. By definition, the object E_Γ is in $\mathrm{Ku}(Y)$, so the object $\Psi\rho^*(E_\Gamma))$ is in $\mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0)$. Therefore (5.13) holds for $i = 1, 2, 3$.

When $i \geq 4$, we may apply $\mathrm{Hom}(\mathcal{B}_i[1], -)$ to the triangles (5.8), (5.11) and (5.12). The vanishing holds for every factor, therefore (5.13) holds. $\square$

5.3. Potential destabilizing objects for $\Psi(\rho^* E_\Gamma)$ and $\Psi(\rho^* E_C)$ in $\mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$

In this section, we prove some lemmas which will be useful to characterize the potential destabilizing objects of $\Psi(\rho^* E_\Gamma)$ and $\Psi(\rho^* E_C)$. In order to do this, we need the following natural definition.

Definition 5.5. Let F be an object in $\mathrm{Coh}(\mathbb{P}^3, \mathcal{B}_0)$, we define

$$F^* := \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^3}}(F, \mathcal{O}_{\mathbb{P}^3})$$

as the dual of F. Note that $\mathcal{O}_{\mathbb{P}^3}$ is the center of the algebra $\mathcal{B}_0$. The dual sheaf F^* becomes a left $\mathcal{B}_0$-module. The double dual F^{**} is a right $\mathcal{B}_0$-module. When F is a torsion-free $\mathcal{O}_{\mathbb{P}^3}$-module, its double dual F^{**} is reflexive as a $\mathcal{O}_{\mathbb{P}^3}$-module and we have the natural inclusion

$$F \hookrightarrow F^{**} \to F_{\mathrm{s}}$$

as a right $\mathcal{B}_0$-module. Here F_{s} is a torsion $\mathcal{B}_0$-module and $\dim \operatorname{supp}(F_{\mathrm{s}}) \leq 1$.

Recall that the tilt-stability condition $\sigma_{\alpha,\beta}$ is defined in Proposition 2.13.

Lemma 5.6 *([45, Lemma 3.2]). Let E be a $\sigma_{\alpha_0,\beta_0}$-semistable object in $\mathrm{Coh}^{\beta_0}(\mathbb{P}^3, \mathcal{B}_0)$ for some $\alpha_0 > 0$ and $\beta_0 \in \mathbb{R}$. Assume that $\Delta_{\mathcal{B}_0}(E) = 0$ and $\mathrm{rk}(E) < 0$. Then*

$$E = \mathcal{B}_i^{\oplus n}[1] \quad \textit{for some } i \in \mathbb{Z} \textit{ and } n \in \mathbb{N}.$$

Lemma 5.7. *Let $F \in \mathrm{Coh}(\mathbb{P}^3, \mathcal{B}_0)$ be reflexive as an $\mathcal{O}_{\mathbb{P}^3}$-module with rank 4, then $F \cong \mathcal{B}_i$ for some $i \in \mathbb{Z}$.*

Proof. As F is a reflexive $\mathcal{O}_{\mathbb{P}^3}$-module, we may choose a general hyperplane section $\mathbb{P}^2 \cong H \subset \mathbb{P}^3$ such that the restricted sheaf $F|_H$ is a $\mathcal{B}_0|_H$-module and locally free as a $\mathcal{O}_H$-module. Note that $F|_H$ is of rank 4 and torsion free. By [15, Proposition 2.12], the rank of a $\mathcal{B}_0|_H$-module is multiple of 4. It follows that $F|_H$ is a slope stable $\mathcal{B}_0|_H$-module in $\mathrm{Coh}(H, \mathcal{B}_0|_H)$. By [15, Proposition 2.12], the numerical character

$$\mathrm{ch}(F|_H) = \mathrm{ch}(\mathcal{B}_i|_H) + (0, 0, -m)$$

for some $i, m \in \mathbb{Z}$. By [44, Remark 2.2 and Lemma 2.4], we have

$$1 \geq \chi_{\mathcal{B}_0|_H}(F|_H, F|_H) = \chi(\mathcal{B}_i|_H, \mathcal{B}_i|_H) - 2m = 1 - 2m.$$

Hence, $m \geq 0$. We denote by $\mathfrak{M}$ the moduli space of semistable $\mathcal{B}_0|_H$-modules with numerical class $[F|_H]$. By the same argument as that for [44, Theorem 2.12], this moduli space is irreducible and smooth of dimension $2m$. A generic point in $\mathfrak{M}$ stands for the object $\mathrm{Ker}(\mathcal{B}_i \twoheadrightarrow \mathcal{O}_Z)$, where Z is a 0-dimensional subscheme with length m on H. By the semi-continuity property, we have

$$\mathrm{Hom}_{\mathcal{B}_0|_H}(F|_H, \mathcal{B}_i|_H) \neq 0.$$

As $F|_H$ is locally free with the smallest possible rank as a $\mathcal{B}_0$-module, we have $F|_H \cong \mathcal{B}_i|_H$. In particular, we have $\mathrm{ch}(F|_H) = \mathrm{ch}(\mathcal{B}_i|_H)$, which implies

$$\mathrm{ch}^{-1}_{\mathcal{B}_0,\leq 2}(F|_H) = \mathrm{ch}^{-1}_{\mathcal{B}_0,\leq 2}(\mathcal{B}_i) \implies \Delta_{\mathcal{B}_0}(F) = 0. \tag{5.14}$$

For any $T \in \mathrm{Coh}(\mathbb{P}^3, \mathcal{B}_0)$ such that $\dim \mathrm{supp}(T) \leq 1$, we have $\mathrm{Hom}_{\mathcal{B}_0}(T, F[1]) = 0$ as otherwise, by [15, Lemma 2.15] (same statement holds for $(\mathbb{P}^3, \mathcal{B}_0)$-algebra), F is strictly contained in another torsion-free $\mathcal{B}_0$-module F' with the same rank and degree. This contradicts the assumption that F is a reflexive $\mathcal{O}_{\mathbb{P}^3}$-module.

The object $F[1]$ is therefore $\sigma_{\alpha,\beta}$-stable for $\alpha \gg 0$ and $\beta > \mu_{\mathrm{slope}}(\mathcal{B}_i)$. By (5.14) and Lemma 5.6, we have $F \cong \mathcal{B}_i$. □

Notation: For an object F in $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$, we denote $\mathcal{H}^i(F) := \mathcal{H}^i_{\mathrm{Coh}(\mathbb{P}^3,\mathcal{B}_0)}(F)$ for $i \in \mathbb{Z}$.

Corollary 5.8. *Let F be an object in $\mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$ with rank -4 such that F is $\sigma_{\alpha,-1}$-stable for $\alpha \gg 0$. Then $\mathcal{H}^{-1}(F)$ is $\mathcal{B}_i$, and $\mathcal{H}^0(F)$ is either 0 or a torsion sheaf such that $\dim \mathrm{supp}(\mathcal{H}^0(F)) = 1$.*

Proof. By [16, Lemma 2.7(c)], the sheaf $\mathcal{H}^{-1}(F)$ is torsion-free of rank 4, and the sheaf $\mathcal{H}^0(F)$ is either 0 or torsion supported in dimension ≤ 1. Consider the double dual of $\mathcal{H}^{-1}(F)$:

$$0 \to \mathcal{H}^{-1}(F) \to (\mathcal{H}^{-1}(F))^{**} \to F_s \to 0. \tag{5.15}$$

If F_s is non-zero, then we have $\mu_{\alpha,-1}(F_s) = +\infty$ and the injective map

$$0 \to \mathrm{Hom}_{\mathcal{B}_0}(F_s, \mathcal{H}^{-1}(F)[1]) \to \mathrm{Hom}_{\mathcal{B}_0}(F_s, F).$$

As (5.15) is non-split, we have $\mathrm{Hom}_{\mathcal{B}_0}(F_s, \mathcal{H}^{-1}(F)[1]) \neq 0$. In particular, $\mathrm{Hom}_{\mathcal{B}_0}(F_s, F) \neq 0$. This contradicts the stability of F. Therefore, the sheaf $\mathcal{H}^{-1}(F)$ is reflexive as an $\mathcal{O}_{\mathbb{P}^3}$-module of rank 4. By Lemma 5.7, $\mathcal{H}^{-1}(F) \cong \mathcal{B}_i$ for some $i \leq 0$.

If $\mathcal{H}^0(F)$ is non-zero, we must have $\mathrm{Hom}_{\mathcal{B}_0}(\mathcal{H}^0(F), \mathcal{H}^{-1}(F)[2]) \neq 0$, since otherwise $F = \mathcal{H}^0(F) \oplus \mathcal{H}^{-1}(F)[1]$. Note that $\mathcal{H}^{-1}(F) = \mathcal{B}_i$ is locally free, so the dimension of the support of $\mathcal{H}^0(F)$ must be 1. □

5.4. Tilt-stability of $\Psi(\rho^* E_\Gamma)$

We are now ready to show the stability of $\Psi(\rho^* E_\Gamma)$ with respect to $\sigma_{\alpha,-1}$ for α large enough. The following basic commutative algebra lemma will be useful.

