Particle-surface interactions in a uniform electric field

Zhanwen Wang [1], Michael J. Miksis [2], and Petia M. Vlahovska [2,*]

[1] *Theoretical and Applied Mechanics Program, Northwestern University, Evanston, Illinois 60208, USA*

[2] *Engineering Sciences and Applied Mathematics, Northwestern University, Evanston, Illinois 60208, USA*

(Received 23 May 2022; accepted 23 August 2022; published 12 September 2022)

The electrostatic force on a spherical particle near a planar surface is calculated for the cases of a uniform electric field applied in either normal or tangential direction to the surface. The particle and suspending media are assumed to be weakly conducting, so that that the leaky dielectric model applies. The Laplace equation for the electric potential is solved in bipolar coordinate system and the potential is obtained in terms of a series expansion of Legendre polynomials. The force on the particle is calculated using the Maxwell tensor. We find that in the case of normal electric field, which corresponds to a particle near an electrode, the force is always attractive but at a given separation it varies nontrivially with particle-suspending medium conductivity ratio; the force on a particle that is more conducting than the suspending medium is much larger compared to the force on a particle less conducing than the suspending medium. In the case of tangential electric field, which corresponds to a particle near an insulating boundary, the force is always repulsive.

DOI: 10.1103/PhysRevE.106.034607

I. INTRODUCTION

Electric fields are a classic means to manipulate and assemble colloidal particles [1–4]. More recently, electric fields have become a popular tool to energize and create self-propelled particles [5–7] due to field-induced charge electrophoresis [8–10] or torque (due to the Quincke effect), which drives colloids to roll on a surface [11–15]. In these applications particles are in close proximity to boundaries, and the electrostatic force (and torque) exerted on the particle is significantly influenced by confinement. The canonical problem of a particle near an electrode in the presence of a uniform electric field has been analyzed in the literature theoretically in the two limiting cases of a conducting or an insulating sphere. The surface of a conducting particle is equipotential, and consequently, the electric field inside vanishes. The net charge and force on a spherical particle are found using the method of images [16,17] or the equivalent problem of two spheres in a uniform electric field [18]. If the particle is perfect dielectric, the boundary condition on the particle-medium interface are continuity of the electric potential and a jump of the displacement field due to given surface charge. The electrostatic force has been found either in terms of series expansion in eigenfunctions of the Laplace equation in bispherical coordinate system [19–22] or from a multipole-moment theory for the pair-wise dielectrophoretic interactions of dielectric spheres [23].

In this paper, we solve for the electrostatic force on a particle with arbitrary conductivity. In this case, the boundary conditions on the particle interface are continuity of the electric potential and the normal electric current, and the bulk media is assumed to be charge free. The surface charge density needed to satisfy the charge flux continuity is thus determined as part of the problem. This is the so called "leaky-dielectric" model [24]. Using this approach, the force on a sphere straddling a planar interface subjected to either a normal or tangential uniform electric field has been calculated [25]. However, the force on a sphere near an interface has not be derived, even though the electric field for the equivalent problem of two leaky-dielectric spheres with line-of-centers parallel or perpendicular to the applied field has been derived [22,26]. This paper is organized as follows. The problem is formulated in Sec. II. In Sec. III, the solution methodology using bispherical coordinates is presented. First, the solution of the electric field is expressed in terms of a series of Legendre polynomials, and an algorithm to determine its coefficients is given. Then, the electric field strength, surface charge distribution on the interface, and the force on the particle are calculated. Details are presented in the Appendix. In Sec. IV, we validate the solution with the published results for the interaction force between two identical dielectric spheres in a uniform electric field [23] and a conducting sphere near an electrode [16,17], and study the force dependence on separation and particle electric properties.

II. PROBLEM FORMULATION

Consider a spherical particle with radius R, conductivity σ_1, and permittivity ε_1 suspended in a medium with conductivity σ_2 and permittivity ε_2. The mismatch in the particle and medium conductivities is characterized by the parameter $\beta_{12} = (\sigma_1 - \sigma_2)/(\sigma_1 + \sigma_2)$, which ranges from -1 for a perfectly insulating particle to $+1$ for a for a perfectly conducting particle. The particle is in the vicinity of a boundary and the surface-to-surface distance is s. A uniform electric field is applied in a direction either normal or tangential to the boundary. The problem is sketched in Fig. 1(a).

*petia.vlahovska@northwestern.edu

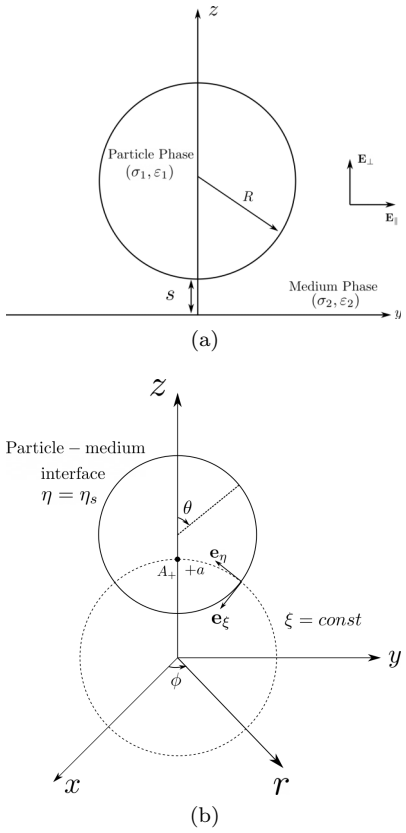


FIG. 1. (a) A spherical particle with radius R is suspended above a planar wall. The gap between the particle and wall surfaces is s. The applied electric field is either in the normal direction $\mathbf{E}_\perp = E_0\hat{\mathbf{z}}$ with a conducting wall or in the tangential direction $\mathbf{E}_{||} = E_0\hat{\mathbf{y}}$ with an insulating wall. (b) Sketch of Cartesian coordinates (x, y, z), cylindrical coordinates (r, z, ϕ) and bipolar coordinates (ξ, η, ϕ) with unit coordinate vectors $\boldsymbol{e}_\xi$ and $\boldsymbol{e}_\eta$, and a pole A_+. The isosurfaces of coordinates η and ξ are obtained by rotating about z axis.

We adopt the leaky dielectric model, which assumes an irrotational electric field ($\mathbf{E} = -\nabla\Phi$, where Φ is the electric potential), electroneutral bulk fluids, and that electric current obeys Ohm's law [27]. Accordingly, the electric potential inside the particle, Φ_1, and in the suspending medium, Φ_2, satisfy the Laplace equation

$$\nabla^2\Phi_1 = 0, \quad \nabla^2\Phi_2 = 0. \tag{1}$$

At the particle-medium interface, the potential and normal electric current are continuous

$$\Phi_1 = \Phi_2, \quad \sigma_1\mathbf{n}\cdot\mathbf{E}_1 = \sigma_2\mathbf{n}\cdot\mathbf{E}_2, \tag{2}$$

where $\mathbf{n}$ is the inward normal to the interface. The current leads to accumulation of charge Q at the surface, which is calculated *a posteriori* from the jump in the electric displacement field at the interface,

$$Q = (\varepsilon_1\mathbf{E}_1 - \varepsilon_2\mathbf{E}_2)\cdot\mathbf{n}.$$

Away from the particle, the electric field is undisturbed. We consider two cases: (i) an electric field applied in a direction normal to a boundary that is an equipotential surface, $\mathbf{E}_\perp = E_0\hat{\mathbf{z}}$ at infinity, and $\Phi_2 = 0$ at $z = 0$, and (ii) an electric field applied in a direction tangential to an insulating boundary, $\mathbf{E}_{||} = E_0\hat{\mathbf{y}}$, and $\mathbf{n}\cdot\mathbf{E}_2 = 0$ at $z = 0$.