Lemma 5.9. *Let V be a smooth proper variety and U be a smooth proper subvariety of dimension n. Denote the embedding map by $\iota : U \to V$. Let $\mathcal{G}$ be a locally free sheaf on U, and $\mathcal{F}$ be a coherent sheaf on V such that* $\dim(\operatorname{supp}(\mathcal{F}) \cap U) = l$. *Then we have*

$$\operatorname{Ext}^i_{\mathcal{O}_V}(\mathcal{F}, \iota_* \mathcal{G}) = 0$$

for $i < n - l$.

Proof. Let m be the dimension of V, by Serre duality, we need to show that

$$\operatorname{Ext}^i_{\mathcal{O}_V}(\iota_* \mathcal{G}, \mathcal{F}) = 0$$

for $i > m - n + l$ and every $\mathcal{F}$ as in the statement. By the local to global spectral sequence, we have

$$E_2^{p,q} = H^p(\mathcal{E}xt^q_{\mathcal{O}_V}(\iota_* \mathcal{G}, \mathcal{F})) \Rightarrow \operatorname{Ext}^{p+q}_{\mathcal{O}_V}(\iota_* \mathcal{G}, \mathcal{F}).$$

Since $\dim(\operatorname{supp}(\mathcal{F}) \cap U) = l$, we have $E_2^{p,q} = 0$ when $p > l$.

For any closed point $x \in U$, since $\mathcal{G}$ is locally free on U, we have $\iota_* \mathcal{G}_x \cong \mathcal{O}_{U,x}^{\oplus r}$ as an $\mathcal{O}_{V,x}$-module. Since U is smooth in V, the quotient module $\mathcal{O}_{U,x}$ admits a free resolution of $m - n + 1$ terms. Therefore,

$$\mathcal{E}xt^q_{\mathcal{O}_V}(\iota_* \mathcal{G}, \mathcal{F})_x \cong \operatorname{Ext}^q_{\mathcal{O}_{V,x}}(\iota_* \mathcal{G}_x, \mathcal{F}_x) = 0,$$

when $q \geq m - n + 1$.

As a consequence, the term $E_2^{p,q} = 0$ when $p + q > m - n + l$, so we get the Ext vanishing as in the statement. $\square$

Applying Lemma 5.9, we obtain the following result which allows to rule out some destabilizing objects for $\Psi \rho^* E_\Gamma$.

Lemma 5.10. *Let Γ be a non-degenerate smooth elliptic quintic spanning a smooth cubic threefold X. For any $F \in \operatorname{Coh}(\mathbb{P}^3, \mathcal{B}_0)$ such that $\dim \operatorname{supp}(F) \leq 1$, we have*

$$\operatorname{Hom}_{\mathcal{B}_0}(F, \Psi \rho^* E_\Gamma) = 0.$$

Proof. By using the property of adjoint functors and Serre duality, we have

$$\operatorname{Hom}_{\mathcal{B}_0}(F, \Psi \rho^* E_\Gamma) \cong (\operatorname{Hom}_{\mathcal{B}_0}(\Psi \rho^* E_\Gamma, F \otimes_{\mathcal{B}_0} \mathcal{B}_{-3}[3]))^* \tag{5.16}$$

$$\cong (\mathrm{Hom}_{\mathcal{O}_{\tilde{Y}}}(\rho^* E_\Gamma, \Phi(F \otimes_{\mathcal{B}_0} \mathcal{B}_{-3})[3])))^* \tag{5.17}$$

$$\cong \mathrm{Hom}_{\mathcal{O}_{\tilde{Y}}}(\Phi(F \otimes_{\mathcal{B}_0} \mathcal{B}_{-3}), \rho^* E_\Gamma \otimes K_{\tilde{Y}}[1]). \tag{5.18}$$

Recall from (5.4) that Φ is the right adjoint functor of Ψ. By Condition 5.3(c) on the choice of P, the morphism $\pi : \tilde{X} \to \mathbb{P}^3$ is generically finite and only contracts finitely many lines. Thus we have

$$\dim(\mathrm{supp}(\Phi(F \otimes_{\mathcal{B}_0} \mathcal{B}_{-3})) \cap \tilde{X}) \le 1.$$

By Lemma 5.9, we conclude that the $\mathrm{Hom}_{\mathcal{O}_{\tilde{Y}}}$ in the formula (5.18) is 0. $\square$

Proposition 5.11. *Let Γ be a non-degenerate smooth elliptic quintic spanning a smooth cubic threefold. Then the object $\Psi\rho^*(E_\Gamma)$ is in $\mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$ and $\sigma_{\alpha,-1}$-stable for $\alpha \gg 0$.*

Proof. Step 1: By Condition 5.3 on the choice of P, the projection map from $\pi|_{\tilde{X}} : \tilde{X} \to \mathbb{P}^3$ is generically finite except contracting (the transverse image of) finitely many lines that across P on X. Note that $\rho^* E_\Gamma$ is locally free, hence by definition of Ψ in (5.1) and Lemma 5.1, the object $\Psi(\rho^* E_\Gamma)$ is contained in the extension closure of $\{\mathcal{T}\mathrm{or}^{\le 0}, \mathrm{Coh}(\mathbb{P}^3, \mathcal{B}_0)[1]\}$, where $\mathcal{T}\mathrm{or}^{\le 0}$ consists of torsion $\mathcal{B}_0$-modules supported on a 0-dimensional locus. In particular, the object $\Psi\rho^*(E_\Gamma)$ sits in the distinguished triangle

$$G[1] \to \Psi(\rho^* E_\Gamma) \to T \xrightarrow{+}, \tag{5.19}$$

where $G \in \mathrm{Coh}(\mathbb{P}^3, \mathcal{B}_0)$ and T is a torsion $\mathcal{B}_0$-modules supported on a 0-dimensional locus.

Step 2: To show that $\Psi\rho^*(E_\Gamma) \in \mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$, it is enough to show that for any torsion-free $\mathcal{B}_0$-module D with rank 4 and slope $\mu_{\mathrm{slope}}(D) > -1$, the vanishing

$$\mathrm{Hom}_{\mathcal{B}_0}(D, G) = 0$$

holds. Suppose $\mathrm{Hom}_{\mathcal{B}_0}(D, G) \neq 0$, then $\mathrm{Hom}_{\mathcal{B}_0}(D[1], \Psi(\rho^* E_\Gamma)) \neq 0$. Taking the double dual of D as in Definition 5.5, we have the distinguished triangle

$$D_{\mathrm{tor}} \to D[1] \to D^{**}[1] \xrightarrow{+},$$

where D_{tor} is a torsion $\mathcal{B}_0$-module supported at a locus of dimension at most 1. By Lemma 5.10, we have $\mathrm{Hom}_{\mathcal{B}_0}(D^{**}[1], \Psi(\rho^* E_\Gamma)) \neq 0$. By Lemma 5.7 and the fact that $\mu_{\mathrm{slope}}(\mathcal{B}_0) = -\frac{5}{4}$, we may assume

$$D^{**} \cong \mathcal{B}_i \text{ for some } i \ge 1.$$

By Lemma 5.4, this can never happen. As a summary, the object $\Psi(\rho^* E_\Gamma)$ is in $\mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$.

Step 3: To show that $\Psi(\rho^* E_\Gamma)$ is $\sigma_{\alpha,-1}$-stable for $\alpha \gg 0$, we need to rule out the possibility that

(i) $\Psi(\rho^* E_\Gamma)$ has a sub-torsion object which is a torsion $\mathcal{B}_0$-module with support of dimension at most 1;
(ii) $\Psi(\rho^* E_\Gamma)$ has a quotient object F in $\mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$ such that F is a $\sigma_{+\infty,-1}$-stable with rank -4 and $\mu_{\alpha,-1}(F) \leq \mu_{\alpha,-1}(\Psi(\rho^* E_\Gamma))$ for $\alpha \gg 0$.

Indeed, a quotient object F of $\Psi(\rho^* E_\Gamma)$ with $\mathrm{rk}(F) \leq -8$ would not destabilize $\Psi(\rho^* E_\Gamma)$, as for $\alpha \gg 0$ the slope converges to $-\frac{\mathrm{rk}}{\mathrm{ch}_1^{-1}}$. Case (i) cannot happen by Lemma 5.10. As for Case (ii), let K be the kernel of $\Psi(\rho^* E_\Gamma) \twoheadrightarrow F$ in $\mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$. Consider the exact sequence in $\mathrm{Coh}(\mathbb{P}^3, \mathcal{B}_0)$:

$$0 \to \mathcal{H}^{-1}(K) \to \mathcal{H}^{-1}(\Psi(\rho^* E_\Gamma)) \to \mathcal{H}^{-1}(F) \to \mathcal{H}^0(K) \to \mathcal{H}^0(\Psi(\rho^* E_\Gamma)) \to \mathcal{H}^0(F) \to 0.$$

Note that the term $\mathcal{H}^0(\Psi(\rho^* E_\Gamma))$ is supported on a 0-dimensional locus or is zero. By Corollary 5.8, $\mathcal{H}^0(F) = 0$ and the object

$$F = \mathcal{H}^{-1}(F)[1] = \mathcal{B}_i[1]$$

for some $i \leq 0$. By Definition 2.2, Proposition 2.13 and Remark 5.2, we have

$$\mu_{\alpha,-1}(\mathcal{B}_0[1]) = 2\alpha^2 - \frac{1}{8} > \frac{16\alpha^2 + 7}{24} = \mu_{\alpha,-1}(\Psi(\rho^* E_\Gamma))$$

for $\alpha \gg 0$. Therefore $\mathcal{B}_0[1]$ does not codestabilize $\Psi(\rho^* E_\Gamma)$. We may assume $F = \mathcal{B}_i$ for some $i \leq -1$.

Note that $K \in \mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$, so we have $\mathrm{ch}^{-1}_{\mathcal{B}_0,1}(K) \geq 0$ which implies $\mathrm{ch}^{-1}_{\mathcal{B}_0,1}(F) \leq 6$. Therefore, we may assume $F = \mathcal{B}_i$ for some $i \geq -2$.

In either case of $i = -1, -2$, by Serre duality and the fact that $\Psi(\rho^* E_\Gamma)$ is an object in $\mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0)$, we have

$$\mathrm{Hom}_{\mathcal{B}_0}(\Psi(\rho^* E_\Gamma), \mathcal{B}_i[1]) \cong (\mathrm{Hom}_{\mathcal{B}_0}(\mathcal{B}_{i+3}, \Psi(\rho^* E_\Gamma)[2]))^* = 0.$$

Therefore, Case (ii) can neither happen. We conclude that $\Psi(\rho^* E_\Gamma)$ is $\sigma_{\alpha,-1}$-stable for $\alpha \gg 0$. $\square$

5.5. Tilt-stability of $\Psi\rho^*(E_C)$

Let C be a smooth conic curve on Y. Similarly to the case of non-degenerate elliptic quintics, we now study the $\sigma_{\alpha,-1}$-stability of the object $\Psi\rho^*(E_C)$ for α large enough. We choose the blown-up line L for $\rho : \tilde{Y} \to Y$ such that:

Condition 5.12.

(a) The line L does not intersect the projective plane spanned by C;
(b) the plane spanned by L and a generic point on C intersects Y at the union of L and a smooth conic curve.

By Lemma 4.4, the object $\Psi\rho^*(E_C)$ sits in the distinguished triangle:

$$\Psi\rho^*\left(\mathcal{O}_Y(-H)^{\oplus 2}[1]\right) \to \Psi\rho^*(E_C) \to \Psi\rho^*(\mathsf{L}_{\mathcal{O}_Y}(\theta_C(H))[-1]). \tag{5.20}$$

By (5.3), the triangle can be simplified as

$$\mathcal{B}_{-1}^{\oplus 2}[1] \to \Psi\rho^*(E_C) \to T_C, \tag{5.21}$$

where

$$T_C := \Psi\rho^*(\mathsf{L}_{\mathcal{O}_Y}(\theta_C(H))[-1]) = \Psi\rho^*(\theta_C(H)[-1]).$$

The second equality is by noticing that $\Psi\rho^*(\mathcal{O}_Y) = 0$. By the choice of L as in Condition 5.12(a), the image $C' := \pi(\rho^{-1}(C))$ is a smooth conic in $\mathbb{P}^3$. By the definition of Ψ in (5.1) and Lemma 5.1, the object T_C is a torsion $\mathcal{B}_0$-module supported on C'.

Lemma 5.13. *Adopt the notation as above.*

(1) *As a $\mathcal{O}_{\mathbb{P}^3}$-coherent sheaf,* $\mathrm{Forg}(T_C) \cong \mathcal{O}_{C'}^{\oplus 2}$.
(2) *A torsion $\mathcal{B}_0$-module with C' as its support has rank at least* 2.

In particular, the sheaf T_C is indecomposable as a $\mathcal{B}_0$-module.

Proof. (1). Note that $\Psi\rho^*(E_C)$ is an object in $\mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0)$, so we have

$$0 = \mathrm{Hom}_{\mathcal{B}_0}(\mathcal{B}_1, \Psi\rho^*(E_C)[i]) = \mathrm{Hom}_{\mathcal{B}_0}(\mathcal{B}_0, T_C \otimes_{\mathcal{B}_0} \mathcal{B}_{-1}[i])$$

for every $i \in \mathbb{Z}$. Denote the embedding map by $\iota : C' \to \mathbb{P}^3$. By the definition of Ψ in (5.1) and Lemma 5.1, the sheaf $\mathrm{Forg}(T_C) = \iota_*(\mathcal{F}_{C'})$ for some rank 2 locally free sheaf $\mathcal{F}_{C'}$ on C'. By (2.9), we have

$$0 = \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_{\mathbb{P}^3}, \mathrm{Forg}(T_C \otimes_{\mathcal{B}_0} \mathcal{B}_{-1})[i]) = \mathrm{Hom}_{\mathcal{O}_{C'}}(\mathcal{O}_{C'}, \mathcal{F}_{C'} \otimes \theta_{C'}[i])$$

for every $i \in \mathbb{Z}$. This can only happen when $\mathcal{F}_{C'} \cong \mathcal{O}_{C'}^{\oplus 2}$.

(2). By the choice of L as in Condition 5.12(b), the $\mathcal{B}_0$-algebra structure as in (2.8) on a generic point on C is isomorphic to $\mathrm{Mat}_{2\times 2}(\mathbb{C})$, the 2 by 2 complex matrices, as a $\mathbb{C}$-algebra. Since a $\mathrm{Mat}_{2\times 2}(\mathbb{C})$-module is at least of dimension 2 as a $\mathbb{C}$-vector space, there is no torsion $\mathcal{B}_0$-module supported on C' with rank 1. In particular, T_C is indecomposable as a $\mathcal{B}_0$-module. $\square$

Lemma 5.14. *If $F \in \mathrm{Coh}(\mathbb{P}^3, \mathcal{B}_0)$ is such that $\dim \operatorname{supp}(F) \leq m$, then $\mathrm{Hom}_{\mathcal{B}_0}(F, \mathcal{B}_i[j]) = 0$, for $j \leq 2 - m$ and all $i \in \mathbb{Z}$.*

Proof. By Serre duality and (2.9),

$$\begin{aligned}\mathrm{Hom}_{\mathcal{B}_0}(F, \mathcal{B}_i[j]) &\cong (\mathrm{Hom}_{\mathcal{B}_0}(\mathcal{B}_0, F \otimes_{\mathcal{B}_0} \mathcal{B}_{-3-i}[3-j]))^* \\ &\cong (\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_{\mathbb{P}^3}, \mathrm{Forg}(F \otimes_{\mathcal{B}_0} \mathcal{B}_{-3-i})[3-j]))^* = 0\end{aligned}$$

when $3 - j \geq 1 + m$. $\square$

Lemma 5.15. *For any $F \in \mathrm{Coh}(\mathbb{P}^3, \mathcal{B}_0)$ such that $\dim \operatorname{supp}(F) \leq 1$, we have*

$$\mathrm{Hom}_{\mathcal{B}_0}(F, \Psi(\rho^* E_C)) = 0.$$

Proof. We first show that for any sub-$\mathcal{B}_0$-module F of T_C, the statement holds. By Lemma 5.13, the sheaf F is supported on C', locally free with rank 2 as a sheaf on C'. Moreover, any non-zero morphism $f : F \to T_C$ is injective. Applying $\mathrm{Hom}_{\mathcal{B}_0}(-, \mathcal{B}_{-1}^{\oplus 2})$ to the short exact sequence

$$0 \to F \xrightarrow{f} T_C \to F' \to 0, \tag{5.22}$$

by Lemma 5.14, we have

$$\mathrm{Hom}_{\mathcal{B}_0}(F', \mathcal{B}_{-1}^{\oplus 2}[2]) = 0 \text{ and } \mathrm{Hom}_{\mathcal{B}_0}(F, \mathcal{B}_{-1}^{\oplus 2}[1]) = 0.$$

Thus the morphism

$$f \circ -\colon \mathrm{Hom}_{\mathcal{B}_0}(T_C, \mathcal{B}_{-1}^{\oplus 2}[2]) \to \mathrm{Hom}_{\mathcal{B}_0}(F, \mathcal{B}_{-1}^{\oplus 2}[2])$$

is injective. In other words, the composition of $\mathsf{ev} : T_C \to \mathcal{B}_{-1}^{\oplus 2}[2]$ in (5.21) with any non-zero f is a non-zero morphism in $\mathrm{Hom}_{\mathcal{B}_0}(F, \mathcal{B}_{-1}^{\oplus 2}[2])$. Therefore, in (5.21), any non-zero morphism $f : F \to T_C$ cannot lift to a morphism from F to $\Psi\rho^*(E_C)$. Note that $\mathrm{Hom}_{\mathcal{B}_0}(F, \mathcal{B}_{-1}[1]) = 0$ by Lemma 5.14, so the statement holds for any sub-$\mathcal{B}_0$-module F of T_C.