We introduce disturbance fields $\hat{\Phi}_1$ and $\hat{\Phi}_2$,

$$\hat{\Phi}_1 = \Phi_1 + E_0x_\alpha, \ \hat{\Phi}_2 = \Phi_2 + E_0x_\alpha, \tag{3}$$

where x_α is the direction of the externally applied electric field. In the case of normal electric field, $x_\alpha = z$, and in the case of tangential electric field, $x_\alpha = y$. The governing equation becomes

$$\nabla^2\hat{\Phi}_1 = 0, \ \nabla^2\hat{\Phi}_2 = 0. \tag{4}$$

The boundary condition at infinity is now homogeneous, $\hat{\Phi}_2 \to 0$. At the planar boundary, in the case of a normal electric field,

$$\hat{\Phi}_2 = 0 \text{ at } z = 0 \quad \text{(normal electric field)}, \tag{5}$$

or, in the case of a tangential electric field,

$$\frac{\partial\hat{\Phi}_2}{\partial z} = 0 \text{ at } z = 0 \quad \text{(tangential electric field)}. \tag{6}$$

The boundary conditions on the particle-medium interface become

$$\hat{\Phi}_1 = \hat{\Phi}_2 \tag{7}$$

and

$$\sigma_1\frac{\partial\hat{\Phi}_1}{\partial n} - \sigma_2\frac{\partial\hat{\Phi}_2}{\partial n} = E_0(\sigma_1 - \sigma_2)\frac{\partial x_\alpha}{\partial n}. \tag{8}$$

The problem is solved using separation of variables in bispherical coordinates (ξ, η, ϕ)[21], defined as

$$r = \frac{a}{h}\sin\xi, \ z = \frac{a}{h}\sinh\eta, \ h \equiv \cosh\eta - \cos\xi, \tag{9}$$

where $a = [(2R + s)s]^{1/2}$. Figure 1(b) illustrates the coordinate systems that are used in this analysis. All isocurves of coordinate ξ pass through poles A_+ and A_-, $A_\pm = (0, 0, \pm a)$ in Cartesian coordinates (x, y, z), as $\eta \to \pm\infty$. The particle-medium interface is specified by $\eta = \eta_s$,

$$\eta_s = \ln\left\{1 + \frac{s}{R} + \left[\left(2 + \frac{s}{R}\right)\frac{s}{R}\right]^{1/2}\right\}. \tag{10}$$

The inward normal to the particle surface is $\mathbf{e}_\eta$. Hence, Eqs. (7) and (8) are written as

$$\hat{\Phi}_1 = \hat{\Phi}_2, \text{ at } \eta = \eta_s, \tag{11}$$

$$\sigma_1\frac{\partial\hat{\Phi}_1}{\partial\eta} - \sigma_2\frac{\partial\hat{\Phi}_2}{\partial\eta} = E_0(\sigma_1 - \sigma_2)\frac{\partial x_\alpha}{\partial\eta}, \text{ at } \eta = \eta_s. \tag{12}$$

The planar wall boundary is specified by $\eta = 0$. Accordingly,

$$\hat{\Phi}_2 = 0, \text{ at } \eta = 0 \quad \text{(normal electric field)}, \tag{13}$$

$$\frac{\partial\hat{\Phi}_2}{\partial\eta} = 0, \text{ at } \eta = 0 \quad \text{(tangential electric field)}. \tag{14}$$

In bispherical coordinates, we require an additional boundary condition, that is, $\hat{\Phi}_1$ to be finite at pole A^+, where $\eta \to \infty$,

$$|\hat{\Phi}_1| < \infty, \text{ as } \eta \to \infty. \tag{15}$$

Finally, note that $r \to \infty$ corresponds to $\eta \to 0$ and $\cos\xi \to 1$. Hence, from Eq. (13), we see that the boundary

condition at infinity is satisfied automatically in the case of normal electric field, while for the tangential electric field, an auxiliary boundary condition is needed:

$$\hat{\Phi}_2 \sim 0, \text{ as } \eta \to 0 \text{ and } \cos\xi \to 1. \tag{16}$$

III. ELECTRIC POTENTIAL

In bispherical coordinates, the Laplace's equation can be written as

$$\nabla^2 F = \frac{h^3}{a^2 \sin\xi}\left[\sin\xi \frac{\partial}{\partial\eta}\left(\frac{1}{h}\frac{\partial F}{\partial\eta}\right) + \frac{\partial}{\partial\xi}\left(\frac{\sin\xi}{h}\frac{\partial F}{\partial\xi}\right)\right] + \frac{h^2}{a^2\sin^2\xi}\frac{\partial^2 F}{\partial\phi^2} = 0,$$

where F stands for either $\hat{\Phi}_1$ or $\hat{\Phi}_2$. The general solution is

$$F = \sqrt{h}\sum_{m,n}\left[X_n^m e^{\lambda_n\eta} + Y_n^m e^{-\lambda_n\eta}\right]P_n^m(\cos\xi)\cos(m\phi) + \sum_{m,n}\left[M_n^m e^{\lambda_n\eta} + N_n^m e^{-\lambda_n\eta}\right]P_n^m(\cos\xi)\sin(m\phi),$$

where $\lambda_n \equiv n + 1/2$ and P_n^m stands for associated Legendre polynomials of degree n and order m. $\sum_{m,n}$ denotes a double sum $\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}$.

In the case of a uniform electric field, the solution is greatly simplified. For a normal electric field, the axial symmetry of the problem leads to

$$F = \sqrt{h}\sum_{n=0}^{\infty}\left[X_n^0 e^{\lambda_n\eta} + Y_n^0 e^{-\lambda_n\eta}\right]P_n(\cos\xi). \tag{17}$$

The tangential electric field indicates the dependence on ϕ is $\sin\phi$. In this case, the general solution is

$$F = \sqrt{h}\sum_{n=1}^{\infty}\left[M_n^1 e^{\lambda_n\eta} + N_n^1 e^{-\lambda_n\eta}\right]P_n^1(\cos\xi)\sin\phi. \tag{18}$$

From the general solutions Eqs. (17) and (18), applying the boundary conditions at the wall and at the particle interface for continuous potential leads to

$$\Phi_1 = -E_0 x_\alpha + \sqrt{h}\sum_{n=n_0}^{\infty} A_n[e^{2\lambda_n\eta_s} \mp 1]e^{-\lambda_n\eta}\mathcal{P}_n, \tag{19}$$

$$\Phi_2 = -E_0 x_\alpha + \sqrt{h}\sum_{n=n_0}^{\infty} A_n[e^{\lambda_n\eta} \mp e^{-\lambda_n\eta}]\mathcal{P}_n, \tag{20}$$

where $\mathcal{P}_n(\cos\xi)$ stands for $P_n(\cos\xi)$ and $P_n^1(\cos\xi)\sin\phi$ for the applied normal and tangential electric fields, respectively. The minus signs in the second term in Eqs. (19) and (20) are for normal electric field. Note that $P_n^1(1) = 0$. Accordingly, Eq. (16) is automatically satisfied. The sum starts at $n_0 = 0$ and 1 for normal and tangential electric fields, respectively. Plugging Eqs. (19) and (20) into the boundary condition Eq. (12), we get the following recurrence formula of A_n after some algebra:

$$L_{n,1}A_{n-1} + L_{n,2}A_n + L_{n,3}A_{n+1} = G_n. \tag{21}$$

Details are listed in Appendix B. Equation (21) is solved numerically as follows. First, we choose a sufficiently large number, N, which set the number of terms retained in the series solution. For $n = N$, the last equation in the system Eq. (21) is

$$L_{N-1,1}A_{N-1} + L_{N,2}A_N = G_N - L_{N,3}A_{N+1}, \tag{22}$$

where A_{N+1} comes from the asymptotic behavior Eqs. (B15) and (B18). Then, coefficients A_n ($n \leqslant N$) are found from the three-diagonal system [Eq. (21)], whose matrix form is presented below:

$$[\mathbf{L}]\cdot\mathbf{A} = \mathbf{G}, \tag{23}$$

where the left-hand-side coefficient matrix $[\mathbf{L}]$ is a triple-diagonal matrix, $\mathbf{A}$ is the column vector of coefficients A_n, and $\mathbf{G}$ is the column vector of the right-hand-side terms.

For the normal applied electric field, their components are listed below:

$$[\mathbf{L}] = \begin{bmatrix} L_{0,2} & L_{0,3} & & & & \\ L_{1,1} & L_{1,2} & L_{1,3} & & & \\ & L_{2,1} & L_{2,2} & L_{2,3} & & \\ & & \ddots & \ddots & \ddots & \\ & & & L_{N-1,1} & L_{N-1,2} & L_{N-1,3} \\ & & & & L_{N,1} & L_{N,2} \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_{N-1} \\ A_N \end{bmatrix}, \mathbf{G} = \begin{bmatrix} G_0 \\ G_1 \\ G_2 \\ \vdots \\ G_{N-1} \\ G_N - L_{N,3}A_{N+1} \end{bmatrix}.$$

The expressions for L, G, and A_{N+1} are given by Eqs. (B7), (B14), and (B15).

For the tangential electric field, subscripts start from 1,

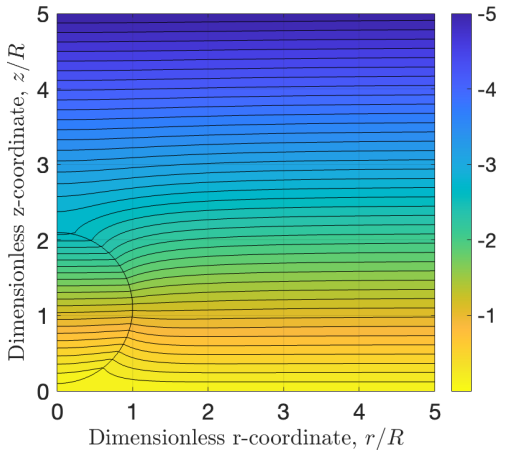
$$[\mathbf{L}] = \begin{bmatrix} L_{1,2} & L_{1,3} & & & & \\ L_{2,1} & L_{2,2} & L_{2,3} & & & \\ & L_{3,1} & L_{3,2} & L_{3,3} & & \\ & & \ddots & \ddots & \ddots & \\ & & & L_{N-1,1} & L_{N-1,2} & L_{N-1,3} \\ & & & & L_{N,1} & L_{N,2} \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_{N-1} \\ A_N \end{bmatrix}, \mathbf{G} = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ \vdots \\ G_{N-1} \\ G_N - L_{N,3}A_{N+1} \end{bmatrix}.$$

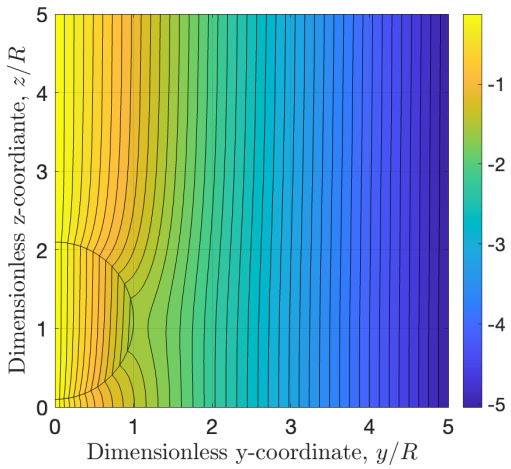
The expressions for L, G, and A_{N+1} are given by Eqs. (B16)–(B18).

Figure 2 illustrates the electric potential computed from Eqs. (19)–(23). The solution appears spectrally convergent at sufficiently high N. As N increases, we observe that the error decreases as $1/N^2$ and approaches near plateau as N increases [28]. We find that the number of terms N needed for the force coefficient $C_f = F_z/(\varepsilon_2 E_0^2 R^2)$ to reach 10^{-5} relative error,

$$\frac{|C_f(N+1) - C_f(N)|}{|C_f(N)|} \leqslant 10^{-5},$$



(a)Normal electric field



(b)Tangential electric field

FIG. 2. Equipotential contours for particle-wall separation $s/R = 0.1$ and perfectly insulating particle $\beta_{12} = -1$. For the tangential electric field, the equipotential contours are on $x = 0$ plane. ($N = 50$).

scales as $10\delta^{-1/2}$, independently of the conductivity and permittivity ratios. We validated our solution for a conducting particle with the result derived using electrostatic images [16], see Fig. 4. We also compared with the two-sphere solution of [23]. In both cases, the agreement is excellent. The numerical computation is very fast, taking a fraction of a minute even at small separations, $\delta \sim 10^{-5}$.

IV. ELECTRIC FIELD AND INDUCED SURFACE CHARGE DISTRIBUTION

The electric field is obtained by calculating the gradient of the electric potential,

$$\mathbf{E}_i = -\nabla\Phi_i = E_{i,\eta}\mathbf{e}_\eta + E_{i,\xi}\mathbf{e}_\xi + E_{i,\phi}\mathbf{e}_\phi, \tag{24}$$

where the subscript i is the index of the phases. $\mathbf{E}_1$ and $\mathbf{E}_2$ are the electric field strengths inside and outside the particle, respectively. Substituting Eq. (3) into Eq. (24) yields the component form

$$E_{i,\eta} = -\frac{h}{a}\left(\frac{\partial\hat{\Phi}_i}{\partial\eta} - E_0\frac{\partial x_\alpha}{\partial\eta}\right), \tag{25}$$

$$E_{i,\xi} = -\frac{h}{a}\left(\frac{\partial\hat{\Phi}_i}{\partial\xi} - E_0\frac{\partial x_\alpha}{\partial\xi}\right), \tag{26}$$

$$E_{i,\phi} = -\frac{h}{a\sin\xi}\left(\frac{\partial\hat{\Phi}_i}{\partial\phi} - E_0\frac{\partial x_\alpha}{\partial\phi}\right). \tag{27}$$

The discontinuity of the electric field across the particle-medium interface leads to accumulation of charge Q. The surface density of this induced charge is calculated from

$$Q = (\varepsilon_1 E_{1,\eta} - \varepsilon_2 E_{2,\eta})|_{\eta=\eta_s}. \tag{28}$$

In the case a normal applied field, due to the symmetry of the problem, $E_{i,\phi} = 0$. Explicit expressions for $E_{i,\eta}$ and $E_{i,\xi}$ are listed in Appendix A. For a net charge-neutral particle, we find the following identity satisfied by the coefficients A_n:

$$\sum_{n=0}^{\infty} A_n = 0, \tag{29}$$

see Appendix C for details.

In the case of the tangential applied electric field, the electric field components are given by

$$E_{i,\eta} = \hat{E}_{i,\eta}\sin\phi,$$
$$E_{i,\xi} = \hat{E}_{i,\xi}\sin\phi,$$
$$E_{i,\phi} = \hat{E}_{i,\phi}\cos\phi,$$

and $\hat{E}_{i,\eta}$, $\hat{E}_{i,\xi}$, and $\hat{E}_{i,\phi}$ are listed in Appendix A. Equation (28) shows that Q for a tangential electric field has the sine dependence of ϕ,

$$Q = \hat{Q}\sin\phi. \tag{30}$$

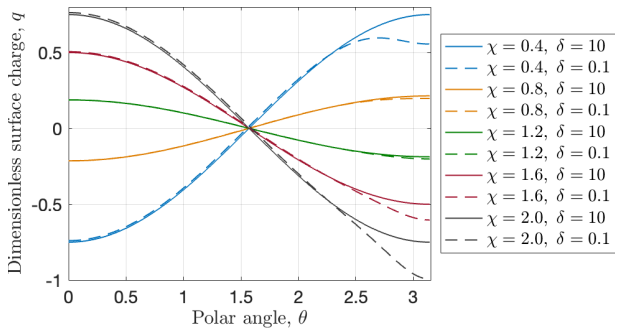
Figure 3 illustrates the dependence of the dimensionless surface charge distribution, $q = Q/(\varepsilon_2 E_0)$, on dimensionless particle-wall separation $\delta = s/R$ and conductivity ratio $\chi = \sigma_1/\sigma_2$ with permittivity ratio $\kappa = \varepsilon_1/\varepsilon_2 = 1$. If $\kappa > \chi$, i.e., $\varepsilon_1/\sigma_1 > \varepsilon_2/\sigma_2$, the surface charge distribution indicates a free-charge dipole which is antiparallel to the applied electric field. In the opposite case, $\kappa < \chi$, the dipole is in the same direction as the applied electric field. At large separations ($s/R = 10$), the distribution is symmetric. The wall proximity introduces asymmetry in the charge distribution. However, the surface integral over the particle-medium interface (a unit sphere in dimensionless form), is zero and the particle is net charge neutral.