As for an arbitrary F with $\dim \operatorname{supp} F \leq 1$, we make induction on its ch_2. Let g be a morphism in $\mathrm{Hom}_{\mathcal{B}_0}(F, \Psi(\rho^* E_C))$. Applying $\mathrm{Hom}_{\mathcal{B}_0}(F, -)$ to (5.20), we have

$$.. \to \mathrm{Hom}_{\mathcal{B}_0}(F, \mathcal{B}_{-1}[1]^{\oplus 2}) \to \mathrm{Hom}_{\mathcal{B}_0}(F, \Psi(\rho^* E_C)) \to \mathrm{Hom}_{\mathcal{B}_0}(F, T_C) \to ..$$

and since $\mathrm{Hom}_{\mathcal{B}_0}(F, \mathcal{B}_{-1}[1]) = 0$, the morphism g is mapped to a morphism g'.

Suppose $g \neq 0$, then $g' \neq 0$ and $\mathrm{Hom}_{\mathcal{B}_0}(\mathrm{im}(g'), \Psi(\rho^* E_C)) = 0$ as $\mathrm{im}(g')$ is a submodule of T_C. By induction, $\mathrm{Hom}_{\mathcal{B}_0}(\ker(g'), \Psi(\rho^* E_C)) = 0$. Therefore, $\mathrm{Hom}_{\mathcal{B}_0}(F, \Psi(\rho^* E_C)) = 0$, which contradicts $g \neq 0$. □

Proposition 5.16. *Let C be a smooth conic curve. Then the object $\Psi\rho^*(E_C)$ is in $\mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$ and $\sigma_{\alpha,-1}$-stable for $\alpha \gg 0$.*

Proof. By (5.21), the object $\Psi\rho^*(E_C)$ is in $\mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$. To show that $\Psi(\rho^* E_C)$ is $\sigma_{\alpha,-1}$-stable for $\alpha \gg 0$, we need to rule out the possibility that

(i) $\Psi(\rho^* E_C)$ has a sub-torsion object which is a torsion $\mathcal{B}_0$-module with support of dimension at most 1;
(ii) $\Psi(\rho^* E_C)$ has a quotient object F in $\mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$ such that F is a $\sigma_{+\infty,-1}$-stable with rank -4 and $\mu_{\alpha,-1}(F) \leq \mu_{\alpha,-1}(\Psi(\rho^* E_\Gamma))$ for $\alpha \gg 0$.

Case (i) cannot happen by Lemma 5.15. As for Case (ii), let K be the kernel of $\Psi(\rho^* E_C) \twoheadrightarrow F$ in $\mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$, then we have the exact sequence in $\mathrm{Coh}(\mathbb{P}^3, \mathcal{B}_0)$:

$$0 \to \mathcal{H}^{-1}(K) \to \mathcal{B}_{-1}^{\oplus 2} \to \mathcal{H}^{-1}(F) \to \mathcal{H}^0(K) \to T_C \to \mathcal{H}^0(F) \to 0.$$

By Corollary 5.8, we may assume $\mathcal{H}^{-1}(F)$ is $\mathcal{B}_i$ for some $i \leq 0$. Note that $\mathcal{B}_0[1]$ has a larger slope than $\Psi(\rho^* E_\Gamma)$ with respect to $\sigma_{\alpha,-1}$, so we have $i \leq -1$. Since $\mathrm{Hom}_{\mathcal{B}_0}(\mathcal{B}_{-1}^{\oplus 2}, \mathcal{B}_i) = 0$ for every $i \leq -2$, the sheaf $\mathcal{H}^{-1}(F)$ can only be $\mathcal{B}_{-1}$ as well as $\mathcal{H}^{-1}(K)$. Hence we have the sequence in $\mathrm{Coh}(\mathbb{P}^3, \mathcal{B}_0)$:

$$0 \to \mathcal{H}^0(K) \to T_C \to \mathcal{H}^0(F) \to 0.$$

By Lemma 5.13, the sheaf $\mathcal{H}^0(F)$ is either with 0-dimensional support, or supported on C', locally free of rank 2 as a sheaf on C'. The second case cannot happen since the slope of F would be larger than that of $\Psi(\rho^* E_C)$. By Corollary 5.8, $\mathcal{H}^0(F) = 0$. In other words, $F = \mathcal{B}_{-1}[1]$.

Since $\Psi(\rho^* E_C) \in \mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0)$, by Serre duality, we have

$$\mathrm{Hom}_{\mathcal{B}_0}(\Psi(\rho^* E_C), F) \cong (\mathrm{Hom}_{\mathcal{B}_0}(\mathcal{B}_2, \Psi(\rho^* E_C)[2]))^* = 0.$$

Therefore, Case (ii) can neither happen. The object $\Psi(\rho^* E_C)$ is $\sigma_{\alpha,-1}$-stable for $\alpha \gg 0$. □

5.6. No actual walls for $\Psi\rho^*(E_\Gamma)$ and $\Psi\rho^*(E_C)$

By Propositions 5.11 and 5.16, we have the $\sigma_{\alpha,-1}$-stability of $\Psi\rho^*(E_\Gamma)$ ($\Psi\rho^*(E_C)$) for $\alpha \gg 0$. In this section, we show that $\sigma_{\alpha,-1}$-stable objects in $\mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0)$ with this character cannot be destabilized when α decreases.

We first list the character $\mathrm{ch}^{-1}_{\mathcal{B}_0,\leq 2}$ of all possible destabilizing objects with respect to the weak stability conditions $\sigma_{\alpha,-1}$. Recall that the rank of $\mathcal{B}_0$-modules on $\mathbb{P}^3$ is always a multiple of 4. We can write the characters of potential destabilizing subobjects and quotient objects for $\Psi\rho^*(E_\Gamma)$ and $\Psi\rho^*(E_C)$ as

$$\mathrm{ch}^{-1}_{\mathcal{B}_0,\leq 2}(\Psi\rho^*(2\lambda_1+2\lambda_2)) = (-8, 6, \frac{14}{8}) = (4a, b, \frac{c}{8}) + (-8-4a, 6-b, \frac{14}{8}-\frac{c}{8}), \quad (5.23)$$

where $a, b, c \in \mathbb{Z}$. These characters have to satisfy the following conditions:

(a) The two characters have non-negative discriminant $\Delta_{\mathcal{B}_0}$ by Proposition 2.13.

(b) The two characters should be integral combinations of the characters of $\mathrm{ch}^{-1}_{\mathcal{B}_0,\leq 2}(\mathcal{B}_i)$ for $i = -1, 0, 1$ by restriction to $\mathrm{Coh}(\mathbb{P}^2, \mathcal{B}_0|_{\mathbb{P}^2})$ and [15, Proposition 2.12]. In particular, the set

$$\{(4, 1, \frac{1}{8}), (0, 2, 0), (0, 0, 1)\} \quad (5.24)$$

forms a $\mathbb{Z}$-linear basis for all possible characters.

(c) There exists $\alpha > 0$ such that the two characters have the same slope with respect to $\sigma_{\alpha,-1}$. In particular, both b and $6-b > 0$. Indeed, as the objects are in $\mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$ we have $b \geq 0$, $6-b \geq 0$. Then the inequalities are strict since we are assuming the objects have the same slope.

(d) Without loss of generality, we may assume that the character $(4a, b, \frac{c}{8})$ is the character of a destabilizing subobject. The equivalent numerical assumption is

$$\frac{4a}{b} > \frac{\mathrm{rk}(\Psi\rho^*(2\lambda_1+2\lambda_2))}{\mathrm{ch}^{-1}_{\mathcal{B}_0,1}(\Psi\rho^*(2\lambda_1+2\lambda_2))} = -\frac{4}{3},$$

as $\Psi\rho^*(E_\Gamma)$ and $\Psi\rho^*(E_C)$ are $\sigma_{\alpha,-1}$-stable for $\alpha \gg 0$.

Using these conditions, by a standard computation we obtain the following result.

Proposition 5.17. *All possible solutions of* (5.23) *are:*

(1) *for* $\alpha = \frac{\sqrt{17}}{4}$, $a = 0$, $b = 2$, $c = 16$;

(2) *for* $\alpha = \frac{\sqrt{5}}{4}$,

 (i) $a = -1$, $b = 5$, $c = 15$;

 (ii) $a = 0$, $b = 4$, $c = 16$;

 (iii) $a = 0$, $b = 2$, $c = 8$;

(3) *for* $\alpha = \frac{1}{4}$, $a = 1$, $b = 3$, $c = 9$.

Proof. We sketch the steps of the computation here. The first step is to rule out the 'higher rank' wall case. Namely, by the non-negativity condition (a), (c) and (d), we have:

- $b^2 - ac \geq 0$ and $(6-b)^2 + (\frac{c}{4} - \frac{7}{2})(-8-4a) \geq 0$;
- $0 < b < 6$;
- $3c - 7b > 0$.

Combining these inequalities, one may deduce from a standard computation that $-12 \leq -8 - 4a \leq 0$. Then by condition (b) and (d), the possible pairs (a, b) are $(-1, 5)$, $(0, 2)$, $(0, 4)$, $(1, 1)$, $(1, 3)$, and $(1, 5)$. By condition (a) again, one can list all possible triples of (a, b, c). $\square$

Proposition 5.18. *Let E be a $\sigma_{\alpha_0,-1}$-stable object in $\mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0)$ and $\mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$ with $\mathrm{ch}^{-1}_{\mathcal{B}_0,\leq 2}(E) = (-8, 6, \frac{14}{8})$. Then E is $\sigma_{\alpha,-1}$-stable for any $\alpha \leq \alpha_0$.*

Proof. Suppose E becomes strictly semistable with respect to $\sigma_{\alpha,-1}$ for some $0 < \alpha < \alpha_0$. By Proposition 5.17, this may happen when $\alpha = \frac{1}{4}, \frac{\sqrt{5}}{4}$ or $\frac{\sqrt{17}}{4}$. Let us denote the destabilizing sequence in $\mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$ as follows:

$$0 \to S \to E \to Q \to 0, \tag{5.25}$$

where S and Q are $\sigma_{\alpha,-1}$-semistable objects with characters as those in Proposition 5.17.

Step I: We get rid of two cases when the destabilizing object is $\mathcal{B}_0^{\oplus a}[1]$.

When $\alpha = \frac{\sqrt{5}}{4}$, if Case (2.i) or (2.ii) in Proposition 5.17 happens, then the Chern character of the quotient object Q is

$$\mathrm{ch}^{-1}_{\mathcal{B}_0,\leq 2}(Q) = (-4, 1, -\frac{1}{8}) \text{ or } (-8, 2, -\frac{1}{4}).$$

By Lemma 5.6, the quotient object Q is either $\mathcal{B}_0[1]$ or $\mathcal{B}_0^{\oplus 2}[1]$. In either case, we would have

$$\mathrm{Hom}(\mathcal{B}_3, E[2]) \cong (\mathrm{Hom}(E, \mathcal{B}_0[1]))^* \neq 0,$$

which contradicts the assumption that $E \in \mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0)$.

Step II: We show that $\mathrm{Hom}(\mathcal{B}_j, Q[i]) = 0$ for $i \geq 1$ and $j = 1, 2, 3$.

Now there are three cases in Proposition 5.17 left. In Case (1) and (3), it is possible to show that any $\sigma_{\frac{\sqrt{17}}{4},-1}$-semistable (resp. $\sigma_{\frac{1}{4},-1}$) objects with character $(0, 2, 2)$ and $(-8, 4, -\frac{1}{4})$ (resp. $(4, 3, \frac{9}{8})$ and $(-12, 3, \frac{5}{8})$) are $\sigma_{\frac{\sqrt{17}}{4},-1}$-stable (resp. $\sigma_{\frac{1}{4},-1}$), using Definition 2.2 and similar computations as in Proposition 5.17. Both S and Q are $\sigma_{\alpha,-1}$-stable in these two cases. In Case (2.iii), the object S with character $(0, 2, 8)$ is also $\sigma_{\frac{\sqrt{5}}{4},-1}$-stable. If Q is strictly $\sigma_{\frac{\sqrt{5}}{4},-1}$-semistable, we may reduce to either Case (2.i) or (2.ii). Therefore, in any of the remaining cases, we may assume both S and Q are $\sigma_{\alpha,-1}$-stable.

For each $\mathcal{B}_j$, $1 \leq j \leq 3$, apply $\mathrm{Hom}(\mathcal{B}_j, -)$ to the sequence (5.25). Since $E \in \mathrm{Ku}(\mathbb{P}^3, \mathcal{B}_0)$, we have

$$\mathrm{Hom}(\mathcal{B}_j, S[i+1]) \cong \mathrm{Hom}(\mathcal{B}_j, Q[i]) \tag{5.26}$$

for all $i \in \mathbb{Z}$.

We next show that $\mathrm{Hom}(\mathcal{B}_j, S[i+1]) = 0$ for $i \geq 1$ and $j \leq 0$. In Case (1) of Proposition 5.17, the object S is $\sigma_{\frac{\sqrt{17}}{4},-1}$-stable. Note that the character $\mathrm{ch}^{-1}_{\mathcal{B}_0,\leq 2}(S) = (0,2,2)$, by a standard computation on the potential walls, the object S is $\sigma_{\alpha,-1}$-stable for $\alpha \in (0, \frac{\sqrt{17}}{4}]$. We may let $\alpha_1 = \frac{1}{4}$ in this case. In Case (2.iii) of Proposition 5.17, as S is $\sigma_{\frac{\sqrt{5}}{4},-1}$-stable, the object S is $\sigma_{\alpha_1,-1}$-stable for some $\alpha_1 < \frac{\sqrt{5}}{4}$. In Case (3) of Proposition 5.17, we may let $\alpha_1 = \frac{1}{4}$. By the choice of α_1 in each case, we always have

$$\mu_{\alpha_1,-1}(S) = \begin{cases} 1 & \text{Case (1)} \\ \frac{1}{2} & \text{Case (2.iii)} \\ \frac{1}{3} & \text{Case (3)} \end{cases} > \frac{16\alpha_1^2 - 1}{8} = \mu_{\alpha_1,-1}(\mathcal{B}_0[1]) \geq \mu_{\alpha_1,-1}(\mathcal{B}_j[1]) \tag{5.27}$$

for $j \leq 0$. Note that both S and $\mathcal{B}_j[1]$ are in $\mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$ and $\sigma_{\alpha_1,-1}$-stable for $j \leq 0$. By (5.27) and Serre duality,

$$\mathrm{Hom}(\mathcal{B}_{j+3}, S[i+1]) \cong (\mathrm{Hom}(S, \mathcal{B}_j[2-i]))^* = 0 \tag{5.28}$$

for any $j \leq 0$, $i \geq 1$. By (5.26), we have $\mathrm{Hom}(\mathcal{B}_j, Q[i]) = 0$ for $i \geq 1$ and $1 \leq j \leq 3$.

Step III: We show that $\mathrm{Hom}(\mathcal{B}_j, Q[i]) = 0$ for $i \leq -1$ and $j = 1,2,3$, or $i = 0$ and $j = 2,3$.

As $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ and Q are in the heart $\mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$, we have $\mathrm{Hom}(\mathcal{B}_j, Q[i]) = 0$ for any $j = 1,2,3$ and $i \leq -1$. Together with Step II, this implies $\mathrm{Hom}(\mathcal{B}_j, Q[i])$ may be nonzero only when $i = 0$. In Case (1) of Proposition 5.17, as $\mathrm{ch}^{-1}_{\mathcal{B}_0,\leq 2}(Q) = (-8, 4, -\frac{1}{4})$, by a standard computation on the potential walls, the object Q is $\sigma_{\alpha_1,-1}$-stable for $\alpha_1 \in (\frac{1}{4}, \frac{\sqrt{17}}{4}]$, we may let $\alpha_1 = \frac{1}{2}$. In Case (2.iii) of Proposition 5.17, the object Q is $\sigma_{\frac{\sqrt{5}}{4},-1}$-stable. Note that $\mathrm{ch}^{-1}_{\mathcal{B}_0,\leq 2}(Q) = (-8, 4, \frac{3}{4})$, by a standard computation on the potential walls, the object Q is $\sigma_{\alpha_1,-1}$-stable for all $\alpha_1 \in (0, \frac{\sqrt{5}}{4}]$, we may let $\alpha_1 = \frac{1}{4}$. In Case (3) of Proposition 5.17, the object Q is $\sigma_{\alpha_1,-1}$-stable for some $\alpha_1 < \frac{1}{4}$. By the choice of α_1 in each case, we always have

$$\mu_{\alpha_1,-1}(Q) = \begin{cases} \frac{16\alpha_1^2-1}{16} & \text{Case (1)} \\ \frac{16\alpha_1^2+3}{16} & \text{Case (2.iii)} \\ \frac{48\alpha_1^2+5}{24} & \text{Case (3)} \end{cases} < \frac{-16\alpha_1^2 + 9}{24} = \mu_{\alpha_1,-1}(\mathcal{B}_2) \leq \mu_{\alpha_1,-1}(\mathcal{B}_j) \tag{5.29}$$

for $j \geq 2$. Therefore,

$$\mathrm{Hom}(\mathcal{B}_j, Q) = 0 \text{ for } j \geq 2. \tag{5.30}$$

Step IV: We show that the character of Q or $\mathsf{L}_{\mathcal{B}_1}Q$ cannot be in $\mathrm{Ku}(\mathbb{P}^3,\mathcal{B}_0)$.