V. INTERACTION FORCE

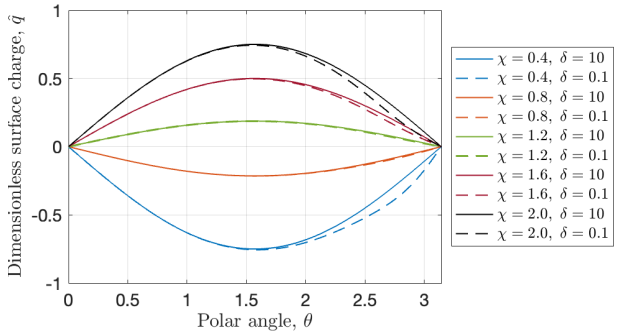
The electrostatic force on the particle is calculated from the Maxwell stress

$$\mathbf{F} = \int_{S_p} (\mathbf{T}_1 - \mathbf{T}_2)\cdot\mathbf{e}_\eta dS, \tag{31}$$

where $\mathbf{e}_\eta$ is the inward normal on the interface, and $\mathbf{T}_1$ and $\mathbf{T}_2$ are Maxwell stress tensor in the particle and media phases,



(a)Normal electric field



(b)Tangential electric field

FIG. 3. Dimensionless surface charge density, $q = Q/(\varepsilon_2 E_0)$, as a function of the polar angle θ for various values of the conductivity ratio $\chi = \sigma_1/\sigma_2$ and particle-wall separations $\delta = s/R$. In the case of the tangential applied field, $q = \hat{q}\sin\phi$ and the θ dependence of $\hat{q}$ is shown. θ is the angle between the positive z axis and the line from the particle center to a point on the interface, see Fig. 1(b). Permittivity ratio $\kappa = \varepsilon_1/\varepsilon_2 = 1$.

respectively:

$$\mathbf{T}_i = \varepsilon_i\left[\mathbf{E}_i\mathbf{E}_i - \frac{1}{2}(\mathbf{E}_i\cdot\mathbf{E}_i)\mathbf{I}\right]. \tag{32}$$

The divergence of the Maxwell strength is zero, $\nabla\cdot\mathbf{T}_i = 0$, because of the bulk media are charge-free. Using the divergence theorem leads to

$$\mathbf{F} = -\int_{S_p}\mathbf{T}_2^D\cdot\mathbf{e}_\eta dS,$$

where S_B stands for the wall boundary and $\mathbf{T}_2^D$ is the disturbance Maxwell stress tensor,

$$\mathbf{T}_2^D = \varepsilon_2\Big[\mathbf{E}_0\mathbf{E}_2^D + \mathbf{E}_2^D\mathbf{E}_0 + \mathbf{E}_2^D\mathbf{E}_2^D - \frac{1}{2}\big(2\mathbf{E}_0\cdot\mathbf{E}_2^D + \mathbf{E}_2^D\cdot\mathbf{E}_2^D\big)\mathbf{I}\Big].$$

Here, $\mathbf{E}_0$ is the applied electric field and $\mathbf{E}_2^D$ is the disturbance field due to the particle. The detailed calculation process can be found in Appendix C. For both normal and tangential applied fields, the force has only a component in the direction normal to the wall surface (in the z direction).

For a normal electric field, the force on the particle is

$$F_z = -4\varepsilon_2\pi\sum_{n=0}^{\infty}\left[\lambda_n A_n^2 - (n+1)A_nA_{n+1}\right]. \tag{33}$$

For a tangential electric field, the force on the particle is

$$F_z = \tfrac{1}{2}\varepsilon_2\pi\left(I_1^T + I_2^T\right), \tag{34}$$

where

$$I_1^T = \sum_{n=1}^{\infty}\frac{A_n^2}{\lambda_n}n(n+1) - 3\sum_{n=1}^{\infty}\frac{A_nA_{n+1}}{\lambda_n\lambda_{n+1}}n(n+1)(n+2) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_nA_m\frac{n^2m^2}{\lambda_n\lambda_m}I_{n+1,m+1} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_nA_m\frac{(n+1)^2(m+1)^2}{\lambda_n\lambda_m}I_{n-1,m-1} - 2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_nA_m\frac{(n+1)^2m^2}{\lambda_n\lambda_m}I_{n-1,m+1}, \tag{35}$$

$$I_2^T = 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_nA_mI_{n,m}. \tag{36}$$

$I_{n,m}$ stands for the following integral:

$$I_{n,m} = \int_{-1}^{1}\frac{P_n^1(u)P_m^1(u)}{1+u}du = \begin{cases} n(n+1) & n = m \\ (-1)^{p-q}q(q+1) & n \neq m \end{cases},$$

where $p = \max(n, m)$ and $q = \min(n, m)$.

A. Special case: Perfectly conducting sphere in normal electric field

A perfectly conducting sphere occurs in the limit of $\sigma_1 \to \infty$. This requires a special solution because $\beta_{12} \to 1$ and causes the denominator in Eq. (B15) to become zero. Here, we outline this special solution.

A perfectly conducting sphere implies a constant potential Φ_p along the interface. In this case, boundary condition Eq. (2) is replaced by $\Phi_2 = \Phi_p$. Here, Φ_p is an unknown which we will determine using the zero net charge condition.

The solution approach for Φ_2 parallels the approach at the beginning of this section. Equation (17) still represents the solution $\hat{\Phi}_2$. Using the constant potential boundary condition, it is straightforward to show that

$$\hat{\Phi}_2 = \sqrt{h}\sum_{n=0}^{\infty}B_n\sinh(\lambda_n\eta)P_n(\cos\xi),$$

where

$$B_n = \sqrt{2}(\Phi_p + 2E_0a\lambda_n)\frac{\exp(-\lambda_n\eta_s)}{\sinh(\lambda_n\eta_s)}.$$

Now, paralleling the charge calculation in Appendix C, we find the net charge on the particle,

$$Q_{net} = 4\pi\varepsilon_2 a\Phi_p S_1(\eta_s) + 8\pi\varepsilon_2 a^2 E_0 S_2(\eta_s) = 0,$$

where S_1 and S_2 are two sums

$$S_1(\eta_s) = \sum_{n=0}^{\infty} \frac{\exp(-\lambda_n\eta_s)}{\sinh(\lambda_n\eta_s)},$$

$$S_2(\eta_s) = \sum_{n=0}^{\infty} \frac{\lambda_n \exp(-\lambda_n\eta_s)}{\sinh(\lambda_n\eta_s)}.$$

Setting Q_{net} to zero we can solve for Φ_p to find

$$\Phi_p = -2E_0 a \frac{S_2(\eta_s)}{S_1(\eta_s)}.$$

From Eq. (31) with $\mathbf{T}_1 = 0$, we can calculate the vertical force on the particle to find

$$F_z = -\varepsilon_2\pi \sum_{n=0}^{\infty} \left[\lambda_n B_n^2 - (n+1)B_n B_{n+1}\right]. \tag{37}$$

B. Variation of the force with separation and media electric properties

In the absence of a wall, the force on a charge-free particle in a uniform electric field is zero. The wall breaks the symmetry and, as shown below, gives rise to attraction in the case of a normal applied field and repulsion in the case of a tangential applied field. In Fig. 4, we plot the force coefficient $C_f = F_z/(\varepsilon_2 E_0^2 R^2)$ as a function of particle-wall separation δ. In both cases, the magnitude of the force increases with decreasing separation.

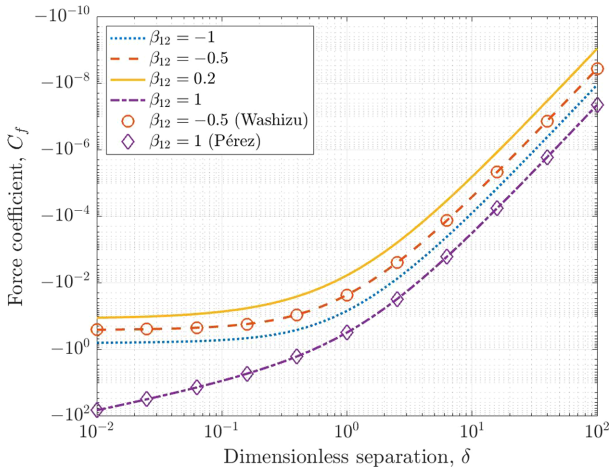
The dependence on the dimensionless conductivity mismatch β_{12}, however, is nonmonotonic. Figure 5 shows that the force vanishes for $\beta_{12} = 0$, as expected since the particle and suspending media have the same electric properties. The variation with β_{12} is asymmetric with respect to $\beta_{12} = 0$. In the case of a normal applied field, a conducting particle is attracted more strongly by the electrode compared to an insulating particle. A similar trend is observed also in the case of a tangential applied field: the conducting particle experiences stronger repulsion compared to the insulating particle.

C. Asymptotic behavior of the force at large particle-wall separations

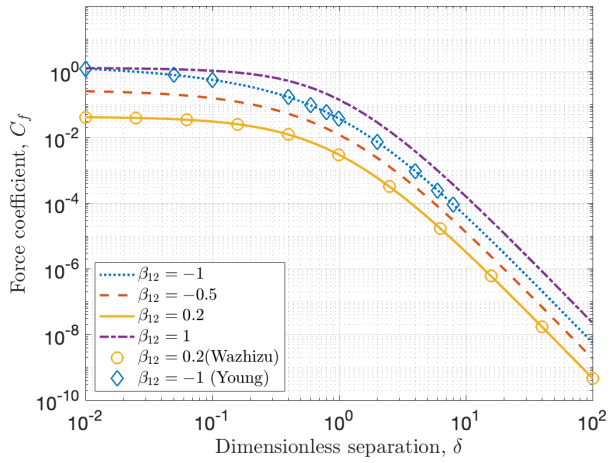
In this section, we present the asymptotic behavior of the force coefficient C_f for large separations. In the case of normal electric field, the numerical solution for A_n suggests a two-term approximation. For $\delta = s/R \gg 1$, from Eq. (10), $\eta_s \sim \ln(2\delta)$. Applying this result in Eqs. (B7) and (B14), we find the leading-order asymptotic form of the system Eq. (21):

$$\sqrt{2}(\beta_{12}-1)A_0 + 2\sqrt{2}A_1 \sim E_0\beta_{12}R\frac{1}{\delta^2}, \tag{38}$$

$$\sqrt{2}A_0 + 2\sqrt{2}(\beta_{12}-3)\delta^2 A_1 \sim -2E_0\beta_{12}R. \tag{39}$$



(a)Normal electric field



(b)Tangential electric field

FIG. 4. Force coefficient $C_f = F_z/(\varepsilon_2 E_0^2 R^2)$ as a function of dimensionless separation $\delta = s/R$. Circles: Washizu's solution using the equivalent multipole method [23]. Diamonds in (a): Pérez solution using the method of images for a conducting particle [16] Diamonds in (b): Young's solution using bispherical coordinates for a insulating particle [29].

The asymptotic behavior of coefficients A_0 and A_1 is found from these two equations by balancing the order of δ,

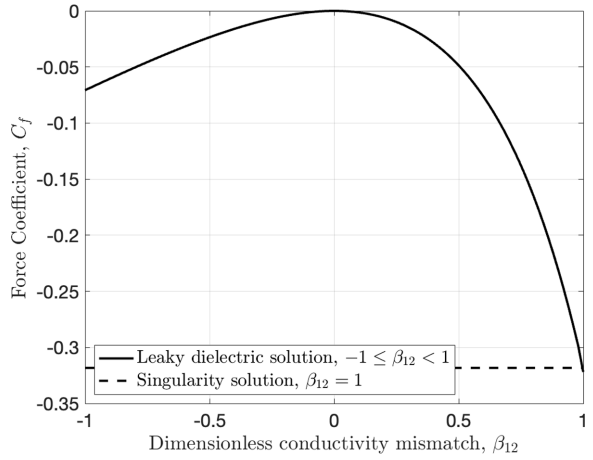
$$A_0 \sim -A_1 \sim -\frac{E_0\beta_{12}R}{\sqrt{2}(3-\beta_{12})}\frac{1}{\delta^2}. \tag{40}$$

Substituting into Eq. (33) yields the asymptotic behavior of the force coefficient C_f for the normal electric field,

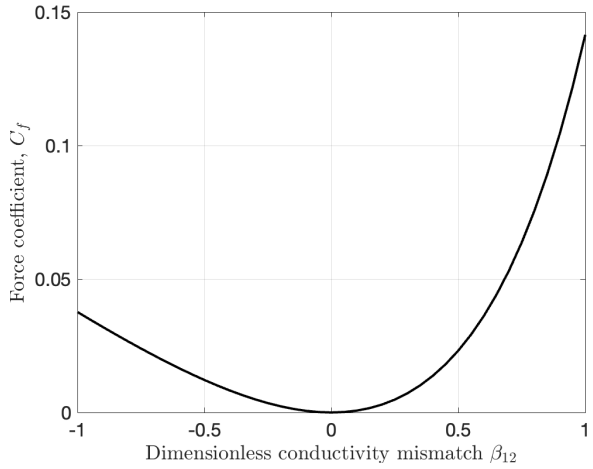
$$C_f \sim -\frac{6\pi\beta_{12}^2}{(3-\beta_{12})^2}\frac{1}{\delta^4}. \tag{41}$$

In the case of a tangential applied electric field, the numerical solution for A_n suggests a one-term approximation. The asymptotic form of Eq. (21) is

$$\sqrt{2}(\beta_{12}-3)\delta^{5/2}A_1 \sim E_0\beta_{12}R\delta^{1/2}, \tag{42}$$

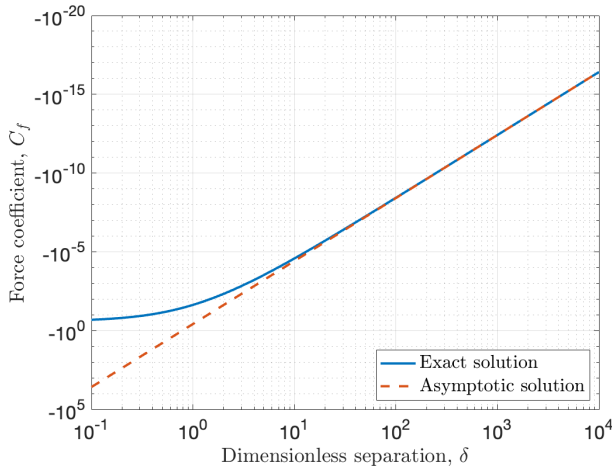


(a)Normal electric field

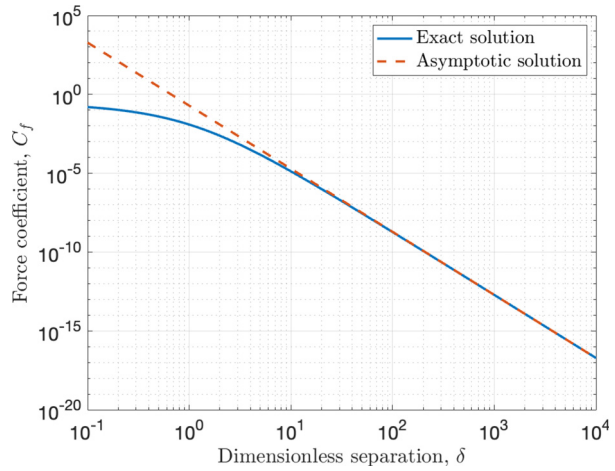


(b)Tangential electric field

FIG. 5. Force coefficient as a function of β_{12} (separation $s/R = 1$). The dashed line corresponds to the perfectly conducting solution ($\beta_{12} = 1$).



(a)Normal electric field



(b)Tangential electric field

FIG. 6. Force coefficient asymptotic ($\beta_{12} = -0.5$).

from which we find the asymptotic behavior of A_1,

$$A_1 \sim \frac{E_0 \beta_{12} R}{\sqrt{2}(\beta_{12} - 3)} \frac{1}{\delta^2}. \tag{43}$$

Substituting into Eq. (34) yields the asymptotic behavior of the force coefficient C_f for the tangential electric field

$$C_f \sim \frac{3\pi \beta_{12}^2}{(\beta_{12} - 3)^2} \frac{1}{\delta^4}. \tag{44}$$

To a leading order, the effect of the wall is equivalent to placing a mirror image dipole. The $1/\delta^4$ dependence is consistent with the dipole-dipole interaction [30].

The exact and asymptotic solutions are compared in Fig. 6. The agreement is excellent when $\delta > 10$. Equation (41) is also valid for the special case $\beta_{12} = 1$.

VI. CONCLUSIONS

In this paper, the interaction of a spherical particle and a planar wall in the presence of a uniform electric field is studied in the framework of the leaky dielectric model. The bulk media are assumed to be charge free; charge brought by conduction is only present at the surfaces separating media with different conductivities and permittivities. Analytical solutions of the Laplace equation for the electric potential are derived using separation of variables in bispherical coordinates. The force on the particle is calculated using the Maxwell stress tensor. We find that in the case of normal electric field, which corresponds to a particle near an electrode, the force is always attractive but at a given separation it varies nonmonotonically with particle-suspending medium conductivity ratio; the force on a particle that is more conducting than the suspending medium is much larger compared to the force on a particle less conducing than the suspending medium. In the case of tangential electric field, which corresponds to a particle near an insulating boundary, the force is always repulsive. In both cases, the force decreases with increasing separation between the particle and the wall and its asymptotic behavior at large separation is given by the force between the particle dipole and its mirror image. The results provide a comprehensive understanding of the effect of particle conductivity on the wall-induced electrostatic force.