Now $\mathrm{Hom}(\mathcal{B}_1,Q)$ is the only possible non-zero space among all $\mathrm{Hom}(\mathcal{B}_j,Q[i])$ for $j=1,2,3$, $i\in\mathbb{Z}$. Therefore, the object

$$\mathsf{L}_{\mathcal{B}_1}Q=\mathrm{Cone}(\mathcal{B}_1\otimes\mathrm{Hom}(\mathcal{B}_1,Q)\to Q)$$

is in $\mathrm{Ku}(\mathbb{P}^3,\mathcal{B}_0)$. By (2.22) in Definition and Proposition 2.15, the character

$$\mathrm{ch}^{-1}_{\mathcal{B}_0,\leq 2}(\mathsf{L}_{\mathcal{B}_1}Q)\in\mathrm{ch}^{-1}_{\mathcal{B}_0,\leq 2}(\mathrm{Ku}(\mathbb{P}^3,\mathcal{B}_0))\subset\{(a,b,-\frac{7}{32}a)|a,b\in\mathbb{Z}\}.$$

On the other hand, in any case of Proposition 5.17, we have $\mathrm{rk}(Q)<0$ and $\frac{\mathrm{ch}^{-1}_{\mathcal{B}_0,2}(Q)}{\mathrm{rk}(Q)}\geq -\frac{3}{32}$. Note that $\mathrm{ch}^{-1}_{\mathcal{B}_0,\leq 2}(\mathcal{B}_1[1])=(-4,-1,-\frac{1}{8})$, so we have

$$\frac{\mathrm{ch}^{-1}_{\mathcal{B}_0,2}(\mathsf{L}_{\mathcal{B}_1}Q)}{\mathrm{rk}(\mathsf{L}_{\mathcal{B}_1}Q)}=\frac{\mathrm{ch}^{-1}_{\mathcal{B}_0,2}(Q)+\hom(\mathcal{B}_1,Q)\,\mathrm{ch}^{-1}_{\mathcal{B}_0,2}(\mathcal{B}_1[1])}{\mathrm{rk}(Q)+\hom(\mathcal{B}_1,Q)\,\mathrm{rk}(\mathcal{B}_1[1])}\geq-\frac{3}{32}>-\frac{7}{32}.$$

We get the contradiction. Therefore, the object E does not become strictly $\sigma_{\alpha,-1}$-semistable for any $\alpha\leq\alpha_0$. $\square$

Theorem 5.19. *Let Γ be a non-degenerate quintic elliptic curve spanning a smooth cubic threefold on Y. Let C be a smooth conic curve on Y. Let σ be the stability condition on* $\mathrm{Ku}(Y)$ *as in Proposition and Definition 2.15 and Remark 2.18. Then the objects E_Γ and E_C are σ-stable.*

Proof. By Proposition 5.11, 5.16 and 5.18, the objects $\Psi\rho^*(E_\Gamma)$ and $\Psi\rho^*(E_C)$ are in the heart $\mathrm{Coh}^{-1}(\mathbb{P}^3,\mathcal{B}_0)$ and $\sigma_{\alpha,-1}$-stable for every $\alpha>0$. Note that

$$\mu_{Z_{\alpha,-1}}(E_\Gamma)=\mu_{Z_{\alpha,-1}}(E_C)=\frac{16\alpha^2+7}{24}>0.$$

Thus $\Psi\rho^*(E_\Gamma)$ and $\Psi\rho^*(E_C)$ are in $\left(\mathrm{Coh}^{-1}(\mathbb{P}^3,\mathcal{B}_0)\right)^0_{\sigma_{\alpha,-1}}\bigcap\mathrm{Ku}(\mathbb{P}^3,\mathcal{B}_0)$.

By Lemma 5.10 and Lemma 5.15, both $\Psi\rho^*(E_\Gamma)$ and $\Psi\rho^*(E_C)$ satisfy the conditions as those in Lemma 2.16. Therefore, they are stable with respect to the stability condition defined in Proposition 2.15. By Proposition 2.10 and Remark 2.18, both of them are σ-stable. $\square$

5.7. Example of $\mathcal{C}_{12}$

We give an example when E_Γ is not expected to be σ-stable.

Denote by $\mathcal{C}_{12}$ the divisor in the moduli space of cubic fourfolds parametrizing cubic fourfolds containing a rational cubic scroll [23, Section 4.1.2]. Let Γ be a non-degenerate quintic elliptic curve in $\mathbb{P}^5$; then Γ is contained in a rational cubic scroll $\Sigma\subset\langle\Gamma\rangle$ (see [25, Lemma 6.11]). Assume that $\Sigma \subset Y$ for some smooth cubic fourfold in $\mathbb{P}^5$, in particular, the fourfold Y is in $\mathcal{C}_{12}$. Consider the cubic threefold $X := \langle \Gamma \rangle \cap Y$, which contains Σ by our assumption. We point out that such X cannot be smooth.

Lemma 5.20. *Let Γ be a non-degenerate quintic elliptic curve contained in a cubic scroll Σ in Y. Then the object $\mathcal{I}_{\Sigma/X}(H)$ is in* $\mathrm{Ku}(Y)$. *If $\mathcal{I}_{\Sigma/X}(H)$ is σ-stable, then E_Γ is not σ-stable.*

Proof. Consider the exact sequence

$$0 \to \mathcal{I}_{\Sigma/X}(H) \to \mathcal{O}_X(H) \to \mathcal{O}_\Sigma(H) \to 0.$$

Note that Σ is defined by a quadric in $\mathbb{P}^4 = \langle \Gamma \rangle \subset \mathbb{P}^5$, so its canonical bundle is $\omega_\Sigma \cong \mathcal{O}_\Sigma(-2H)$. Moreover, we have $\chi(\mathcal{O}_\Sigma(tH)) = \frac{3}{2}t^2 + \frac{5}{2}t + 1$. Then by Kodaira vanishing we have $H^i(\mathcal{O}_\Sigma(-H)) = 0$ for every i. Similarly we compute $H^i(\mathcal{O}_\Sigma) = H^i(\mathcal{O}_\Sigma(H)) = 0$ for $i \neq 0$, $H^0(\mathcal{O}_\Sigma) = \mathbb{C}$, $H^0(\mathcal{O}_\Sigma(H)) = \mathbb{C}^5$.

Applying $\mathrm{Hom}(\mathcal{O}_Y(mH), -)$ to the sequence for $m = 0, 1, 2$, we observe that $\mathcal{I}_{\Sigma/X}(H)$ is in $\mathrm{Ku}(Y)$. In particular, by Serre duality, we have

$$\mathcal{I}_{\Sigma/X}(H) \in {}^{\perp}\langle \mathcal{O}_Y(-2H), \mathcal{O}_Y(-H) \rangle. \tag{5.31}$$

Recall from Definition 4.3 that:

$$E_\Gamma = \mathsf{pr}(\mathcal{I}_\Gamma(H)) = \mathsf{R}_{\mathcal{O}_Y(-H)} \mathsf{R}_{\mathcal{O}_Y(-2H)} \mathsf{L}_{\mathcal{O}_Y}(\mathcal{I}_\Gamma(H)) = \mathsf{R}_{\mathcal{O}_Y(-H)} \mathsf{R}_{\mathcal{O}_Y(-2H)}(\mathcal{I}_{\Gamma/X}(H)).$$

By (5.31),

$$\begin{aligned} \mathrm{Hom}(\mathcal{I}_{\Sigma/X}(H), E_\Gamma) &= \mathrm{Hom}(\mathcal{I}_{\Sigma/X}(H), \mathsf{R}_{\mathcal{O}_Y(-H)} \mathsf{R}_{\mathcal{O}_Y(-2H)}(\mathcal{I}_{\Gamma/X}(H))) \\ &\cong \mathrm{Hom}(\mathcal{I}_{\Sigma/X}(H), \mathcal{I}_{\Gamma/X}(H)) \neq 0. \end{aligned}$$

Since $\mathrm{ch}(\mathcal{O}_\Sigma) = \Sigma - \frac{2}{3}H^3 + \frac{1}{12}H^4$, we have

$$\mathrm{ch}(\mathcal{I}_{\Sigma/X}(H)) = \mathrm{ch}(\lambda_1) + \mathrm{ch}(\lambda_2) + s,$$

where $s = H^2 - \Sigma$ is a class in $H^{2,2}(Y, \mathbb{Z})_{\mathrm{prim}}$. In particular, $H^2 s = 0$ and $\mu_\sigma(\mathcal{I}_{\Sigma/X}(H)) = \mu_\sigma(E_\Gamma)$. Therefore, if $\Psi\rho^*(\mathcal{I}_{\Sigma/X}(H))$ is in $\mathrm{Coh}^{-1}(\mathbb{P}^3, \mathcal{B}_0)$, then it will destabilize E_Γ with respect to σ. $\square$

In [52, Section 5], the authors give a classification of walls for stability for objects with non-primitive Mukai vector with square 2 and divisibility 2 on a K3 surface. In our more general noncommutative setting, we expect similar results hold for the singular moduli space $M_\sigma(2\lambda_1 + 2\lambda_2)$.

Question 5.21. Let Y be in $\mathcal{C}_{12}$, Γ, Σ and X be as those in the lemma. We expect that $\mathcal{I}_{\Sigma/X}(H)$ is always σ-stable. Moreover, the object E_Γ is strictly σ-semistable and σ is on the flopping wall predicted by [52].

6. Application: Lagrangian fibration and twisted family of intermediate Jacobians

We are now ready to prove Theorem 1.3. Let Y be a smooth cubic fourfold, and fix

$$v_0 = \lambda_1 + \lambda_2,\ v = 2\lambda_1 + 2\lambda_2.$$

By [12], we have full numerical stability conditions on $\mathrm{Ku}(Y)$. In particular, we choose σ_0 which is generic with respect to v, and also is in a chamber whose closure contains the stability condition σ. By Theorem 3.1, there exists a projective moduli space $M := M_{\sigma_0}(v)$, which admits a projective hyperkähler resolution $\widetilde{M}$, deformation equivalent to O'G10.

For a very general Y, we can just take $\sigma_0 = \sigma$. However, the example in Section 5.7 shows that a change of the stability condition is necessary in special cases.

Recall from Section 4 that, for every elliptic quintic Γ contained in a *smooth* hyperplane section of Y, we have an object $E_\Gamma \in \mathrm{Ku}(Y)$. We further denote by M_0 the locus of the objects of the form E_Γ in M, which is identified with an open subvariety $i: M_0 \to \widetilde{M}$. Note that M_0 is non-empty, since by Theorem 5.19 and our choice of σ_0, we know that E_Γ is σ_0-stable. Similarly, by Theorem 5.19 we have a σ_0-stable object E_C for every smooth conic C in Y.

Recall that by Proposition 4.5(2), each E_Γ is supported on a smooth cubic threefold. We define a morphism

$$\pi_0 : M_0 \to \mathbb{P}_0 \subset \mathbb{P}^{5\vee},$$

which sends each E_Γ to its support. Here $\mathbb{P}_0$ parametrizes smooth hyperplane sections of Y. By definition, this morphism is induced by a linear series on M_0: for every point $x \in \mathbb{P}^5$, consider the linear divisor in $\mathbb{P}_0$ given by the cubic threefolds which span hyperplanes containing x in $\mathbb{P}^5$. Its preimage in M_0 is the divisor

$$D_x := \{E_\Gamma \mid \mathrm{Supp}(E_\Gamma) \text{ spans a hyperplane containing } x \text{ in } \mathbb{P}^5\}.$$

Denote this linear series on M_0 by $|D|$. Note that the fiber of π_0 over a point corresponding to a smooth cubic threefold is the moduli of E_Γ on that threefold, hence is affine and irreducible by Remark 4.2 and Proposition 4.5(2). Hence the elements in $|D|$ are prime divisors.

Now define a line bundle on $\widetilde{M}$ as follows: by taking closure, each divisor in $|D|$ extends to a prime divisor in $\widetilde{M}$. The closures of generic elements in $|D|$ are algebraically equivalent divisors on $\widetilde{M}$. Since $\widetilde{M}$ is simply-connected, we know that the closures of generic elements in $|D|$ are linearly equivalent. This defines a line bundle $\mathcal{L}$. We use $|\mathcal{L}|$ to denote the complete linear series associated to $\mathcal{L}$, which is at least 5-dimensional by our construction.

Recall that two birational hyperkähler manifolds are isomorphic outside a locus with codimension at least 2 (see [26, Section 2.2]), hence the line bundles on each are naturally identified. Now we have the following result by Matsushita:

Proposition 6.1. *There exists a projective hyperkähler manifold N birational to $\widetilde{M}$, with the following properties:*

a) *the birational map restricts to an isomorphism away from* $\mathrm{Bs}(\mathcal{L})$*;*
b) *the induced line bundle $\mathcal{L}'$ on N is nef.*

Proof. Note that $|\mathcal{L}|$ on $\widetilde{M}$ has no fixed divisor (fixed component), as it contains prime divisors given as closures of elements in $|D|$. Now the existence of N with a) and b) follows from [49, Prop. 1]. □

The aim of this section is to prove the following theorem:

Theorem 6.2. *The line bundle $\mathcal{L}'$ on N is semiample. A multiple of it induces a Lagrangian fibration $\pi : N \to B$.*

Remark 6.3. We do not know whether $B \cong \mathbb{P}^5$, though by construction B contains the open subset $\mathbb{P}_0$. It is in general a conjecture that the base of a Lagrangian fibration on a hyperkähler manifold is always isomorphic to a projective space.

To prove this theorem we need to introduce one more construction. Denote by $\mathcal{X} \to \mathbb{P}_0$ the family of smooth hyperplane sections of Y. In [58], the twisted family of intermediate Jacobians of $p : J \to \mathbb{P}_0$ was constructed, where the fiber J_t is the intermediate Jacobians of the cubic threefold X_t for each point $t \in \mathbb{P}_0$. Note that the relative Hilbert scheme of conics naturally maps to J, and we denote the image by Q.

Now for the family $\mathcal{X} \to \mathbb{P}_0$, consider the relative moduli space $\widetilde{J} \to \mathbb{P}_0$ of semistable instanton sheaves. By Remark 4.2, each fiber $\widetilde{J}_t$ is isomorphic to the blowup of J_t along the involution of the Fano surface. We have the following relationship of $\widetilde{J}$ and J.

Proposition 6.4. *The space $\widetilde{J}$ is isomorphic to the blowup of J along the image Q of the relative Hilbert scheme of conics.*

Proof. Note that there exists a quasi-universal family on $\widetilde{J}$ of instanton sheaves with second Chern classes given by 1-cycles of degree 2. By [21, Theorem 4.8], there exists a morphism $\widetilde{J} \to J$ induced by taking the second Chern class. By the previous discussion, we know this morphism is birational, with exceptional divisor in $\widetilde{J}$ mapped to $Q \subset J$. Now the result follows from the universal property of blowup. □

Now we have the following observation.

Lemma 6.5. *The variety $J_0 := J \backslash C$ is isomorphic to an open subset of $\widetilde{M}$. More precisely, it is isomorphic to the union of M_0 and the open subset of the exceptional divisor over the locus parametrizing objects of the form $P_{\ell_1} \oplus P_{\ell_2}$ for disjoint lines. Moreover, this open set is disjoint from the base locus of $|\mathcal{L}|$ on $\widetilde{M}$.*

Proof. Proposition 6.4 implies that J_0 can be identified with the moduli space parametrizing instanton sheaves E_Γ and $\mathcal{I}_{\ell_1/X} \oplus \mathcal{I}_{\ell_2/X}$ for *disjoint* lines on any smooth cubic threefold $X \subset Y$. Recall that E_Γ, viewed as a torsion sheaf on Y, is a σ_0-stable object in $\mathrm{Ku}(Y)$. By Proposition 4.5(3), the sheaf $\mathcal{I}_{\ell_1/X} \oplus \mathcal{I}_{\ell_2/X}$ projects to the σ_0-semistable object $P_{\ell_1} \oplus P_{\ell_2}$ defined in (4.1).

Hence the projection functor $\mathsf{pr}\iota_*$ induces a morphism $J_0 \to M$, which is an isomorphism over M_0. For the object $P_{\ell_1} \oplus P_{\ell_2} \in M$ with disjoint lines, the fiber of the morphism is the open set of the $\mathbb{P}^1$ parametrizing smooth cubic threefolds containing ℓ_1 and ℓ_2. Recall that $\widetilde{M}$ is given by the blowup of M along the singular locus. At $P_{\ell_1} \oplus P_{\ell_2}$, it is locally an A_1-singularity, and the resolution produces a $\mathbb{P}^1$-fiber. Now the result follows from the universal property of blowup.

The last assertion follows from the construction of $|\mathcal{L}|$: it generically consists of the closure of $D_x \subset M_0$ in $\widetilde{M}$. A point in the exceptional divisor of $\widetilde{M}$ in the fiber over the point $P_{\ell_1} \oplus P_{\ell_2}$, for disjoint lines, is identified with a point in J_0, hence is associated to a cubic threefold X. Now choose $x \notin X$, then the point is not contained in the closure of D_x. This proves the statement. $\square$

This implies the following result.

Lemma 6.6. *The line bundle $\mathcal{L}'$ on N is not big.*

Proof. As $J_0 \subset \widetilde{M}$ is away from $\mathrm{Bs}(\mathcal{L})$, by Proposition 6.1 a), we regard J_0 as an open subset of N. The important observation is that $p : J \to \mathbb{P}_0$ is a projective morphism and Q is of relative codimension three. Hence the open set J_0 can be covered by *proper* curves that are contracted by p.