ACKNOWLEDGMENTS

This research was supported by the National Science Foundation award nos. DMS-2108502 and DMR-2004926.

APPENDIX A: COMPONENTS OF THE ELECTRIC FIELD STRENGTH

In the case of the normal electric field, the components of the electric field strength are given by

$$E_{1,\eta} = \frac{E_0}{h}(1-\cosh\eta\cos\xi) - \frac{\sqrt{h}\sinh\eta}{2a}\sum_{n=0}^{\infty}A_n[e^{2\lambda_n\eta_s}-1]e^{-\lambda_n\eta}P_n(\cos\xi) + \frac{h^{3/2}}{a}\sum_{n=0}^{\infty}A_n\lambda_n[e^{2\lambda_n\eta_s}-1]e^{-\lambda_n\eta}P_n(\cos\xi) \tag{A1}$$

$$E_{1,\xi} = -\frac{E_0}{h}\sinh\eta\sin\xi - \frac{\sqrt{h}\sin\xi}{2a}\sum_{n=0}^{\infty}A_n[e^{2\lambda_n\eta_s}-1]e^{-\lambda_n\eta}P_n(\cos\xi) - \frac{h^{3/2}}{a}\sum_{n=0}^{\infty}A_n[e^{2\lambda_n\eta_s}-1]e^{-\lambda_n\eta}\frac{d}{d\xi}P_n(\cos\xi) \tag{A2}$$

$$E_{2,\eta} = \frac{E_0}{h}(1-\cosh\eta\cos\xi) - \frac{\sqrt{h}\sinh\eta}{a}\sum_{n=0}^{\infty}A_n\sinh(\lambda_n\eta)P_n(\cos\xi) - \frac{2h^{3/2}}{a}\sum_{n=0}^{\infty}A_n\lambda_n\cosh(\lambda_n\eta)P_n(\cos\xi) \tag{A3}$$

$$E_{2,\xi} = -\frac{E_0}{h}\sinh\eta\sin\xi - \frac{\sqrt{h}\sin\xi}{a}\sum_{n=0}^{\infty}A_n\sinh(\lambda_n\eta)P_n(\cos\xi) - \frac{2h^{3/2}}{a}\sum_{n=0}^{\infty}A_n\sinh(\lambda_n\eta)\frac{d}{d\xi}P_n(\cos\xi) \tag{A4}$$

$E_{1,\phi}$ and $E_{2,\phi}$ are both zero due to the symmetry.
In the case of the tangential electric field, the components of the electric field strength are given by

$$E_{i,\eta} = \hat{E}_{i,\eta}\sin\phi,\ E_{i,\xi} = \hat{E}_{i,\xi}\sin\phi,\ E_{i,\phi} = \hat{E}_{i,\phi}\cos\phi,$$

$$\hat{E}_{1,\eta} = -\frac{E_0}{h}\sinh\eta\sin\xi - \frac{\sqrt{h}\sinh\eta}{2a}\sum_{n=1}^{\infty}A_n[1+e^{2\lambda_n\eta_s}]e^{-\lambda_n\eta}P_n^1(\cos\xi) + \frac{h^{3/2}}{a}\sum_{n=1}^{\infty}A_n\lambda_n[1+e^{2\lambda_n\eta_s}]e^{-\lambda_n\eta}P_n^1(\cos\xi), \tag{A5}$$

$$\hat{E}_{1,\xi} = \frac{E_0}{h}(\cosh\eta\cos\xi-1) - \frac{\sqrt{h}\sin\xi}{2a}\sum_{n=1}^{\infty}A_n[1+e^{2\lambda_n\eta_s}]e^{-\lambda_n\eta}P_n^1(\cos\xi) - \frac{h^{3/2}}{a}\sum_{n=1}^{\infty}A_n[1+e^{2\lambda_n\eta_s}]e^{-\lambda_n\eta}\frac{d}{d\xi}P_n^1(\cos\xi), \tag{A6}$$

$$\hat{E}_{1,\phi} = E_0 - \frac{h^{3/2}}{a\sin\xi}\sum_{n=1}^{\infty}A_n[1+e^{2\lambda_n\eta_s}]e^{-\lambda_n\eta}P_n^1(\cos\xi), \tag{A7}$$

$$\hat{E}_{2,\eta} = -\frac{E_0}{h}\sinh\eta\sin\xi - \frac{\sqrt{h}\sinh\eta}{a}\sum_{n=1}^{\infty}A_n\cosh(\lambda_n\eta)P_n^1(\cos\xi) - \frac{2h^{3/2}}{a}\sum_{n=1}^{\infty}A_n\lambda_n\sinh(\lambda_n\eta)P_n^1(\cos\xi), \tag{A8}$$

$$\hat{E}_{2,\xi} = \frac{E_0}{h}(\cosh\eta\cos\xi-1) - \frac{\sqrt{h}\sin\xi}{a}\sum_{n=1}^{\infty}A_n\cosh(\lambda_n\eta)P_n^1(\cos\xi) - \frac{2h^{3/2}}{a}\sum_{n=1}^{\infty}A_n\cosh(\lambda_n\eta)\frac{d}{d\xi}P_n^1(\cos\xi), \tag{A9}$$

$$\hat{E}_{2,\phi} = E_0 - \frac{2h^{3/2}}{a\sin\xi}\sum_{n=1}^{\infty}A_n\cosh(\lambda_n\eta)P_n^1(\cos\xi). \tag{A10}$$

APPENDIX B: RECURRENCE FORMULA FOR A_n

Here, we outline the derivation of the recurrence formula which is used to determine coefficients A_n in the expression of the electric potential.

1. Normal electric field

For the normal applied electric field, the disturbance fields $\hat{\Phi}_1$ and $\hat{\Phi}_2$ can be written as

$$\hat{\Phi}_1 = \sqrt{h}\tilde{\Phi}_1,\ \hat{\Phi}_2 = \sqrt{h}\tilde{\Phi}_2, \tag{B1}$$

where

$$\tilde{\Phi}_1 = \sum_{n=0}^{\infty}A_n[e^{2\lambda_n\eta_s}-1]e^{-\lambda_n\eta}P_n(\cos\xi), \tag{B2}$$

$$\tilde{\Phi}_2 = \sum_{n=0}^{\infty}2A_n\sinh(\lambda_n\eta)P_n(\cos\xi). \tag{B3}$$

Substituting Eq. (B1) into the boundary condition for the continuity of the normal electric current,

$$\sigma_1\frac{\partial\hat{\Phi}_1}{\partial\eta} - \sigma_2\frac{\partial\hat{\Phi}_2}{\partial\eta} = E_0(\sigma_1-\sigma_2)\frac{\partial z}{\partial\eta},$$

leads to

$$2(\cosh\eta_s - u)\left(\frac{\sigma_1}{\sigma_1+\sigma_2}\frac{\partial\tilde{\Phi}_1}{\partial\eta} - \frac{\sigma_2}{\sigma_1+\sigma_2}\frac{\partial\tilde{\Phi}_2}{\partial\eta}\right) + \sinh\eta_s\beta_{12}\tilde{\Phi}_2 = 2E_0\beta_{12}(\cosh\eta_s - u)^{1/2}\frac{\partial z}{\partial\eta}, \tag{B4}$$

where $u = \cos\xi$. Plugging Eqs. (B2) and (B3) into the left-hand side (L.H.S.) of Eq. (B4), one gets

$$\begin{aligned}\text{L.H.S.} = &\sum_{n=0}^{\infty}[(\beta_{12}\sinh\eta_s - 2\lambda_n\cosh\eta_s)\exp(\lambda_n\eta_s) - \beta_{12}(\sinh\eta_s - 2\lambda_n\cosh\eta_s)\exp(-\lambda_n\eta_s)]A_nP_n(u)\\ &+\sum_{n=0}^{\infty}A_n[\exp(\lambda_n\eta_s) - \beta_{12}\exp(-\lambda_n\eta_s)]2\lambda_nuP_n(u).\end{aligned}$$

Note that $\lambda_n \equiv n + 1/2$ and $\beta_{12} = (\sigma_1 - \sigma_2)/(\sigma_1 + \sigma_2)$. With the help of the recurrence formula for Legendre polynomials,

$$(2n+1)uP_n(u) = (n+1)P_{n+1}(u) + nP_{n-1}(u), \tag{B5}$$

the second sum of L.H.S. could be written as

$$\begin{aligned}&\sum_{n=1}^{\infty}n[\exp(\lambda_{n-1}\eta_s) - \beta_{12}\exp(-\lambda_{n-1}\eta_s)]A_{n-1}P_n(u)\\ &+\sum_{n=0}^{\infty}(n+1)[\exp(\lambda_{n+1}\eta_s) - \beta_{12}\exp(-\lambda_{n+1}\eta_s)]A_{n+1}P_n(u).\end{aligned}$$