Now recall from [41, Corollary 2.2.7] that a divisor D is big if and only if

$$nD = A + E,$$

for some positive integer n such that A is an ample divisor, and E is an effective divisor. In our case, for any effective divisor E, we can always choose a p-exceptional proper curve $C \subset J_0 \subset N$ not contained in E. With this choice we have

$$\mathcal{L}'.C = 0,\ A.C > 0,\ E.C \geq 0,$$

so $\mathcal{L}'$ is not big. $\square$

Now we are ready to prove Theorem 6.2.

Proof of Theorem 6.2. The theorem follows from several results in the literature. Note that $|\mathcal{L}|$ induces a rational map $\widetilde{M} \dashrightarrow \mathbb{P}^{5\vee}$, which sends an object E_Γ to the point in $\mathbb{P}^{5\vee}$ corresponding to the support of E_Γ. We know that for any point in $\mathbb{P}^{5\vee}$ corresponding to a smooth hypersurface, there exist E_Γ supported on it. Hence the induced rational map $\widetilde{M} \dashrightarrow \mathbb{P}^{5\vee}$ is dominant. So the Iitaka dimension $\kappa(\mathcal{L}') = \kappa(\mathcal{L}) \geq 5$.

On the other hand, since $\mathcal{L}'$ is nef but not big on N, by [51, Cor 3.2], we have $q(\mathcal{L}') = 0$, where q is the Beauville-Bogomolov form. By [22, Prop 24.1], this implies that the numerical dimension $\nu(\mathcal{L}') = 5$. Hence we have

$$\kappa(\mathcal{L}') = \nu(\mathcal{L}').$$

By [31, Theorem 6.1], $\mathcal{L}'$ is semiample. Now the assertion follows from [48, Thm. 1]. $\square$

The following result completes the proof of Theorem 1.3.

Proposition 6.7. *The hyperkähler manifold N provides a compactification of J, i.e.*

$$J \cong \pi^{-1}(\mathbb{P}_0) \subset N.$$

Proof. We know that J, $\widetilde{J}$, M, $\widetilde{M}$ and N are birational to each other. Note that both $\pi : \pi^{-1}(\mathbb{P}_0) \to \mathbb{P}_0$ and $p : J \to \mathbb{P}_0$ are projective morphisms. Now since both N and J have symplectic structures, they are both relative minimal models. By [34, Theorem 3.52], $\pi^{-1}(\mathbb{P}_0)$ and J are isomorphic in codimension 1, hence related by a sequence of relative flops by [32]. Moreover, the exceptional loci are covered by rational curves contracted by π and p. However, as $J \to \mathbb{P}_0$ is a family of abelian varieties, such a relative flop cannot exist. Hence, we know that $J \cong \pi^{-1}(\mathbb{P}_0)$. $\square$

Note that this provides a different construction of the results of [58] and [57, Remark 1.10] on the existence of a hyperkähler compactification of J.

It remains an interesting question to determine all birational models of N for very general Y, similarly to the work [57]. We plan to study this in future work. In this paper, we focus on one flop between N and $\widetilde{M}$, which can be explicitly described by our construction.

Example 6.8. Recall that an open subset of the exceptional divisor of $\widetilde{J}$ parametrizes the sheaves of the form F_C where $C \subset X$ is a conic. The blowdown morphism to $Q \subset J \subset N$ is defined by taking the residual line of C in X. Hence the fiber of this morphism is isomorphic to the $\mathbb{P}^2$ parametrizing all conics contained in a fixed X and residual to a fixed line.

On the other hand, the projection of F_C into $\mathrm{Ku}(Y)$ gives the object E_C, which is σ_0-stable by Theorem 5.19, and defines a point in $\widetilde{M}$. Hence the fiber of this projection is isomorphic to the $\mathbb{P}^2$ parametrizing all cubic threefolds containing a fixed conic C. This explicitly describes a flop between $\widetilde{M}$ and N.

Remark 6.9. For a very general cubic fourfold Y, it is easy to see that the Picard rank of $\widetilde{M}$ and N is two. In this case, we know that their movable cones are identified, with boundaries given by the blow up and the Lagrangian fibration. This implies that for such Y, there exists a unique hyperkähler compactification of the twisted family of intermediate Jacobians with a Lagrangian fibration structure. In particular, $\widetilde{M}$ and N are not isomorphic and N is isomorphic to Voisin's construction in [58].

7. Application: elliptic quintics and MRC quotients

In this section we prove Proposition 1.5. Let Y be a smooth cubic fourfold, recall that we can write the semiorthogonal decomposition

$$\mathrm{D^b}(Y) = \langle \mathcal{O}_Y(-2H), \mathcal{O}_Y(-H), \mathrm{Ku}(Y), \mathcal{O}_Y \rangle.$$

Let $\Gamma \subset Y$ be an elliptic quintic, whose ideal sheaf is denoted by $\mathcal{I}_{\Gamma/Y}$. Recall from Definition 4.3 that we have the following projection E_Γ in $\mathrm{Ku}(Y)$:

$$E_\Gamma := \mathsf{pr}(\mathcal{I}_{\Gamma/Y}(H)) = \mathsf{R}_{\mathcal{O}_Y(-H)} \mathsf{R}_{\mathcal{O}_Y(-2H)} \mathsf{L}_{\mathcal{O}_Y} \mathcal{I}_{\Gamma/Y}(H) \in \mathrm{Ku}(Y).$$

By Proposition 4.5, if Γ is non-degenerate and spanning a smooth cubic threefold $X \subset Y$, then $E_\Gamma = \iota_* F_\Gamma$ where $\iota_* F_\Gamma$ was defined in (4.4). In particular, it sits in the following short exact sequence in $\mathrm{Coh}(Y)$:

$$0 \to \mathcal{O}_X(-H) \to E_\Gamma \to \mathcal{I}_{\Gamma/X}(H) \to 0.$$

Moreover, by Theorem 5.19, the object E_Γ is stable in the moduli $M = M_{\sigma_0}(2\lambda_1 + 2\lambda_2)$, where σ_0 is as chosen in Section 6.

Recall that for a variety X, a rational map $\rho : X \dashrightarrow Y$ is the maximally rationally connected (MRC) fibration of X, if the general fibers of ρ are rationally connected and any rational curve in X intersecting a fiber over a general point of Y is contained in the fiber. Let $\mathcal{C}$ be the connected component of the Hilbert scheme $\mathrm{Hilb}^{5m}(Y)$ containing elliptic quintics in Y.

Proposition 7.1. *There is a rational map $\rho : \mathcal{C} \dashrightarrow M$ defined by the projection of $\mathcal{I}_{\Gamma/Y}(H)$ in $\mathrm{Ku}(Y)$, which is the MRC fibration of $\mathcal{C}$.*

Proof. Consider the open subset $U \subset \mathcal{C}$ parametrizing non-degenerate quintic elliptic curves on Y. Let $\mathcal{I}$ be the universal family on $Y \times U$ parametrizing the objects $\mathcal{I}_{\Gamma/Y}(H)$. Then the projection of $\mathcal{I}$ in $\mathrm{Ku}(Y \times U)$ is a flat family of σ_0-stable objects in $\mathrm{Ku}(Y)$.

Thus there is an induced morphism from U to M, defining the rational map ρ in the statement.

Recall that E_Γ is an instanton bundle over its support X, and Γ can be identified with the vanishing locus of a section of $E_\Gamma(H)$. As mentioned in Remark 4.2, for a generic section, the vanishing locus is a locally complete intersection, connected and reduced elliptic quintic. Hence we know that the general fibers of ρ are identified with open subset of global sections of $E_\Gamma(H)$, hence the general fibers are rational.

To see that ρ is the MRC fibration of $\mathcal{C}$, it is enough to note that by Proposition 6.1 there exists a hyperkähler compactification of the locus M_0 parametrizing E_Γ. Now [19, Lemma 1.4] proves the claim. □

A closely related question is about rational quartics on cubic fourfolds. The following was conjectured by Castravet [19, Page 416], and follows from our results in Section 6.

Proposition 7.2. *For any smooth cubic fourfold Y, the MRC quotient of the main component of the Hilbert scheme of rational quartics on Y is (birational to) the twisted family of intermediate Jacobians J of Y.*

Proof. It was observed in [19] that it is enough to show that J is not uniruled. This follows from the existence of the hyperkähler compactification of the twisted family in Proposition 6.7. □

This is the only remaining case of MRC quotients of rational curves on cubic fourfolds: the case of degree $d \leq 3$ is classical, while $d \geq 5$ was treated in [19].

Remark 7.3. Here we briefly recall the connection between elliptic quintics and rational quartics. It was proved in [25, Section 8] that for a generic elliptic quintic in a generic cubic threefold, we can choose a generic cubic scroll surface containing the curve, such that the residual curve is a smooth rational quartic. Along this line, it should be possible to show that the main components of the Hilbert schemes corresponding to these two cases are stably birational. We do not need this result and leave it as an open question.

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