Putting these terms together, we have

$$\begin{aligned}\text{L.H.S.} = &\sum_{n=1}^{\infty}L_{n,1}A_{n-1}P_n(u) + \sum_{n=0}^{\infty}L_{n,2}A_nP_n(u)\\ &+\sum_{n=0}^{\infty}L_{n,3}A_{n+1}P_n(u).\end{aligned} \tag{B6}$$

Coefficients $L_{n,1}$, $L_{n,2}$, and $L_{n,3}$ are listed below:

$$\begin{aligned}L_{n,1} &= n(e^{\lambda_{n-1}\eta_s} - \beta_{12}e^{-\lambda_{n-1}\eta_s}),\\ L_{n,2} &= (\beta_{12}\sinh\eta_s - 2\lambda_n\cosh\eta_s)e^{\lambda_n\eta_s}\\ &\quad - \beta_{12}(\sinh\eta_s - 2\lambda_n\cosh\eta_s)e^{-\lambda_n\eta_s},\\ L_{n,3} &= (n+1)(e^{\lambda_{n+1}\eta_s} - \beta_{12}e^{-\lambda_{n+1}\eta_s}).\end{aligned} \tag{B7}$$

The next step is to expand the right-hand side of Eq. (B4) in terms of Legendre polynomials. Note that

$$z = \frac{a}{h}\sinh\eta.$$

Substituting it into the right-hand side (R.H.S.) of Eq. (B4), we have

$$\text{R.H.S.} = 2E_0a\beta_{12}\frac{1 - u\cosh\eta_s}{h_s^{3/2}}, \tag{B8}$$

where $h_s = \cosh\eta_s - u$. The generating function of Legendre polynomials is

$$\frac{1}{(1-2ut+t^2)^{1/2}} = \sum_{n=0}^{\infty}t^nP_n(u). \tag{B9}$$

Letting $t = \exp(-\eta_s)$ yields

$$\frac{1}{h_s^{1/2}} = \sqrt{2}\sum_{n=0}^{\infty}\exp(-\lambda_n\eta_s)P_n(u). \tag{B10}$$

Calculating the derivative with respect to η_s, we get

$$\frac{1}{h_s^{3/2}} = \frac{2\sqrt{2}}{\sinh\eta_s}\sum_{n=0}^{\infty}\lambda_n\exp(-\lambda_n\eta_s)P_n(u). \tag{B11}$$

Multiplying Eq. (B11) by u and using the recurrence formula Eq. (B5), we obtain

$$\frac{u}{h_s^{3/2}} = \frac{\sqrt{2}}{\sinh\eta_s}\sum_{n=0}^{\infty}[ne^{-\lambda_{n-1}\eta_s} + (n+1)e^{-\lambda_{n+1}\eta_s}]P_n(u). \tag{B12}$$

Using Eqs. (B11) and (B12), we could expand Eq. (B8) as

$$\text{R.H.S.} = \sum_{n=0}^{\infty}G_nP_n(u), \tag{B13}$$

where G_n is

$$\begin{aligned}G_n = -2\sqrt{2}E_0a\beta_{12}\Bigg\{&\frac{\cosh\eta_s}{\sinh\eta_s}[ne^{-\lambda_{n-1}\eta_s} + (n+1)e^{-\lambda_{n+1}\eta_s}]\\ &- \frac{2}{\sinh\eta_s}\lambda_ne^{-\lambda_n\eta_s}\Bigg\}.\end{aligned} \tag{B14}$$

Equating Eqs. (B6) and (B13) leads to the recurrence formula we need.

For $n \gg 1$, Eq. (21) has the following asymptotic form:

$$\begin{aligned}&-2nt^2A_{n-1} + [(2n+1)(1+t^2) - \beta_{12}(1-t^2)]\\ &\times A_n - 2(n+1)A_{n+1} \sim 4\sqrt{2}\frac{E_0a\beta_{12}}{1-t^2}t^{2n+1}\\ &\times [n - (2n+1)t^2 + (n+1)t^4],\end{aligned}$$

where $t = \exp(-\eta_s)$. It gives the asymptotic behavior of coefficients A_n for large $n \gg 1$,

$$A_n \sim 4\sqrt{2}\frac{E_0a\beta_{12}}{1-t^2}t^{2n+1}\left[\frac{n}{3-\beta_{12}}(1-t^2) - \frac{1}{1-\beta_{12}}t^2\right]. \tag{B15}$$

2. Tangential electric field

For the tangential electric field, the recurrence formula could be obtained following similar steps. Results are listed below:

$$L_{n,1} = \frac{n-1}{2}\left(e^{\lambda_{n-1}\eta_s} + \beta_{12}e^{-\lambda_{n-1}\eta_s}\right),$$

$$L_{n,2} = \left(\frac{\beta_{12}}{2}\sinh\eta_s - \lambda_n\cosh\eta_s\right)e^{\lambda_n\eta_s} + \beta_{12}\left(\frac{1}{2}\sinh\eta_s - \lambda_n\cosh\eta_s\right)e^{-\lambda_n\eta_s},$$

$$L_{n,3} = \frac{n+2}{2}\left(e^{\lambda_{n+1}\eta_s} + \beta_{12}e^{-\lambda_{n+1}\eta_s}\right), \tag{B16}$$

$$G_n = 2\sqrt{2}E_0 a\beta_{12}\sinh\eta_s e^{-\lambda_n\eta_s}, \tag{B17}$$

where $n \geqslant 1$. For $n \gg 1$,

$$A_n \sim \frac{4\sqrt{2}E_0 a\beta_{12}}{\beta_{12}-3}t^{2n+1}. \tag{B18}$$

APPENDIX C: ELECTRIC FORCE

Consider the case of a normal applied field $\mathbf{E}_0 = E_0\hat{\mathbf{z}}$. At $\eta = 0$, the disturbance field is given by

$$E^D_{2,\eta} = -\frac{h_0^{3/2}}{a}\sum_{n=0}^{\infty}2A_n\lambda_n P_n(\cos\xi), \tag{C1}$$

$$E^D_{2,\xi} = E^D_{2,\phi} = 0, \tag{C2}$$

where $h_0 = 1-\cos\xi$. Due to the symmetry of this problem, the force is in z direction,

$$\begin{aligned} F_z &= -\int_{S_B}\mathbf{e}_z\cdot\mathbf{T}^D_2\cdot\mathbf{n}dS \\ &= -\varepsilon_2\int_{S_B}E_0E^D_{2,\eta} + \frac{1}{2}\left(E^D_{2,\eta}\right)^2 dS \\ &= -\varepsilon_2\pi a^2\int_0^{\pi}\frac{\sin\xi}{h_0^2}\left[2E_0E^D_{2,\eta} + \left(E^D_{2,\eta}\right)^2\right]d\xi \\ &= \varepsilon_2\pi a^2\left(I_1^N - I_2^N\right), \end{aligned} \tag{C3}$$

where I_1^N and I_2^N are two integrals

$$I_1^N = -2\int_0^{\pi}\frac{\sin\xi}{h_0^2}E_0E^D_{2,\eta}d\xi, \tag{C4}$$

$$I_2^N = \int_0^{\pi}\frac{\sin\xi}{h_0^2}\left(E^D_{2,\eta}\right)^2 d\xi. \tag{C5}$$

Substituting Eq. (C1) into Eq. (C4), we have

$$\begin{aligned} I_1^N &= \int_{-1}^{1}\frac{2E_0}{a\sqrt{1-u}}\sum_{n=0}^{\infty}2A_n\lambda_n P_n(u)du \\ &= \frac{4E_0}{a}\sum_{n=0}^{\infty}A_n\lambda_n\int_{-1}^{1}\frac{P_n(u)}{\sqrt{1-u}}du. \end{aligned}$$

From the generating function Eq. (B9), we have

$$\int_{-1}^{1}\frac{1}{(1-2ut+t^2)^{1/2}}P_n du = \sum_{m=0}^{\infty}t^m\int_{-1}^{1}P_m(u)P_n(u)du = \frac{t^n}{\lambda_n}.$$

Here, the orthogonality of Legendre polynomials is used,

$$\int_{-1}^{1}P_m(u)P_n(u)du = \frac{2}{2n+1}\delta_{n,m} = \frac{\delta_{n,m}}{\lambda_n}. \tag{C6}$$

Letting $t = 1$ yields

$$\int_{-1}^{1}\frac{P_n(u)}{\sqrt{1-u}}du = \frac{\sqrt{2}}{\lambda_n}.$$

Substituting it into I_1^N, we get

$$I_1^N = 4\sqrt{2}\frac{E_0}{a}\sum_{n=0}^{\infty}A_n. \tag{C7}$$

For I_2^N, substituting Eq. (C1) into Eq. (C5) yields

$$\begin{aligned} I_2^N &= \int_{-1}^{1}\frac{1-u}{a^2}\left(\sum_{n=0}^{\infty}2A_n\lambda_n P_n(u)\right)^2 du \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{4}{a^2}A_mA_n\lambda_m\lambda_n\int_{-1}^{1}(1-u)P_n(u)P_m(u)du. \end{aligned}$$

Using the recurrence formula Eq. (B5) and orthogonality Eq. (C6), the integral in I_2^N could be calculated explicitly,

$$\begin{aligned} &\int_{-1}^{1}(1-u)P_n(u)P_m(u)du \\ &= \int_{-1}^{1}P_n(u)P_m(u)du - \frac{n+1}{2n+1}\int_{-1}^{1}P_{n+1}(u)P_m(u)du \\ &\quad - \frac{n}{2n+1}\int_{-1}^{1}P_{n-1}(u)P_m(u)du \\ &= \frac{\delta_{n,m}}{\lambda_n} - \frac{n+1}{2n+1}\frac{\delta_{n,m+1}}{\lambda_m} - \frac{n}{2n+1}\frac{\delta_{n,m-1}}{\lambda_m}. \end{aligned}$$

Plugging into I_2^N, we have

$$I_2^N = \frac{4}{a^2}\sum_{n=0}^{\infty}\left[\lambda_n A_n^2 - (n+1)A_nA_{n+1}\right]. \tag{C8}$$

From Eq. (C3), we could get the solution for the electric force. A similar calculation can be done for a tangential electric field.

It is interesting to note that because $\mathbf{E}$ is divergence free, plus the boundary condition (2), the net charge on the particle is

$$Q_{net} = \oint_{S_p}(\varepsilon_1\mathbf{E}_1 - \varepsilon_2\mathbf{E}_2)\cdot\mathbf{e}_\eta dS = 0.$$

Also since $\oint_{S_p}\mathbf{E}_2\cdot\mathbf{e}_\eta dS = \int_{S_B}\mathbf{E}^D_2\cdot\mathbf{e}_\eta dS = 0$, we have, from integrating Eq. (C1), that $\sum_{n=0}^{\infty}A_n = 0$. Hence, $I_1^N = 0$.

[1] M. Motosuke, Electric Field-Induced Arrangement of Colloidal Materials in Microfluidic Devices, in *Stimuli-Responsive Interfaces* (Springer, Berlin, 2017), pp. 297–313.

[2] D. C. Prieve, P. J. Sides, and C. L. Wirth, 2-d assembly of colloidal particles on a planar electrode, Curr. Opin. Colloid Interface Sci. **15**, 160 (2010).

[3] O. D. Velev and K. H. Bhatt, On-chip micromanipulation and assembly of colloidal particles by electric fields, Soft Matter **2**, 738 (2006).
[4] B. Bharti and O. D. Velev, Assembly of reconfigurable colloidal structures by multidirectional field-induced interactions, Langmuir **31**, 7897 (2015).
[5] J. Yan, M. Han, J. Zhang, C. Xu, E. Luijten, and S. Granick, Reconfiguring active particles by electrostatic imbalance, Nat. Mater. **15**, 1095 (2016).
[6] K. Han, C. Wyatt Shields IV, and O. D. Velev, Engineering of self-propelling microbots and microdevices powered by magnetic and electric fields, Adv. Funct. Mater. **28**, 1705953 (2018).
[7] M. Driscoll and B. Delmotte, Leveraging collective effects in externally driven colloidal suspensions: Experiments and simulations, Curr. Opin. Colloid Interface Sci. **40**, 42 (2019).
[8] S. Gangwal, O. J. Cayre, M. Z. Bazant, and O. D. Velev, Induced-Charge Electrophoresis of Metallodielectric Particles, Phys. Rev. Lett. **100**, 058302 (2008).
[9] F. Ma, X. Yang, H. Zhao, and N. Wu, Inducing Propulsion of Colloidal Dimers by Breaking the Symmetry in Electrohydrodynamic Flow, Phys. Rev. Lett. **115**, 208302 (2015).
[10] D. Nishiguchi and M. Sano, Mesoscopic turbulence and local order in janus particles self-propelling under an ac electric field, Phys. Rev. E **92**, 052309 (2015).
[11] A. Bricard, J.-B. Caussin, N. Desreumaux, O. Dauchot, and D. Bartolo, Emergence of macroscopic directed motion in populations of motile colloids, Nature (London) **503**, 95 (2013).
[12] A. Bricard, J.-B. Caussin, D. Das, C. Savoie, Vijayakumar Chikkadi, K. Shitara, O. Chepizhko, F. Peruani, D. Saintillan, and D. Bartolo, Emergent vortices in populations of colloidal rollers, Nat. Commun. **6**, 7470 (2015).
[13] H. Karani, G. E. Pradillo, and P. M. Vlahovska, Tuning the Random Walk of Active Colloids: From Individual Run-and-Tumble to Dynamic Clustering, Phys. Rev. Lett. **123**, 208002 (2019).
[14] G. E. Pradillo, H. Karani, and P. M. Vlahovska, Quincke rotor dynamics in confinement: rolling and hovering, Soft Matter **15**, 6564 (2019).
[15] B. Zhang, H. Karani, P. M. Vlahovska, and A. Snezhko, Persistence length regulates emergent dynamics in active roller ensembles, Soft Matter **17**, 4818 (2021).
[16] A. T. Pérez, Charge and force on a conducting sphere between two parallel electrodes, J. Electrost. **56**, 199 (2002).
[17] A. M. Drews, M. Kowalik, and K. J. M. Bishop, Charge and force on a conductive sphere between two parallel electrodes: A Stokesian dynamics approach, J. Appl. Phys. **116**, 074903 (2014).
[18] M. H. Davis, Two charged spherical conductors in a uniform electric field: Forces and field strength, Q. J. Mech. Appl. Math. **17**, 499 (1964).
[19] W. Lin, A critical study of a dielectric sphere near a grounded plane, J. Electromagn. Waves. Appl. **8**, 195 (1994).
[20] C. Xiaoping, The electrostatic problem of a dielectric sphere near a plane, J. Electrost. **19**, 201 (1987).
[21] K. D. Danov, P. A. Kralchevsky, K. P. Ananthapadmanabhan, and A. Lips, Particle-interface interaction across a nonpolar medium in relation to the production of particle-stabilized emulsions, Langmuir **22**, 106 (2006).
[22] R. D. Stoy, Solution procedure for the laplace equation in bispherical coordinates for two spheres in a uniform external field: Parallel orientation, J. Appl. Phys. **65**, 2611 (1989).
[23] M. Washizu and T. B. Jones, Dielectrophoretic interaction of two spherical particles calculated by equivalent multipole-moment method, IEEE Transactions on Industry Applications **32**, 233 (1996).
[24] G. I. Taylor, Studies in electrohydrodynamics. I. The circulation produced in a drop by an electric field, Proc. R. Soc. London A **291**, 159 (1966).
[25] Y. Hu, P. M. Vlahovska, and M. J. Miksis, Dielectric spherical particle on an interface in an applied electric field, SIAM J. Appl. Math. **79**, 850 (2019).
[26] R. D. Stoy, Solution procedure for the laplace equation in bispherical coordinates for two spheres in a uniform external field: Perpendicular orientation, J. Appl. Phys. **66**, 5093 (1989).
[27] J. R. Melcher and G. I. Taylor, Electrohydrodynamics: a review of the role of interfacial shear stresses, Annu. Rev. Fluid Mech. **1**, 111 (1969).
[28] Zhanwen. Wang, Ph.D. thesis, Northwestern University, in preparation.
[29] E. W. K. Young and D. Li, Dielectrophoretic force on a sphere near a planar boundary, Langmuir **21**, 12037 (2005).
[30] E. Yariv, "force-free" electrophoresis? Phys. Fluids **18**, 031702 (2006).